TAUBERIAN TYPE THEOREM FOR OPERATORS
WITH INTERPOLATION SPECTRUM FOR HÖLDER CLASSES

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Abstract. We consider an invertible operator $T$ on a Banach space $X$ whose spectrum is an interpolating set for Hölder classes. We show that if $\|T^n\| = O(n^p)$, $p \geq 1$, $\|T^{-n}\| = O(w_n)$ with $n^q = o(w_n)$ $\forall q \in \mathbb{N}$ and $\sum_n n^1/(n^1 - (\log n)^{1+q}) = +\infty$, then $\|T^{-n}\| = O(n^{p/s})$ for all $s > \frac{1}{2}$, assuming that $(w_n)_{n \geq 1}$ satisfies suitable regularity conditions. When $X$ is a Hilbert space and $p = 0$ (i.e. $T$ is a contraction), we show that under the same assumptions, $T$ is unitary and this is sharp.

1. Introduction

In this note, we are interested in invertible operators $T$ on a Banach space $X$ with polynomial growth and whose spectrum, denoted by $\sigma(T)$, is a $K$-set. We study growth of the norms of the negative iterates of $T$. A closed set $E$ of the unit circle $T$ is said to be a $K$-set if there exists $c_E > 0$ such that for all arcs $L \subset T$,

$$\sup_{\zeta \in E} d(\zeta, E) \geq c_E |L|,$$

where $|L|$ denotes the length of $L$. Dynkin [4] showed that $K$-sets are the interpolating sets for Hölder classes: if we denote by $A(D)$ the disc algebra and set, for $s \in (0, 1),

$$A^s = \{ f \in C(T) : \|f\|_s = \|f\|_{C(T)} + \sup_{h \not= 0, t \in \mathbb{R}} \frac{|f(e^{i(t+h)}) - f(e^{it})|}{|h|^s} < +\infty \}$$

and $A^e = A^e \cap A(D)$, then $E$ is $K$-set iff $A^e|E = A^e|E$.

We also need the following definition: let $w = (w_n)_{n \geq 1}$ be a sequence of positive real numbers; we say that $w$ satisfies condition $(R)$, and we write $w \in (R)$, if it satisfies:

1. $(\log w_n)_{n \geq 1}$ is non-decreasing, and $(w_{n+1}/w_n)_{n \geq 1}$ is non-increasing;
2. $n^q = o(w_n)$ for all $q \geq 0$;
3. the sequence $(\log w_n/n^{\beta})_{n \geq n_0}$ is non-increasing for some $\beta < 1/2$.

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Theorem 1.1. Let \( w \in (R) \) and let \( T \) be an operator on a Banach space \( X \) such that \( \sigma(T) \) is a \( K \)-set, \( \|T^n\| = O(n^p) \) for some \( p \geq 1 \) and \( \|T^{-n}\| = O(w_n) \). If for all \( \alpha \in (0,1) \),

\[
\sum_{n \geq 1} \frac{1}{n^{1-\alpha}(\log w_n)^{1+\alpha}} = +\infty,
\]

then, for all \( \varepsilon > 0 \),

\[
\|T^{-n}\| = O(n^{p+\frac{1}{2}+\varepsilon}), \quad n \to +\infty.
\]

In [5], Theorem 1.1 was obtained when norms of the negative powers of \( T \) satisfy the condition \( \sum_{n \geq 1} \frac{1}{(n \log w_n)^{1+\alpha}} = +\infty \) instead of (1) and the spectrum was an arbitrary \( K \)-set. In [1], Theorem 1.1 was obtained when \( \sigma(T) = E_\zeta \) is a perfect symmetric set with constant of ratio \( \xi \in (0,1/2) \) (special classes of \( K \)-sets) and under the condition \( \|T^{-n}\| = O(e^{\alpha n}) \) with \( \beta < \frac{1}{\log 2} |\log 2\zeta|/|\log 2\zeta^2| \). Theorem 1.1 extends results of [1, 5]. For contractions on a Hilbert space we improve Theorem 1.1 to obtain the following result.

