Solutions to the Landau-Lifshitz system with nonhomogenous Neumann boundary conditions arising from surface anisotropy and super-exchange interactions in a ferromagnetic media

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Abstract

We study the solutions to the Landau-Lifshitz system in a bilayered ferromagnetic body when super-exchange and surface anisotropy interactions are present at the interface between the layers. We prove the existence of weak solutions in infinite time and strong solutions in finite time.

Key words: Ferromagnetism; Micromagnetism; LandauâĂŞLifshitz equation; Nonlinear PDE

1 Introduction

Ferromagnetic materials have long been the subject of scientific studies. Nowadays, they play an important role in the industry. Optimizing the form of a ferromagnetic body is an important goal since its magnetic macroscopic properties depend strongly on it. Among the possible configurations, multi-layers have been the focus of research in recent years. Physicists [1] have modeled short-range interactions, such as super-exchange and surface anisotropy that are able to cross a nonferromagnetic interface that split two ferromagnetic bodies. These interactions modify the Neumann boundary condition in a nonlinear way.

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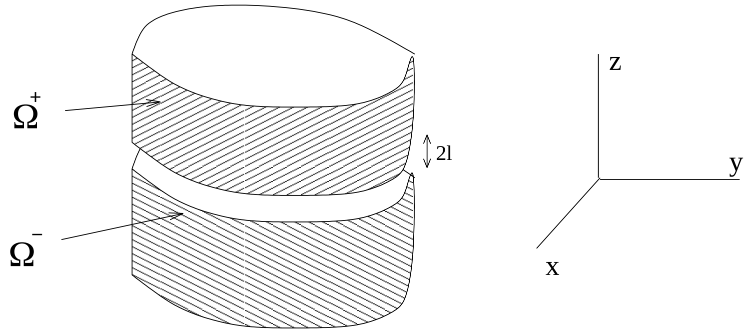

Figure 1. Geometry of the problem

The magnetic state of a ferromagnetic body [1] can be represented by a vector field over $\mathbb{R}^3$ called the magnetization $\boldsymbol{M}$. The local norm of $\boldsymbol{M}$ is constant and equal to M_s inside the ferromagnetic body and 0 outside. The value of M_s depends only on the material and its temperature which will be considered constant throughout this paper. We work with the dimensionless vector field $\boldsymbol{m} = \boldsymbol{M}/M_s$ with a local norm $|\boldsymbol{m}| = 1$. The behavior of $\boldsymbol{m}$ may be modeled by the Landau-Lifshitz equation

$$\frac{\partial \boldsymbol{m}}{\partial t} = -\boldsymbol{m} \times \boldsymbol{h} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{h}),$$

where $\boldsymbol{h}$ is the magnetic excitation. This equation has been widely studied. The existence of weak solutions over infinite time and of strong solutions over finite time with the homogenous Neumann boundary condition has been established in [4], [5], [6], [7], and [8]. Existence of weak solutions has also been established with the surface anisotropy interaction in [9].

We prove here the existence of both strong and weak solutions to the Landau-Lifshitz system with the nonhomogenous Neumann boundary conditions that arise from the presence of super-exchange and surface anisotropy.

We work on a geometry similar to the one presented in figure 1. Let B be a regular convex bounded open set of $\mathbb{R}^2$. Let L^+ and L^- be two positive real number and l be a small given positive real number, such that $l \ll \min(L^+, L^-)$. We now introduce some notations

- $\mathcal{I}_l^+ = (l, L^+)$, $\mathcal{I}_l^- = (-L^-, -l)$, and $\mathcal{I}_l = \mathcal{I}_l^+ \cup \mathcal{I}_l^-$.
- $\Omega^+ = B \times \mathcal{I}_l^+$, $\Omega^- = B \times \mathcal{I}_l^-$, and $\Omega = \Omega^+ \cup \Omega^-$.
- $Q_T = \Omega \times (0, T)$.
- $\Gamma^+ = B \times \{+l\}$, $\Gamma^- = B \times \{-l\}$ and $\Gamma^\pm = \Gamma^+ \cup \Gamma^-$.
- γ is the map that sends $\boldsymbol{m}$ to its trace on $\Gamma^\pm$.
- γ' is the trace map that sends $\boldsymbol{m}$ to $\gamma(\boldsymbol{m} \circ \sigma)$ where σ is the application that sends (x, y, z, t) to $(x, y, -z, t)$.

