Modelization of a Split in a Ferromagnetic Body by an Equivalent Boundary Condition: Part II.

THE INFLUENCE OF SUPER-EXCHANGE AND SURFACE ANISOTROPY.

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Abstract. We continue the study of equivalent boundary conditions in ferromagnetic domains crossed by a thin split. In this second part, we add nonhomogenous boundary conditions arising from interactions such as surface anisotropy and super-exchange. We expand the problem up to the first order and establish equivalent boundary conditions in presence of surface anisotropy and super-exchange. In particular, the well-posedness of the expansion problem with equivalent boundary condition and the convergence in some meaning of the expansion are proven.

Introduction

Following part I [8], we study the behavior of a ferromagnetic domain crossed by a thin split. In order to efficiently compute the evolution of the magnetization on such a geometry, we expanded the magnetization $\boldsymbol{m}^{(0),\varepsilon} = \boldsymbol{m}^{(0)} + \varepsilon\boldsymbol{m}^{(1)}$ on Ω_ε and derived an equivalent boundary condition on the contact surface:

$$\frac{\partial \boldsymbol{m}^{(1)}}{\partial \boldsymbol{\nu}} = \frac{\partial^2 \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}^2}.$$

In this second part of the article, we extend our results when boundary interactions such as super-exchange and surface anisotropy are present [4]. The mathematical effect of these interactions is to modify the Neumann boundary condition in a nonlinear way. The new terms will be described in section 1. In the same paper, the existence, but not the uniqueness, of infinite time weak solutions is also proved.

We denote by $\boldsymbol{m}$ the dimensionless magnetization. In the expansion, $\boldsymbol{m}^{(0)}$ represents the term of order 0 and $\boldsymbol{m}^{(1)}$ the term of order 1. Formally $\boldsymbol{m} = \boldsymbol{m}^{(0)} + \varepsilon\boldsymbol{m}^{(1)}$, where ε is the half-thickness of the split. The considered geometry is presented in Figure 1. We use the same notations as in part I. Let

- ε the half thickness of the split, always verifying $\varepsilon \ll \min(L^+, L^-)$.
- B a bounded convex open set of $\mathbb{R}^2$, with a smooth boundary.
- L^+, L^- be two nonzero positive numbers.
- $\Omega_\varepsilon^+ = B \times (\varepsilon, L^+)$ and $\Omega_\varepsilon^- = B \times (-L^-, -\varepsilon)$ for all $\varepsilon < \min(L^-, L^+)/2$ are the domains filled with the ferromagnetic material.
- $\Omega^+ = B \times (0, L^+)$ and $\Omega^- = B \times (-L^-, 0)$.

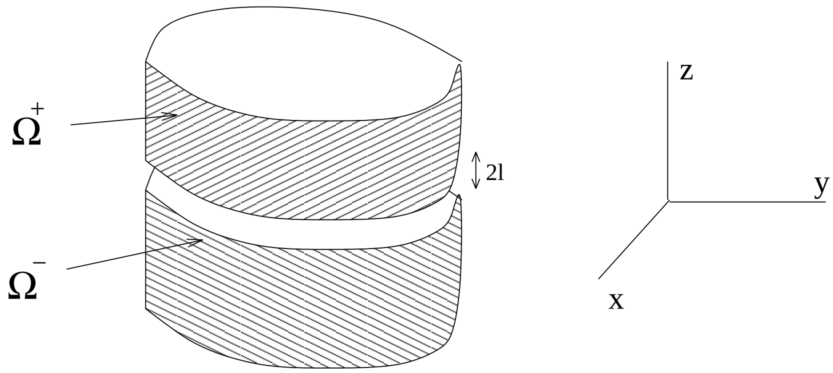

Figure 1: Geometry of the problem

- $\Omega = \Omega^+ \cup \Omega^-$ and $\Omega_\varepsilon = \Omega_\varepsilon^+ \cup \Omega_\varepsilon^-$ for all $\varepsilon < \min(L^-, L^+)/2$.
- $Q_T^\varepsilon = \Omega_\varepsilon \times (0,T)$, for all $\varepsilon < \min(L^-, L^+)/2$ and $Q_T = \Omega \times (0,T)$.
- $\Gamma_\varepsilon^+ = B \times \{+\varepsilon\}$, $\Gamma_\varepsilon^- = B \times \{-\varepsilon\}$ and $\Gamma_\varepsilon^\pm = \Gamma_\varepsilon^+ \cup \Gamma_\varepsilon^-$. When ε is omitted, it corresponds to $\varepsilon = 0$.
- γ_ε^0 is the map that sends $\boldsymbol{m}$ to its trace on $\Gamma_\varepsilon^\pm$.
- $\gamma_\varepsilon^{0,\prime}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_\varepsilon^0(\boldsymbol{m} \circ \sigma)$, where σ is the application that sends (x,y,z,t) to $(x,y,-z,t)$.
- We define the surface $\Gamma = B \times \{0\}$. $\gamma_\varepsilon^{0,+}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_0^0(\boldsymbol{m} \circ \tau_{-\varepsilon})$ on Γ^+, where $\tau_{-\varepsilon}(x,y,z,t) = (x,y,z+\varepsilon,t)$. $\gamma_\varepsilon^{0,-}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_0^0(\boldsymbol{m} \circ \tau_{+\varepsilon})$ on Γ^-.
- γ_ε^1 is the map that sends $\boldsymbol{m}$ to its normal trace $\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}}$ on $\Gamma_\varepsilon^\pm$.
- $\gamma_\varepsilon^{1,\prime}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_\varepsilon^1(\boldsymbol{m} \circ \sigma)$. (x,y,z,t) to $(x,y,-z,t)$.
- $\gamma_\varepsilon^{1,+}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_0^1(\boldsymbol{m} \circ \tau_{-\varepsilon})$ on Γ^+. $\gamma_\varepsilon^{1,-}$ is the trace map that sends $\boldsymbol{m}$ to $\gamma_0^1(\boldsymbol{m} \circ \tau_\varepsilon)$ on Γ^-.
- $\boldsymbol{\nu}$ represents the unitary exterior normal to the surface boundary of an open set, usually Ω_ε or Ω.

In this second part, we use the same notations concerning Sobolev spaces as in section 2.1 of [8]. In particular, $\mathrm{H}^s(\Omega)$ are the classical Sobolev spaces as defined in [1], $\mathbb{H}^s(\Omega) = (\mathrm{H}^s(\Omega))^3$, and

$$\mathbb{H}^{p,q}(\Omega \times (0,T)) = \mathrm{H}^q(0,T;\mathbb{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathbb{H}^p(\Omega)). \tag{0.1}$$

as in Lions-Magenes [5]. We make an extensive use of the spaces $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ and $\mathbb{H}^{2,1}(\Omega \times (0,T))$. By $\mathbb{L}^p(\Omega)$, we denote $(\mathrm{L}^p(\Omega))^3$.

In section 1, we describe the super-exchange and surface anisotropy interactions along with their energies and operators. The description of the other interactions can be found in section 1 of [8]. We also give the complete description of the Landau-Lifshitz system and its linearization. We also state in this section the well posedness of the Landau-Lifchitz system with super-exchange and surface anisotropy as proved in [9]. In section 2, we formally derive the equivalent boundary condition. In section 3, we give additional inequalities which are not contained in section 2.3.1 of [8]. In section 4, we prove the existence of the first order term. Then, in section 5, we prove the weak $\mathbb{H}^{2,1}$ convergence of the expansion at first order. Finally, in section 6, we provide the results of some numerical simulations which compute the effect of super-exchange and surface anisotropy to the magnetization terms of both order 0 and 1.

1 The mathematical model

We only describe the additional material compared to part I section 1. Readers should refer to it for more information.

1.1 The surface anisotropy interaction

The surface anisotropy energy, see [4], and the associated operator are

$$\mathrm{E}_{sa} = \frac{K_s}{2}\int_{\Gamma_\varepsilon^-\cup\Gamma_\varepsilon^+}(1-(\gamma_0^0\boldsymbol{m}\cdot\boldsymbol{\nu})^2)\mathrm{d}\sigma(\boldsymbol{x}),$$
$$\mathcal{H}_{sa} = K_s((\gamma_0^0\boldsymbol{m}\cdot\boldsymbol{\nu})\boldsymbol{\nu}-\gamma_0^0\boldsymbol{m}) \text{ on } \Gamma_\varepsilon^-\cup\Gamma_\varepsilon^+.$$

when the ferromagnetic material fills Ω_ε. This operator has basically the same form as the volume anisotropy uniaxis operator but with $\boldsymbol{u}=\boldsymbol{\nu}$.

1.2 The super-exchange interaction

This interaction has its roots in quantum mechanics—see [2]. We use the mathematical model found in [4]. If the ferromagnetic material fills Ω_ε, the energy and the operator associated to the super-exchange operator are

$$\mathrm{E}_{se}(\boldsymbol{m}) = J_1\int_\Gamma(1-\gamma_\varepsilon^{0+}\boldsymbol{m}\cdot\gamma_\varepsilon^{0-}\boldsymbol{m})\mathrm{d}\sigma(\boldsymbol{x}) + J_2\int_\Gamma(1-|\gamma_\varepsilon^{0+}\boldsymbol{m}\cdot\gamma_\varepsilon^{0-}\boldsymbol{m}|^2)\mathrm{d}\sigma(\boldsymbol{x}),$$
$$\mathcal{H}_{se} = J_1(\gamma_\varepsilon^{0,\prime}\boldsymbol{m}-\gamma_\varepsilon^{0}\boldsymbol{m}) + 2J_2((\gamma_\varepsilon^{0}\boldsymbol{m}\cdot\gamma_\varepsilon^{0,\prime}\boldsymbol{m})\gamma_\varepsilon^{0,\prime}\boldsymbol{m} - |\gamma_\varepsilon^{0,\prime}\boldsymbol{m}|^2\gamma_\varepsilon^{0}\boldsymbol{m}) \text{ on } \Gamma^-\cup\Gamma^+.$$

where J_1, J_2 are positive real numbers. In reality, J_1 and J_2 depend on the distance ε, but as they converge as ε tends to 0, we will consider them to be constant throughout this article.

1.3 The boundary condition

For the remaining part of the article, we define

$$\mathcal{H}_{d,a} = \mathcal{H}_d+\mathcal{H}_a, \qquad \mathcal{H}_v = \mathcal{H}_d+\mathcal{H}_a+\mathcal{H}_e, \qquad \mathcal{H}_s = \mathcal{H}_{sa}+\mathcal{H}_{se}. \tag{1.1}$$

where $\mathcal{H}_d, \mathcal{H}_a, \mathcal{H}_e$ are defined in section 1.2 in part I. The boundary conditions on $\boldsymbol{m}$ are

$$\frac{\partial\boldsymbol{m}}{\partial\boldsymbol{\nu}} = 0 \text{ on } \partial\Omega_\varepsilon\setminus\Gamma_\varepsilon^\pm, \tag{1.2a}$$
$$\frac{\partial\boldsymbol{m}}{\partial\boldsymbol{\nu}} = \frac{1}{A}\left(\mathcal{H}_s(\boldsymbol{m}) - \left(\gamma_0^0\boldsymbol{m}\cdot\mathcal{H}_s(\boldsymbol{m})\right)\gamma_0^0\boldsymbol{m}\right) \text{ on } \Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-. \tag{1.2b}$$

These are obtained as the Euler-Lagrange conditions on the boundary.

1.4 The Landau-Lifshitz system

The Landau-Lifshitz system is

$$\frac{\partial \boldsymbol{m}^\varepsilon}{\partial t} = -\boldsymbol{m}^\varepsilon \times \mathcal{H}_v(\boldsymbol{m}^\varepsilon) - \alpha \boldsymbol{m}^\varepsilon \times (\boldsymbol{m}^\varepsilon \times \mathcal{H}_v(\boldsymbol{m}^\varepsilon)) \text{ in } \Omega_\varepsilon \times (0,T), \tag{1.3a}$$

$$|\boldsymbol{m}^\varepsilon| = 1 \text{ a.e. in } \Omega_\varepsilon \times (0,T), \tag{1.3b}$$

$$\boldsymbol{m}^\varepsilon(\cdot,0) = \boldsymbol{m}_0^\varepsilon, \tag{1.3c}$$

$$\frac{\partial \boldsymbol{m}^\varepsilon}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma, \\ Q^+(\gamma_0^0 \boldsymbol{m}^\varepsilon, \gamma_0^{0,\prime} \boldsymbol{m}^\varepsilon) & \text{on } \Gamma_\varepsilon^+, \\ Q^-(\gamma_0^0 \boldsymbol{m}^\varepsilon, \gamma_0^{0,\prime} \boldsymbol{m}^\varepsilon) & \text{on } \Gamma_\varepsilon^-, \end{cases} \tag{1.3d}$$

where $Q^\pm(\gamma_\varepsilon^0 \boldsymbol{m}, \gamma_\varepsilon^{0,\prime} \boldsymbol{m}) = Q_r^\pm(\gamma_\varepsilon^0 \boldsymbol{m}, \gamma_\varepsilon^{0,\prime} \boldsymbol{m}) - (Q_r^\pm(\gamma_\varepsilon^0 \boldsymbol{m}, \gamma_\varepsilon^{0,\prime} \boldsymbol{m}) \cdot \gamma_\varepsilon^0 \boldsymbol{m})\gamma_\varepsilon^0 \boldsymbol{m}$ with Q_r being a polynomial in two variables. It is left to the reader to verify that conditions (1.2) are a particular case of conditions (1.3d). In [9], we proved the following theorem of existence.

Theorem 1.1. *If the initial condition $\boldsymbol{m}_0^\varepsilon$ belongs to $\mathbb{H}^2(\Omega_\varepsilon)$ and satisfies the boundary condition*

$$\frac{\partial \boldsymbol{m}_0^\varepsilon}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma_\varepsilon^\pm, \\ Q^+(\gamma_\varepsilon^0 \boldsymbol{m}_0^\varepsilon, \gamma_\varepsilon^{0,\prime} \boldsymbol{m}_0^\varepsilon) & \text{on } \Gamma_\varepsilon^+, \\ Q^-(\gamma_\varepsilon^0 \boldsymbol{m}_0^\varepsilon, \gamma_\varepsilon^{0,\prime} \boldsymbol{m}_0^\varepsilon) & \text{on } \Gamma_\varepsilon^-, \end{cases} \tag{1.4}$$

and $|\boldsymbol{m}_0| = 1$, then there exists a unique $T^* > 0$ and $\boldsymbol{m}^\varepsilon$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$ satisfying (1.3). *The system is well-posed.*

Anticipating the expansion of the Landau-Lifshitz system, we introduce a general form of the linearized Landau-Lifshitz system.

$$\begin{aligned}\frac{\partial \boldsymbol{w}}{\partial t} &= -\boldsymbol{m} \times \mathcal{H}_v(\boldsymbol{w}) - \boldsymbol{w} \times \mathcal{H}_v(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}_v(\boldsymbol{w})) \\ &\quad - \alpha \boldsymbol{m} \times (\boldsymbol{w} \times \mathcal{H}_v(\boldsymbol{m})) - \alpha \boldsymbol{w} \times (\boldsymbol{m} \times \mathcal{H}_v(\boldsymbol{m})) + \theta \text{ on } \Omega \times (0,T^*),\end{aligned} \tag{1.5a}$$

$$\boldsymbol{m} \cdot \boldsymbol{w} = 0, \tag{1.5b}$$

$$\boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0. \tag{1.5c}$$

$$\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma \times (0,T), \\ \mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{w}, \gamma_0^{0,\prime} \boldsymbol{w}) + \beta^+ & \text{on } \Gamma^+ \times (0,T), \\ \mathrm{D}Q^-(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{w}, \gamma_0^{0,\prime} \boldsymbol{w}) + \beta^- & \text{on } \Gamma^- \times (0,T). \end{cases} \tag{1.5d}$$

where $\boldsymbol{w}$ is the unknown. β^+, β^-, θ, $\boldsymbol{w}$, $\boldsymbol{m}$ are in appropriate functions spaces with values in $\mathbb{R}^3$. Q^+, Q^- are the polynomials defined in (1.3d). D is the differentiation operator.

Definition 1.2. We define $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma \times (0,T))$ as the subset of $\mathbb{H}^{\frac{3}{2},\frac{3}{4}}(\Gamma \times (0,T))$ containing all functions g such that

$$\int_0^T \int_\Gamma |g(\boldsymbol{x},t)|^2 \frac{\mathrm{d}\boldsymbol{x}}{\rho(\boldsymbol{x})} \mathrm{d}\sigma(\boldsymbol{x}) \mathrm{d}t < +\infty,$$

where $\rho(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Gamma)$.

This is to ensure compatibility relations, see Lions-Magenes [5]. The existence and uniqueness problem for the $\boldsymbol{m}^{(1)}$ are stated by the next two theorems.

