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A self-contained proof of the strong-type capacitary inequality for the Dirichlet space

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ABSTRACT. In this expository article, we give a self-contained proof of the strong-type capacitary inequality for functions in the Dirichlet space. We also show that such functions possess boundary limits in exponentially tangential approach regions at almost every point on the unit circle.

1. Introduction

Let f be a function holomorphic in the unit disk $\mathbb{D}$. The *Dirichlet integral* of f is defined by

$$\mathcal{D}(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2\,dA(z),$$

where dA denotes area measure. The *Dirichlet space* is the vector space of all f holomorphic in $\mathbb{D}$ for which $\mathcal{D}(f) < \infty$.

An easy calculation shows that, if $f(z) = \sum_{k\geq 0} a_k z^k$, then $\mathcal{D}(f) = \sum_{k\geq 1} k|a_k|^2$. It follows that $\mathcal{D}$ is a subspace of the Hardy space H^2, and that it is a Hilbert space with respect to the norm $\|\cdot\|_{\mathcal{D}}$ given by

$$\|f\|_{\mathcal{D}}^2 := \|f\|_{H^2}^2 + \mathcal{D}(f) = \sum_{k\geq 0}(k+1)|a_k|^2.$$

We denote the corresponding inner product by $\langle\cdot,\cdot\rangle_{\mathcal{D}}$.

It is well known that, if $f \in H^2$, then the non-tangential limit $f^*(\zeta)$ exists at almost every point ζ of the unit circle $\mathbb{T}$, and $f^* \in L^2(\mathbb{T})$. Since $\mathcal{D}$ is contained in H^2, it obviously inherits the same property. But in fact much more is true: if $f \in \mathcal{D}$, then

- the non-tangential limit $f^*(\zeta)$ exists for all $\zeta \in \mathbb{T}$ outside a set of logarithmic capacity zero;

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- the boundary limit function f^* satisfies the capacitary strong-type inequality
$$\int_0^\infty c(|f^*| > t)t\,dt \le A\|f\|_{\mathcal{D}}^2,$$
where c denotes logarithmic capacity, and A is an absolute constant;
- for all $\zeta \in \mathbb{T}$ outside a set of Lebesgue measure zero, $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each region
$$|z-\zeta| < \kappa\Big(\log\frac{1}{1-|z|}\Big)^{-1} \quad (\kappa > 0).$$

Our aim in this expository article is to give self-contained proofs of these results. In particular, we assume no prior knowledge of capacity. Our approach follows an idea of A. Borichev, which appears to lead to simpler proofs than the ones commonly found in the literature.

2. Capacity

Throughout this section, we fix a compact metric space (X, d) and a continuous decreasing function $K : (0, \infty) \to [0, \infty)$. The function K is called a *kernel*. We extend the definition of K to 0 by defining $K(0) := \lim_{t\to 0^+} K(t)$. It may well happen that $K(0) = \infty$, and in fact this is the case for most interesting kernels, though we do not insist upon it. However, in order to avoid trivialities, we do assume that $K \not\equiv 0$.

Definition 2.1. Let μ be a finite positive Borel measure on X. Its *K-potential* is the function $K\mu : X \to [0, \infty]$ defined by
$$K\mu(x) := \int_X K(d(x, y))\,d\mu(y) \qquad (x \in X).$$
The *K-energy* of μ is defined as
$$I_K(\mu) := \int K\mu\,d\mu = \iint K(d(x, y))\,d\mu(x)\,d\mu(y).$$

Clearly $I_K(\mu) \in [0, \infty]$. In fact it is not hard to see that $I_K(\mu) > 0$. However, it is quite possible that $I_K(\mu) = \infty$.

Definition 2.2. Let F be a compact subset of X. We write $\mathcal{P}(F)$ for the set of Borel probability measures on F. The *K-capacity* of F is defined by
$$c_K(F) := 1/\inf\{I_K(\mu) : \mu \in \mathcal{P}(F)\}. \tag{2.1}$$

In particular $c_K(F) = 0$ if and only if there is no measure $\mu \in \mathcal{P}(F)$ with $I_K(\mu) < \infty$. The following result gives some basic properties of capacity as a set function.

Theorem 2.3. *(i) $c_K(\emptyset) = 0$.*
(ii) If $F_1 \subset F_2$, then $c_K(F_1) \le c_K(F_2)$.
(iii) $c_K(F_1 \cup \cdots \cup F_n) \le c_K(F_1) + \cdots + c_K(F_n)$.

Proof. Parts (i) and (ii) are obvious. For (iii), set $F := \cup_1^n F_k$. Let $\mu \in \mathcal{P}(F)$. For each k, let μ_k be the measure given by $\mu_k(S) := \mu(S \cap F_k \setminus (F_1 \cup \cdots \cup F_{k-1}))$. Then $\mu = \sum_{k=1}^n \mu_k$, so
$$I_K(\mu) \ge \sum_{k=1}^n I_K(\mu_k).$$

On the other hand, each measure μ_k is supported on F_k, so, by the definition of capacity and a simple scaling argument, $I_K(\mu_k)c_K(F_k) \geq \mu_k(F_k)^2$. Hence, using Schwarz's inequality, we have

$$1 = \mu(F) = \sum_{k=1}^n \mu_k(F_k) \leq \sum_{k=1}^n \sqrt{I_K(\mu_k)c_K(F_k)} \leq \Bigl(\sum_{k=1}^n I_K(\mu_k)\Bigr)^{1/2}\Bigl(\sum_{k=1}^n c_K(F_k)\Bigr)^{1/2}.$$

Combining these estimates, we obtain $I_K(\mu)\sum_{k=1}^n c_K(F_k) \geq 1$. As this holds for all $\mu \in \mathcal{P}(F)$, we deduce that $\sum_{k=1}^n c_K(F_k) \geq c_K(F)$, as required. □

Next, here is a simple upper bound for capacity in terms of diameter.

Theorem 2.4. *If F is a compact subset of X, then $c_K(F) \leq 1/K(\operatorname{diam}(F))$.*

Proof. If $\mu \in \mathcal{P}(F)$, then

$$I_K(\mu) \geq \iint K(\operatorname{diam}(F))\,d\mu(x)\,d\mu(y) = K(\operatorname{diam}(F)).$$

It follows that $c_K(F) \leq 1/K(\operatorname{diam}(F))$. □

Corollary 2.5. *For every compact set F we have $c_K(F) < \infty$.*

Proof. By assumption $K \not\equiv 0$, so there exists $d_0 > 0$ such that $K(d_0) > 0$. A compact set F can be covered by finitely many compact sets $F_1, \dots, F_n$ of diameter at most d_0, so $c_K(F) \leq c_K(F_1) + \cdots + c_K(F_n) \leq n/K(d_0) < \infty$. □

The next result is sometimes expressed by saying that capacity is upper semi-continuous.

Theorem 2.6. *Let $(F_n)_{n\geq 1}$ be a decreasing sequence of compact subsets of X, and let $F := \cap_n F_n$. Then $c_K(F) = \lim_n c_K(F_n)$.*

The proof makes use of the notion of weak*-convergence of measures. Recall that a sequence of finite measures (μ_n) on X is weak*-convergent to μ if $\int g\,d\mu_n \to \int g\,d\mu$ for every continuous function g on X.

Lemma 2.7. *Let (μ_n) be a sequence in $\mathcal{P}(X)$ that is weak*-convergent to $\mu \in \mathcal{P}(X)$. Then $\liminf_{n\to\infty} I_K(\mu_n) \geq I_K(\mu)$.*

Proof. We first claim that, if $f : X \times X \to \mathbb{R}$ is a continuous function, then $\iint f\,d\mu_n\,d\mu_n \to \iint f\,d\mu\,d\mu$ as $n \to \infty$. Indeed, this is clearly true if f has the form $f(x,y) = g(x)h(y)$, and, using the Stone–Weierstrass theorem, a general continuous f may be uniformly approximated by finite sums of functions of this special form.

If $K(0) < \infty$, then we can apply the claim with $f(x,y) := K(d(x,y))$, and deduce that $I_K(\mu_n) \to I_K(\mu)$. For the general case, consider $K_T(t) := \min\{K(t), T\}$. By what we have just proved, $I_{K_T}(\mu) = \lim_{n\to\infty} I_{K_T}(\mu_n)$. Clearly $I_{K_T}(\mu_n) \leq I_K(\mu_n)$ for all n, and so $I_{K_T}(\mu) \leq \liminf_{n\to\infty} I_K(\mu_n)$. Also, by the monotone convergence theorem $I_{K_T}(\mu) \to I_K(\mu)$ as $T \to \infty$. Hence $I_K(\mu) \leq \liminf_{n\to\infty} I_K(\mu_n)$, as required. □

Proof of Theorem 2.6. By Theorem 2.3 (ii), the sequence $c_K(F_n)$ decreases and $c_K(F_n) \geq c_K(F)$ for all n. We need to show that $\lim_{n\to\infty} c_K(F_n) \leq c_K(F)$.