[1] For an introduction to ferromagnetism, refer to [2] or [3].

- $\Gamma = B \times \{0\}$. γ_l^+ is the trace map that sends $\boldsymbol{m}$ to $(\gamma \boldsymbol{m}) \circ \tau_{-l}$ on Γ where $\tau_{-l}(x, y, z, t) = (x, y, z+l, t)$. γ_l^- is the trace map that sends $\boldsymbol{m}$ to $(\gamma \boldsymbol{m}) \circ \tau_{+l}$ on Γ.
- $\boldsymbol{\nu}$ is the extension of the unitary exterior normal to Ω on $\Gamma^{\pm}$, thus $\boldsymbol{\nu}(\boldsymbol{x}) = -\boldsymbol{e}_z$ if $z > 0$ or if $\boldsymbol{x}$ belongs to Γ^+, and $\boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{e}_z$ if $z < 0$ or if $\boldsymbol{x}$ belongs to Γ^-.

In section 2, we describe the mathematical model of micromagnetism in a ferromagnetic body, introducing the physical interactions and their associated energies and operators. In section 3, we state the theorems we establish in this article. These theorems assert the existence of both weak and strong solutions of the Landau-Lifshitz system with an uniqueness result for strong solutions. We then proceed to prove these theorems in section 5 and 6. Both proofs use Galerkin's method. As the basis verifies the homogenous Neumann condition, other methods are required to obtain a nonhomogenous Neumann condition on the boundary. We proceed by penalization for weak solutions and by extension results for strong solutions.

2 Energies and associated operators

There are some interactions associated with the magnetization state of a ferromagnetic body. To each interaction corresponds an energy E_p and an operator $\mathcal{H}_p$ given by the formulae

$$\begin{aligned} \mathrm{E}_p(0) &= 0, \\ \mathrm{DE}_p(\boldsymbol{m}) \cdot \boldsymbol{v} &= -\int_\Omega \mathcal{H}_p(\boldsymbol{m}) \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{v} \in \mathbb{H}^1(\Omega). \end{aligned}$$

Conversely, to each operator, we can associate an energy by the same formulae. In the frequent case of $\mathcal{H}_p$ being a self-adjoint linear operator, $\mathrm{E}_p(\boldsymbol{m}) = -\frac{1}{2}\int_\Omega \mathcal{H}_p(\boldsymbol{m}) \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{x}$.

2.1 Volume interactions

2.1.1 The anisotropy interaction

The anisotropic energy is $\mathrm{E}_a(\boldsymbol{m}) = \frac{1}{2}\int_\Omega (\mathbf{K}\boldsymbol{m}) \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{x}$, where $\mathbf{K}$ is a $\mathcal{C}^1(\overline{\Omega})$ map onto the set of symmetric positive matrices. Thus, $\mathcal{H}_a(\boldsymbol{m}) = -\mathbf{K}\boldsymbol{m}$. A most common form of anisotropy is the uniaxis anisotropy with $\mathbf{K}\boldsymbol{m} = K_v((\boldsymbol{m} \cdot \boldsymbol{u})\boldsymbol{m} - \boldsymbol{m})$ where $\boldsymbol{u}$ is a $\mathbb{R}^3$ vector field and $K_v > 0$.

2.1.2 *The exchange interaction*

The exchange energy is $\mathrm{E}_e(\boldsymbol{m}) = \frac{A}{2}\int_\Omega |\nabla \boldsymbol{m}|^2 \,\mathrm{d}\boldsymbol{x}$, where $A > 0$. The exchange operator is defined as $\mathcal{H}_e(\boldsymbol{m}) = A\triangle \boldsymbol{m}$.