Theorem 1.3. *Let $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ be a solution to the Landau-Lifshitz system with upper time T^*. Let θ be in $\mathbb{H}^{1,\frac{1}{2}}(\Omega \times (0,T))$ for all $T < T^*$ such that θ is orthogonal to $\boldsymbol{m}$ almost everywhere. Let β^+, β^- in $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma^\pm \times (0,T))$, orthogonal almost everywhere on the boundary $\Gamma \times (0,T)$ to $\boldsymbol{m}$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^2(\Omega)$ orthogonal almost everywhere in Ω to $\boldsymbol{m}$, such that*

$$\begin{aligned} \frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} &= 0 \text{ on } \partial\Omega \setminus \Gamma, \\ \frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} &= \mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}_0, \gamma_0^{0,\prime} \boldsymbol{m}_0)(\gamma_0^0 \boldsymbol{w}_0, \gamma_0^{0,\prime} \boldsymbol{w}_0) + \beta^+(\cdot, 0) \text{ on } \Gamma^+, \\ \frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} &= \mathrm{D}Q^-(\gamma_0^0 \boldsymbol{w}_0, \gamma_0^{0,\prime} \boldsymbol{w}_0)(\gamma_0^0 \boldsymbol{w}_0, \gamma_0^{0,\prime} \boldsymbol{w}_0) + \beta^-(\cdot, 0) \text{ on } \Gamma^-. \end{aligned} \tag{1.6}$$

Then, there exists a unique $\boldsymbol{w}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ solution to system (1.5).

The following theorem solves the existence problem of $\boldsymbol{m}^{(1)}$. It lowers the requirements of the previous theorem on the regularity of the data as well as the regularity of the solution.

Theorem 1.4. *Let $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ be a solution to Landau-Lifshitz system with upper time T^*. Let θ be in $\mathbb{L}^2(\Omega \times (0,T))$ for all $T < T^*$ such that θ is orthogonal to $\boldsymbol{m}$ almost everywhere. Let β^+, β^- belonging to $\mathbb{H}^{\frac{1}{2},\frac{1}{4}}(\Gamma^\pm \times (0,T))$, $\beta^\pm \cdot \boldsymbol{m} = 0$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^1(\Omega)$ such that $\boldsymbol{m}_0 \cdot \boldsymbol{w}_0 = 0$. Then, there exists a unique $\boldsymbol{w}$ in $\mathbb{H}^{2,1}(\Omega \times (0,T))$ solution to system* (1.5).

The complete proofs of both theorems can be found at section 4.

Remark 1.5. Theorems 1.4 and 1.3 also hold if we replace Ω by Ω_ε.

2 The equivalent boundary condition

As in the first part, we consider for ε small enough an initial condition $\boldsymbol{m}_0^\varepsilon$ belonging to $\mathbb{H}^2(\Omega)$, satisfying (1.4) on $\Gamma_\varepsilon^\pm$. We suppose that there exists $\boldsymbol{m}_0^{(1)}$ such that

$$\|\boldsymbol{m}_0^{(0)} - \boldsymbol{m}_0^{\varepsilon,(0)}\|_{\mathbb{H}^2(\Omega_\varepsilon)} = O(1), \quad \|\boldsymbol{m}_0^{(0)} - \boldsymbol{m}_0^{\varepsilon,(0)}\|_{\mathbb{H}^1(\Omega_\varepsilon)} = O(\varepsilon), \tag{2.1a}$$

$$\frac{\boldsymbol{m}_0^{(0),\varepsilon} - \boldsymbol{m}_0^{(0)}}{\varepsilon} \to \boldsymbol{u}^1_{|\Omega_{\varepsilon_0}} = \boldsymbol{m}_0^{(1)} \text{ weakly in } \mathbb{H}^1(\Omega_{\varepsilon_0}) \text{ for all } \varepsilon_0 > 0. \tag{2.1b}$$

We then define $\boldsymbol{m}^\varepsilon$ as the solution to the Landau-Lifshitz system (1.3), with initial condition $\boldsymbol{m}_0^\varepsilon$ on Ω_ε. If we formally expand $\boldsymbol{m}^\varepsilon = \boldsymbol{m}^{(0)} + \varepsilon \boldsymbol{m}^{(1)}$ up to the first order, we obtain as in [8], equations on $\boldsymbol{m}^{(1)}$ and $\boldsymbol{m}^{(0)}$. Formally, $\boldsymbol{m}^{(0)} = \boldsymbol{m}^0$ and is the solution $\boldsymbol{m}$ to the Landau-Lifshitz system (1.3) with initial condition $\boldsymbol{m}_0^{(0)} = \boldsymbol{m}_0^0$ on domain Ω. $\boldsymbol{m}^{(1)}$ formally satisfy

$$\begin{aligned} \frac{1}{\varepsilon} \frac{\partial(\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{(0)})}{\partial z}(\cdot,\cdot,\varepsilon,\cdot) \approx& -\frac{1}{\varepsilon} Q^+(\gamma_\varepsilon^{0+} \boldsymbol{m}^\varepsilon, \gamma_\varepsilon^{0-} \boldsymbol{m}^\varepsilon) - Q^+(\gamma_{0\,\varepsilon}^{0+} \boldsymbol{m}^{(0)}, \gamma_{0\,\varepsilon}^{0-} \boldsymbol{m}^{(0)}) \\ &- \frac{1}{\varepsilon} Q^+(\gamma_{0\,\varepsilon}^{0+} \boldsymbol{m}^{(0)}, \gamma_{0\,\varepsilon}^{0-} \boldsymbol{m}^{(0)}) - Q^+(\gamma_0^{0+} \boldsymbol{m}^{(0)}, \gamma_0^{0-} \boldsymbol{m}^{(0)}) \\ &- \frac{1}{\varepsilon} \frac{\partial(\gamma_{0\,\varepsilon}^{0+} \boldsymbol{m}^{(0)} - \gamma_0^{0+} \boldsymbol{m}^{(0)})}{\partial z} \\ \approx -\mathrm{D}Q^+(\gamma_0^{0+} \boldsymbol{m}^{(0)}, \gamma_0^{0-} \boldsymbol{m}^{(0)}) \cdot &\left(\gamma_0^{0+} \boldsymbol{m}^{(1)} + \frac{\partial \boldsymbol{m}^{(0)}}{\partial z}, \gamma_0^{0-} \boldsymbol{m}^{(1)} - \frac{\partial \boldsymbol{m}^{(0)}}{\partial z} \right) - \frac{\partial^2 \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}^2}. \end{aligned}$$

Hence, on $\Gamma^{\pm}$ the boundary condition is

$$\begin{aligned}\frac{\partial \boldsymbol{m}^{(1)} - \frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}}{\partial \boldsymbol{\nu}} &= \mathrm{D}Q^{\pm}(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)}) \cdot \left(\gamma_0^0 \boldsymbol{m}^{(1)} - \gamma_0^1 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(1)} - \gamma_0^{1,\prime} \boldsymbol{m}^{(0)}\right) \\ &= \mathrm{D}Q^{\pm}(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)}) \cdot (\gamma_0^0 \boldsymbol{m}^{(1)} - \gamma_0^1 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(1)} - \gamma_0^{1,\prime} \boldsymbol{m}^{(0)}). \quad (2.2)\end{aligned}$$

Thus, formally $\boldsymbol{m}^{(1)}$ is the solution $\boldsymbol{w}$ of the linearized Landau-Lifshitz system (1.5) with $\boldsymbol{m} = \boldsymbol{m}^{(0)}$ and

$$\beta^{+} = -\mathrm{D}Q^{+}(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)}) \cdot \left(\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}, \frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}\right) \quad (2.3a)$$

$$\beta^{-} = -\mathrm{D}Q^{\pm}(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)}) \cdot \left(\frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}, \frac{\partial \boldsymbol{m}^{(0)}}{\partial \boldsymbol{\nu}}\right). \quad (2.3b)$$

Also, as in part I, formally $\theta = \boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma_0^0 \boldsymbol{m}^{(0)} \mathrm{d}\sigma(\Gamma)) + \boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma_0^0 \boldsymbol{m}^{(0)} \mathrm{d}\sigma(\Gamma)))$.

3 Miscellaneous inequalities and Sobolev spaces

We recall here without proof some well-known properties of Sobolev spaces. It can be verified [8] that the considered domain Ω is regular enough for those inequalities to hold. In particular, Sobolev embeddings hold.

Lemma 3.1. *The space* $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *is continuously embedded in* $\mathcal{C}^0(0,T;\mathrm{H}^2(\Omega))$ *and in* $\mathrm{L}^\infty(\Omega \times (0,T))$. *Besides, the gradient application is linear continuous from* $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *to* $\mathrm{L}^4(0,T;\mathbb{L}^\infty(\Omega))$.

PROOF: This is a consequence of theorem 4.2 Lions-Magenes [6]. and Maz'ya [7] page 274. □

Lemma 3.2. $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *is an algebra for the pointwise multiplication. Moreover, the pointwise multiplication of a function in* $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *and a function in* $\mathbb{H}^{2,1}(\Omega \times (0,T))$ *is in* $\mathbb{H}^{2,1}(\Omega \times (0,T))$.

The constants involved depend strongly on T when T tends to 0.

Definition 3.3. We define $\mathrm{H}^{m-\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)$ as the subset of $\mathrm{L}^2(\partial\Omega)$ of functions whose restrictions on $\partial B \times (0,L)$, $B \times \{0\}$ et $B \times \{L\}$ are in $\mathrm{H}^{m-\frac{1}{2}}$.

The following regularity properties hold.

Proposition 3.4 (Elliptic regularity)**.**
The space $\left\{v \in \mathrm{H}^1(\Omega) \mid \triangle v \in \mathrm{L}^2(\Omega), \frac{\partial v}{\partial \boldsymbol{\nu}} \in \mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)\right\}$ *is equal to* $\mathrm{H}^2(\Omega)$ *and there exists a constant* C *such that for all* v *in* $\mathrm{H}^2(\Omega)$

$$\|v\|_{\mathrm{H}^2(\Omega)} \le C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\triangle v\|_{\mathrm{L}^2(\Omega)} + \left\|\frac{\partial v}{\partial \boldsymbol{\nu}}\right\|_{\mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)}\right). \quad (3.1a)$$

Proposition 3.5. *The space* $\left\{v \in \mathrm{H}^1(\Omega), \nabla\triangle v \in \mathbb{L}^2(\Omega), \frac{\partial v}{\partial \boldsymbol{\nu}} \in \mathrm{H}^{\frac{3}{2}}_{\mathrm{morc}}(\partial\Omega)\right\}$ *is equal to* $\mathrm{H}^3(\Omega)$ *and there exists a constant* C *such that for all* v *in* $\mathrm{H}^3(\Omega)$

$$\|v\|_{\mathrm{H}^3(\Omega)} \le C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\nabla\triangle v\|_{\mathbb{L}^2(\Omega)} + \left\|\frac{\partial v}{\partial \boldsymbol{\nu}}\right\|_{\mathrm{H}^{\frac{3}{2}}(\partial\Omega)}\right). \quad (3.1b)$$

PROOF: This proposition is a trivial consequence of Proposition 2.8 in part I of this article. □

Lemma 3.6. *If A is a continuous bilinear operator from spaces X, Y into space Z then A is bilinear continuous from the spaces $(\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(X), (\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(Y)$ into $(\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(Z)$.*

PROOF: The proof is left to the reader. It is similar to the proof of Lemma 2.14 in the first part [8]. □

This lemma allows to recover the $\mathrm{H}^{\frac{1}{2}}$ regularity in time from the Landau-Lifshitz equation and the $\mathrm{H}^1(0,T;\mathbb{H}^1(\Omega)) \cap \mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))$ regularity of the solution.

4 Proof of the existence theorems

In this section, we prove the existence Theorems 1.3 and 1.4.

4.1 An equivalent problem

We first develop the second term of Landau-Lifshitz equation. This gives us a system equivalent to (1.5). This new system includes equations (1.5c), (1.5d), (1.5b), and the following developed form of Landau-Lifshitz.

$$\begin{aligned}\frac{\partial \boldsymbol{w}}{\partial t} = &-A\boldsymbol{m} \times \triangle \boldsymbol{w} - A\boldsymbol{w} \times \triangle \boldsymbol{m} + A\alpha \triangle \boldsymbol{w} + A\alpha |\boldsymbol{m}| \boldsymbol{w} + 2A\alpha (\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{w})\boldsymbol{m} \\ &- \boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{w}) - \boldsymbol{w} \times \mathcal{H}_{d,a}(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{w})) \\ &- \alpha \boldsymbol{m} \times (\boldsymbol{w} \times \mathcal{H}_{d,a}(\boldsymbol{m})) - \alpha \boldsymbol{w} \times (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{m})) + \theta.\end{aligned} \tag{4.1}$$

Lemma 4.1. *Let β^+, β^-, θ, and $\boldsymbol{w}_0$ satisfy the hypothesis of Theorem 1.3. If $\boldsymbol{w}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ satisfies equations* (1.5c), (1.5d) *and* (4.1)*, then $\boldsymbol{w}$ also satisfies the local orthogonality* (1.5b)*. Thus, $\boldsymbol{w}$ is a solution to Theorem 1.3.*

PROOF: The time equation is

$$\frac{\partial(\boldsymbol{w}\cdot\boldsymbol{m})}{\partial t} = \alpha A \left(\triangle(\boldsymbol{w}\cdot\boldsymbol{m}) + 2|\nabla \boldsymbol{m}|^2(\boldsymbol{w}\cdot\boldsymbol{m})\right).$$

Moreover, the boundary condition is

$$\begin{aligned}\frac{\partial(\boldsymbol{m}\cdot\boldsymbol{w})}{\partial \boldsymbol{\nu}} &= 0 \text{ on } \partial\Omega \setminus \Gamma \times (0,T), \\ \frac{\partial(\boldsymbol{m}\cdot\boldsymbol{w})}{\partial \boldsymbol{\nu}} &= -2Q_r^+(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m})\gamma_0^0 \boldsymbol{m} \cdot \gamma_0^0 \boldsymbol{w} \text{ on } \Gamma^+ \times (0,T), \\ \frac{\partial(\boldsymbol{m}\cdot\boldsymbol{w})}{\partial \boldsymbol{\nu}} &= -2Q_r^-(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m})\gamma_0^0 \boldsymbol{m} \cdot \gamma_0^0 \boldsymbol{w} \text{ on } \Gamma^- \times (0,T).\end{aligned}$$

We multiply equation (4.1) by $\boldsymbol{w}\cdot\boldsymbol{m}$ and integrate over $\Omega \times (0,T)$.

$$\left[\int_\Omega |\boldsymbol{w}\cdot\boldsymbol{m}|^2 \mathrm{d}\boldsymbol{x}\right]_0^T + (\alpha A - \eta)\int_{Q_T} \left|\nabla(\boldsymbol{w}\cdot\boldsymbol{m})\right|^2 \mathrm{d}\boldsymbol{x} \leq \\ 2\alpha A \int_0^T \|\nabla \boldsymbol{m}\|^2_{\mathbb{L}^\infty(\Omega)} \int_\Omega |\boldsymbol{m}\cdot\boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x} + \frac{C}{\eta^3}\int_{Q_T} |\boldsymbol{m}\cdot\boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x}.$$

By Gronwall's inequality, $\boldsymbol{m}\cdot\boldsymbol{w} = 0$. □

We need to prove the existence and uniqueness of solutions to this new system. We basically prove the theorem in four steps.

1. Existence with the homogenous Neumann boundary condition.
2. Extension of the boundary condition, then existence with an affine Neumann boundary condition.
3. Particular case with an affine Neumann boundary condition that allow the construction of a sequence $\boldsymbol{w}^{n+1}$ satisfying (4.1), (1.5c) and a Neumann boundary condition involving $\boldsymbol{w}^n$.

$$\frac{\partial \boldsymbol{w}^{n+1}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma \times (0,T), \\ \mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{w}^n, \gamma_0^{0,\prime}\boldsymbol{w}^n) + \beta^+ & \text{on } \Gamma^+ \times (0,T), \\ \mathrm{D}Q^-(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{w}^n, \gamma_0^{0,\prime}\boldsymbol{w}^n) + \beta^- & \text{on } \Gamma^- \times (0,T). \end{cases} \tag{4.2}$$

4. Convergence of the sequence to the solution by proving that the difference between two elements in the sequence tend to 0.

4.2 The affine Landau-Lifshitz system

Theorem 4.2. *Let $0 < T^*$. For all $T < T^*$, let $\boldsymbol{m}$ and $\boldsymbol{a}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ and θ in $\mathbb{H}^{1,\frac{1}{2}}(\Omega \times (0,T))$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^2(\Omega)$, $\frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega$. Then, there exists a unique $\boldsymbol{w}$ satisfying $\boldsymbol{w}(\cdot,0) = \boldsymbol{w}_0$, $\frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega \times (0,T^*)$, and equation* (4.1).