For each n, pick $\mu_n \in \mathcal{P}(F_n)$ such that $I_K(\mu_n) < 1/c_K(F_n) + 1/n$. By the Banach–Alaoglu theorem, there exists a subsequence of the μ_n which is weak*-convergent to some $\mu \in \mathcal{P}(X)$. Relabeling, if necessary, we may as well suppose that this is the whole sequence. Note that μ is supported in F_n for each n, so $\mu \in \mathcal{P}(F)$ and $I_K(\mu) \geq 1/c_K(F)$. On the other hand, by Lemma 2.7, we have $I_K(\mu) \leq \liminf_{n\to\infty} I_K(\mu_n) \leq \lim_{n\to\infty} 1/c_K(F_n)$. It follows that $\lim_{n\to\infty} c_K(F_n) \leq c_K(F)$. □

Now we extend the definition of capacity to non-compact sets.

Definition 2.8. Let E be an arbitrary subset of X.

- The *inner K-capacity* of E is defined by

$$c_K(E) := \sup\{c_K(F) : F \subset E,\ F \text{ compact}\}.$$

- The *outer K-capacity* of E is defined by

$$c_K^*(E) := \inf\{c_K(U) : U \supset E,\ U \text{ open in } X\}.$$

Evidently $c_K(E) \leq c_K^*(E)$. A set E is called *capacitable* if $c_K(E) = c_K^*(E)$. It is clear that open sets are capacitable, and Theorem 2.6 shows that compact sets are capacitable too. In fact, it can be shown that every Borel subset of X is capacitable (Choquet's theorem), but we do not need that result here.

The next theorem summarizes some basic properties of c_K^*.

Theorem 2.9. (i) *$c_K^*(\emptyset) = 0$.*
(ii) *If $E_1 \subset E_2$, then $c_K^*(E_1) \leq c_K^*(E_2)$.*
(iii) *$c_K^*(\cup_{k\geq 1} E_k) \leq \sum_{k\geq 1} c_K^*(E_k)$.*

In particular, the outer capacity of set does not change if we adjoin to it a set of outer capacity zero. Also, a countable union of sets of outer capacity zero is still of outer capacity zero.

Proof. Parts (i) and (ii) are obvious.

For part (iii), we consider first the case when each E_k is an open set U_k. Let F be a compact subset of $\cup_k U_k$. By compactness, there exists $n \geq 1$ such that $F \subset U_1 \cup \cdots \cup U_n$. We can write $F = F_1 \cup \cdots \cup F_n$, where each F_k is a compact subset of U_k. Then, using Theorem 2.3 (iii), we have

$$c_K(F) \leq \sum_{k=1}^n c_K(F_k) \leq \sum_{k=1}^n c_K(U_k) \leq \sum_{k\geq 1} c_K(U_k).$$

As this holds for each such F, we obtain $c_K(\cup_k U_k) \leq \sum_k c_K(U_k)$, proving the result in this case.

For the general case, we may suppose that $c_K^*(E_k) < \infty$ for all k, otherwise there is nothing to prove. Let $\epsilon > 0$, and, for each k, let U_k be an open set such that $U_k \supset E_k$ and $c(U_k) < c^*(E_k) + \epsilon/2^k$. Then $\cup_k U_k$ is an open set containing $\cup_k E_k$, and, by what we have already proved,

$$c_K(\cup_k U_k) \leq \sum_k c_K(U_k) \leq \sum_k c_K^*(E_k) + \epsilon.$$

Thus $c_K^*(\cup_k E_k) \leq \sum_k c_K^*(E_k) + \epsilon$. This holds for all $\epsilon > 0$, whence the result. □

Next, we consider the important class of measures for which the infimum is attained in (2.1).

Definition 2.10. Let F be a compact subset of X such that $c_K(F) > 0$. An *equilibrium measure* for F is a measure $\nu \in \mathcal{P}(F)$ such that $I_K(\mu) \geq I_K(\nu)$ for all $\mu \in \mathcal{P}(F)$.

Clearly, if ν is an equilibrium measure for F, then $c_K(F) = 1/I_K(\nu)$.

Theorem 2.11. *Let F be a compact subset of X with $c_K(F) > 0$. Then F has an equilibrium measure.*

Proof. For each $n \geq 1$, choose $\mu_n \in \mathcal{P}(F)$ such that $I_K(\mu_n) < 1/c_K(F) + 1/n$. A subsequence of the μ_n converges weak* to some $\nu \in \mathcal{P}(F)$. By Lemma 2.7, this ν must satisfy $I_K(\nu) \leq 1/c_K(F)$, so it is an equilibrium measure for F. □

For many kernels K, it turns out that the equilibrium measure is unique. We shall not pursue this here. We do however need the following version of a theorem of Frostman, sometimes called the fundamental theorem of potential theory, which details some special properties of the potential of an equilibrium measure. We write $\operatorname{supp}\mu$ for the (closed) support of a measure μ.

Theorem 2.12. *Let F be a compact subset of X such that $c_K(F) > 0$. Let ν be an equilibrium measure for F. Then:*

(i) *$K\nu(x) \leq I_K(\nu)$ for all $x \in \operatorname{supp}\nu$,*
(ii) *$K\nu(x) \geq I_K(\nu)$ for all $x \in F \setminus E$, where $c_K^*(E) = 0$.*

Lemma 2.13. *If μ is a finite positive Borel measure on X, then its potential $K\mu$ is a lower semicontinuous function on X.*

Proof. Let $x_n \to x_0$ in X. By Fatou's lemma, we have

$$\liminf_{n\to\infty} \int K(d(x_n, y))\, d\mu(y) \geq \int K(d(x_0, y))\, d\mu(y).$$

In other words $\liminf_{n\to\infty} K\mu(x_n) \geq K\mu(x_0)$, as required. □

Proof of Theorem 2.12. We first claim that, if $\mu \in \mathcal{P}(F)$ and $I_K(\mu) < \infty$, then $\int K\nu\, d\mu \geq I_K(\nu)$. In showing this, we may as well suppose that $\int K\nu\, d\mu < \infty$, otherwise there is nothing to prove. For each $t \in (0,1)$ consider the measure $\mu_t := t\mu + (1-t)\nu$. A simple calculation shows that

$$I_K(\mu_t) = I_K(\nu) + 2t\Big(\int K\nu\, d\mu - I_K(\nu)\Big) + t^2\Big(I_K(\mu) + I_K(\nu) - 2\int K\nu\, d\mu\Big).$$

Also, since $\mu_t \in \mathcal{P}(F)$, it must satisfy $I_K(\mu_t) \geq I_K(\nu)$ for all $t \in (0,1)$. It follows that $\int K\nu\, d\mu \geq I_K(\nu)$, as claimed.

(i) Suppose, if possible, that $K\nu(x) > I_K(\nu)$ for some $x \in \operatorname{supp}\nu$. As $K\nu$ is lower semicontinuous, there exists a neighborhood N of x such that $K\nu > I_K(\nu)$ on N. As $x \in \operatorname{supp}\nu$, we must have $\nu(N) > 0$, and hence

$$\int_N K\nu\, d\nu > I_K(\nu)\nu(N).$$

Also, applying the claim at the beginning of the proof to the measure $\mu(S) := \nu(S \setminus N)/\nu(X \setminus N)$, we have

$$\int_{X\setminus N} K\nu\, d\nu \geq I_K(\nu)\nu(X \setminus N).$$

Adding together the two inequalities, we obtain

$$\int K\nu\, d\nu > I_K(\nu),$$

which is clearly a contradiction. Hence $K\nu(x) \leq I_K(\nu)$ for all $x \in \operatorname{supp}\nu$.