2.1.3 *The demagnetization field interaction*

We use here the quasi-static approximation of Maxwell's system. We define $\mathcal{H}_d$ as the operator that sends any $\boldsymbol{m}$ to the solution $\boldsymbol{h}_d$ of the system

$$\begin{aligned} \operatorname{div}(\boldsymbol{h}_d) &= -\operatorname{div}(\boldsymbol{m}), \\ \mathbf{rot}(\boldsymbol{h}_d) &= 0, \\ \boldsymbol{m} &= 0 \qquad \text{on } \mathbb{R}^3 \setminus \Omega, \end{aligned}$$

in the sense of distributions. The demagnetization field energy is $\mathrm{E}_d(\boldsymbol{m}) = -\frac{1}{2}\int_\Omega \boldsymbol{h}_d \cdot \boldsymbol{m} \,\mathrm{d}\boldsymbol{x} = \frac{1}{2}\int_\Omega |\boldsymbol{h}_d|^2 \,\mathrm{d}\,\boldsymbol{x}$.

Regarding the regularity of operator $\mathcal{H}_d$, we have the following result

Theorem 1 *For all $1 \leq p < +\infty$, $\mathcal{H}_d$ is a continuous operator from $\mathbb{L}^p(\Omega)$ to $\mathbb{L}^p(\Omega)$, and from $\mathbb{W}^{1,p}(\Omega)$ to $\mathbb{W}^{1,p}(\Omega)$.*

PROOF. See [10] or [11].

2.2 *Interaction on the boundary*

2.2.1 *The surface anisotropy interaction*

We use the model described in [1]. The surface anisotropy energy and the associated operator are

$$\begin{aligned} \mathrm{E}_{sa} &= \frac{K_s}{2}\int_{\Gamma^- \cup \Gamma^+} (1 - (\gamma \boldsymbol{m} \cdot \boldsymbol{\nu})^2) \,\mathrm{d}\boldsymbol{x}, \\ \mathcal{H}_{sa} &= K_s((\gamma \boldsymbol{m} \cdot \boldsymbol{\nu})\boldsymbol{\nu} - \gamma \boldsymbol{m})\delta\Gamma^\pm \qquad \text{on } \Gamma^- \cup \Gamma^+. \end{aligned}$$

This operator has the same form as the volume anisotropy uniaxis operator but with $\boldsymbol{u} = \boldsymbol{\nu}$.

2.2.2 *The super-exchange interaction*

This interaction has its roots in quantum mechanics, see [2]. We use the mathematical model described in [1]. In this model, the energy and the operator associated with the super-exchange operator are

$$
\begin{aligned}
\mathrm{E}_{se}(\boldsymbol{m}) = J_1 \int_\Gamma (1 - \gamma_l^+ \boldsymbol{m} \cdot \gamma_l^- \boldsymbol{m}) \, \mathrm{d}\boldsymbol{x} + J_2 \int_\Gamma (1 - |\gamma_l^+ \boldsymbol{m} \cdot \gamma_l^- \boldsymbol{m}|^2) \, \mathrm{d}\boldsymbol{x}, \\
\mathcal{H}_{se} = J_1(\gamma' \boldsymbol{m} - \gamma \boldsymbol{m}) + 2J_2((\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma' \boldsymbol{m} - |\gamma' \boldsymbol{m}|^2 \gamma \boldsymbol{m}) \delta\Gamma^{\pm},
\end{aligned}
$$

where J_1, J_2 are positive numbers.

2.2.3 Modification of the boundary condition

The boundary conditions verified by $\boldsymbol{m}$ are

$$
\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = 0 \qquad \text{on } \partial\Omega \setminus \Gamma, \tag{2.1a}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = \frac{Ks}{A}(\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})\gamma \boldsymbol{m}) + \frac{J_1}{A}(\gamma' \boldsymbol{m} - (\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma \boldsymbol{m}) \\
+ 2\frac{J_2}{A}(\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})(\gamma' \boldsymbol{m} - (\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma \boldsymbol{m}) \qquad \text{on } \Gamma^+,
\end{aligned} \tag{2.1b}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = \frac{Ks}{A}(\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})\gamma \boldsymbol{m}) + \frac{J_1}{A}(\gamma' \boldsymbol{m} - (\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma \boldsymbol{m}) \\
+ 2\frac{J_2}{A}(\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})(\gamma' \boldsymbol{m} - (\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma \boldsymbol{m}) \qquad \text{on } \Gamma^-.
\end{aligned} \tag{2.1c}
$$

These are obtained from the stationary conditions on the boundary, see [1].