PROOF: We use Galerkin's method as in 2.24 in the first part [8]. We introduce the orthonormal base $w_1, \dots, w_n, \dots$ of $\mathrm{L}^2(\Omega)$ made of the eigenvectors of the Laplace operator with Neumann boundary conditions. We denote by V_n the subspace generated by $w_1, \dots, w_n$ and by $\mathcal{P}_n$ the orthogonal projection on V_n in $\mathrm{L}^2(\Omega)$. $\mathcal{P}_n$ is also an orthogonal projector in $\mathrm{H}^1(\Omega)$ and in $\{u \in \mathrm{H}^2(\Omega), \frac{\partial u}{\partial \boldsymbol{\nu}} = 0\}$. First, we introduce for each contribution p,

$$\begin{aligned} \boldsymbol{F}^{p,\mathrm{lin}}_{\boldsymbol{m}(0)}(\boldsymbol{w}) = -\boldsymbol{w} \times \mathcal{H}_p(\boldsymbol{m}(0)) - \boldsymbol{m}(0) \times \mathcal{H}_p(\boldsymbol{w}) - \alpha \boldsymbol{w} \times (\boldsymbol{m}(0) \times \mathcal{H}_p(\boldsymbol{m}(0))) \\ - \alpha \boldsymbol{m}(0) \times (\boldsymbol{w} \times \mathcal{H}_p(\boldsymbol{m}(0))) - \alpha \boldsymbol{m}(0) \times (\boldsymbol{m}(0) \times \mathcal{H}_p(\boldsymbol{w})). \end{aligned} \tag{4.3}$$

We look for $\boldsymbol{w}^n$ in $\mathbb{H}^1(0,T) \otimes V_n$ such that

$$\begin{aligned} \frac{\partial \boldsymbol{w}^n}{\partial t} - \alpha A \triangle \boldsymbol{w}^n = \mathcal{P}_n(-A\boldsymbol{w}^n \times \triangle \boldsymbol{m} - A\boldsymbol{m} \times \triangle \boldsymbol{w}^n + 2\alpha A(\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{w}^n)\boldsymbol{m}) \\ + \alpha A \mathcal{P}_n(|\nabla \boldsymbol{m}|^2 \boldsymbol{w}^n + \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w}^n) + \theta), \end{aligned} \tag{4.4}$$

$$\boldsymbol{w}^n(\cdot,0) = \mathcal{P}_n(\boldsymbol{w}_0) \tag{4.5}$$

In subsequent estimates, η will be a positive real that can be chosen arbitrarily small. C will be a generic constant depending only on domain Ω. To make these estimates, we need the inequalities

$$\|\boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w})\|_{\mathbb{L}^2(\Omega)} \le C'(1 + \|\boldsymbol{m}\|_{\mathbb{H}^1(\Omega)})\|\boldsymbol{w}\|_{\mathbb{H}^1(\Omega)}, \tag{4.6a}$$

$$\|\nabla \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w})\|_{\mathbb{L}^2(\Omega)} \le C'(1 + \|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})^2\|\boldsymbol{w}\|_{\mathbb{H}^2(\Omega)}. \tag{4.6b}$$

We already know from the proof of Theorem 2.24 in part I.that $\boldsymbol{w}^n$ exists over $[0,T^*)$ and is unique. Besides, $\boldsymbol{w}^n$ is bounded in $\mathbb{H}^{2,1}(\Omega \times (0,T))$. We only need the following additional estimate. We multiply (4.4) by $\triangle^2 \boldsymbol{w}^n$ and integrate. Since, $\triangle^2 \boldsymbol{w}^n$ belongs V_n

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\nabla\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} = \\ A \underbrace{\int_\Omega (\nabla \boldsymbol{w}^n \times \triangle \boldsymbol{m}) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{I} + A \underbrace{\int_\Omega (\boldsymbol{w}^n \times \nabla\triangle \boldsymbol{m}) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{II} \\ + A \underbrace{\int_\Omega (\nabla \boldsymbol{m} \times \triangle \boldsymbol{w}^n) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{III} - 2\alpha A \underbrace{\int_\Omega (\nabla \boldsymbol{m} \cdot \mathrm{D}^2 \boldsymbol{w}^n)\boldsymbol{m} \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{IV} \\ - 2\alpha A \underbrace{\int_\Omega (\mathrm{D}^2 \boldsymbol{m} \cdot \nabla \boldsymbol{w}^n)\boldsymbol{m} \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{V} - 2\alpha A \underbrace{\int_\Omega (\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{w}^n)\nabla \boldsymbol{m} \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{VI} \\ - 2\alpha A \underbrace{\int_\Omega (\mathrm{D}^2 \boldsymbol{m} \cdot \nabla \boldsymbol{m})\boldsymbol{w}^n \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{VII} - \alpha A \underbrace{\int_\Omega |\nabla \boldsymbol{m}|^2 \nabla \boldsymbol{w}^n \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{VIII} \\ - \underbrace{\int_\Omega \nabla\theta \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{IX} - \underbrace{\int_\Omega \nabla \boldsymbol{F}_{\boldsymbol{m}}^{a,d,\mathrm{lin}}(\boldsymbol{w}^n) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}}_{X}\end{aligned} \tag{4.7}$$

We evaluate $I = \int_\Omega (\nabla \boldsymbol{w}^n \times \triangle \boldsymbol{m}) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|I| &\le \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^6(\Omega)} \|\triangle \boldsymbol{m}\|_{\mathbb{L}^3(\Omega)} \|\nabla\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\le \frac{1}{4\eta}\|\triangle \boldsymbol{m}\|^2_{\mathbb{L}^3(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{4.8a}$$

Then, we estimate $II = \int_\Omega (\boldsymbol{w}^n \times \nabla\triangle \boldsymbol{m}) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|II| &\le \|\boldsymbol{w}^n\|_{\mathbb{L}^\infty(\Omega)} \|\nabla\triangle \boldsymbol{m}\|_{\mathbb{L}^2(\Omega)} \|\nabla\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\le \frac{1}{4\eta}\|\nabla\triangle \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{4.8b}$$

Estimating $III = \int_\Omega (\nabla \boldsymbol{m} \times \triangle \boldsymbol{w}^n) \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$ yields

$$\begin{aligned}|III| &\le \|\nabla \boldsymbol{m}\|_{\mathbb{L}^\infty(\Omega)} \|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\nabla\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\le \frac{1}{4\eta}\|\mathrm{D}^3 \boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega)} \|\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{4.8c}$$

Then, we estimate $IV = \int_\Omega (\nabla \boldsymbol{m} \cdot \mathrm{D}^2 \boldsymbol{w}^n)\boldsymbol{m} \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$.

$$\begin{aligned}|IV| &\le \|\boldsymbol{m}\|_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{m}\|_{\mathbb{L}^\infty(\Omega)} \|\mathrm{D}^2 \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \|\nabla\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\le \frac{1}{4\eta}\|\boldsymbol{m}\|^2_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{m}\|^2_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{4.8d}$$

If we estimate $V = \int_\Omega (\mathrm{D}^2 \boldsymbol{m} \cdot \nabla \boldsymbol{w}^n)\boldsymbol{m} \cdot \nabla\triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$, we obtain

$$\begin{aligned}|V| &\le \|\mathrm{D}^2 \boldsymbol{m}\|_{\mathbb{L}^6(\Omega)} \|\nabla \boldsymbol{w}^n\|_{\mathbb{L}^3(\Omega)} \|\nabla\triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\le \frac{1}{4\eta}\|\boldsymbol{m}\|^2_{\mathbb{H}^3(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla\triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}.\end{aligned} \tag{4.8e}$$

Estimating $VI = \int_\Omega (\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{w}^n) \nabla \boldsymbol{m} \cdot \nabla \triangle \boldsymbol{w}^n$ and $VIII = \int_\Omega |\nabla \boldsymbol{m}|^2 \nabla \boldsymbol{w}^n \cdot \nabla \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$ yields

$$|VI|, |VIII| \leq \frac{1}{4\eta} \|\nabla \boldsymbol{m}\|^4_{\mathbb{L}^\infty(\Omega)} \|\nabla \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{4.8f}$$

When we estimate $VII = \int_\Omega (\mathrm{D}^2 \boldsymbol{m} \cdot \nabla \boldsymbol{m}) \boldsymbol{w}^n \cdot \nabla \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$, we obtain

$$|VII| \leq \frac{1}{4\eta} \|\mathrm{D}^2 \boldsymbol{m}\|^2_{\mathbb{L}^3(\Omega)} \|\nabla \boldsymbol{m}\|^2_{\mathbb{L}^6(\Omega)} \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla \triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{4.8g}$$

Estimating $IX = \int_\Omega \nabla \theta \cdot \nabla \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$ yields

$$|IX| \leq \frac{1}{4\eta} \|\nabla \theta\|^2_{\mathbb{L}^2(\Omega)} + \eta \|\nabla \triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{4.8h}$$

Eventually, we estimate $X = \int_\Omega \nabla \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w}^n) \cdot \nabla \triangle \boldsymbol{w}^n \mathrm{d}\boldsymbol{x}$ using inequality (4.6).

$$\begin{aligned} |X| &\leq \|\nabla \boldsymbol{F}^{a,d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w}^n)\|_{\mathbb{L}^2(\Omega)} \|\nabla \triangle \boldsymbol{w}^n\|_{\mathbb{L}^2(\Omega)} \\ &\leq \frac{C}{4\eta}(1 + \|\boldsymbol{m}\|^2_{\mathbb{H}^2(\Omega)}) \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla \triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

We combine inequalities (4.8).

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} + \alpha A \int_\Omega |\nabla \triangle \boldsymbol{w}^n|^2 \mathrm{d}\boldsymbol{x} \leq \frac{C}{\eta} g(t) \|\boldsymbol{w}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\nabla \triangle \boldsymbol{w}^n\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla \theta\|^2_{\mathbb{L}^2(\Omega)}. \tag{4.9}$$

where g belongs to $\mathrm{L}^1(0,T)$. Choosing η small enough, we can apply Gronwall's lemma.

$$\|\triangle \boldsymbol{w}^n\|_{\mathbb{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq C_T, \qquad \|\nabla \triangle \boldsymbol{w}^n\|_{\mathbb{L}^2((0,T)\times\Omega)} \leq C_T. \tag{4.10}$$

The previous estimate, the regularity inequalities 3.1 and Lemma 3.6 prove that the sequence $\boldsymbol{w}^n$ remains bounded in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$. Thus, there exists a subsequence such that, for all $T < T^*$, $\boldsymbol{w}^n_k$ converges weakly to $\boldsymbol{w}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$. This limit satisfies $\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = 0$, $\boldsymbol{w}(\cdot, 0) = \boldsymbol{w}_0$, and

$$\begin{aligned} \iint_{Q_T} \frac{\partial \boldsymbol{w}}{\partial t} \boldsymbol{\psi} \mathrm{d}\boldsymbol{x}\mathrm{d}t = -A \int_{Q_T} (\boldsymbol{w} \times \triangle \boldsymbol{m} + \boldsymbol{m} \times \triangle \boldsymbol{w} \mathrm{d}\boldsymbol{x}\mathrm{d}t - \alpha \triangle \boldsymbol{w} - 2\alpha (\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{w}) \boldsymbol{m}) \boldsymbol{\psi} \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ + \alpha A \int_{Q_T} \left(|\nabla \boldsymbol{m}|^2 \boldsymbol{w} \mathrm{d}\boldsymbol{x}\mathrm{d}t + \boldsymbol{F}^{a,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w}) + \boldsymbol{F}^{d,\mathrm{lin}}_{\boldsymbol{m}}(\boldsymbol{w}) + \theta \right) \boldsymbol{\psi}, \end{aligned}$$

for all $\boldsymbol{\psi}$ in $\mathcal{C}^1(0,T;\mathbb{R}^3) \otimes \bigcup_{i=1}^\infty V_n$. Since this space is dense in $\mathbb{L}^2(\Omega \times (0,T))$, $\boldsymbol{w}$ is a solution.

We now prove the uniqueness. Let $\boldsymbol{w}$ and $\boldsymbol{w}'$ be solutions to the system (4.1) then $\delta \boldsymbol{w} = \boldsymbol{w}' - \boldsymbol{w}$ is solution to (4.1) affine term $\theta = 0$ and initial condition $\boldsymbol{w}_0 = 0$. After multiplying this equation by $\delta \boldsymbol{w}$ and integrating over $\Omega \times (0,T)$, we obtain the following estimate.

$$\begin{aligned} \left[\int_\Omega |\delta \boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \right]_0^T + (\alpha A - \eta) \iint_{Q_T} |\delta \nabla \boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}t \\ \leq C(\eta) \int_0^T (\|\boldsymbol{m}\|^2_{\mathbb{H}^1(\Omega)} \|\nabla \boldsymbol{m}\|^2_{\mathbb{L}^\infty(\Omega)}) \|\delta \boldsymbol{w}\|^2_{\mathbb{L}^2(\Omega)} \mathrm{d}t. \end{aligned}$$

The uniqueness follows from choosing $\eta = \alpha A/2$ and Gronwall's lemma. □

Corollary 4.3. *Let $T < T^*$. Let $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$. Let β^+, β^- be in $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma \times (0,T))$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^2(\Omega)$ satisfy equation* (1.6) *with $Q^+, Q^- = 0$. Then, there exists a unique $\boldsymbol{w}$ solution to equations* (1.5c), (4.1) *and* (1.5d) *with $Q^+, Q^- = 0$.*

PROOF: β^+ and β^- satisfy the compatibility relations of Theorem A.6. Thus, there exists an extension $\boldsymbol{v}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$, such that $\boldsymbol{v}$ satisfy the boundary conditions of $\boldsymbol{w}$ in (1.5d) with $Q^+, Q^- = 0$. We then define

$$\begin{aligned}\widetilde{\theta} = -\frac{\partial \boldsymbol{v}}{\partial t} - A\boldsymbol{m} \times \triangle \boldsymbol{w} - A\boldsymbol{v} \times \triangle \boldsymbol{m} + A\alpha \triangle \boldsymbol{v} + A\alpha |\boldsymbol{m}| \boldsymbol{w} + 2A\alpha(\nabla \boldsymbol{m} \cdot \nabla \boldsymbol{v})\boldsymbol{m} \\ - \boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{v}) - \boldsymbol{v} \times \mathcal{H}_{d,a}(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{v}) - \alpha \boldsymbol{m} \times (\boldsymbol{v} \times \mathcal{H}_{d,a}(\boldsymbol{m})) \\ - \alpha \boldsymbol{v} \times (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{m})) + \theta.\end{aligned}$$

$\widetilde{\theta}$ belongs to $\mathbb{H}^{1,\frac{1}{2}}(\Omega \times (0,T))$, $\boldsymbol{w}_0 - \boldsymbol{v}(\cdot, 0)$ satisfy the Neumann boundary conditions. By Theorem 4.2, there exists $\boldsymbol{u}$ such that $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is the solution to our theorem. Since $\boldsymbol{u}$ is unique, so is $\boldsymbol{w}$. □

Corollary 4.4. *Let $0 < T^*$. Let $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$. Let $\boldsymbol{a}$ be in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$. Let β^+, β^- be in $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma \times (0,T))$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^2(\Omega)$ satisfying*

$$\frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma \times (0,T), \\ \mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}_0, \gamma_0^{0,\prime} \boldsymbol{m}_0) \cdot (\gamma_0^0 \boldsymbol{a}, \gamma_0^{0,\prime} \boldsymbol{a}) + \beta^+ & \text{on } \Gamma^+ \times (0,T), \\ \mathrm{D}Q^-(\gamma_0^0 \boldsymbol{m}_0, \gamma_0^{0,\prime} \boldsymbol{m}_0) \cdot (\gamma_0^0 \boldsymbol{a}, \gamma_0^{0,\prime} \boldsymbol{a}) + \beta^- & \text{on } \Gamma^- \times (0,T). \end{cases} \tag{4.11}$$

Then, there exists a unique $\boldsymbol{w}$ solution to equations (1.5c), (4.1) *and*

$$\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma \times (0,T), \\ \mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{a}, \gamma_0^{0,\prime} \boldsymbol{a}) + \beta^+ & \text{on } \Gamma^+ \times (0,T), \\ \mathrm{D}Q^-(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{a}, \gamma_0^{0,\prime} \boldsymbol{a}) + \beta^- & \text{on } \Gamma^- \times (0,T). \end{cases} \tag{4.12}$$

PROOF: $\mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}, \gamma_0^{0,\prime} \boldsymbol{m}) \cdot (\gamma_0^0 \boldsymbol{a}, \gamma_0^{0,\prime} \boldsymbol{a})$ is in $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma \times (0,T))$. The easiest way to prove it is to construct the extension explicitly. First, we recall that $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ is an algebra. Given χ in $\mathcal{C}_c^\infty(-\infty, +\infty; \mathbb{R}^+)$ such that $\chi(z) = 0$ if $z > \frac{3}{4}\min(L^+, L^-)$ and $\chi(z) = 1$ if $z < \frac{\min(L^+,L^-)}{2}$. We define

$$\boldsymbol{g} = \begin{cases} \mathrm{D}Q^+(\boldsymbol{m}, \boldsymbol{m} \circ \sigma) \cdot (\boldsymbol{a}, \boldsymbol{a} \circ \sigma) & \text{in } \Omega^+, \\ \mathrm{D}Q^-(\boldsymbol{m}, \boldsymbol{m} \circ \sigma) \cdot (\boldsymbol{a}, \boldsymbol{a} \circ \sigma) & \text{in } \Omega^-, \end{cases}$$

where σ is the application that maps (x, y, z, t) to $(x, y, -z, t)$. Then, the function $\boldsymbol{v}$ that maps (x, y, z, t) to $\int_0^z \chi(s)\boldsymbol{g}(x, y, s, t)\mathrm{d}s$ is in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ and has the required properties. We apply corollary 4.3. □

We need the following lemma to provide a first element to our converging sequence.