(ii) For each $n \geq 1$, set

$$E_n := \{x \in F : K\nu(x) \leq I_K(\nu) - 1/n\}.$$

As $K\nu$ is lower semicontinuous, E_n is compact. Suppose, if possible, that $c_K(E_n) > 0$ for some n. Then E_n possesses an equilibrium measure ν_n. By the claim at the beginning of the proof, we have

$$\int K\nu \, d\nu_n \geq I_K(\nu).$$

On the other hand, since $K\nu \leq I_K(\nu) - 1/n$ on E_n, we have

$$\int K\nu \, d\nu_n \leq I_K(\nu) - 1/n.$$

This is clearly a contradiction. Thus, in fact, $c_K(E_n) = 0$ for all n. Setting $E := \cup_{n \geq 1} E_n$, we therefore have $c^*(E) = 0$ and $K\nu(x) \geq I_K(\nu)$ for all $x \in F \backslash E$. □

Though the notion of capacity has been developed in some generality, we shall be interested mostly in the following special case.

Definition 2.14. Let $X = \mathbb{T}$ with the metric $d(z, w) := |z - w|$, and let $K(t) := \log^+(2/t)$. The corresponding capacity is called *logarithmic capacity*. We shall denote it simply by $c(\cdot)$.

The choice of 2 in $\log^+(2/t)$ is largely for convenience, since $\mathbb{T}$ has diameter 2. It could be replaced by any other constant $A \geq 2$, and the resulting capacity c_A would be equivalent in the sense that $1/c_A(F) - 1/c(F) = \log(A/2)$ for all F.

Another possibility, often seen in the literature, is to take $K(t) = \log(1/t)$. This gives a non-positive kernel, but one can nevertheless define

$$\widetilde{c}(F) := \exp\Big(-\inf\{I_K(\mu) : \mu \in \mathcal{P}(F)\}\Big).$$

The capacities c and $\widetilde{c}$ are related via the formula $1/c = \log(2/\widetilde{c})$. In particular, $\widetilde{c}(F) = 0$ if and only if $c(F) = 0$.

We shall need two results specific to logarithmic capacity on the circle. The first is an estimate in terms of Lebesgue measure. We write $|E|$ for arc-length measure of E.

Theorem 2.15. *Let E be a Borel subset of $\mathbb{T}$ with $|E| > 0$. Then*

$$c(E) \geq \frac{1}{\log(2\pi e/|E|)}.$$

Proof. Let $K(t) := \log^+(2/t)$. Let F be a compact subset of $\mathbb{T}$ with $|F| > 0$. Let μ be the probability measure on F defined by $\mu(S) := |F \cap S|/|F|$. Then

$$K\mu(z) = \frac{1}{|F|} \int_{e^{i\theta} \in F} \log \frac{2}{|z - e^{i\theta}|} \, d\theta \qquad (z \in \mathbb{T}).$$

For a fixed value of $z \in \mathbb{T}$, the integral is increased by replacing F with an arc in $\mathbb{T}$ of the same length $|F|$, centred at z. Thus

$$\begin{aligned} K\mu(z) &\leq \frac{1}{|F|} \int_{-|F|/2}^{|F|/2} \log \frac{2}{|2\sin(\theta/2)|} \, d\theta \leq \frac{1}{|F|} \int_{-|F|/2}^{|F|/2} \log \frac{\pi}{|\theta|} \, d\theta \\ &= \frac{2\pi}{|F|} \int_0^{|F|/2\pi} \log(1/t) \, dt = \log(2\pi e/|F|). \end{aligned}$$

It follows that $I_K(\mu) \le \log(2\pi e/|F|)$, and therefore $c_K(F) \ge 1/\log(2\pi e/|F|)$. This proves the result for compact sets.

The result for a general Borel set E now follows easily using the inner regularity of Lebesgue measure, namely the fact that $|E| = \sup\{|F| : F \subset E,\ F \text{ compact}\}$. □

It is also possible to give an upper bound for the logarithmic capacity of an arc in terms of its measure. However, no such bound is possible for general closed subsets of $\mathbb{T}$. Indeed, there exist Cantor-type sets E such that $|E| = 0$ but $c(E) > 0$.

The second result that we need is sometimes called the maximum principle for potentials.

Theorem 2.16. *Let $X = \mathbb{T}$ with the metric $d(z,w) := |z - w|$, and let $K : (0,\infty) \to [0,\infty)$ be a decreasing convex function. If μ is a finite positive Borel measure on $\mathbb{T}$ and $K\mu \le M$ on $\operatorname{supp}\mu$, then $K\mu \le M$ on $\mathbb{T}$.*

In particular, this result applies if $K(t) := \log^+(2/t)$.

Proof. Let I be a connected component of $\mathbb{T}\setminus\operatorname{supp}\mu$, say $I = (e^{i\alpha}, e^{i\beta})$, where $0 < \beta - \alpha \le 2\pi$. We shall prove that $K\mu \le M$ on I.

If $\alpha \le \gamma \le \beta$, then

$$K\mu(e^{i\gamma}) = \int K(|e^{i\theta} - e^{i\gamma}|)\, d\mu(e^{i\theta}) = \int_{[\beta,\alpha+2\pi]} K\Big(2\sin\frac{\theta-\gamma}{2}\Big)\, d\mu(\theta).$$

Now sin is concave on $[0,\pi]$, and K is convex and decreasing on $[0,\infty)$. Hence, for each $\theta \in [\beta, \alpha+2\pi]$, the function $\gamma \mapsto K(2\sin(\theta-\gamma)/2))$ is convex on $[\alpha,\beta]$ (perhaps infinite at the endpoints). Integrating with respect to μ, we deduce that $\gamma \mapsto K\mu(e^{i\gamma})$ is convex on $[\alpha,\beta]$. In particular, $K\mu(e^{i\gamma}) \le \max\{K\mu(e^{i\alpha}), K\mu(e^{i\beta})\}$ for all $\gamma \in (\alpha,\beta)$. Now $K\mu \le M$ at $e^{i\alpha}, e^{i\beta}$, because both these points belong to $\operatorname{supp}\mu$. Therefore $K\mu(e^{i\gamma}) \le M$ for all $\gamma \in (\alpha,\beta)$. In other words $K\mu \le M$ on I, which is what we wanted to prove. □

Combining this result with Theorem 2.12, we obtain the following important corollary.

Corollary 2.17. *Let $X = \mathbb{T}$ with the metric $d(z,w) := |z-w|$, and let $K : (0,\infty) \to [0,\infty)$ be a decreasing convex function. Let F be a compact subset of $\mathbb{T}$ such that $c_K(F) > 0$. Let ν be an equilibrium measure for F. Then:*

(i) *$K\nu(x) \le I_K(\nu)$ for all $x \in \mathbb{T}$,*
(ii) *$K\nu(x) = I_K(\nu)$ for all $x \in F \setminus E$, where $c_K^*(E) = 0$.* □

3. Cauchy transform

The rest of this article is based upon a representation formula for functions in $\mathcal{D}$ in terms of the Cauchy transform, which we shall establish in this section. We begin with some notation.

Definition 3.1. Let $\mathbb{A} := \{z \in \mathbb{C} : 1 < |z| < 2\}$. We write $L^2(\mathbb{A})$ for the Hilbert space of measurable functions $g : \mathbb{A} \to \mathbb{C}$ such that

$$\|g\|^2_{L^2(\mathbb{A})} := \frac{1}{\pi}\int_{\mathbb{A}} |g(w)|^2\, dA(w) < \infty.$$

Given $g \in L^2(\mathbb{A})$, we define its *Cauchy transform* $\mathcal{C}g : \mathbb{D} \to \mathbb{C}$ by

$$\mathcal{C}g(z) := \frac{1}{\pi}\int_{\mathbb{A}} \frac{g(w)}{w-z}\,dA(w) \qquad (z \in \mathbb{D}).$$

Using Schwarz's inequality, we see that $\mathcal{C}g(z)$ is well-defined for all $z \in \mathbb{D}$, and that $\mathcal{C}g$ is holomorphic in $\mathbb{D}$. In fact, as the next result shows, it belongs to the Dirichlet space.