2.3 Some notations on the interaction operators and energies

We introduce the following notations,

$$
\mathcal{H}_{d,a} \equiv \mathcal{H}_d + \mathcal{H}_a, \qquad \mathrm{E}_{d,a} \equiv \mathrm{E}_d + \mathrm{E}_a, \tag{2.2a}
$$

$$
\mathcal{H}_v \equiv \mathcal{H}_d + \mathcal{H}_a + \mathcal{H}_e, \qquad \mathrm{E}_v \equiv \mathrm{E}_a + \mathrm{E}_d + \mathrm{E}_e, \tag{2.2b}
$$

$$
\mathrm{E}_s \equiv \mathrm{E}_{se} + \mathrm{E}_{sa}, \qquad \mathcal{H}_s \equiv \mathcal{H}_{se} + \mathcal{H}_{sa}. \tag{2.2c}
$$

The total energy is

$$
\mathrm{E} \equiv \mathrm{E}_v + \mathrm{E}_s. \tag{2.2d}
$$

3 Definitions of solutions and main theorems

Formally, the solutions verify the following system of equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{m}}{\partial t} &= -\boldsymbol{m} \times \mathcal{H}_v(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}_v(\boldsymbol{m})) \qquad \text{in } \Omega \times (0,T), && (3.1\text{a})\\
|\boldsymbol{m}| &= 1, && (3.1\text{b})\\
\boldsymbol{m}(\cdot, 0) &= \boldsymbol{m}_0, && (3.1\text{c})
\end{aligned}
$$

in $\Omega \times (0,T)$, with the Neumann boundary conditions, such as in (2.1).

Throughout this article, the notation $\mathrm{H}^s(\Omega)$ represents the classical Sobolev spaces as defined in [12] or in [13]. We denote by $\mathbb{H}^s(\Omega)$ the vector Sobolev spaces $(\mathrm{H}^s(\Omega))^3$. We also use the notation $\mathbb{L}^p(\Omega)$ to represent $(\mathrm{L}^p(\Omega))^3$.

3.1 Weak solutions: definitions and main theorem

We study the system (3.1), with boundary conditions (2.1). First, we define the concept of weak solutions of the Landau-Lifshitz system, see [4].

Definition 2 *Given $\boldsymbol{m}_0$ in $\mathbb{H}^1(\Omega)$, with $|\boldsymbol{m}_0| = 1$ a.e. in Ω, we call $\boldsymbol{m}$ a weak solution to Landau-Lifshitz system if*

(1) For all $T > 0$, $\boldsymbol{m}$ belongs to $\mathbb{H}^1(\Omega \times (0,T))$, and $|\boldsymbol{m}| = 1$ almost everywhere in $\Omega \times (0,T)$.
(2) For all $\boldsymbol{\phi}$ in $\mathbb{H}^1(\Omega \times (0,T))$,

$$
\begin{aligned}
\int_{Q_T} \frac{\partial \boldsymbol{m}}{\partial t} \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t - \alpha \int_{Q_T} \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial t} \right) \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
= (1+\alpha^2) A \int_{Q_T} \sum_{i=1}^{3} \left(\boldsymbol{m} \times \frac{\partial \boldsymbol{m}}{\partial x_i} \right) \cdot \frac{\partial \boldsymbol{\phi}}{\partial x_i} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
- (1+\alpha^2) \int_{Q_T} (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{m})) \cdot \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
- (1+\alpha^2) K_s \int_{(\Gamma^\pm)\times(0,T)} (\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})(\gamma \boldsymbol{m} \times \boldsymbol{\nu}) \cdot \gamma \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
- (1+\alpha^2) J_1 \int_{(\Gamma^\pm)\times(0,T)} (\gamma \boldsymbol{m} \times \gamma' \boldsymbol{m}) \cdot \gamma \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
- 2(1+\alpha^2) J_2 \int_{(\Gamma^\pm)\times(0,T)} (\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})(\gamma \boldsymbol{m} \times \gamma' \boldsymbol{m}) \cdot \gamma \boldsymbol{\phi} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \quad (3.2)
\end{aligned}
$$

(3) $\boldsymbol{m}(\cdot, 0) = \boldsymbol{m}_0$ in the sense of traces.