Lemma 4.5. *Let $\boldsymbol{m}_0$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ such that $\frac{\partial \boldsymbol{w}_0}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega \setminus \Gamma$. Then, there exists $\boldsymbol{a}$ in $\mathbb{H}^{3,\frac{3}{2}}$ such that $\boldsymbol{a}(\cdot, 0) = \boldsymbol{w}_0$ and $\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\nu}} = 0$ on $\partial\Omega \setminus \Gamma \times (0,T)$.*

PROOF: All the compatibility relations of Theorem A.6 are satisfied. So $\boldsymbol{a}$ exists. □

4.3 Existence and convergence of the sequence

Theorem 1.3 is a consequence of Lemma 4.1 and the following result.

Theorem 4.6. *Let $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$. Let $\boldsymbol{w}_0$ in $\mathbb{H}^2(\Omega)$ satisfying* (1.6). *Then, there exists a unique $\boldsymbol{w}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ solution to system* (4.1), (1.5c), *and* (1.5d).

PROOF: We define $\boldsymbol{w}^n$ by induction. Let $\boldsymbol{w}^{-1}$ in $\mathbb{H}^{3,\frac{3}{2}}$ be the $\boldsymbol{a}$ of Lemma 4.5. Then, knowing $\boldsymbol{w}^n$, we define $\boldsymbol{w}^{n+1}$ as the unique solution to equations (4.1), (1.5c), and (4.2). This is possible by corollary (4.4). We need estimates on the size of $\boldsymbol{w}^n$. Let $\boldsymbol{v}_0$ be in $\mathbb{H}^2(\Omega)$ such that $\frac{\partial \boldsymbol{v}_0}{\partial \boldsymbol{\nu}}$ vanishes on $\partial\Omega\setminus\Gamma$, and is equal to $\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}_0,\gamma_0^{0,\prime}\boldsymbol{w}_0)$ on Γ. The existence of such $\boldsymbol{v}_0$ is given by the same construction as the one found in the proof of corollary 4.4. We define $\boldsymbol{u}_0=\boldsymbol{w}_0-\boldsymbol{v}_0$. We first define $\boldsymbol{u}$ as the $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ solution to equations (1.5c), (4.1) and (1.5d) with $Q^\pm=0$, $\theta=0$, and initial condition $\boldsymbol{u}_0$, but with same $\beta^\pm$. Note that if $\beta^\pm=0$ then $\boldsymbol{u}=0$. This solution exists by corollary (4.3). We make all of our estimates on $\boldsymbol{v}^n=\boldsymbol{w}^n-\boldsymbol{u}$. $\boldsymbol{v}^n$ satisfy (1.5c), (4.1) and (4.2) with same initial condition, $Q^\pm$, and θ but $\beta^\pm=\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{u},\gamma_0^{0,\prime}\boldsymbol{u})$. These are basically the same estimates as those of the proof of Theorem 4.2. But, the nonhomogenous Neumann boundary conditions force us to use more complicated forms of inequalities (3.1). Thus, an upper bound on the $\mathbb{L}^2$ norm of $\triangle\boldsymbol{v}$ does not yield an upper bound on the $\mathbb{H}^2$ norm. We introduce our preliminary estimates.

$$\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{H}^2(\Omega)}\leq C\big(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}+\|\triangle\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}+\|\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}^n,\gamma_0^{0,\prime}\boldsymbol{w}^n)\|^2_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}\big)$$

But,

$$\|\mathrm{D}Q^\pm(\boldsymbol{m},\boldsymbol{m}\circ\sigma)\cdot(\boldsymbol{w}^n,\boldsymbol{w}^n\circ\sigma)\|^2_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}\leq P\left(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)}\right)\|\boldsymbol{v}^n+\boldsymbol{u}\|^2_{\mathbb{H}^1(\Omega)}$$

But $\boldsymbol{m}$ is in $\mathcal{C}([0,T^*);\mathbb{H}^2(\Omega))$. Thus,

$$\sup_{1\leq n\leq N}\left\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)}\right\}\leq C\Bigg(\sup_{1\leq n\leq N}\left\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^1(\Omega)}\right\}+\sup_{1\leq n\leq N}\left\{\|\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\right\}+\|\boldsymbol{u}\|^2_{\mathbb{H}^1(\Omega)}+\|\boldsymbol{v}^0\|^2_{\mathbb{H}^2(\Omega)}\Bigg)\tag{4.13a}$$

We do the same estimate for the $\mathbb{H}^3$ norm.

$$\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{H}^3(\Omega)}\leq C\big(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}+\|\nabla\triangle\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}+\|\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}^n,\gamma_0^{0,\prime}\boldsymbol{w}^n)\|^2_{\mathbb{H}^{\frac{3}{2}}(\Gamma)}\big)$$

But

$$\|\mathrm{D}Q^\pm(\boldsymbol{m},\boldsymbol{m}\circ\sigma)\cdot(\boldsymbol{w}^n,\boldsymbol{w}^n\circ\sigma)\|^2_{\mathbb{H}^{\frac{3}{2}}(\Gamma)}\leq P\left(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)}\right)\|\boldsymbol{v}^n+\boldsymbol{u}\|^2_{\mathbb{H}^2(\Omega)}$$

Thus,

$$\sup_{1\leq n\leq N}\left\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^3(\Omega)}\right\}\leq C\Bigg(\sup_{1\leq n\leq N}\left\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^1(\Omega)}\right\}+\sup_{1\leq n\leq N}\left\{\|\nabla\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\right\}+\|\boldsymbol{u}\|^2_{\mathbb{H}^2(\Omega)}+\|\boldsymbol{v}^0\|^2_{\mathbb{H}^3(\Omega)}\Bigg)\tag{4.13b}$$

With inequalities (4.13), we can make useful estimates. We make three estimates on the norm of $\boldsymbol{v}^n$ for $n\geq 1$.

1. We multiply equation (4.1) by $\boldsymbol{v}^n$ and integrate over Ω.

2. We take the gradient of equation (4.1), multiply it by $\nabla\triangle\boldsymbol{v}^n$ and integrate over Ω.

3. We take the gradient of equation (4.1), multiply it by $\nabla\boldsymbol{v}^n$ and integrate over Ω.

The first two estimates have mostly been made in the proof of Theorem 4.2, except for the nonzero Neumann boundary condition. The third is a simplification of the second estimate. We do not detail the volume terms as it was done in the proof of Theorem 4.2. In these estimates η represents a small positive real.

First estimate Using trace theorems, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\|\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t}+\alpha A\|\nabla\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} &\leq P_0(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}+\alpha A\left|\int_\Gamma\frac{\partial\boldsymbol{v}^n}{\partial\boldsymbol{\nu}}\cdot\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x})\\ &\leq P_1(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}),\end{aligned} \tag{4.14}$$

where P_0, P_1 are given polynomial.

Second estimate We obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}\|\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t}+\alpha A\|\nabla\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} \leq C\left(1+\frac{1}{\eta}\right)P_1(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})\|\mathrm{D}^3\boldsymbol{m}\|^2_{\mathbb{L}^2(\Omega)}\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)}\\ +\eta\|\mathrm{D}^3\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}+\alpha A\left|\int_{\Gamma^\pm}\frac{\partial^2\boldsymbol{v}^n}{\partial\boldsymbol{\nu}\partial t}\cdot\triangle\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x}).\end{aligned} \tag{4.15}$$

After integrating over $(0,T)$, the boundary term has a meaning even before using the boundary condition. If $\boldsymbol{v}$ belongs to $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$, then $\frac{\partial^2\boldsymbol{v}}{\partial t\partial\boldsymbol{\nu}}$ belongs to the space $\mathrm{H}^{-\frac{1}{4}}(0,T;\mathbb{L}^2(\Omega))$ and $\gamma_0^0\triangle\boldsymbol{v}$ belongs to $\mathrm{H}^{\frac{1}{4}}(0,T;\mathbb{L}^2(\Omega))$. The evaluation of the integral on Γ^+ for $n\geq 1$ gives

$$\begin{aligned}\int_{\Gamma^\pm}\left|\frac{\partial^2\boldsymbol{v}^n}{\partial\boldsymbol{\nu}\partial t}\cdot\triangle\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x}) &\leq \frac{1}{2}\|\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)}\\ &\quad+\frac{1}{2}\left\|\frac{\partial\big(\mathrm{D}Q^+(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}^{n-1},\gamma_0^{0,\prime}\boldsymbol{w}^{n-1})\big)}{\partial t}\right\|^2_{\mathbb{L}^2(\Gamma^+)},\end{aligned}$$

$$\|\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)}\leq C\|\triangle\boldsymbol{v}^n\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)}(\|\triangle\boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)}+\|\nabla\triangle\boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)})^{\frac{3}{2}}.$$

And,

$$\begin{aligned}&\left\|\frac{\partial\big(\mathrm{D}Q^+(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}^{n-1},\gamma_0^{0,\prime}\boldsymbol{w}^{n-1})\big)}{\partial t}\right\|_{\mathbb{L}^2(\Gamma^+)}\\ &\qquad\leq P_3(\|\boldsymbol{m}\|_{\mathbb{L}^\infty(\Omega)})\left\|\frac{\partial\boldsymbol{m}}{\partial t}\right\|_{\mathbb{H}^1(\Omega)}\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)}\\ &\qquad\quad+P_4(\|\boldsymbol{m}\|_{\mathbb{L}^\infty(\Omega)})\left\|\frac{\partial(\boldsymbol{v}^{n-1}+\boldsymbol{u})}{\partial t}\right\|^{\frac{3}{4}}_{\mathbb{H}^1(\Omega)}\left\|\frac{\partial(\boldsymbol{v}^{n-1}+\boldsymbol{u})}{\partial t}\right\|^{\frac{1}{4}}_{\mathbb{L}^2(\Omega)}.\end{aligned}$$

By (4.1) if $n \geq 1$, there exists C depending on the $\mathbb{H}^{3,\frac{3}{2}}$ norm of $\boldsymbol{m}$ such that

$$\left\|\frac{\partial \boldsymbol{v}^{n-1}}{\partial t}\right\|_{\mathbb{H}^1(\Omega)} \leq C\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^3(\Omega)}, \qquad \left\|\frac{\partial \boldsymbol{v}^{n-1}}{\partial t}\right\|_{\mathbb{L}^2(\Omega)} \leq C\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)}.$$

Thus,

$$\left|\int_{\Gamma^\pm} \frac{\partial^2 \boldsymbol{v}^n}{\partial \boldsymbol{\nu} \partial t} \cdot \triangle \boldsymbol{v}^n\right| \mathrm{d}\sigma(\boldsymbol{x}) \leq C(\|\boldsymbol{m}\|_{\mathbb{L}^\infty}) \left\|\frac{\partial \boldsymbol{m}}{\partial t}\right\|^2_{\mathbb{H}^1(\Omega)} \left(1+\frac{1}{\eta}\right)(\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^2(\Omega)}) + \eta(\|\boldsymbol{v}^n\|^2_{\mathbb{H}^3(\Omega)} + \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^3(\Omega)}) + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2_{\mathbb{H}^1(\Omega)}. \tag{4.16}$$

Third estimate This is a simplification of the previous estimate

$$\frac{1}{2}\frac{\mathrm{d}\|\nabla \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} \leq \left(1+\frac{1}{\eta}\right) P_6(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta\|\mathrm{D}^3\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} + \alpha A\left|\int_{\Gamma^\pm} \frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}} \cdot \triangle \boldsymbol{v}^n\right| \mathrm{d}\sigma(\boldsymbol{x}). \tag{4.17}$$

But,

$$\begin{aligned}\left|\int_{\Gamma^+} \frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}} \cdot \triangle \boldsymbol{v}^n\right| \mathrm{d}\sigma(\boldsymbol{x}) &\leq \frac{1}{2}\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)} + \frac{1}{2}\left\|\frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}}\right\|^2_{\mathbb{L}^2(\Gamma^+)} \\ &\leq C\|\triangle \boldsymbol{v}^n\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)}\|\nabla\triangle \boldsymbol{v}^n\|^{\frac{3}{2}}_{\mathbb{L}^2(\Omega)} + P_3(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^{n-1}+\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\ &\leq \frac{1}{4\eta}\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} + P_4(\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^{n-1}+\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \eta\|\nabla\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)},\end{aligned}$$

where the P_i are polynomials and C a constant only depending on the $\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))$ norm of $\boldsymbol{m}$ which is bounded for all $T < T^*$.

We choose η small enough, combine all estimates, and obtain for all $n \geq 1$

$$\|\boldsymbol{v}^n\|^2_{\mathbb{H}^1(\Omega)} + \|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} + \int_0^t \|\nabla\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\mathrm{d}t \leq \eta\int_0^t \left(\|\mathrm{D}^3\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} + \|\mathrm{D}^3\boldsymbol{v}^{n-1}\|^2_{\mathbb{L}^2(\Omega)}\right)\mathrm{d}t \left(1+\frac{1}{\eta}\right)\int_0^t \Psi(t)(\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^2(\Omega)})\mathrm{d}t + C\left(\|\boldsymbol{u}\|^2_{\mathbb{H}^1(\Omega)} + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2_{\mathbb{H}^1(\Omega)}\right). \tag{4.18}$$

We take the upper bound for $1 \geq n \geq N$, use inequalities (4.13), and obtain

$$\begin{aligned}&\sup_{1\leq n\leq N}\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^1(\Omega)} + \|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\} + \sup_{1\leq n\leq N}\left\{\int_0^T \|\nabla\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\mathrm{d}t\right\} \\ &\leq \int_0^T C(\eta)(\sup_{1\leq n\leq N}(\|\boldsymbol{v}^n\|_{\mathbb{H}^1(\Omega)})^2 + \|\boldsymbol{v}^0\|^2_{\mathbb{H}^1(\Omega)} + \|\boldsymbol{v}^{-1}\|^2_{\mathbb{H}^1(\Omega)})\mathrm{d}t + \int_0^T P_5^\eta(\|\boldsymbol{v}^0\|_{\mathbb{H}^2(\Omega)})\mathrm{d}t \\ &\quad + \eta\int_0^T \|\nabla\triangle \boldsymbol{v}^0\|^2_{\mathbb{L}^2(\Omega)}\mathrm{d}t + \eta\int_0^T \|\nabla\triangle \boldsymbol{v}^{-1}\|^2_{\mathbb{L}^2(\Omega)}\mathrm{d}t + C\left(\|\boldsymbol{u}\|^2_{\mathbb{H}^2(\Omega)} + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|^2_{\mathbb{H}^1(\Omega)}\right).\end{aligned} \tag{4.19}$$

Reusing inequalities (4.13), we obtain by Gronwall's lemma that the $\mathbb{L}^\infty(0,T;\mathbb{L}^2(\Omega))$ of $\boldsymbol{v}^n$, $\triangle\boldsymbol{v}^n$, and $\nabla\boldsymbol{v}^n$ are bounded by a constant $C(T)$ independent of n. By inequalities (4.13), there exists a constant C_T such that, for all $T < T^*$,

$$\|\boldsymbol{v}\|_{\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))} \leq C_T, \qquad \|\mathrm{D}^3\boldsymbol{v}\|_{\mathbb{L}^2(\Omega\times(0,T))} \leq C_T. \tag{4.20}$$