Theorem 3.2. *If $g \in L^2(\mathbb{A})$, then $\mathcal{C}g \in \mathcal{D}$ and $\|\mathcal{C}g\|_{\mathcal{D}} \le \sqrt{3/2}\|g\|_{L^2(\mathbb{A})}$.*

Proof. Let $g \in L^2(\mathbb{A})$. For each $z \in \mathbb{D}$, we have

$$\mathcal{C}g(z) = \sum_{k\ge 0}\Big(\frac{1}{\pi}\int_{\mathbb{A}}\frac{g(w)}{w^{k+1}}\,dA(w)\Big)z^k = \sum_{k\ge 0}\langle g,\phi_k\rangle_{L^2(\mathbb{A})}z^k, \tag{3.1}$$

where $\phi_k(w) := 1/\overline{w}^{k+1}$, and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{A})}$ denotes the inner product on $L^2(\mathbb{A})$. Hence

$$\|\mathcal{C}g\|_{\mathcal{D}}^2 = \sum_{k\ge 0}(k+1)|\langle g,\phi_k\rangle_{L^2(\mathbb{A})}|^2.$$

Now $(\phi_k)_{k\ge 0}$ is an orthogonal sequence in $L^2(\mathbb{A})$, so, by Bessel's inequality,

$$\sum_{k\ge 0}(k+1)|\langle g,\phi_k\rangle_{L^2(\mathbb{A})}|^2 \le B\|g\|_{L^2(\mathbb{A})}^2,$$

where $B := \sup_{k\ge 0}(k+1)\|\phi_k\|_{L^2(\mathbb{A})}^2$. A calculation gives

$$\|\phi_k\|_{L^2(\mathbb{A})}^2 = \frac{1}{\pi}\int_{\mathbb{A}}\frac{1}{|w|^{2k+2}}\,dA(w) = 2\int_1^2\frac{dr}{r^{2k+1}} = \begin{cases}\log 4, & k=0,\\ (1-4^{-k})/k, & k\ge 1.\end{cases}$$

Therefore $B = 3/2$ and the result follows. (Note that the constant $\sqrt{3/2}$ in the theorem is sharp: just take $g(w) = \phi_1(w) = 1/\overline{w}^2$.) □

In fact the Cauchy transform maps $L^2(\mathbb{A})$ *onto* $\mathcal{D}$. This is the next result, which may be viewed as a representation theorem for the Dirichlet space.

Theorem 3.3. *Given $f \in \mathcal{D}$, there exists $g \in L^2(\mathbb{A})$ such that $f = \mathcal{C}g$ and $\|g\|_{L^2(\mathbb{A})} \le \|f\|_{\mathcal{D}}$.*

Proof. Let $f \in \mathcal{D}$, say $f(z) = \sum_{k\ge 0} a_k z^k$. Let $\phi_k(w) := 1/\overline{w}^{k+1}$, as in the previous proof, and consider $g := \sum_{k\ge 0}(a_k/\|\phi_k\|_{L^2(\mathbb{A})}^2)\phi_k$. As (ϕ_k) is an orthogonal sequence in $L^2(\mathbb{A})$, and as

$$\sum_{k\ge 0}|a_k|^2/\|\phi_k\|_{L^2(\mathbb{A})}^2 \le \sum_{k\ge 0}(k+1)|a_k|^2 = \|f\|_{\mathcal{D}}^2 < \infty,$$

the series defining g converges in $L^2(\mathbb{A})$, and $\|g\|_{L^2(\mathbb{A})} \le \|f\|_{\mathcal{D}}$. Furthermore, by (3.1), we have

$$\mathcal{C}\phi_k(z) = \sum_{j\ge 0}\langle\phi_k,\phi_j\rangle_{L^2(\mathbb{A})}z^j = \|\phi_k\|_{L^2(\mathbb{A})}^2 z^k \qquad (z\in\mathbb{D},\ k\ge 0),$$

and therefore

$$\mathcal{C}g(z) = \sum_{k\ge 0}(a_k/\|\phi_k\|_{L^2(\mathbb{A})}^2)\mathcal{C}\phi_k(z) = \sum_{k\ge 0}a_k z^k = f(z) \qquad (z\in\mathbb{D}).$$

This completes the proof. □

4. Beurling's theorem

Recall that $c(E)$ denotes the logarithmic capacity of $E \subset \mathbb{T}$, and that $c^*(E)$ is the corresponding outer capacity. Our main goal in this section is to prove the following theorem of Beurling.

Theorem 4.1. *Let $f \in \mathcal{D}$. Then there exists $E \subset \mathbb{T}$ with $c^*(E) = 0$ such that, if $\zeta \in \mathbb{T} \setminus E$, then $f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$ exists, and $f(z) \to f^*(\zeta)$ as $z \to \zeta$ inside each region $|z-\zeta| < \kappa(1-|z|)$ $(\kappa > 0)$.*

A property is said to hold *quasi-everywhere* (q.e.) on $\mathbb{T}$, if it holds everywhere on $\mathbb{T} \setminus E$ where $c^*(E) = 0$. Thus Beurling's theorem can be summarized by saying that each $f \in \mathcal{D}$ has non-tangential limits quasi-everywhere on $\mathbb{T}$.

For the proof of Beurling's theorem, it is helpful to extend the Cauchy transform to the unit circle as follows.

Definition 4.2. Given $g \in L^2(\mathbb{A})$, we define $\widetilde{\mathcal{C}}g : \mathbb{T} \to [0,\infty]$ by

$$\widetilde{\mathcal{C}}g(\zeta) := \frac{1}{\pi}\int_{\mathbb{A}} \frac{|g(w)|}{|w-\zeta|}\, dA(w) \qquad (\zeta \in \mathbb{T}).$$

If $\zeta \in \mathbb{T}$ and $\widetilde{\mathcal{C}}g(\zeta) < \infty$, then we set

$$\mathcal{C}g(\zeta) := \frac{1}{\pi}\int_{\mathbb{A}} \frac{g(w)}{w-\zeta}\, dA(w).$$

Theorem 4.3. *Let $g \in L^2(\mathbb{A})$. Then $\widetilde{\mathcal{C}}g$ is lower semicontinuous on $\mathbb{T}$.*

Proof. Let $\zeta_n \to \zeta_0$ in $\mathbb{T}$. By Fatou's lemma,

$$\liminf_{n\to\infty} \frac{1}{\pi}\int_{\mathbb{A}} \frac{|g(w)|}{|w-\zeta_n|}\, dA(w) \geq \frac{1}{\pi}\int_{\mathbb{A}} \frac{|g(w)|}{|w-\zeta_0|}\, dA(w).$$

In other words, $\liminf_{n\to\infty} \widetilde{\mathcal{C}}g(\zeta_n) \geq \widetilde{\mathcal{C}}g(\zeta_0)$, as required. □

The next theorem shows that $\widetilde{\mathcal{C}}g$ plays the role of a sort of maximal function (for more on this theme, see e.g. [26, p.332]).

Theorem 4.4. *Let $g \in L^2(\mathbb{A})$, let $\zeta \in \mathbb{T}$, and suppose that $\widetilde{\mathcal{C}}g(\zeta) < \infty$. Then*

$$|\mathcal{C}g(z)| \leq \Big(1 + \frac{|z-\zeta|}{1-|z|}\Big)\widetilde{\mathcal{C}}g(\zeta) \qquad (z \in \mathbb{D}), \tag{4.1}$$

and $\mathcal{C}g(z) \to \mathcal{C}g(\zeta)$ as $z \to \zeta$ in each region $|z-\zeta| < \kappa(1-|z|)$.

Proof. For $z \in \mathbb{D}$, we have

$$|\mathcal{C}g(z)| \leq \frac{1}{\pi}\int_{\mathbb{A}} \frac{|g(w)|}{|w-z|}\, dA(w) \leq \sup_{w\in\mathbb{A}} \frac{|w-\zeta|}{|w-z|}\widetilde{\mathcal{C}}g(\zeta).$$

Now, if $z \in \mathbb{D}$ and $w \in \mathbb{A}$, then

$$\frac{|w-\zeta|}{|w-z|} \leq \frac{|w-z|+|z-\zeta|}{|w-z|} \leq 1 + \frac{|z-\zeta|}{|w-z|} \leq 1 + \frac{|z-\zeta|}{1-|z|}.$$

This gives (4.1).