Theorem 1.2. Let \( w \in (R) \) and let \( T \) be an invertible contraction on a Hilbert space \( X \), such that \( \sigma(T) \) is a \( K \)-set and \( \|T^{-n}\| = O(w_n) \). If condition (1) is satisfied for all \( \alpha \in (0,1) \), then \( T \) is unitary.

On the other hand, if there exists \( \alpha \in (0,1) \) such that

\[
\sum_{n \geq 1} \frac{1}{n^{1-\alpha}(\log w_n)^{1+\alpha}} < +\infty,
\]

then there exists an invertible contraction on a Hilbert space \( T \) such that \( \sigma(T) \) is a \( K \)-set, \( \|T^{-n}\| = O(w_n) \) and \( \|T^{-n}\| \to +\infty \).

Theorem 1.2 is not valid for contractions on general Banach spaces. Indeed, Esterle constructed in [7] a contraction \( T \) on a Banach space such that \( \sigma(T) \) is a \( K \)-set (a perfect symmetric set with constant of ratio \( \zeta \) such that \( 1/\zeta \) is not a Pisot number) and \( \|T^{-n}\| \to +\infty \). Observe also that Theorem 1.2 is not valid when \( \sigma(T) \) is a null measure set (see [10]). Similar results of Tauberian type were obtained in [1, 2, 5, 6, 7, 8, 10, 14].

2. Proofs

2.1. Hausdorff measure of \( K \)-sets. A non-decreasing continuous function on \([0,+\infty)\) such that \( h(0) = 0 \) is said to be a Hausdorff function, and the \( h \)-measure of Hausdorff of a closed set \( E \subset \mathbb{T} \) is defined by

\[
H_h(E) = \lim \inf_{t \to 0} \sum_i h(|\Delta_i|),
\]

where the infimum is taken over all the coverings \( (\Delta_i) \) of \( E \) by arcs of \( \mathbb{T} \) with length \( |\Delta_i| \leq t \). Dynkin showed in [4] that if \( E \) is a \( K \)-set, then there exists \( \alpha_E > 0 \) such that

\[
\int_0^1 \frac{|E_t|}{t^{1+\alpha_E}} dt < +\infty,
\]

where

\[
E_t = \{ \zeta \in \mathbb{T} : d(\zeta, E) \leq t \}, \quad t > 0,
\]
$|E_t|$ denotes the length of $E_t$ and $\alpha_E \geq \log(1/(1-c_E))/\log(2/(1-c_E))$. Note that a $K$-set is a Beurling–Carleson set since
\[ \int_0^1 \frac{|E_t|}{t} \, dt < +\infty. \]

Shapiro gave in [12] a complete characterisation of Beurling–Carleson sets of null $h$-Hausdorff measure: he showed that $H_h(E) = 0$ for all Beurling–Carleson sets $E$ if and only if $\int_0^1 dt/h(t) = +\infty$. Let $(\zeta_n)_{n \geq 1}$ be a sequence of real numbers such that $0 < \zeta_n < 1/2$. We set
\[ E_{(\zeta_n)} = \left\{ \exp \left[ 2\pi \sum_{n \geq 1} \varepsilon_n \zeta_1 \cdots \zeta_n (1 - \zeta_n) \right], \varepsilon_n = 0 \text{ or } 1 \right\}. \]

When $\zeta_n = \zeta$ for all $n$, $E_\zeta$ is the perfect symmetric set of constant ratio $\zeta$ (as $E_{1/3}$ is the usual Cantor triadic) and $E_\zeta$ is a $K$-set of Hausdorff dimension $d_E = \log 2/\log 2^2$ (see [9]). When $\limsup_{n \to \infty} \zeta_n < 1/2$, Esterle showed in [7] (Proposition 2.5) that $E_{(\zeta_n)}$ is still also a $K$-set. The following lemma gives a complete description of a $K$-set of null $h$-Hausdorff measure.