(4) For all $T > 0$*,*

$$\mathrm{E}(\boldsymbol{m}(T)) + \frac{\alpha}{1+\alpha^2} \int_{Q_T} \left| \frac{\partial \boldsymbol{m}}{\partial t} \right|^2 \mathrm{d}t \, \mathrm{d}\boldsymbol{x} \leq \mathrm{E}(\boldsymbol{m}(0)), \tag{3.3}$$

with E *defined in* (2.2) *and in section 2.*

Any classical solution to the Landau-Lifshitz equation is also a weak solution. Any weak solutions of class $\mathcal{C}^2$ is also a classical solution.

We prove in section 5 the following result

Theorem 3 *Given any* $\boldsymbol{m}_0$ *in* $\mathbb{H}^1(\Omega)$*,* $|\boldsymbol{m}_0| = 1$ *almost everywhere on* Ω*, there exists at least one weak solution to the Landau-Lifshitz system with non-homogenous Neumann condition.*

The solution is probably not unique. This is proven when only the exchange interaction is present, see [4].

3.2 Strong solutions: definitions and main theorems

We study the strong solutions to the Landau-Lifshitz system in the presence of super-exchange and surface anisotropy. The existence and uniqueness of such strong solutions without the boundary terms have been established in [14]. We prove the existence with the more general boundary condition

$$\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma, \\ Q^+(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) & \text{on } \Gamma^+, \\ Q^-(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) & \text{on } \Gamma^+, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} Q^+(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) &= Q_r^+(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) - (Q_r^+(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) \cdot \gamma \boldsymbol{m})\gamma \boldsymbol{m}, \\ Q^-(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) &= Q_r^-(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) - (Q_r^-(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) \cdot \gamma \boldsymbol{m})\gamma \boldsymbol{m}, \end{aligned}$$

and where Q_r^+ and Q_r^- are two polynomials in two variables. The particular case of super-exchange and surface anisotropy interactions is given by

$$\begin{aligned} Q_r^+(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) &= \frac{Ks}{A}(\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})\boldsymbol{\nu} + \frac{J_1}{A}\gamma' \boldsymbol{m} + 2\frac{J_2}{A}(\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma' \boldsymbol{m}, \\ Q_r^-(\gamma \boldsymbol{m}, \gamma' \boldsymbol{m}) &= \frac{Ks}{A}(\boldsymbol{\nu} \cdot \gamma \boldsymbol{m})\boldsymbol{\nu} + \frac{J_1}{A}\gamma' \boldsymbol{m} + 2\frac{J_2}{A}(\gamma \boldsymbol{m} \cdot \gamma' \boldsymbol{m})\gamma' \boldsymbol{m}. \end{aligned}$$

Definition 4 *We say that* $\boldsymbol{m}$ *in* $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *is a strong solution to the Landau-Lifshitz equation with generalized boundary condition* (3.4) *and initial* *condition* $\boldsymbol{m}_0$ *if it satisfies boundary condition* (3.4) *and equations* (3.1) *almost everywhere in* $\Omega \times (0,T)$.

We prove in section 6 the following result.

Theorem 5 *Given an initial condition* $\boldsymbol{m}_0$ *belonging to* $\mathbb{H}^2(\Omega)$, $|\boldsymbol{m}_0| = 1$ *almost everywhere on* Ω, *and satisfying the boundary condition*

$$\frac{\partial \boldsymbol{m}_0}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma^{\pm}, \\ Q^{+}(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) & \text{on } \Gamma^{+}, \\ Q^{-}(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) & \text{on } \Gamma^{-}. \end{cases} \tag{3.5}$$

then, there exist a positive time T *and a strong solution to the Landau-Lifshitz equation over* $(0,T)$. *Moreover, the time interval is bounded from below by a function that only depends on the size of the initial condition.*

The proof is based on Galerkin's method with a modified Neumann operator on the interface.

4 Some properties of Sobolev spaces and other useful results

First, we recall the reader about some properties of Sobolev spaces. As in Lions-Magenes [15], we define the spaces

$$\mathrm{H}^{s_1,s_2}(\Omega \times (0,T)) = \mathrm{L}^2(0,T;\mathrm{H}^{s_1}(\Omega)) \cap \mathrm{H}^{s_2}(0,T;\mathrm{L}^2(\Omega)).$$

and we define

$$\mathbb{H}^{s_1,s_2}(\Omega \times (0,T)) = \mathrm{L}^2(0,T;\mathbb{H}^{s_1}(\Omega)) \cap \mathrm{H}^{s_2}(0,T;\mathbb{L}^2(\Omega)) = (\mathrm{H}^{s_1,s_2}(\Omega \times (0,T)))^3.$$

In particular, we make an extensive use of the space $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$.