Reusing equation (4.1) and Lemma 3.6, we obtain that $\boldsymbol{v}^n$ is bounded in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$. We can extract a subsequence that converge weakly in $\mathbb{H}^{3,\frac{3}{2}}$. We denote by $\boldsymbol{v}$ this limit. Suppose that[1]

$$\left(\boldsymbol{v}^{n_k} \text{ and } \boldsymbol{v}^{n_k+1} \text{ both weakly converge in } \mathbb{H}^{2,1}(\Omega\times(0,T))\right) \Longrightarrow \lim_{k\to\infty}\boldsymbol{v}^{n_k} = \lim_{k\to\infty}\boldsymbol{v}^{n_k+1}$$

We extract a further subsequence such that $\boldsymbol{v}^{n_k+1}$ converges. According to our supposition, the limit is $\boldsymbol{v}$ for both subsequences. On Γ^+,

$$\frac{\partial\boldsymbol{v}^{n_k+1}}{\partial\boldsymbol{\nu}} = \mathrm{D}Q^+(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{v}^{n_k}+\boldsymbol{u},\gamma_0^{0,\prime}\boldsymbol{v}^{n_k}+\boldsymbol{u}). \tag{4.21}$$

We take the limit and obtain

$$\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{\nu}} = \mathrm{D}Q^+(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{v}+\boldsymbol{u},\gamma_0^{0,\prime}\boldsymbol{v}+\boldsymbol{u}). \tag{4.22}$$

Thus, $\boldsymbol{w} = \boldsymbol{v}+\boldsymbol{u}$ is a solution. The corresponding result holds on Γ^-. We now prove our previous assumption. For this, we define $\delta^n\boldsymbol{v} = \boldsymbol{v}^{n+1}-\boldsymbol{v}^n$, then we can obtain the same estimate on $\delta^n\boldsymbol{v}$ as obtained in equation (4.18) on $\boldsymbol{v}$, but with $\boldsymbol{u}=0$ and null initial condition. Instead of taking the upper bound, we sum these estimates. The initial conditions of this sum is null. By Gronwall's inequality, obtain that, for all $T<T^*$, there exists $C_T>0$ such that, for all $n\geq 1$,

$$\sum_{k=1}^{n}\|\delta^n\boldsymbol{v}\|^2_{\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))} \leq C_T, \qquad \sum_{k=1}^{n}\|\delta^n\boldsymbol{v}\|^2_{\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))} \leq C_T.$$

Thus, our assertion is proved. □

Proving Theorem 1.4 is equivalent to proving

Theorem 4.7. *Let $\boldsymbol{w}_0$ in $\mathbb{H}^1(\Omega)$, θ in $\mathbb{L}^2(\Omega\times(0,T))$, and $\beta^\pm$ in $\mathbb{H}^{\frac{1}{2},\frac{1}{4}}(\Gamma^\pm\times(0,T))$. There exists a unique $\boldsymbol{w}$ solution to system* (4.1)*,* (1.5c)*, and* (1.5d).

PROOF: We begin by proving the uniqueness. Let $\boldsymbol{w}$ and $\boldsymbol{w}'$ be two solutions in $\mathbb{H}^{2,1}(\Omega\times(0,T))$. Then, $\delta\boldsymbol{w} = \boldsymbol{w}-\boldsymbol{w}'$ satisfy system (4.1), (1.5c), and (1.5d) with $\beta^\pm = 0$, $\theta=0$ and $\boldsymbol{w}_0 = 0$. We notice that $\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m}')(\gamma_0^0\delta\boldsymbol{w},\gamma_0^{0,\prime}\delta\boldsymbol{w})$ belongs to $\mathbb{H}^{\frac{3}{2},\frac{3}{4}}_{00,}(\Gamma^\pm\times(0,T))$. Let $\boldsymbol{w}''$ be the $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ solution to (4.1), (1.5c), and (1.5d) with $\beta^\pm = \mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m}')(\gamma_0^0\delta\boldsymbol{w},\gamma_0^{0,\prime}\delta\boldsymbol{w})$, with the $Q^\pm$ of (1.5d) taken null, and $\theta=0$. $\boldsymbol{w}''$ exists by Theorem 4.2. Thus, $\delta\boldsymbol{w}-\boldsymbol{w}''$ satisfy (4.1), (1.5c),and (1.5d) with $\beta^\pm=0$, the $Q^\pm$ of (1.5d) taken null, and $\theta=0$. Thus, by the uniqueness part of Theorem 2.24 of [8], $\delta\boldsymbol{w}=\boldsymbol{w}''$. $\delta\boldsymbol{w}$ belongs to $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ and is null by the uniqueness part of corollary 4.3.

We now prove the existence. Since $\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}_0,\gamma_0^{0,\prime}\boldsymbol{w}_0)$ belongs to $\mathbb{H}^{\frac{3}{2},\frac{3}{2}}(B\times(-L^-,L^+)\setminus\{0\})$, there exists, by Lemma 2.7 in part I, $\boldsymbol{v}_0$ in $\mathbb{H}^2(\Omega)$ such that $\frac{\partial\boldsymbol{v}_0}{\partial\boldsymbol{\nu}}$ is null on $\partial\Omega\setminus\Gamma^\pm$ and is equal to $\mathrm{D}Q^\pm(\gamma_0^0\boldsymbol{m},\gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{w}_0,\gamma_0^{0,\prime}\boldsymbol{w}_0)$ on $\Gamma^\pm$. Besides, there exists a

[1] We will prove this assertion later

constant C independent of $\boldsymbol{w}_0$ such that $\boldsymbol{v}_0$ can be chosen so that $\|\boldsymbol{v}_0\|_{\mathbb{H}^2(\Omega)} \leq C\|\boldsymbol{w}_0\|_{\mathbb{H}^1(\Omega)}$. We define $\boldsymbol{u}_0$ as $\boldsymbol{w}_0 - \boldsymbol{v}_0$. There exists $\boldsymbol{u}$ in $\mathbb{H}^{2,1}(\Omega\times(0,T))$ that satisfy system (4.1), (1.5c), and (1.5d) with same parameters except $Q^\pm = 0$ and initial condition $\boldsymbol{u}_0$. This was proved by the author in the first part [8], Theorem 2.24. By Theorem (4.6), there exists $\boldsymbol{v}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ with initial condition $\boldsymbol{v}_0$, same $Q^\pm$, $\beta^\pm = \mathrm{d}Q^\pm(\gamma_0^0\boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m})\cdot(\gamma_0^0\boldsymbol{u}, \gamma_0^{0,\prime}\boldsymbol{u})$ belonging to $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}$ and $\theta = 0$. $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}$ is the solution. □

5 Convergence

We recall the reader about our problem stated in section 2. We have a continuum of solutions $\boldsymbol{m}^\varepsilon$ to the Landau-Lifshitz system (1.3) on domain Ω_ε with initial conditions $\boldsymbol{m}_0^\varepsilon$ satisfying conditions (2.1). We know by [9] that, for all $T < T^*$, the $\mathbb{H}^{3,\frac{3}{2}}(\Omega_\varepsilon\times(0,T))$ norm of $\boldsymbol{m}_0^\varepsilon$ is bounded, uniformly in ε. The aim of this section is to prove the convergence of the expansion $\boldsymbol{m}^{(0)} + \varepsilon\boldsymbol{m}^{(1)}$ to $\boldsymbol{m}^\varepsilon$. We do that in two steps.

1. We derive an upper bound of the $\mathbb{H}^{2,1}(\Omega_\varepsilon\times(0,T))$ norm of $\frac{\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{(0)}}{\varepsilon}$.
2. We prove that the weak limit in $\mathbb{H}^{2,1}(\Omega)$ of $\frac{\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{(0)}}{\varepsilon}$ is $\boldsymbol{m}^{(1)}$.

We denote by $\Delta Q^\pm$ the polynomial in four variables such that $Q^\pm(a,b) - Q^\pm(c,d) = \Delta Q^\pm(a,b,c,d)\cdot(a-c, b-d)$. Given $\boldsymbol{m}$, $\boldsymbol{m}'$, $\boldsymbol{u}_0$, β^+ and β^-, and θ, we consider the following auxiliary system in $\boldsymbol{u}$.

$$\begin{aligned}\frac{\partial\boldsymbol{u}}{\partial t} - \alpha A\triangle\boldsymbol{u} = {} & -A\boldsymbol{u}\times\triangle\boldsymbol{m} - A\boldsymbol{m}'\times\triangle\boldsymbol{u} + \alpha A|\nabla\boldsymbol{m}|^2\boldsymbol{u} + \alpha A((\nabla\boldsymbol{m}+\nabla\boldsymbol{m}')\cdot\nabla\boldsymbol{u})\boldsymbol{m}' \\ & - \boldsymbol{u}\times\mathcal{H}_{d,a}(\boldsymbol{m}) - \boldsymbol{m}'\times\mathcal{H}_{d,a}(\boldsymbol{u}) - \alpha\boldsymbol{m}'\times(\boldsymbol{m}'\times\mathcal{H}_{d,a}(\boldsymbol{u})) \\ & - \alpha\boldsymbol{m}'\times(\boldsymbol{u}\times\mathcal{H}_{d,a}(\boldsymbol{m})) - \alpha\boldsymbol{u}\times(\boldsymbol{m}\times\mathcal{H}_{d,a}(\boldsymbol{m})) + \theta,\end{aligned} \tag{5.1a}$$

$$\boldsymbol{u}(\cdot,\cdot,0) = \boldsymbol{u}_0, \tag{5.1b}$$

(5.1c)

and the initial condition

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} = \begin{cases}\Delta Q^+(\gamma_0^0\boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}, \gamma_0^0\boldsymbol{m}', \gamma_0^{0,\prime}\boldsymbol{m})(\gamma_0^0\boldsymbol{u}, \gamma_0^{0,\prime}\boldsymbol{u}) + \beta^+ & \text{on } B\times\{+\varepsilon\}\times(0,T^*),\\ \Delta Q^-(\gamma_0^0\boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}, \gamma_0^0\boldsymbol{m}', \gamma_0^{0,\prime}\boldsymbol{m})(\gamma_0^0\boldsymbol{u}, \gamma_0^{0,\prime}\boldsymbol{u}) + \beta^- & \text{on } B\times\{-\varepsilon\}\times(0,T^*),\\ 0 & \text{on } \partial\Omega_\varepsilon\setminus(B\times\{\pm\varepsilon\}\times(0,T^*)).\end{cases} \tag{5.1d}$$

First, we state a well-posedness result.

Theorem 5.1. *Let $\boldsymbol{m}$, $\boldsymbol{m}'$ be in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$, let β^+, β^- be in $\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma\times(0,T))$, θ in $\mathbb{H}^{1,\frac{1}{2}}(\Omega\times(0,T))$, and $\boldsymbol{u}_0$ be in $\mathbb{H}^2(\Omega)$ satisfying*

$$\frac{\partial\boldsymbol{u}_0}{\partial\boldsymbol{\nu}} = \begin{cases}\Delta Q^+(\gamma_0^0\boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}, \gamma_0^0\boldsymbol{m}', \gamma_0^{0,\prime}\boldsymbol{m})(\gamma_0^0\boldsymbol{u}_0, \gamma_0^{0,\prime}\boldsymbol{u}_0) + \beta^+ & \text{on } B\times\{+\varepsilon\},\\ \Delta Q^-(\gamma_0^0\boldsymbol{m}, \gamma_0^{0,\prime}\boldsymbol{m}, \gamma_0^0\boldsymbol{m}', \gamma_0^{0,\prime}\boldsymbol{m})(\gamma_0^0\boldsymbol{u}_0, \gamma_0^{0,\prime}\boldsymbol{u}_0) + \beta^- & \text{on } B\times\{-\varepsilon\},\\ 0 & \text{on } \partial\Omega_\varepsilon\setminus(B\times\{\pm\varepsilon\}).\end{cases} \tag{5.2}$$

Then, there exists a unique solution $\boldsymbol{u}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$. Furthermore, there exists a constant C depending on the $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ norms of $\boldsymbol{m}, \boldsymbol{m}'$ such that

$$\|\boldsymbol{u}\|_{\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))} \leq C\left(\|\beta^\pm\|_{\mathbb{H}_{00,}^{\frac{3}{2},\frac{3}{4}}(\Gamma\times(0,T))} + \|\theta\|_{\mathbb{H}^{1,\frac{1}{2}}(\Omega\times(0,T))} + \|\boldsymbol{u}_0\|_{\mathbb{H}^2(\Omega)}\right).$$

Besides, suppose that θ only belongs to $\mathbb{L}^2(\Omega \times (0,T))$, β^+, β^- only belongs to $\mathbb{H}^{\frac{1}{2},\frac{1}{4}}(\Gamma \times (0,T))$, and $\boldsymbol{u}_0$ only belongs to $\mathbb{H}^2(\Omega)$. Then, there exists a unique solution $\boldsymbol{u}$ in $\mathbb{H}^{2,1}(\Omega \times (0,T))$. The problem is also well-posed, there exists a constant C depending on the $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ norms of $\boldsymbol{m}, \boldsymbol{m}'$ such that

$$\|\boldsymbol{u}\|_{\mathbb{H}^{2,1}(\Omega\times(0,T))} \leq C\left(\|\beta^{\pm}\|_{\mathbb{H}^{\frac{1}{2},\frac{1}{4}}(\Gamma\times(0,T))} + \|\theta\|_{\mathbb{L}^2(\Omega\times(0,T))} + \|\boldsymbol{u}_0\|_{\mathbb{H}^1(\Omega)}\right).$$

PROOF: We just adapt the proofs of Theorems 4.2, 4.6, and 4.7. □
The theorem holds for the domains Ω_ε, and the constant $C(\boldsymbol{m}, \boldsymbol{m}')$ while depending on the domain can be chosen independently of ε for sufficiently small ε.

Proposition 5.2. *The $\mathbb{H}^{2,1}(\Omega_\varepsilon \times (0,T))$ norm of $\frac{\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{(0)}}{\varepsilon}$ remains bounded for small enough ε.*

PROOF: $\frac{\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{(0)}}{\varepsilon}$ is solution to system (5.1) with $\boldsymbol{m} = \boldsymbol{m}^{(0)}{}_{|\Omega_\varepsilon}$, $\boldsymbol{m}' = \boldsymbol{m}^\varepsilon$, the initial condition $\frac{\boldsymbol{m}_0^\varepsilon - \boldsymbol{m}_0^{(0)}}{\varepsilon}$, the boundary conditions

$$\beta^{\pm} = \frac{1}{\varepsilon}\left(\gamma_\varepsilon^1 \boldsymbol{m}^{(0)} - \gamma_0^1 \boldsymbol{m}^{(0)}\right) + \Delta Q^{\pm}(\gamma_0^0\boldsymbol{m}^{(0)}, \gamma_0^{0,\prime}\boldsymbol{m}^{(0)}, \gamma_\varepsilon^0\boldsymbol{m}^{(0)}, \gamma_\varepsilon^{0,\prime}\boldsymbol{m}^{(0)}) \cdot \frac{(\gamma_\varepsilon^0\boldsymbol{m}^{(0)} - \gamma_\varepsilon^{0,\prime}\boldsymbol{m}^{(0)})}{\varepsilon},$$

on $\Gamma^{\pm} \times (0,T)$ and the affine term

$$\theta = \frac{1}{\varepsilon}\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\chi_{B\times(-\varepsilon,+\varepsilon)}\boldsymbol{m}^{(0)}) + \frac{1}{\varepsilon}\alpha\boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_{d,a}(\chi_{B\times(-\varepsilon,+\varepsilon)}\boldsymbol{m}^{(0)})).$$

The $\mathbb{H}^{\frac{1}{2},\frac{1}{4}}(\Gamma_\varepsilon^{\pm} \times (0,T))$ norm of $\beta^{\pm}$, the $\mathbb{L}^2(\Omega_\varepsilon \times (0,T))$ norm of θ, and the $\mathbb{H}^1(\Omega_\varepsilon)$ norm of the initial condition are $O(\varepsilon)$. This latter fact has the same proof as Proposition 3.2 in part I. The $\mathbb{H}^1(\Omega_\varepsilon \times (0,T))$ norm of the initial condition is $O(\varepsilon)$ by hypothesis (2.1). We apply Theorem 5.1. □

We now extend $\boldsymbol{m}^\varepsilon$ by reflections on Ω,

$$\widetilde{\boldsymbol{m}^\varepsilon} = \begin{cases} \boldsymbol{m}^\varepsilon & \text{on } \Omega_\varepsilon \times (0,T), \\ 3\boldsymbol{m}^\varepsilon(\cdot,\cdot,2\varepsilon - \cdot,\cdot) - 2\boldsymbol{m}^\varepsilon(\cdot,\cdot,3\varepsilon - 2\cdot,\cdot) & \text{on } (B \times (-\varepsilon,0)) \times (0,T), \\ 3\boldsymbol{m}^\varepsilon(\cdot,\cdot,-2\varepsilon - \cdot,\cdot) - 2\boldsymbol{m}^\varepsilon(\cdot,\cdot,-3\varepsilon - 2\cdot,\cdot) & \text{on } (B \times (0,+\varepsilon)) \times (0,T). \end{cases}$$

Then, by the same kind of considerations as those of Lemma 3.4 [8], the $\mathbb{H}^{2,1}(\Omega \times (0,T))$ norm of $\frac{1}{\varepsilon}(\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)})$ is bounded independently of ε as ε tends to 0.