Turning now to the second part of the theorem, for $\delta > 0$ let us define $g_\delta(w) := g(w)1_{\{1<|w|<1+\delta\}}$. Then, for $z \in \mathbb{D}$, we have

$$|\mathcal{C}g(z) - \mathcal{C}g(\zeta)| \leq |\mathcal{C}(g-g_\delta)(z) - \mathcal{C}(g-g_\delta)(\zeta)| + |\mathcal{C}g_\delta(z)| + |\mathcal{C}g_\delta(\zeta)|.$$

Now $\mathcal{C}(g-g_\delta)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}$, so in particular

$$\lim_{\substack{z\to\zeta\\ z\in\mathbb{D}}} \mathcal{C}(g-g_\delta)(z)=\mathcal{C}(g-g_\delta)(\zeta).$$

Also, by (4.1),

$$|\mathcal{C}g_\delta(z)|\le\Bigl(1+\frac{|z-\zeta|}{1-|z|}\Bigr)\widetilde{\mathcal{C}}g_\delta(\zeta),$$

and clearly $|\mathcal{C}g_\delta(\zeta)|\le\widetilde{\mathcal{C}}g_\delta(\zeta)$. Putting these facts together, we deduce that, for each $\kappa>0$,

$$\limsup_{\substack{z\to\zeta\\ |z-\zeta|\le\kappa(1-|z|)}} |\mathcal{C}g(z)-\mathcal{C}g(\zeta)|\le(\kappa+2)\widetilde{\mathcal{C}}g_\delta(\zeta).$$

This inequality holds for each $\delta>0$. Since $\widetilde{\mathcal{C}}g(\zeta)<\infty$, the dominated convergence theorem implies that $\widetilde{\mathcal{C}}g_\delta(\zeta)\to0$ as $\delta\to0$. Hence, for each $\kappa>0$,

$$\limsup_{\substack{z\to\zeta\\ |z-\zeta|\le\kappa(1-|z|)}} |\mathcal{C}g(z)-\mathcal{C}g(\zeta)|=0.$$

This gives the result. □

Theorem 4.4 begs the question: on how large a set can $\widetilde{\mathcal{C}}g(\zeta)$ be infinite? This is where capacity enters the picture.

Theorem 4.5. *Let $g\in L^2(\mathbb{A})$. Then*

$$c(\widetilde{\mathcal{C}}g>t)\le A\|g\|^2_{L^2(\mathbb{A})}/t^2\qquad(t>0),$$

where A is an absolute constant.

The key to the proof of this theorem is the following elementary lemma.

Lemma 4.6. *There exists $B>0$ such that, for all $\zeta_1,\zeta_2\in\mathbb{T}$,*

$$\frac1\pi\int_{\mathbb{A}}\frac{dA(w)}{|w-\zeta_1||w-\zeta_2|}\le2\log\frac{B}{|\zeta_1-\zeta_2|}.$$

Proof. Making the change of variable $z:=(w-\zeta_1)/(\zeta_2-\zeta_1)$, we obtain

$$\int_{1<|w|<2}\frac{dA(w)}{|w-\zeta_1||w-\zeta_2|}=\int_{1<|\zeta_1+(\zeta_2-\zeta_1)z|<2}\frac{dA(z)}{|z||z-1|}\le\int_{|z|<4/|\zeta_2-\zeta_1|}\frac{dA(z)}{|z||z-1|}.$$

Now

$$\begin{aligned}\int_{|z|<2}\frac{dA(z)}{|z||z-1|}&\le\int_{|z|<2}\Bigl(\frac{1}{|z|}+\frac{1}{|z-1|}\Bigr)\,dA(z)\\ &\le\int_{|z|<2}\frac{dA(z)}{|z|}+\int_{|z|<3}\frac{dA(z)}{|z|}=4\pi+6\pi.\end{aligned}$$

Also, for $R>2$,

$$\int_{2\le|z|<R}\frac{dA(z)}{|z||z-1|}\le2\pi\int_2^R\frac{dr}{r-1}\le2\pi\log R.$$

If we put all these inequalities together, then we obtain

$$\frac1\pi\int_{\mathbb{A}}\frac{dA(w)}{|w-\zeta_1||w-\zeta_2|}\le10+2\log\frac{4}{|\zeta_2-\zeta_1|}.$$

This gives the required inequality with $B=4e^5$. □

Proof of Theorem 4.5. For the time being, we work with the capacity c_K, where K is the kernel $K(t) := \log^+(B/t)$, and B is the constant in Lemma 4.6. At the end of the proof we shall make the necessary adjustments to obtain the standard logarithmic capacity c.

Let $t > 0$ and let F be a compact subset of $\{\zeta \in \mathbb{T} : \widetilde{\mathcal{C}}g(\zeta) > t\}$. Let μ be a Borel probability measure on F. Then, by Fubini's theorem and Schwarz's inequality, we have

$$\begin{aligned} t &\le \int_F \widetilde{\mathcal{C}}g(\zeta)\,d\mu(\zeta) = \frac{1}{\pi}\int_{\mathbb{A}}\int_F \frac{|g(w)|}{|w-\zeta|}\,d\mu(\zeta)\,dA(w) \\ &\le \|g\|_{L^2(\mathbb{A})}\Big(\frac{1}{\pi}\int_{\mathbb{A}}\Big(\int_F \frac{1}{|w-\zeta|}\,d\mu(\zeta)\Big)^2\,dA(w)\Big)^{1/2}. \end{aligned}$$

Now, using Lemma 4.6, we have

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{A}}\Big(\int_F \frac{1}{|w-\zeta|}\,d\mu(\zeta)\Big)^2\,dA(w) &= \int_F\int_F \frac{1}{\pi}\int_{\mathbb{A}} \frac{dA(w)}{|w-\zeta_1||w-\zeta_2|}\,d\mu(\zeta_1)\,d\mu(\zeta_2) \\ &\le \int_F\int_F 2\log\frac{B}{|\zeta_1-\zeta_2|}\,d\mu(\zeta_1)\,d\mu(\zeta_2) = 2I_K(\mu). \end{aligned}$$

If we put all this together, then we obtain $t \le \|g\|_{L^2(\mathbb{A})}(2I_K(\mu))^{1/2}$, whence

$$1/I_K(\mu) \le 2\|g\|^2_{L^2(\mathbb{A})}/t^2.$$

As this holds for each $\mu \in \mathcal{P}(F)$, it follows that

$$c_K(F) \le 2\|g\|^2_{L^2(\mathbb{A})}/t^2.$$

And as this holds for each compact F in $\{\widetilde{\mathcal{C}}g > t\}$, we get

$$c_K(\widetilde{\mathcal{C}}g > t) \le 2\|g\|^2_{L^2(\mathbb{A})}/t^2.$$

It remains to relate c_K to c. For this, note that the difference between their respective kernels is $\log B - \log 2 = \log(B/2)$, and so $1/c_K(F) - 1/c(F) = \log(B/2)$ for all compact sets F. Hence

$$c(F)/c_K(F) = 1 + \log(B/2)c(F) \le 1 + \log(B/2)c(\mathbb{T}).$$

It follows that $c(\widetilde{\mathcal{C}}g > t) \le A\|g\|^2_{L^2(\mathbb{A})}/t^2$, where $A := 2(1 + \log(B/2)c(\mathbb{T}))$. □

Corollary 4.7. *If $g \in L^2(\mathbb{A})$, then $\widetilde{\mathcal{C}}g < \infty$ quasi-everywhere on $\mathbb{T}$.*

Proof. Since $\widetilde{\mathcal{C}}g$ is lower semicontinuous, for each $t > 0$ the set $\{\zeta \in \mathbb{T} : \widetilde{\mathcal{C}}g(\zeta) > t\}$ is open in $\mathbb{T}$. Thus $c^*(\widetilde{\mathcal{C}}g = \infty) \le c(\widetilde{\mathcal{C}}g > t) \le A\|g\|^2_{L^2(\mathbb{A})}/t^2$ for all $t > 0$. Letting $t \to \infty$, we get $c^*(\widetilde{\mathcal{C}}g = \infty) = 0$. □

Finally, we are in a position to prove Beurling's theorem.

Proof of Theorem 4.1. Let $f \in \mathcal{D}$. From Theorem 3.3 we know that $f = \mathcal{C}g$ for some $g \in L^2(\mathbb{A})$. By Theorem 4.4, $\mathcal{C}g$ has non-tangential limits at every $\zeta \in \mathbb{T}$ for which $\widetilde{\mathcal{C}}g(\zeta) < \infty$. And by Corollary 4.7, we have $\widetilde{\mathcal{C}}g(\zeta) < \infty$ quasi-everywhere. The proof is complete. □

5. Weak-type and strong-type inequalities

The proof of Beurling's theorem above yields a lot more information about the boundary values of functions in $\mathcal{D}$. In particular, the following important result is more or less an immediate consequence.

Theorem 5.1 (Capacitary weak-type inequality). *Let $f \in \mathcal{D}$. Then*

$$c^*(|f^*| > t) \le A\|f\|_{\mathcal{D}}^2/t^2 \qquad (t > 0), \tag{5.1}$$

where A is an absolute constant.