Lemma 2.1. Let $h$ be a Hausdorff function such that $h(t)/t$ is strictly decreasing. Then the following two conditions are equivalent.

(i) For all $K$-sets $E$, $H_h(E) = 0$.

(ii) For all $\alpha \in (0, 1)$,
\[ \int_0^1 \frac{dt}{t^\alpha h(t)} = +\infty. \]

Proof. (ii) $\Rightarrow$ (i). Suppose that there exists a $K$-set $E$ such that $H_h(E) = c > 0$. For all $t > 0$, $E_t$ is a disjoint union of arcs $\Delta_i$ with $|\Delta_i| \geq 2t$: $E_t = \bigcup_{1 \leq i \leq N} \Delta_i$, and so
\[ c \leq \sum_{1 \leq i \leq N} h(|\Delta_i|) \leq \sum_{1 \leq i \leq N} \frac{h(|\Delta_i|)}{|\Delta_i|} |\Delta_i| \leq \frac{h(2t)}{2t} |E_t|. \]

Since $E$ is a $K$-set, there exists $\alpha \in (0, 1)$ such that $\int_0^1 |E_t|/t^{1+\alpha} \, dt < +\infty$, and we deduce from (3) that
\[ \int_0^1 \frac{dt}{t^\alpha h(t)} < +\infty. \]

(i) $\Rightarrow$ (ii). Suppose that there exists $\alpha \in (0, 1)$ such that
\[ \int_0^1 \frac{dt}{t^\alpha h(t)} < +\infty. \]

We will construct a $K$-set $E$ satisfying $H_h(E) > 0$. In order to do that, we define $(\lambda_n)_{n \geq 0}$ by $\lambda_0 = 1$ and $h(\lambda_n) = 2^{-n}$, $n \geq 1$. Let $E = E_{(\lambda_n)}$ be the perfect symmetric set associated with $(\zeta_n)_{n \geq 0} := (\lambda_n/\lambda_{n-1})_{n \geq 1}$. The set $E$ is as described in [9], $E = \bigcap_{n \geq 0} E_n$, where $E_n$ is a disjoint union of $2^n$ closed arcs $E_{i,n}$ with
\[ |E_{i,n}| = 2\pi (\zeta_1 \cdots \zeta_n) = 2\pi \lambda_n, 1 \leq i \leq 2^n. \] For all \( N \geq 0 \),
\[
+ \infty > (1 - \alpha) \int_0^1 \frac{dt}{t^\alpha h(t)} = (1 - \alpha) \int_0^{\lambda_{N+1}} \frac{dt}{t^\alpha h(t)} + (1 - \alpha) \sum_{0 \leq n \leq N} \int_{\lambda_{n+1}}^{\lambda_n} \frac{dt}{t^\alpha h(t)}
\geq 2^{N+1} \lambda_{N+1}^{1-\alpha} + \sum_{0 \leq n \leq N} 2^n (\lambda_n^{1-\alpha} - \lambda_{n+1}^{1-\alpha})
\geq 2^{N+1} \lambda_{N+1}^{1-\alpha} + \sum_{1 \leq n \leq N} 2^{n-1} \lambda_n^{1-\alpha} + 1 - 2^N \lambda_{N+1}^{1-\alpha} \geq \sum_{1 \leq n \leq N} 2^{n-1} \lambda_n^{1-\alpha}.
\]
Hence \( \sum_{n \geq 1} 2^{n-1} \lambda_n^{1-\alpha} < +\infty \) and so
\[
\limsup_{n \to \infty} \zeta_n = \limsup_{n \to \infty} \frac{\lambda_n}{\lambda_{n-1}} \leq \frac{1}{2^{1/(1-\alpha)}}.
\]
The perfect symmetric set \( E = E(\zeta_n) \) is a \( K \)-set and \( H_h(E) = \lim_{n \to \infty} 2^n h(\lambda_n) = 1 \). □