Theorem 5.3. *The quantity $\frac{\widetilde{\boldsymbol{m}^{\varepsilon,(0)})} - \boldsymbol{m}^{(0)}}{\varepsilon}$ converges weakly to $\boldsymbol{m}^{(1)}$, defined in section 2, in $\mathbb{H}^{2,1}(\Omega \times (0,T))$.*

PROOF: We extract a decreasing subsequence ε_n tending to 0 such that $\boldsymbol{m}^{\varepsilon_n}$ tends weakly in $\mathbb{H}^{2,1}(\Omega)$ to $\overline{\boldsymbol{m}^{(1)}}$. We only need to prove that $\overline{\boldsymbol{m}^{(1)}}$ is $\boldsymbol{m}^{(1)}$. Since $\boldsymbol{m}^{(1)}$ is the unique solution to system (1.5), it is sufficient to prove that $\overline{\boldsymbol{m}^{(1)}}$ is solution to (1.5) with same parameters. As in the proof of 3.5 in part I, $\overline{\boldsymbol{m}^{(1)}}$ satisfy equation (4.1), with $\theta = \boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma_0^0\boldsymbol{m}^{(0)}\mathrm{d}\sigma(\Gamma)) + \boldsymbol{m}^{(0)} \times (\boldsymbol{m}^{(0)} \times \mathcal{H}_d(\gamma_0^0\boldsymbol{m}^{(0)}\mathrm{d}\sigma(\Gamma)))$. Besides, $\overline{\boldsymbol{m}^{(1)}}(\cdot,0) = \boldsymbol{m}_0^{(1)}$, see (2.1) for the definition of $\boldsymbol{m}_0^{(1)}$, $\overline{\boldsymbol{m}^{(1)}}$ obviously also verify the homogenous Neumann boundary condition on $\partial\Omega \setminus \Gamma^{\pm} \times (0, T^*)$. On Γ^+, we have

$$\int_0^T \int_\Gamma \left| \frac{\partial(\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)})}{\partial z}(\boldsymbol{x}, \varepsilon, t) - \frac{\partial(\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)})}{\partial z}(\boldsymbol{x}, 0^+, t) \right|^2 \mathrm{d}\sigma(\boldsymbol{x})\mathrm{d}t$$
$$\leq \iint \left| \int_0^\varepsilon \frac{\partial^2(\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)})}{\partial z^2}(\boldsymbol{x}, z, t)\mathrm{d}z \right|^2 \mathrm{d}\sigma(\boldsymbol{x})\mathrm{d}t$$
$$\leq \varepsilon \iint \int_0^\varepsilon \left| \frac{\partial^2(\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)})}{\partial z^2}(\boldsymbol{x}, z, t) \right|^2 \mathrm{d}z\mathrm{d}\sigma(\boldsymbol{x})\mathrm{d}t \leq \varepsilon^3 \left\| \frac{\widetilde{\boldsymbol{m}^\varepsilon} - \boldsymbol{m}^{(0)}}{\varepsilon} \right\|^2_{\mathbb{H}^{0,2,0}}.$$

Thus, on Γ^+, we have

$$\frac{\partial \overline{\boldsymbol{m}^{(1)}}}{\partial z}(\cdot, 0^+, \cdot) = \lim_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k} \frac{\partial(\widetilde{\boldsymbol{m}^{\varepsilon_k}} - \boldsymbol{m}^{(0)})}{\partial z}(\cdot, \varepsilon_k, \cdot),$$
$$= -\lim_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k} \left(Q^+(\gamma_0^0 \widetilde{\boldsymbol{m}^{\varepsilon_k}}, \gamma_0^{0,\prime} \widetilde{\boldsymbol{m}^{\varepsilon_k}})(\cdot, \varepsilon_k, \cdot) - Q^+(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)})(\cdot, \varepsilon_k, \cdot) \right)$$
$$- \lim_{\varepsilon_k \to 0} \frac{1}{\varepsilon_k} \left(Q^+(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)})(\cdot, \varepsilon_k, \cdot) - Q^+(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)})(\cdot, 0^+, \cdot) \right) - \frac{\partial^2 \boldsymbol{m}^{(0)}}{\partial z^2}$$
$$= -\mathrm{D}Q^+(\gamma_0^0 \boldsymbol{m}^{(0)}, \gamma_0^{0,\prime} \boldsymbol{m}^{(0)}) \left(\overline{\boldsymbol{m}^{(1)}} + \frac{\partial \boldsymbol{m}^{(0)}}{\partial z}, \overline{\boldsymbol{m}^{(1)}} + \frac{\partial \boldsymbol{m}^{(0)}}{\partial z} \right) - \frac{\partial^2 \boldsymbol{m}^{(0)}}{\partial z^2}.$$

Hence, $\overline{\boldsymbol{m}^{(1)}}$ satisfy (2.2) on Γ^+ and by symmetry also on Γ^-. Thus, $\overline{\boldsymbol{m}^{(1)}}$ is the $\boldsymbol{m}^{(1)}$ solution of system (1.5) with the $\beta^{\pm}$ of relations (2.3). Since $\boldsymbol{m}^{(1)}$ is unique, the whole sequence converges. □

6 Simulations: schemes and numerical results

We use the same schemes as those found in section 4. of [8]. The computation of the discretized demagnetization field operator is done by the method found in [3]. The only differences are found in the computation of the discretized exchange operator as $\boldsymbol{m}^{(0)}$ and $\boldsymbol{m}^{(1)}$ satisfy the nonhomogenous Neumann boundary conditions arising from super-exchange and surface-anisotropy. The discretization of the exchange operator for order 0 gives

$$\mathcal{H}_{e,h}^0(\boldsymbol{m})_i = \frac{A}{h^2} \sum_{j \in V(i)} (\boldsymbol{m}_i - \boldsymbol{m}_j) + \frac{A}{h} \sum_{j \in VC(i)} (Q(\boldsymbol{m}_i, \boldsymbol{m}_j)), \tag{6.1}$$

where $V(i)$ is the set of all the neighbors of cell i in the mesh and $VC(i)$ is the set of all neighbors of cell i across the split. We also define the discretization of the the exchange operator of order 1.

$$\mathcal{H}_{e,h}^1(\boldsymbol{m}^0, \boldsymbol{m}^1)_i = \frac{A}{h^2} \sum_{j \in V(i)} \left(\boldsymbol{m}_i^1 - \boldsymbol{m}_j^1 \right)$$
$$+ \delta(i) \frac{A}{h^2} \left(\frac{\boldsymbol{m}_{N(i)}^0 - \boldsymbol{m}_i^0}{h} + Q(\boldsymbol{m}_i^0, \boldsymbol{m}_{N'(i)}^0) \right)$$
$$+ \frac{A}{h} \sum_{j \in VC(i)} \mathrm{D}Q(\boldsymbol{m}_i^0, \boldsymbol{m}_j^0) \cdot (\boldsymbol{m}_i^1 - Q(\boldsymbol{m}_i, \boldsymbol{m}_j), \boldsymbol{m}_j^1 - Q(\boldsymbol{m}_j, \boldsymbol{m}_i)), \tag{6.2}$$

where $\delta(i)$ is 1 if cell i is adjacent to the interface Γ, and 0 otherwise. In the former case, cell $N(i)$ is the adjacent cell to cell i such that cell i is between cell $N(i)$ and Γ. Cell $N'(i)$ being the cell such that cell i is between cells $N(i)$ and $N'(i)$. This discretization requires at least two cells in the z direction on each side of the split.

Our aim in these simulations is to compute equilibrium states. We stop the simulation when the derivative of the discrete energy crosses a threshold.

6.0.1 Physical parameters

We use the same physical parameters A, K as in section 4 of [8]. We consider a thin plate with a mesh $256\times128\times1$, hence 32768 grid points, with a step size of 2.3nm. Their magnetic parameters are

$$M_s = 1.4 * 10^6, \qquad A = 10^{-11}/\mu_0, \qquad K = 0.$$

We also take $K_s = 0$ — no surface anisotropy — and $J_2 = 0$. In the geometry considered in

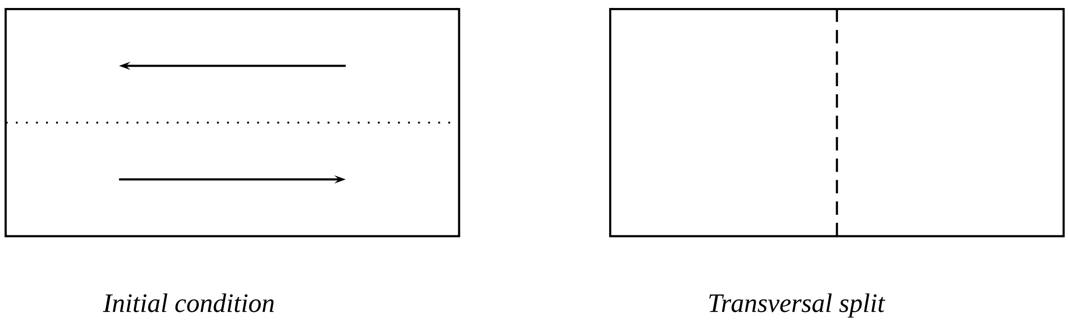

Figure 2: Possible position of the spacer

our computations, the split is transversal and crosses the domain in the middle . The initial condition is given by a magnetization parallel to the longest side of the thin plate. Those are represented in Figure 2. In the numerical results, we prefer to represent $h\boldsymbol{m}^{(1)}$ instead of $\boldsymbol{m}^{(1)}$. We present the results of the simulations corresponding to

- a geometry with no split, as a reference.
- the transversal split drawn in Figure 2 with the following values of J_1.

$$1.0\times10^{-5}, \quad 1.0\times10^{-4}, \quad 2.0\times10^{-4}, \quad 5.0\times10^{-4}, \quad 8.0\times10^{-4}, \quad 1.0\times10^{-3}.$$

6.1 Analysis of the results

We analyze the equilibrium states obtained by our simulation. They represent the equilibrium states and are presented in figures 3, 4, 5, 6, 7, 8, 9, and 10. When we raise the value of J_1, the term of order 0 of the equilibrium point magnetization becomes nearer to the equilibrium point of the magnetization with no split. It confirms that a strong super-exchange interaction favors the alignment of the magnetization across the split. When the super-exchange is weak, the reversal of the magnetization across the split is brutal, which was expected. The quantities of order 1 show two unfinished vortices stretched across the transversal split, those two vortices lower in intensity as the super-exchange become stronger.

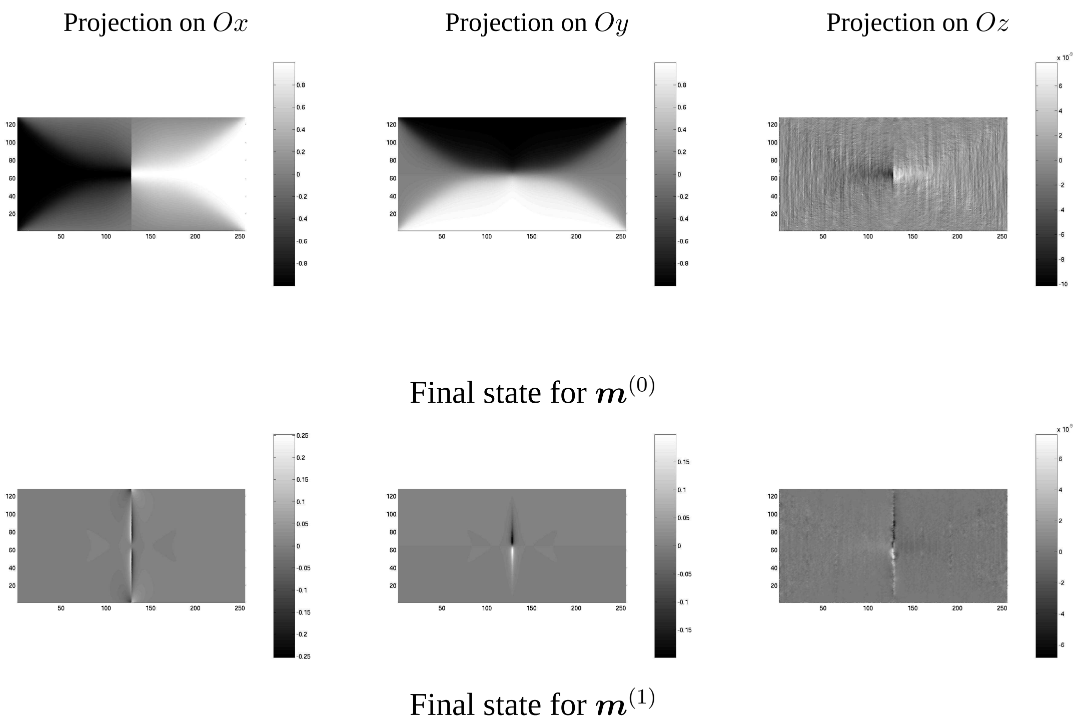

Figure 3: Transversal split, $J_1 = 0$

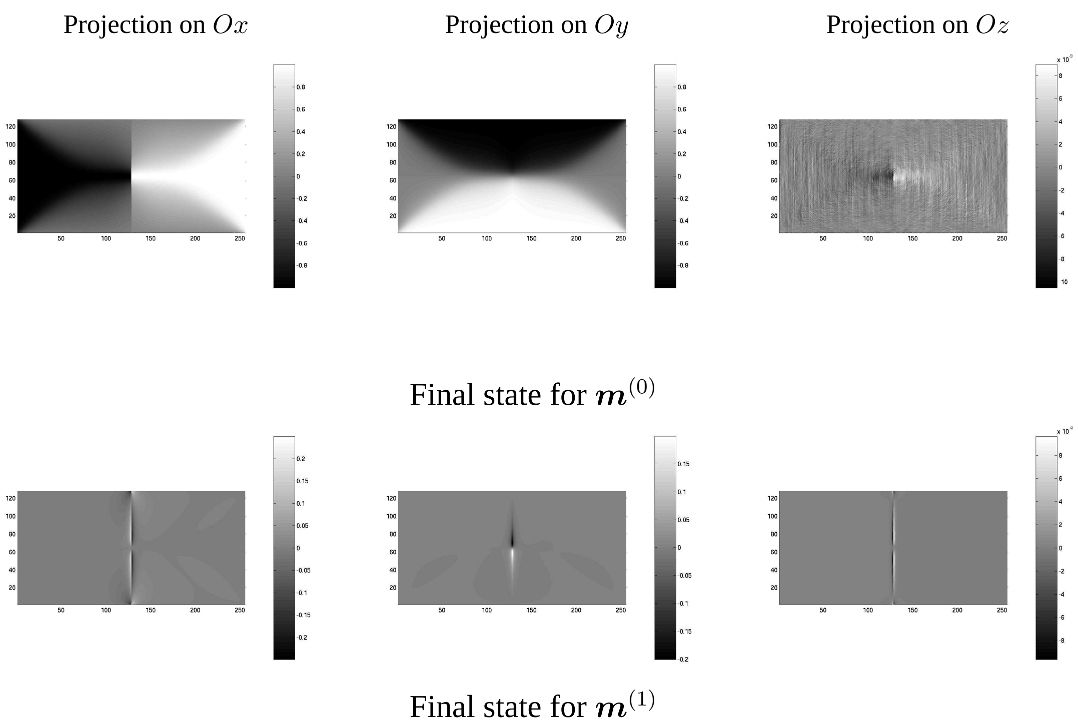

Figure 4: Transversal split, $J_1 = 1.0 \times 10^{-4}$

Conclusion

We have established here non trivial equivalent boundary conditions for a simpler geometry when surface interactions such as surface anisotropy and super-exchange are present, generalizing the results in part I [8]. We can now compute the effect of a split in a ferromagnetic material with a more accurate physical model involving interactions arising near the split.