Proof. Using Theorem 3.3, we can find a function $g \in L^2(\mathbb{A})$ such that $f = \mathcal{C}g$ and $\|g\|_{L^2(\mathbb{A})} \le \|f\|_{\mathcal{D}}$. By Theorem 4.3 the set $\{\widetilde{\mathcal{C}}g > t\}$ is open in $\mathbb{T}$, and by Theorem 4.5 we have $c(\widetilde{\mathcal{C}}g > t) \le A\|g\|_{L^2(\mathbb{A})}^2/t^2$, where A is an absolute constant. Finally, $|f^*| = |\mathcal{C}g| \le \widetilde{\mathcal{C}}g$ q.e. on $\mathbb{T}$. Assembling the pieces, we obtain (5.1). □

Recall that $|\cdot|$ denotes Lebesgue measure on $\mathbb{T}$, and that it is related to logarithmic capacity via Theorem 2.15.

Corollary 5.2. *Let $f \in \mathcal{D}$. Then*

$$|\{|f^*| > t\}| \le Ae^{-Bt^2/\|f\|_{\mathcal{D}}^2} \qquad (t > 0),$$

where $A, B > 0$ are absolute constants.

Proof. Combine Theorems 5.1 and 2.15. □

Corollary 5.3. *Let $f \in \mathcal{D}$. Then* $\exp(|f^*|^2) \in L^1(\mathbb{T})$.

Proof. Assume first that $\|f\|_{\mathcal{D}}^2 < B$, where B is the constant in Corollary 5.2. Then

$$\begin{aligned}\int_{\mathbb{T}} (e^{|f^*(e^{i\theta})|^2} - 1)\, d\theta &= \int_{\mathbb{T}} \int_0^{|f^*(e^{i\theta})|} 2te^{t^2}\, dt\, d\theta = \int_{t=0}^{\infty} \int_{\{|f^*|>t\}} 2te^{t^2}\, d\theta\, dt \\ &= \int_{t=0}^{\infty} 2te^{t^2} |\{|f^*| > t\}|\, dt \le \int_{t=0}^{\infty} 2Ate^{t^2} e^{-Bt^2/\|f\|_{\mathcal{D}}^2}\, dt < \infty.\end{aligned}$$

For the general case, write $f = p + g$, where p is a polynomial and $\|g\|_{\mathcal{D}}^2 < B/2$. Then $|f^*|^2 \le 2|p|^2 + 2|g^*|^2$ q.e. on $\mathbb{T}$, whence

$$\int_{\mathbb{T}} e^{|f^*(e^{i\theta})|^2}\, d\theta \le e^{2\|p\|_\infty^2} \int_{\mathbb{T}} e^{2|g^*(e^{i\theta})|^2}\, d\theta < \infty.$$

□

This result is close to optimal. If $p > 2$ and $f = (\log \frac{1}{1-z})^{1/p}$, then $f \in \mathcal{D}$ but $\exp(|f^*|^p) \notin L^1(\mathbb{T})$.

The weak-type inequality, Theorem 5.1, can be strengthened a little further. This extra strength turns out to be vital for certain applications.

Theorem 5.4 (Capacitary strong-type inequality). *Let $f \in \mathcal{D}$. Then*

$$\int_0^\infty c^*(|f^*| > t)\, dt^2 \le A\|f\|_{\mathcal{D}}^2, \tag{5.2}$$

where A is an absolute constant.

Proof. Once again, it will be convenient to work with the capacity c_K, where $K(t) := \log^+(B/t)$ and B is the constant in Lemma 4.6. We shall prove that, if $g \in L^2(\mathbb{A})$, then

$$\int_0^\infty c_K(\widetilde{\mathcal{C}}g > t)\,dt^2 \le 32\|g\|^2_{L^2(\mathbb{A})}. \tag{5.3}$$

If so, then, since c is majorized by a multiple of c_K, the same inequality holds with c_K by c, and 32 replaced by another absolute constant. We can then finish off the proof by using the representation theorem for $\mathcal{D}$, exactly as in the proof of Theorem 5.1. Thus, it remains to prove (5.3).

Let $n \ge 1$ and, for $k = -n, \dots, n$, let F_k be a compact subset of $\{\widetilde{\mathcal{C}}g > 2^k\}$. Set $\mu := \sum_{k=-n}^n 2^k c_K(F_k)\nu_k$, where ν_k is an equilibrium measure for F_k. By Fubini's theorem and Schwarz's inequality, we have

$$\begin{aligned}\sum_{k=-n}^n 2^{2k} c_K(F_k) &\le \int_{\mathbb{T}} \widetilde{\mathcal{C}}g(\zeta)\,d\mu(\zeta) = \frac{1}{\pi}\int_{\mathbb{A}}\int_{\mathbb{T}} \frac{|g(w)|}{|w-\zeta|}\,d\mu(\zeta)\,dA(w)\\ &\le \|g\|_{L^2(\mathbb{A})}\Big(\frac{1}{\pi}\int_{\mathbb{A}}\Big(\int_{\mathbb{T}}\frac{1}{|w-\zeta|}\,d\mu(\zeta)\Big)^2 dA(w)\Big)^{1/2}.\end{aligned}$$

Using Lemma 4.6, we have

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{A}}\Big(\int_{\mathbb{T}}\frac{1}{|w-\zeta|}\,d\mu(\zeta)\Big)^2 dA(w) &= \int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{\pi}\int_{\mathbb{A}}\frac{dA(w)}{|w-\zeta_1||w-\zeta_2|}\,d\mu(\zeta_1)\,d\mu(\zeta_2)\\ &\le \int_{\mathbb{T}}\int_{\mathbb{T}} 2\log\frac{B}{|\zeta_1-\zeta_2|}\,d\mu(\zeta_1)\,d\mu(\zeta_2) = 2I_K(\mu).\end{aligned}$$

Now, by Corollary 2.17, we have $K\nu_k \le I_K(\nu_k)$ on $\mathbb{T}$. It follows that $\int K\nu_j\,d\nu_k = \int K\nu_k\,d\nu_j \le I(\nu_k)$ for all j,k. Therefore

$$\begin{aligned}I_K(\mu) &= \sum_{k=-n}^n\sum_{j=-n}^n 2^j c_K(F_j) 2^k c_K(F_k)\int_{\mathbb{T}} K\nu_j\,d\nu_k\\ &\le \sum_{k=-n}^n\sum_{j=-n}^n 2^j c_K(F_j) 2^k c_K(F_k)\min\{I_K(\nu_j), I_K(\nu_k)\}\\ &= \sum_{k=-n}^n\sum_{j=-n}^n 2^j 2^k \min\{c_K(F_j), c_K(F_k)\}\\ &\le 2\sum_{k=-n}^n\sum_{j=-n}^k 2^j 2^k c_K(F_k)\\ &\le 4\sum_{k=-n}^n 2^{2k} c_K(F_k).\end{aligned}$$

If we put all this together, then we obtain

$$\sum_{k=-n}^n 2^{2k} c_K(F_k) \le \|g\|_{L^2(\mathbb{A})}\Big(8\sum_{k=-n}^n 2^{2k} c_K(F_k)\Big)^{1/2},$$

whence

$$\sum_{k=-n}^n 2^{2k} c_K(F_k) \le 8\|g\|^2_{L^2(\mathbb{A})}.$$

As this holds for all such compact sets F_k, we get

$$\sum_{k=-n}^{n} 2^{2k} c_K(\widetilde{\mathcal{C}}g > 2^k) \le 8\|g\|^2_{L^2(\mathbb{A})}.$$

Finally, letting $n \to \infty$, we obtain

$$\sum_{k=-\infty}^{\infty} 2^{2k} c_K(\widetilde{\mathcal{C}}g > 2^k) \le 8\|g\|^2_{L^2(\mathbb{A})},$$

and this implies (5.3). □

6. Tangential approach regions

In this section we shall consider limits along certain exponentially tangential approach regions. Our aim is to prove the following result, due to Nagel, Rudin and Shapiro [28].

Theorem 6.1. *Let $f \in \mathcal{D}$. Then, for a.e. $\zeta \in \mathbb{T}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each region*

$$|z-\zeta| < \kappa\Big(\log\frac{1}{1-|z|}\Big)^{-1} \quad (\kappa > 0).$$

The proof is based on the representation formula of Theorem 3.3, and on a slightly more sophisticated version of the estimate (4.1). In order to state the estimate, we need to recall the notion of maximal function.

Given $h \in L^1(\mathbb{T})$, its *maximal function* $Mh : \mathbb{T} \to [0,\infty]$ is defined by

$$(Mh)(\zeta) := \sup_{\delta>0} \frac{1}{2\delta}\int_{|\theta-\arg\zeta|<\delta} |h(e^{i\theta})|\,d\theta.$$

Mh is lower semicontinuous, and it satisfies the weak-type inequality

$$|\{Mh > t\}| \le A\|h\|_{L^1(\mathbb{T})}/t \qquad (t > 0), \tag{6.1}$$

where A is an absolute constant. For more details, we refer to [32, §7].