2.2. Hyperfunctions supported by a \( K \)-set. A hyperfunction on \( \mathbb{T} \) is a holomorphic function on \( \mathbb{C} \setminus \mathbb{T} \) vanishing at infinity. We denote by \( \mathcal{H}(\mathbb{T}) \) the set of all hyperfunctions. The support of a hyperfunction \( \psi \in \mathcal{H}(\mathbb{T}) \), denoted by \( \text{supp} \psi \), is the smallest closed set \( E \subset \mathbb{T} \) such that \( \psi \) can be analytically extended on \( \mathbb{C} \setminus E \). For a closed set \( E \subset \mathbb{T} \), we set \( \mathcal{H}(E) = \{ \psi \in \mathcal{H}(\mathbb{T}) : \text{supp} \psi \subset E \} \). The Taylor coefficients of \( \psi \) are given by
\[
\left\{ \begin{array}{ll}
\psi^+(z) := \psi|_{\mathbb{B}}(z) = \sum_{n \geq 1} \tilde{\psi}_n z^{n-1}, & |z| < 1, \\
\psi^-(z) := \psi|_{\mathbb{C} \setminus \mathbb{B}}(z) = -\sum_{n \leq 0} \tilde{\psi}_n z^{n-1}, & |z| > 1.
\end{array} \right.
\]
We set
\[
\mathcal{H}_d^2(\mathbb{T}) = \left\{ \psi \in \mathcal{H}(\mathbb{T}) : \sup_{n \geq 1} \frac{|\tilde{\psi}_n|}{w_n} < +\infty \text{ and } \sum_{n \leq 0} |\tilde{\psi}_n|^2 < \infty \right\}
\]
and \( \mathcal{H}_d^2(E) = \mathcal{H}_d^2(\mathbb{T}) \cap \mathcal{H}(E) \). We will need the following lemma, which follows from a result of Hruscev [11].

Lemma 2.2. Let \( w \in (R) \). The following conditions are equivalent.
(i) For all \( K \)-sets \( E \), we have \( \mathcal{H}_d^2(E) = \{0\} \).
(ii) For all \( \alpha \in (0, 1) \), condition [1] is satisfied.
Proof. Define \( F_h(E) \) for a Hausdorff function \( h \) by
\[
F_h(E) = \left\{ \psi \in \mathcal{H}(E) : |\psi^+(z)| = O(\exp \frac{h(1-|z|)}{1-|z|}) \text{ and } \psi^- \in H^2(\mathbb{C} \setminus \mathbb{B}) \right\}.
\]
We set \( h_w(t) = t \log \sup_{n \geq 1} (1-t)^n w_n \). According to Lemma 5.2 of [3], the function \( h_w \) is a Hausdorff function, \( h_w(t)/t \) is strictly decreasing and
\[
\int_0^1 \frac{dt}{t^\alpha h_w(t)} \leq \sum_{n \geq 1} \frac{[(n+1)/\log w_{n+1}]^\alpha - [n/\log w_n]^\alpha}{\log w_n}.
\]
Since \( (\log w_n/\sqrt{n})_{n \geq 0} \) is non-increasing and \( (\log w_n)_{n \geq 0} \) is non-decreasing,
\[
\left( \frac{\sqrt{n}}{\log w_n} \right)^\alpha (n+1)^{\alpha/2} - n^{\alpha/2} \leq \left[ \frac{n+1}{\log w_{n+1}} \right]^\alpha - \left[ \frac{n}{\log w_n} \right]^\alpha \leq \frac{(n+1)^\alpha - n^\alpha}{(\log w_n)^\alpha}.
\]
Remark 2.3. Denote by \( A(\mathbb{D}) \) the disk algebra, denote by \( A^p(\mathbb{D}) \) the algebra of all functions \( f \) such that \( f^{(k)} \in A(\mathbb{D}) \), \( 0 \leq k \leq p \), and let \( A^\infty(\mathbb{D}) = \bigcap_{p \geq 1} A^p(\mathbb{D}) \). First observe that a \( K \)-set \( E \) is a Beurling–Carleson set, and so there exists \( f \in A^\infty(\mathbb{D}) \) with \( f^{(n)}|E| = 0 \) (see [13]). Now set