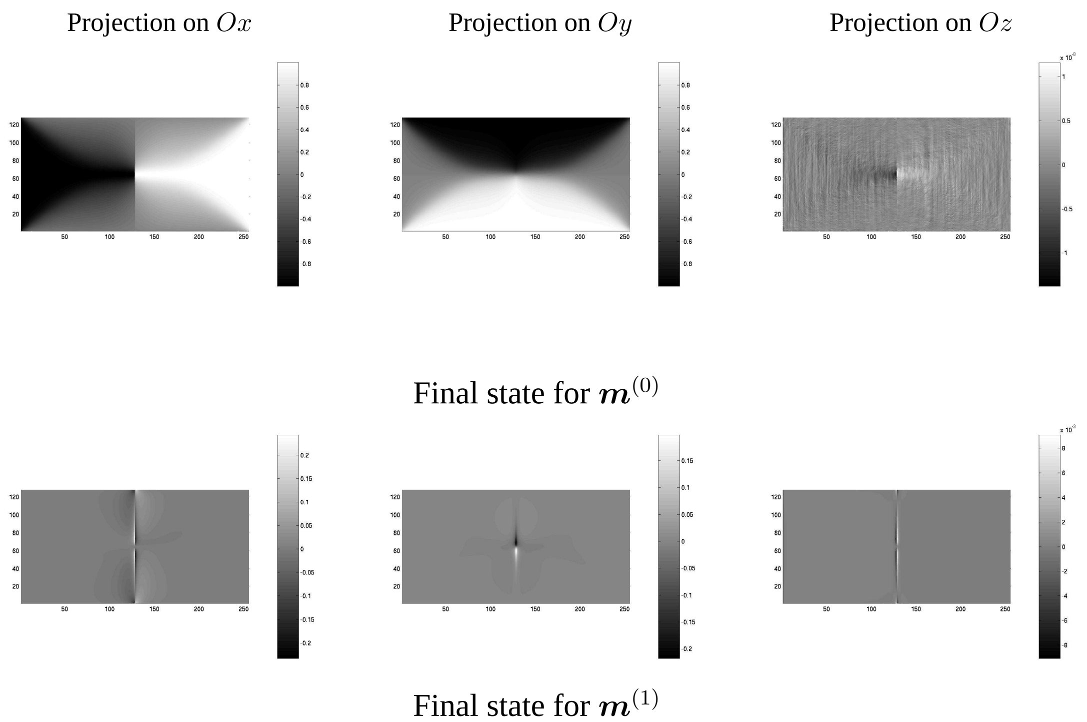

Figure 5: Transversal split, $J_1 = 1.0 \times 10^{-3}$

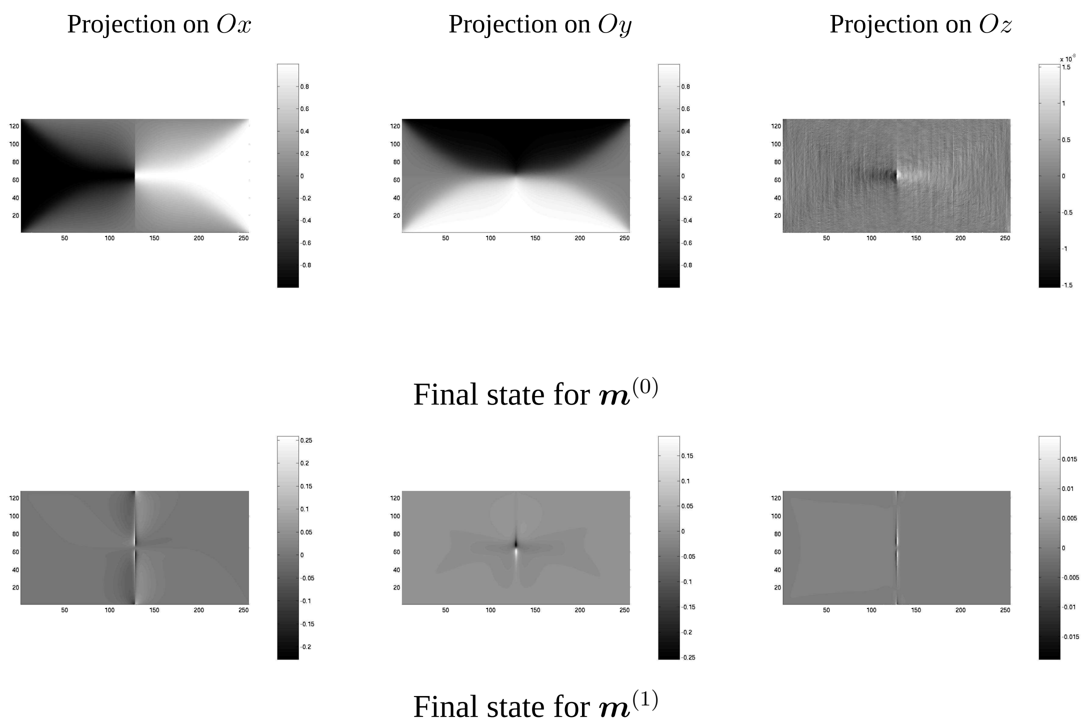

Figure 6: Transversal split, $J_1 = 2.0 \times 10^{-3}$

Future research will be concerned with non-void weak magnetic material filling the split.

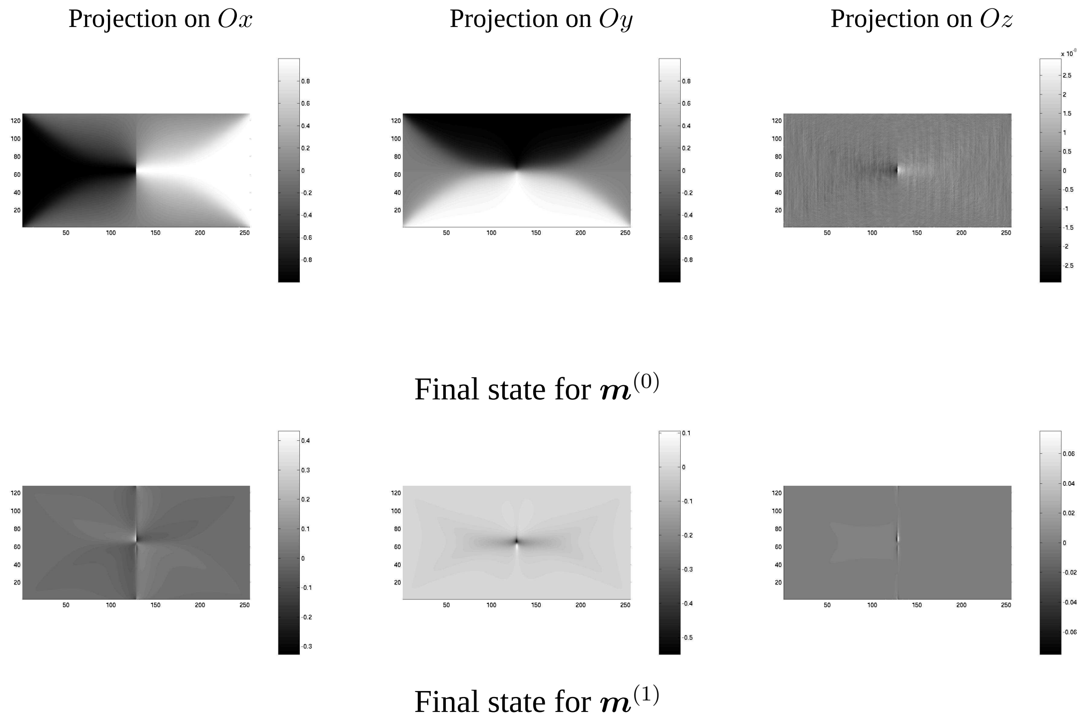

Figure 7: Transversal split, $J_1 = 5.0 \times 10^{-3}$

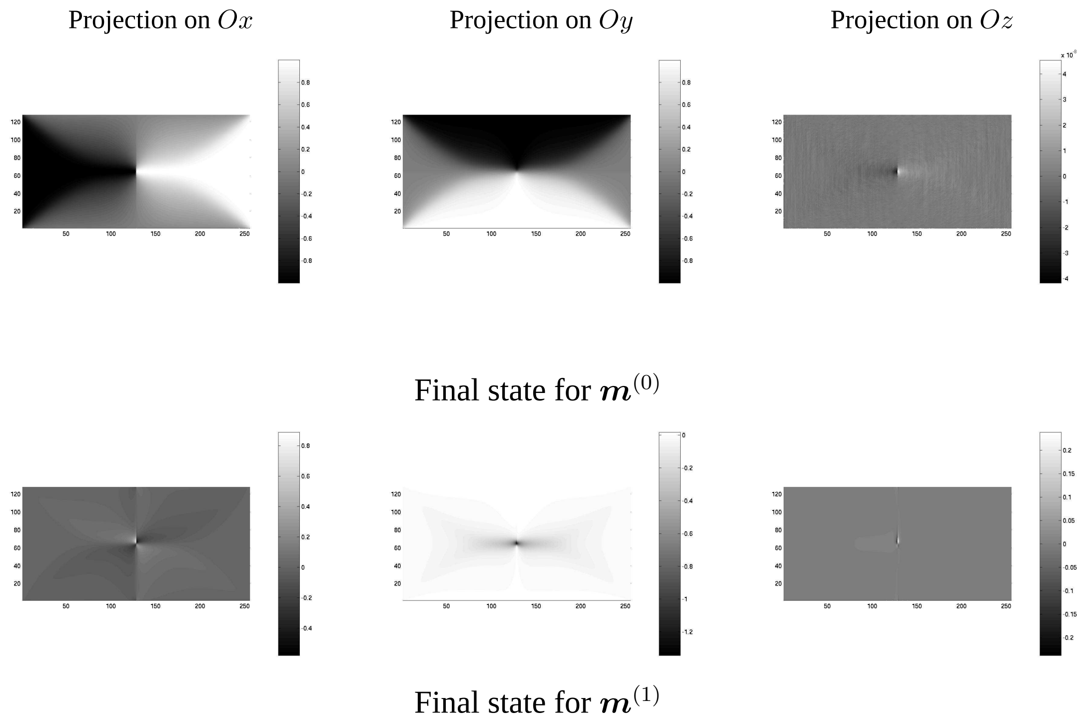

Figure 8: Transversal split, $J_1 = 8.0 \times 10^{-3}$

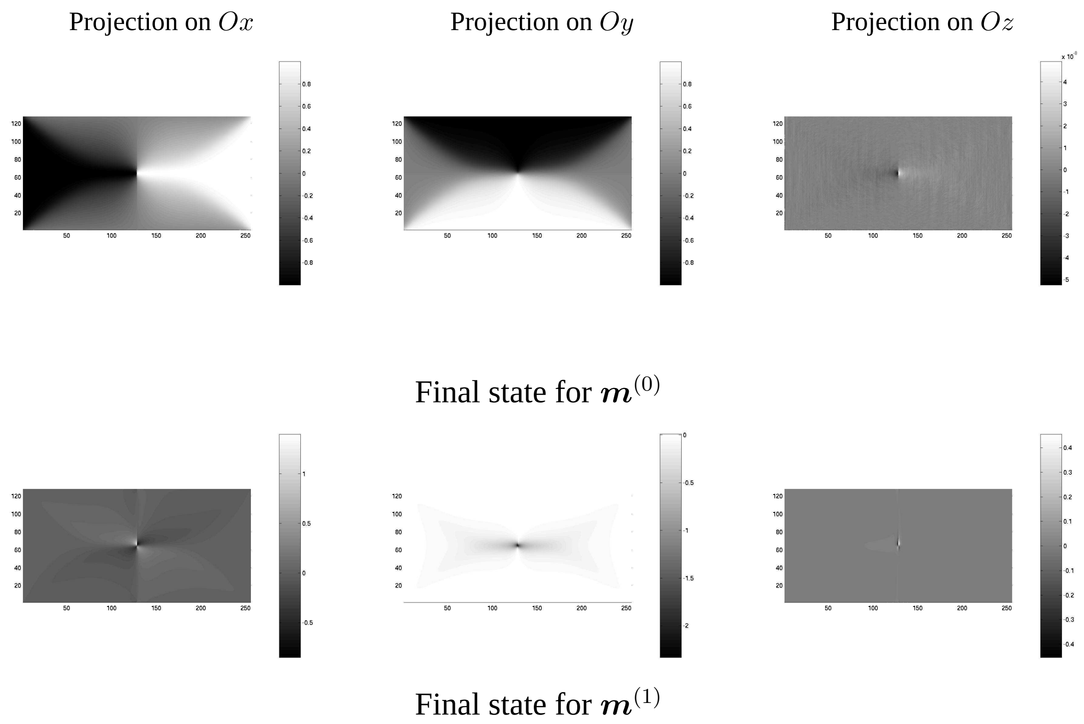

Figure 9: Transversal split, $J_1 = 1.0 \times 10^{-2}$

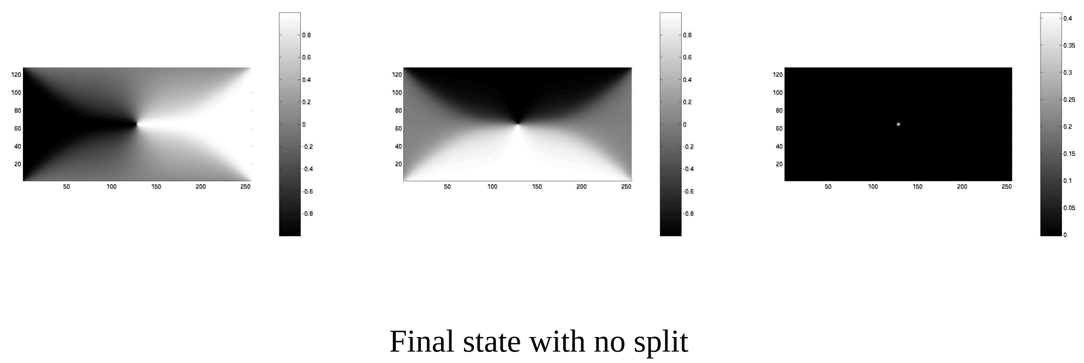

Figure 10: No Split

APPENDIX

A The $\mathrm{H}^{r,s,t}$ spaces

J.L. Lions et E. Magenes have in [5] introduce the $\mathrm{H}^{r,s}$ spaces defined at (0.1) and proved traces theorems. We refer the lector to this book for the details. We adapt this work to study the Sobolev spaces on twice cylindrical domains, once in space and once in time.

A.1 Definition and traces theorems

Let $\mathcal{O} = B \times (0, +L)$ and $Q_T = \mathcal{O} \times (0,T)$. For $r, s, t > 0$, we define the spaces

$$\mathrm{H}^{r,s,t}(Q_T) = \mathrm{H}^t(0,T;\mathrm{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathrm{H}^s(0,L;\mathrm{L}^2(B))) \cap \mathrm{L}^2(0,T;\mathrm{L}^2(0,L;\mathrm{H}^r(B))).$$

As in [5], we also define the spaces

$$\begin{aligned}
\mathrm{H}^{r,s,t}_{00,\,,}(Q_T) &= \mathrm{H}^t(0,T;\mathrm{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathrm{H}^s(0,L;\mathrm{L}^2(B))) \cap \mathrm{L}^2(0,T;\mathrm{L}^2(0,L;\mathrm{H}^r_{00}(B))),\\
\mathrm{H}^{r,s,t}_{,00,}(Q_T) &= \mathrm{H}^t(0,T;\mathrm{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathrm{H}^s_{00}(0,L;\mathrm{L}^2(B))) \cap \mathrm{L}^2(0,T;\mathrm{L}^2(0,L;\mathrm{H}^r(B))),\\
\mathrm{H}^{r,s,t}_{,\,,00}(Q_T) &= \mathrm{H}^t_{00}(0,T;\mathrm{L}^2(\mathcal{O})) \cap \mathrm{L}^2(0,T;\mathrm{H}^s(0,L;\mathrm{L}^2(B))) \cap \mathrm{L}^2(0,T;\mathrm{L}^2(0,L;\mathrm{H}^r(B))),
\end{aligned}$$

and,

$$\begin{aligned}
&\mathrm{H}^{r,s,t}_{00,00,}(Q_T) = \mathrm{H}^{r,s,t}_{00,\,,} \cap \mathrm{H}^{r,s,t}_{,00,}, \quad \mathrm{H}^{r,s,t}_{00,\,,00}(Q_T) = \mathrm{H}^{r,s,t}_{00,\,,} \cap \mathrm{H}^{r,s,t}_{,\,,00},\\
&\mathrm{H}^{r,s,t}_{,00,00}(Q_T) = \mathrm{H}^{r,s,t}_{,00,} \cap \mathrm{H}^{r,s,t}_{,\,,00}.
\end{aligned}$$

Lemma A.1 (Interpolation of the $\mathrm{H}^{r,s,t}$ spaces)**.**

$$[\mathrm{H}^{r_1,s_1,t_1}, \mathrm{H}^{r_2,s_2,t_2}]_\theta = \mathrm{H}^{(1-\theta)r_1+\theta r_2,(1-\theta)s_1+\theta s_2,(1-\theta)t_1+\theta t_2}, \tag{A.1}$$

$$[\mathrm{H}^{r_1,s_1}_{00,}, \mathrm{H}^{r_2,s_2}_{00,}]_\theta = \mathrm{H}^{(1-\theta)r_1+\theta r_2,(1-\theta)s_1+\theta s_2}_{00,}, \tag{A.2}$$

$$[\mathrm{H}^{r_1,s_1}_{00,00}, \mathrm{H}^{r_2,s_2}_{00,00}]_\theta = \mathrm{H}^{(1-\theta)r_1+\theta r_2,(1-\theta)s_1+\theta s_2}_{00,00}. \tag{A.3}$$

Theorem A.2 (Existence of traces)**.** *If v belongs to $\mathrm{H}^{r,s,t}(Q_T)$ then*

1. *If $r > \frac{1}{2}$, and for all $0 \le j$,*

$$\frac{\partial^j v}{\partial \boldsymbol{\nu}^j} \in \mathrm{H}^{\mu_j,\nu_j,\lambda_j}(\partial B \times (0,L) \times (0,T)),$$

with $\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{\lambda_j}{t} = \frac{r-j-\frac{1}{2}}{r}$.