Lemma 6.2. *Let $g \in L^2(\mathbb{A})$ and let $\zeta \in \mathbb{T}$. Then*

$$|\mathcal{C}g(z)| \le 2\widetilde{\mathcal{C}}g(\zeta) + 2\Big((Mh_g)(\zeta)|z-\zeta|\log\Big(\frac{3}{1-|z|}\Big)\Big)^{1/2} \qquad (z \in \mathbb{D}), \tag{6.2}$$

where

$$h_g(\zeta) := \int_1^2 |g(r\zeta)|^2 r\,dr \qquad (\zeta \in \mathbb{T}).$$

Proof. Fix $z \in \mathbb{D}$. Clearly

$$|\mathcal{C}g(z)| \le \frac{1}{\pi}\int_{\mathbb{A}} \frac{|g(w)|}{|w-z|}\,dA(w).$$

We shall estimate the right-hand side by splitting the domain of integration into two regions, $\mathbb{A}_1$ and $\mathbb{A}_2$, say. Let $\delta := |z-\zeta|$ and set $\mathbb{A}_1 := \mathbb{A} \cap \{w : |w-\zeta| \ge 2\delta\}$. For w in this region, we have

$$\Big|\frac{w-\zeta}{w-z}\Big| \le \frac{|w-\zeta|}{|w-\zeta|-\delta} \le 2,$$

and consequently

$$\frac{1}{\pi}\int_{\mathbb{A}_1}\frac{|g(w)|}{|w-z|}\,dA(w)\le\frac{1}{\pi}\int_{\mathbb{A}}\frac{2|g(w)|}{|w-\zeta|}\,dA(w)=2\widetilde{\mathcal{C}}g(\zeta).$$

In the complementary region $\mathbb{A}_2$, we apply Schwarz's inequality:

$$\frac{1}{\pi}\int_{\mathbb{A}_2}\frac{|g(w)|}{|w-z|}\,dA(w)\le\Bigl(\frac{1}{\pi}\int_{\mathbb{A}_2}|g(w)|^2\,dA(w)\Bigr)^{1/2}\Bigl(\frac{1}{\pi}\int_{\mathbb{A}_2}\frac{1}{|w-z|^2}\,dA(w)\Bigr)^{1/2}.$$

Now $\mathbb{A}_2$ is contained in the sector $\{1<r<2,\ |\theta-\arg\zeta|<\pi\delta\}$. Hence

$$\frac{1}{\pi}\int_{\mathbb{A}_2}|g(w)|^2\,dA(w)\le\frac{1}{\pi}\int_{|\theta-\arg\zeta|<\pi\delta}h_g(e^{i\theta})\,d\theta\le 2\delta(Mh_g)(\zeta).$$

Also, we have

$$\frac{1}{\pi}\int_{\mathbb{A}_2}\frac{1}{|w-z|^2}\,dA(w)\le\frac{1}{\pi}\int_{1-|z|<|w|<3}\frac{1}{|w|^2}\,dA(w)=2\log\frac{3}{1-|z|}.$$

Putting all these estimates together, we obtain (6.2). □

Proof of Theorem 6.1. Let $f\in\mathcal{D}$. By Theorem 3.3 we can write $f=\mathcal{C}g$, where $g\in L^2(\mathbb{A})$. We then have $f(z)-f^*(\zeta)=\mathcal{C}g(z)-\mathcal{C}g(\zeta)$ for all $z\in\mathbb{D}$ and a.e. $\zeta\in\mathbb{T}$. For $\delta>0$, let us set $g_\delta(w):=g(w)1_{\{1<|w|<1+\delta\}}$. Just as in the proof of Theorem 4.4, we have

$$|\mathcal{C}g(z)-\mathcal{C}g(\zeta)|\le|\mathcal{C}(g-g_\delta)(z)-\mathcal{C}(g-g_\delta)(\zeta)|+|\mathcal{C}g_\delta(z)|+|\mathcal{C}g_\delta(\zeta)|,$$

where the first term tends to zero as $z\to\zeta$ unrestrictedly in $\mathbb{D}$, and the third term satisfies $|\mathcal{C}g_\delta(\zeta)|\le\widetilde{\mathcal{C}}g(\zeta)$. As for the second term, if we write

$$\Omega_\kappa(\zeta):=\{z\in\mathbb{D}:|z-\zeta|<\kappa/\log(3/(1-|z|))\},$$

then by Lemma 6.2

$$|\mathcal{C}g_\delta(z)|\le 2\widetilde{\mathcal{C}}g_\delta(\zeta)+2\bigl(\kappa(Mh_{g_\delta})(\zeta)\bigr)^{1/2}\qquad(z\in\Omega_\kappa(\zeta)).$$

Putting this all together, we deduce that

$$\limsup_{\substack{z\to\zeta\\ z\in\Omega_\kappa(\zeta)}}|f(z)-f^*(\zeta)|\le 3\widetilde{\mathcal{C}}g_\delta(\zeta)+2\bigl(\kappa(Mh_{g_\delta})(\zeta)\bigr)^{1/2}.$$

Now, by Theorems 4.5 and 2.15, we have

$$|\{\widetilde{\mathcal{C}}g_\delta>t\}|\le Ae^{-Bt^2/\|g_\delta\|^2_{L^2(\mathbb{A})}},$$

and by (6.1)

$$|\{Mh_{g_\delta}>t\}|\le A\|h_{g_\delta}\|_{L^1(\mathbb{T})}/t\le A'\|g_\delta\|^2_{L^2(\mathbb{A})}/t,$$

where A,A',B are absolute constants. Also, clearly, $\|g_\delta\|_{L^2(\mathbb{A})}\to 0$ as $\delta\to 0$. Hence, given $\epsilon>0$, we can choose $\delta>0$ small enough to ensure that both $|\{\widetilde{\mathcal{C}}g_\delta>\epsilon\}|<\epsilon$ and $|\{Mh_{g_\delta}>\epsilon\}|<\epsilon$. It follows that

$$\limsup_{\substack{z\to\zeta\\ z\in\Omega_\kappa(\zeta)}}|f(z)-f^*(\zeta)|\le 3\epsilon+2(\kappa\epsilon)^{1/2}$$

for all $\zeta\in\mathbb{T}$ outside a set of measure 2ϵ. Letting $\epsilon\to 0$, we deduce that

$$\lim_{\substack{z\to\zeta\\ z\in\Omega_\kappa(\zeta)}}f(z)=f^*(\zeta)$$

for all $\zeta \in \mathbb{T}$ outside a set of measure zero. □

7. Further results

In this section we discuss very briefly and, for the most part, without detailed proofs, some extensions of the principal results in this article, as well as some theorems concerning their sharpness.

The two main theorems on boundary limits, namely Beurling's theorem (Theorem 4.1) and the theorem of Nagel–Rudin–Shapiro (Theorem 6.1), are the simplest in a whole scale of results of this kind, with a playoff between the width of the approach region and the size of the exceptional set. The following result is taken from [36], where it is also shown to be optimal in a certain sense.

Theorem 7.1. *Let $f \in \mathcal{D}$.*

(i) For all $\zeta \in \mathbb{T}$ outside a set of logarithmic capacity zero, $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each region

$$\{z : |z-\zeta| < \kappa(1-|z|)^\lambda\} \qquad (\kappa > 0,\ \lambda > 0).$$

(ii) Given $0 < \gamma \le 1$, for all $\zeta \in \mathbb{T}$ outside a set of γ-dimensional Hausdorff content zero, $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in each region

$$\Big\{z : |z-\zeta| < \kappa\Big(\log\frac{1}{1-|z|}\Big)^{-1/\gamma}\Big\} \qquad (\kappa > 0).$$

It is not possible to widen the approach region any further than in (ii). Indeed, given a function $\psi : (0,1) \to \mathbb{R}^+$ with $\lim_{t\to 0}\psi(t)\log(1/t) = \infty$, it is possible to construct $f \in \mathcal{D}, f \not\equiv 0$ with infinitely many zeros in $\Omega(\zeta) := \{z : |z-\zeta| < \psi(1-|z|)\}$ for each $\zeta \in \mathbb{T}$. Consequently, whenever $\lim_{z\to\zeta, z\in\Omega(\zeta)} f(z)$ exists, it must equal zero, and this can only happen for ζ in a set of measure zero. (This argument is taken from [28].)