\[
\mathcal{H}_{w,p}(\mathbb{T}) = \left\{ \psi \in \mathcal{H}(\mathbb{T}) : \sup_{n \geq 1} \frac{|\hat{\psi}_n|}{w_n} < +\infty \text{ and } \sup_{n \leq 0} \frac{|\hat{\psi}_n|}{(1 + |n|)^p} < +\infty \right\}
\]

and set \( \mathcal{H}_{w,p}(E) = \mathcal{H}_{w,p}(\mathbb{T}) \cap \mathcal{H}(E) \). If \( f \in A^\infty(\mathbb{D}) \) and \( \psi \in \mathcal{H}_{w,p}(\mathbb{T}) \), we define the hyperfunction \( f.\psi \) whose Taylor coefficients are given by

\[
\hat{f.\psi}_n = \sum_{m \in \mathbb{Z}} \hat{f}(n)\hat{\psi}_{n-m}, \quad n \in \mathbb{Z}.
\]

If \( \psi \in \mathcal{H}_{w,p}(E) \) and \( f^{(n)}|E| = 0 \), then \( f.\psi \in \mathcal{H}^2_{w}(E) \) (see [5], Proposition 2.1). Hence, if condition (ii) of the lemma is satisfied, then for all \( K \)-sets \( E \) and for all \( p \geq 0 \), \( \psi \in \mathcal{H}_{w,p}(E) \), \( f \in A^\infty(\mathbb{D}) \) with \( f^{(n)}|E| = 0 \) we have \( f.\psi = 0 \).

2.3. Proofs of Theorem 1.1 and Theorem 1.2. Suppose that condition (1) is satisfied. Letting \( x \in X \) and \( l \in X^* \), we set

\[
\phi(z) = \langle (T - zI)^{-1}x, l \rangle, \quad z \notin \sigma(T).
\]

We have \( \phi \in \mathcal{H}_{w,p}(\sigma(T)) \) (\( p = 0 \) for Theorem 1.2). Consider an outer function \( f \in A^\infty(\mathbb{D}) \) such that \( f^{(m)}|\sigma(T) = 0 \) for all \( m \geq 0 \). A standard computation of (5) gives that

\[
f.\phi(z) = \langle (T - zI)^{-1}f(T)x, l \rangle, \quad z \notin \sigma(T).
\]

According to Remark 2.3, \( f.\phi = 0 \), and so \( f(T) = 0 \). The conclusion follows from the proof of Theorem 4.1 of [5] (see also [2]) for Theorem 1.1 and from the proof of Theorem 6.4 of [6] for Theorem 1.2.

Now suppose that condition (2) is satisfied for some \( \alpha \in (0, 1) \). Set \( \bar{w}_n = w_n^{1/2} \). Then \( \bar{w} \) satisfies (R) and (2). According to (4), we have \( \int_0^1 dt/(t^\alpha h_{\bar{w}}(t)) < +\infty \), where \( h_{\bar{w}}(t) = t \log \sup_n (1 - t)\bar{w}_n \) is a Hausdorff function and \( h_{\bar{w}}(t)/t \) is strictly decreasing. Lemma 1 and Frostman’s Theorem [9] give the existence of a \( K \)-set \( E \) and a singular measure \( \mu \) supported by \( E \) which modulus of continuity satisfies \( \rho_\mu(t) = O(h_{\bar{w}}(t)) \). Let \( S_\mu \) be the singular inner function associated with \( \mu \). Consider the operator \( T : H^2 \ominus S_\mu H^2 \rightarrow H^2 \ominus S_\mu H^2 \) defined by \( Tg = P_\mu(zg) \), where \( P_\mu \) is the orthogonal projection on \( H^2 \ominus S_\mu H^2 \). Then \( T \) is an invertible contraction with spectrum \( E \), \( \|T^{-n}\| = O(w_n) \) and \( \|T^{-n}\| \rightarrow \infty \) (see [10] for more details).

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