2. *If $s > \frac{1}{2}$, then for all $0 \le k < s - \frac{1}{2}$,*

$$\frac{\partial^k v}{\partial z^k} \in \mathrm{H}^{p_k,q_k}(B \times (0,T)),$$

with $\frac{p_k}{r} = \frac{q_k}{t} = \frac{s-k-\frac{1}{2}}{s}$.

3. *If $s > \frac{1}{2}$, then for all $0 \le l < t - \frac{1}{2}$,*

$$\frac{\partial^l v}{\partial t^l} \in \mathrm{H}^{\alpha_l,\beta_l}(B \times (0,L)),$$

with $\frac{\alpha_l}{r} = \frac{\beta_l}{s} = \frac{t-l-\frac{1}{2}}{t}$.

Furthermore, the trace maps are linear continuous.

PROOF: We adapt the proof of Lions-Magenes [5]. The theorem is a direct consequence of theorem 4.2 in Lions-Magenes [6]. To apply this theorem, we need interpolation results provided in proposition 2.1 in [5] and Lemma A.1. □

A.2 Conditions of compatibility

Proposition A.3 (First compatibility conditions)**.** *Let*

$$f_l = \frac{\partial^l v}{\partial t^l} \in \mathrm{H}^{\alpha_l,\beta_l}(B \times (0, L)),$$
$$g_k = \frac{\partial^k v}{\partial z^k} \in \mathrm{H}^{p_k,q_k}(B \times (0, T)),$$
$$h_j = \frac{\partial^j v}{\partial \boldsymbol{\nu}^j} \in \mathrm{H}^{\mu_j,\nu_j,\lambda_j}(\partial B \times (0, L) \times (0, T)).$$

Then

1. *If* $1 - \frac{1}{2}(\frac{1}{r} + \frac{1}{s}) > 0$*, then for all* $j, k \geq 0$ *such that* $\frac{j}{r} + \frac{k}{s} < 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right)$*,*
$$\frac{\partial^j g_k}{\partial \boldsymbol{\nu}^j} = \frac{\partial^k h_j}{\partial z^k}$$
2. *If* $1 - \frac{1}{2}(\frac{1}{r} + \frac{1}{t}) > 0$*, then for all* $j, l \geq 0$ *such that* $\frac{j}{r} + \frac{l}{t} < 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{t}\right)$*,*
$$\frac{\partial^j f_l}{\partial \boldsymbol{\nu}^j} = \frac{\partial^l h_j}{\partial t^l}.$$
3. *If* $1 - \frac{1}{2}(\frac{1}{s} + \frac{1}{t}) > 0$*, then for all* $k, l \geq 0$ *verifying* $\frac{k}{s} + \frac{l}{t} < 1 - \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right)$*,*
$$\frac{\partial^j g_k}{\partial t^l} = \frac{\partial^k f_l}{\partial z^k}.$$

PROOF: $\mathcal{D}(\overline{Q_T})$ is dense in $\mathrm{H}^{r,s,t}(Q_T)$ and the trace maps are continuous. □

Proposition A.4 (Second compatibility conditions)**.** *With the same notations as the one used for the first conditions. We suppose that* B *is the semi-space* $B = \mathbb{R}^{n-1} \times \mathbb{R}_+$ *and that* $L, T = +\infty$*.Then*

1. *For all* j, k *such that* $\frac{j}{r} + \frac{k}{s} = 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right)$*,*
$$\int_{\sigma=0}^{+\infty} \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \left| \frac{\partial^k h_j}{\partial z^k}(\boldsymbol{x}', \sigma^r, t) - \frac{\partial^j g_k}{\partial \boldsymbol{\nu}^j}(\boldsymbol{x}', \sigma^s, t) \right|^2 \mathrm{d}t\mathrm{d}\boldsymbol{x}' \frac{\mathrm{d}\sigma}{\sigma} < +\infty.$$
2. *For all* j, l *such that* $\frac{j}{r} + \frac{l}{t} = 1 - \frac{1}{2}\left(\frac{1}{r} + \frac{1}{t}\right)$*,*
$$\int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \left| \frac{\partial^l h_j}{\partial t^l}(\boldsymbol{x}', z, \sigma^r) - \frac{\partial^j f_l}{\partial \boldsymbol{\nu}^j}(\boldsymbol{x}', \sigma^t, z) \right|^2 \mathrm{d}z\mathrm{d}\boldsymbol{x}' \frac{\mathrm{d}\sigma}{\sigma} < +\infty.$$

3. *For all* k, l *such that* $\frac{k}{s} + \frac{l}{t} = 1 - \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right)$,

$$\int_0^{+\infty}\int_{\mathbb{R}^{n-1}}\int_0^{+\infty}\left|\frac{\partial^k f_l}{\partial z^k}(\boldsymbol{x}', x_n, \sigma^t) - \frac{\partial^l g_k}{\partial t^l}(\boldsymbol{x}', x_n, \sigma^s, x')\right|^2 \mathrm{d}x_n \mathrm{d}\boldsymbol{x}' \frac{\mathrm{d}\sigma}{\sigma} < +\infty.$$

PROOF: It is a direct consequence of theorem 2.2 of Lions-Magenes [5]. We do it for one inequality. We differentiate u and obtain that $\frac{\partial^k}{\partial z^k}\left(\frac{\partial^l u}{\partial t^l}\right)$ belongs to $\mathrm{H}^{\mu,\nu,\lambda}(B \times \mathbb{R}_z^+ \times \mathbb{R}_t^+)$, with $\frac{\mu}{r} = \frac{\nu}{s} = \frac{\lambda}{t} = 1 - \left(\frac{k}{s} + \frac{l}{t}\right)$. Since, $\frac{1}{\nu} + \frac{1}{\lambda} = 2$, we can conclude. □

Definition A.5. We define

$$F = \prod_{j<r-\frac{1}{2}} \mathrm{H}^{\mu_j,\nu_j,\lambda_j}(\partial B\times(0,L)\times(0,T))\times \prod_{k<s-\frac{1}{2}} \mathrm{H}^{p_k,q_k}(\partial B\times(0,T))\times \prod_{l<t-\frac{1}{2}} \mathrm{H}^{\alpha_l,\beta_l}(\partial B\times(0,L)).$$

Let F_0 be the subspace of F comprising functions (h_j, g_k, f_l) satisfying both compatibility conditions stated in Propositions A.3 and A.4.

Let $\Sigma = \partial B \times (0, L) \times (0, T)$. We state the principal extension theorem.

Theorem A.6 (Surjectivity of the trace map)**.** *The trace map*

$$\gamma : \mathrm{H}^{r,s,t}(Q_T) \to F_0$$
$$v \mapsto (h_j, g_k, f_l)$$

is onto and has a continuous right inverse.

PROOF: We need interpolations equalities (A.2) and (A.3). We only need to prove the surjectivity. Let (f_l, g_k, h_j) be in F_0. Then, there exists φ in $\mathrm{H}^{r,s,t}(Q_T)$ such that $\frac{\partial^j\varphi}{\partial\boldsymbol{\nu}^j} = h_j$ for all $0 \le j < r - \frac{1}{2}$. And $(f_l - \partial_t^l\varphi, g_k - \partial_z^k\varphi, 0)$ belongs to F_0. Thus, $g_k - \partial_z^k\varphi$ belongs to $\mathrm{H}_{00,}^{p_k,q_k}(\mathbb{R}_t^+ \times B)$. Using theorem 4.2 of Lions-Magenes [6], there exists ψ in $\mathrm{H}^{r,s,t}(Q_T)$ such that $\frac{\partial^j\psi}{\partial\boldsymbol{\nu}^j} = 0$ for all $0 \le j < r - \frac{1}{2}$, and $\frac{\partial^k\psi}{\partial z^k} = g_k - \partial_z^k\varphi$ for all $0 \le k < s - \frac{1}{2}$. And $(f_l - \partial_t^l\varphi - \partial_t\psi, 0, 0)$ belongs to F_0. Thus, $f_l - \partial_t^l\varphi - \partial_t^l\psi$ belongs to $\mathrm{H}_{00,00}^{\alpha_l,\beta_l}(\mathbb{R}_z^+ \times B)$. By Theorem 4.2 of Lions-Magenes [6], there exists Φ in $\mathrm{H}^{r,s,t}(Q_T)$ such that $\frac{\partial^j\psi}{\partial\boldsymbol{\nu}^j} = 0$ for all $0 \le j < r - \frac{1}{2}$, $\frac{\partial^k\psi}{\partial z^k} = 0$ for all $0 \le k < s - \frac{1}{2}$, and $\frac{\partial^l\psi}{\partial t^k} = h_l - \partial_t^l\varphi - \partial_t^l\psi$ for all $0 \le l < t - \frac{1}{2}$. $u = \varphi + \psi + \Phi$ belongs to $\mathrm{H}^{r,s,t}(Q_T)$ and has traces (f_j, g_k, h_l). The construction also provide the continuous right inverse. □

We know for the spaces $\mathrm{H}^{r,s}$ and these spaces $\mathrm{H}^{r,s,t}$ the compatibility relations. However, these relations ensure the surjectivity when all traces are presents. Sometimes, we only wish to extend a subset of all traces. If the direct compatibility relations are verified, can we always complete the mandatory traces by dummy traces such that all compatibility relations are satisfied? We prove in two particular cases that no new indirect compatibility relations are necessary. From now, we denote by $\boldsymbol{x}$ a vector of $\mathbb{R}^n$ with decomposition $\boldsymbol{x} = (\boldsymbol{x}', x_n)$ where $\boldsymbol{x}'$ belongs to $\mathbb{R}^{n-1}$ and x_n is scalar. z is the additional variable of space. t is the time variable.

Theorem A.7. *Let* B *be a bounded open set with a smooth boundary, and* L *and* T *two positive real. Then, the maps*

$$\begin{aligned}\mathrm{H}^{2,2}(B \times (0, L)) &\to \mathrm{H}^{\frac{1}{2}}(B \times \{0\}) \times \mathrm{H}^{\frac{1}{2}}(B \times \{L\}) \times \mathrm{H}^{\frac{1}{2}}(\partial B \times (0, L)),\\ u &\mapsto \left(\frac{\partial u}{\partial\boldsymbol{\nu}}(\boldsymbol{x} \in \partial B), \frac{\partial u}{\partial z}(\cdot, \cdot, 0), -\frac{\partial u}{\partial z}(\cdot, \cdot, L)\right).\end{aligned} \tag{A.4}$$

and

$$\mathrm{H}^{2,2,1}(B\times(0,L)\times(0,T))\to \mathrm{H}^{\frac12,\frac12,\frac34}(\partial B\times(0,L)\times(0,T))\times \mathrm{H}^{\frac12,\frac34}(B\times\{0\}\times(0,T))$$
$$\times\,\mathrm{H}^{\frac12,\frac34}(B\times\{L\}\times(0,T)),$$
$$u\mapsto\left(\frac{\partial u}{\partial\boldsymbol{\nu}}(\boldsymbol{x}\in\partial B),\frac{\partial u}{\partial z}(\cdot,\cdot,0,\cdot),\frac{\partial u}{\partial z}(\cdot,\cdot,L,\cdot)\right). \tag{A.5}$$

are onto and have a continuous right inverse.

PROOF: By abstract consideration on Hilbert spaces, we only need to prove the surjectivity. By local map and partition of the unity, we reduce both problems to the case $L=+\infty$, $T=+\infty$, and $B=\mathbb{R}^{n-1}\times\mathbb{R}^+_{x_n}$. Let f_1,g_1 be in $\mathrm{H}^{\frac12}(\mathbb{R}^{n-1}_{\boldsymbol{x}'}\times\mathbb{R}^+_z)\mathrm{H}^{\frac12}(\mathbb{R}^{n-1}_{\boldsymbol{x}'}\times\mathbb{R}^+_{x_n})$. To apply the surjectivity theorems of Lions-Magenes for map (A.4), we must construct f_0,g_0 such that (g_1,f_1,g_0,f_0) satisfy all compatibility relations. We first notice that there is no direct compatibility condition between f_1 and g_1. Then, we construct g_0 and f_0 with Theorem A.8. We use the same method to prove the surjectivity of map (A.5). To apply Theorem A.6, we use Theorem A.9 to construct g_0 and , h_0 satisfying the compatibility condition. Constructing f_0 is easy and anyway not necessary because of the way we proved Theorem A.6. □

A.3 Completion of traces for the space $\mathrm{H}^{2,2}(\mathbb{R}^{n-1}\times\mathbb{R}^+)$

Theorem A.8. *There exists a linear continuous map* $\mathbf{Y}$ *from* $\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^{n-1}))$ *to the space* $\mathrm{H}^{1,1}(\mathbb{R}^{n-1}\times(0,+\infty))$ *and from* $\mathrm{H}^{\frac12,\frac12}(\mathbb{R}^{n-1}\times(0,+\infty))$ *to the space* $\mathrm{H}^{\frac32,\frac32}(\mathbb{R}^{n-1}\times(0,+\infty))$. *Moreover, there exists a constant* C *such that*$\mathbf{Y}(f)(\cdot,0)=0$, *and*

$$\int_0^{+\infty}\frac{\|\frac{\partial(\mathbf{Y}f)}{\partial z}-f\|^2_{\mathrm{L}^2(\mathbb{R}^{n-1})}}{z}\mathrm{d}z\le C\|f\|_{\mathrm{L}^2(0,+\infty;\mathrm{H}^{\frac12}(\mathbb{R}^{n-1}))}.$$

PROOF: Here $\widehat{u}$ is the partial Fourier transform of u in the first two variables. We define $\widehat{\mathbf{Y}(f)}(\xi,z)=\chi(z\sqrt{1+|\xi|^2})\int_0^z\widehat{f}(\xi,z)\mathrm{d}z$, where χ is a smooth real function satisfying, $0\le\chi\le1$, with $\mathrm{Supp}(\chi)\subset[0,2]$ and $\chi=1$ in $[0,1]$. $\mathbf{Y}$ is the application we were looking for. Verifying it is tedious but straightforward. □

Theorem A.9. *There exists a linear continuous map* $\mathbf{A}$ *from* $\mathrm{H}^{\frac12,\frac12,\frac14}(\mathbb{R}^{n-1}_{\boldsymbol{x}}\times\mathbb{R}^+_z\times\mathbb{R}_t)$ *to* $\mathrm{H}^{\frac32,\frac32,\frac34}(\mathbb{R}^{n-1}_{\boldsymbol{x}}\times\mathbb{R}^+_z\times\mathbb{R}_t)$ *and a constant* $C>0$ *such that* $\mathbf{A}(f)(\cdot,0,\cdot)=0$, *and*

$$\int_{z=0}\int_t\int_{\boldsymbol{x}}\left|\frac{\partial\mathbf{A}(f)}{\partial z}-f\right|^2\mathrm{d}\boldsymbol{x}\mathrm{d}t\frac{\mathrm{d}z}{z}\le C\|f\|^2_{\mathrm{L}^2(\mathbb{R}^+_z\times\mathbb{R}_t;\mathrm{H}^{\frac12}(\mathbb{R}^{n-1}_{\boldsymbol{x}}))},.$$

PROOF: We first define the $\mathbf{B}$ operator with help from Theorem A.8. For all f in $\mathrm{H}^{\frac12,\frac12,\frac14}(\mathbb{R}^{n-1}_{\boldsymbol{x}}\times\mathbb{R}^+_z\times\mathbb{R}_t)$, we define $\mathbf{B}(f)(t)$ as $\mathbf{Y}(f(t))$ for all time t. We then define $\mathbf{A}(f)$ as $\widehat{\mathbf{A}f}=\chi(z(1+|\tau|^2)^{\frac14})\widehat{\mathbf{B}(f)}$, where the Fourier transform is only in time and where χ is smooth and satisfies. $\chi=1$ in $(-L/4,5L/4)$, and $\chi=0$ in $\complement(-L/2,3L/2)$. $\mathbf{A}$ has the required properties. The verification is straightforward but tedious. □

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