The fact that logarithmic capacity is optimal in part (i) can be demonstrated using a construction of Carleson [10] (where it was actually used to prove something a little different). The construction shows that, given a compact subset E of $\mathbb{T}$ of logarithmic capacity zero, there exists $f \in \mathcal{D}$ such that $\lim_{r\to 1}|f(r\zeta)| = \infty$ for all $\zeta \in E$. By refining this construction, one can obtain the following more precise result, taken from [15].

We write d for arclength distance in $\mathbb{T}$ and $E_t := \{\zeta \in \mathbb{T} : d(\zeta, E) \le t\}$.

Theorem 7.2. *Let E be a closed subset of $\mathbb{T}$, and let $\eta : (0,\pi] \to \mathbb{R}^+$ be a decreasing continuous function. The following statements are equivalent:*

(i) there exists $f \in \mathcal{D}$ such that

$$\liminf_{z\to\zeta} |f(z)| \ge \eta(d(\zeta, E)) \quad (\zeta \in \mathbb{T});$$

(ii) there exists $f \in \mathcal{D}$ such that $|\operatorname{Im} f(z)| \le 1$ and

$$\liminf_{z\to\zeta} \operatorname{Re} f(z) \ge \eta(d(\zeta, E)) \quad (\zeta \in \mathbb{T});$$

(iii) E and η satisfy

$$\int_0^\pi c(E_t)\,|d\eta^2(t)| < \infty. \tag{7.1}$$

That (ii) implies (i) is obvious, and that (i) implies (iii) follows easily from the strong-type inequality, Theorem 5.4. The main thrust of the theorem is the implication that (iii) implies (ii), which can be considered as a partial converse to the strong-type inequality.

It is shown in [15] that (7.1) is implied by the condition

$$\int_0^1 |E_t|\eta'(t)^2\,dt<\infty. \tag{7.2}$$

Thus (7.2) is sufficient for (i) and (ii) to hold. We conclude this section by giving a simple direct proof of this fact in the case when η is convex. It is a consequence of the following theorem (in which we write 'dist' for Euclidean distance).

Theorem 7.3. *Let E be a closed subset of $\mathbb{T}$, and let $\eta:(0,2]\to\mathbb{R}^+$ be a decreasing convex function. If (7.2) holds, then there exists $f\in\mathcal{D}$ such that*

$$\operatorname{Re} f(z)\ge \eta(\operatorname{dist}(z,E))\quad (z\in\mathbb{D}). \tag{7.3}$$

Proof. Let $\mathbb{A}:=\{w:1<|w|<2\}$, and define $g:\mathbb{A}\to\mathbb{C}$ by

$$g(w):=-3w\eta'\big(\operatorname{dist}(w,E)/3\big)\qquad (w\in\mathbb{A}).$$

We claim that $g\in L^2(\mathbb{A})$. Indeed, writing $w=re^{i\theta}$ and denoting by d the arclength distance, we have

$$\operatorname{dist}(w,E)\ge \max\big\{(r-1),\ (2/\pi)d(e^{i\theta},E)\big\}\ge (r-1)/2+d(e^{i\theta},E)/\pi.$$

Therefore, since $|\eta'(t)|$ is a decreasing function,

$$\begin{aligned}\int_{\mathbb{A}}|g(w)|^2\,dA(w)&\le \int_{\theta=0}^{2\pi}\int_{r=1}^{2}9r^2\Big|\eta'\Big(\frac{r-1}{6}+\frac{1}{3\pi}d(e^{i\theta},E)\Big)\Big|^2 r\,dr\,d\theta\\ &\le 72\int_{\theta=0}^{2\pi}\int_{t=d(e^{i\theta},E)/3\pi}^{1}|\eta'(t)|^2\,dt\,d\theta\\ &=72\int_{t=0}^{1}\int_{d(e^{i\theta},E)\le 3\pi t}|\eta'(t)|^2\,d\theta\,dt\\ &=72\int_{t=0}^{1}|E_{3\pi t}||\eta'(t)|^2\,dt\\ &\le 216\pi\int_{t=0}^{1}|E_t||\eta'(t)|^2\,dt.\end{aligned}$$

By (7.2), this last integral is finite, thereby justifying our claim. (In fact, a very similar calculation shows that (7.2) is also a necessary condition for $g\in L^2(\mathbb{A})$.)

Now set $f:=\mathcal{C}g+\eta(1/3)$, where $\mathcal{C}$ denotes the Cauchy transform. Explicitly,

$$f(z):=\frac{1}{\pi}\int_{\mathbb{A}}\Big(\frac{3w}{w-z}\Big)\big|\eta'\big(\operatorname{dist}(w,E)/3\big)\big|\,dA(w)+\eta(1/3)\qquad (z\in\mathbb{D}).$$

By Theorem 3.2, we have $f\in\mathcal{D}$. We shall show that f satisfies (7.3). Performing the change of variable $z\mapsto e^{i\theta}z$, and using the obvious rotation-invariance properties of f, it is enough to prove (7.3) in the case when $z=x\in[0,1)$. Furthermore, since $\operatorname{Re}(w/(w-x))$ is always positive for $w\in\mathbb{A}$, it is clear that $\operatorname{Re} f(x)\ge\eta(1/3)$, so (7.3) is certainly true if $\operatorname{dist}(x,E)\ge 1/3$.

Suppose now that $z = x \in [0,1)$ and that $\operatorname{dist}(x,E) < 1/3$. We make the change of variable $w = x + \rho e^{i\phi}$. Note that $w \in \mathbb{A}$ provided that $2\operatorname{dist}(x,E) < \rho < 1$ and $0 < \phi < \pi/3$. Indeed, under these conditions, we have

$$|x + \rho e^{i\phi}| \le x + \rho < 2 \quad \text{and} \quad |x + \rho e^{i\phi}| \ge x + \rho\cos\phi \ge x + \operatorname{dist}(x,E) \ge 1.$$

Also $\operatorname{Re}(3w/(w-x)) = 3 + \operatorname{Re}(3x/\rho e^{i\phi}) \ge 1/\rho$ and $\operatorname{dist}(w,E) \le \operatorname{dist}(x,E) + \rho$. Hence, after making the change of variable, we have

$$\begin{aligned}\operatorname{Re} f(x) &\ge \frac{1}{\pi}\int_0^{\pi/3}\int_{2\operatorname{dist}(x,E)}^1 \frac{1}{\rho}\big|\eta'\big((\operatorname{dist}(x,E)+\rho)/3\big)\big|\,\rho\,d\rho\,d\phi + \eta(1/3)\\ &= \frac{1}{3}\int_{2\operatorname{dist}(x,E)}^1 \big|\eta'\big((\operatorname{dist}(x,E)+\rho)/3\big)\big|\,d\rho + \eta(1/3) \ge \eta(\operatorname{dist}(x,E)).\end{aligned}$$

This gives (7.3) and completes the proof. □

8. Bibliographical remarks

§2. Logarithmic capacity can be defined in many different ways: in terms of energy [2, 5, 19, 21, 24, 31, 33, 34], potentials [1, 3, 11, 22, 27], réduites [4, 13, 20], Green's functions [17, 23, 29], and transfinite diameter [30]. We have followed the energy definition. Though these definitions may give rise to different numbers for the logarithmic capacity of a set, they are all equivalent in the sense that any one of them can be made small if any other is sufficiently small. In particular, the sets of logarithmic capacity zero are the same, whichever definition is used.

§3. The representation theorem 3.3 is due to Dyn′kin [14]. The simple proof given here is taken from Borichev [8].

§4. Beurling's theorem (for radial limits) appeared in [7]. Our approach follows Borichev [8].

§5. The weak-type capacitary inequality is due to Beurling [7]. He had earlier obtained Corollary 5.2 in his thesis [6], showing that, if $f(0) = 0$ and $\mathcal{D}(f) \le 1$, then $|\{|f^*| > t\}| \le 2\pi e^{1-t^2}$. This was later strengthened by Chang and Marshall [12] (see also [16] and [25]), who proved that

$$\sup\Big\{\int_{\mathbb{T}} \exp(|f^*|^2)\,d\theta : f(0) = 0,\ \mathcal{D}(f) \le 1\Big\} < \infty.$$

The strong-type capacitary inequality is due to Hansson [18]. Our proof is adapted from that given in [1].

§6. Theorem 6.1 is due to Nagel, Rudin and Shapiro [28]. Our approach is an adaptation of their ideas and those in [8].

§7. For more information about the playoff between the width of the approach region and the size of the exceptional set, we refer to [8, 9, 35, 36]. Theorem 7.2 is taken from [15], where it is stated in a slightly different form. Theorem 7.3 we believe to be new.

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