We recall here without proof some well-known properties of Sobolev spaces. It can be verified [14], or [16, chap. 5] that the considered domain Ω is regular enough for those inequalities to hold. In particular, Sobolev embeddings hold.

Lemma 6 *The space* $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *is embedded in the space* $\mathcal{C}^0(0,T;\mathrm{H}^2(\Omega))$.

Lemma 7 *Suppose* $\boldsymbol{u}$ *in* $\mathrm{H}^2(\Omega)$, *then* $\boldsymbol{u}$ *belongs to* $\mathrm{L}^\infty(\Omega)$ *and there exists a constant* C *such that*

$$\|\boldsymbol{u}\|_{\mathrm{L}^\infty(\Omega)} \leq C \|\boldsymbol{u}\|^{\frac{1}{2}}_{\mathrm{H}^1(\Omega)} \|\boldsymbol{u}\|^{\frac{1}{2}}_{\mathrm{H}^2(\Omega)}.$$

PROOF. This is true for all open subsets of $\mathbb{R}^3$ satisfying the cone property. See Maz'ya [17], pp. 274.

Definition 8 *We define* $\mathrm{H}^{m-\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)$ *as the subset of* $\mathrm{L}^2(\partial\Omega)$ *of functions whose restrictions on* $\partial B \times (0, L)$*,* $B \times \{0\}$ *et* $B \times \{L\}$ *are in* $\mathrm{H}^{m-\frac{1}{2}}$*.*

We define γ_1 as the trace application that maps $\boldsymbol{m}$ to $\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}}$. Then $\mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}} = \gamma^1(\mathrm{H}^2(\Omega))$. This happens because there is no need of compatibility relations between the normal traces in that case, see [15].

The following regularity properties hold.

Proposition 9 (Elliptic regularity)
The space $\left\{v \in \mathrm{H}^1(\Omega) \mid \triangle v \in \mathrm{L}^2(\Omega), \frac{\partial v}{\partial \boldsymbol{\nu}} \in \mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)\right\}$ *is equal to* $\mathrm{H}^2(\Omega)$ *and there exists a constant* C *such that for all* v *in* $\mathrm{H}^2(\Omega)$

$$\|v\|_{\mathrm{H}^2(\Omega)} \leq C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\triangle v\|_{\mathrm{L}^2(\Omega)} + \left\|\frac{\partial v}{\partial \boldsymbol{\nu}}\right\|_{\mathrm{H}^{\frac{1}{2}}_{\mathrm{morc}}(\partial\Omega)}\right). \tag{4.1a}$$

Proposition 10 *The space* $\left\{v \in \mathrm{H}^1(\Omega), \nabla\triangle v \in \mathbb{L}^2(\Omega), \frac{\partial v}{\partial \boldsymbol{\nu}} \in \mathrm{H}^{\frac{3}{2}}_{\mathrm{morc}}(\partial\Omega)\right\}$ *is equal to* $\mathrm{H}^3(\Omega)$ *and there exists a constant* C *such that for all* v *in* $\mathrm{H}^3(\Omega)$

$$\|v\|_{\mathrm{H}^3(\Omega)} \leq C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\nabla\triangle v\|_{\mathbb{L}^2(\Omega)} + \left\|\frac{\partial v}{\partial \boldsymbol{\nu}}\right\|_{\mathrm{H}^{\frac{3}{2}}(\partial\Omega)}\right). \tag{4.1b}$$

PROOF. See [14], [18], and [19]. For a way to reduce the case of domains such as Ω to the case of domains with a smooth boundary by reflections, see [16, chap. 5]

In particular, for v satisfying the homogenous Neumann boundary condition, we derive from propositions 9 and 10

Corollary 11 *If* v *belongs to* $\mathrm{H}^2(\Omega)$ *and satisfies* $\frac{\partial v}{\partial \boldsymbol{\nu}} = 0$ *then*

$$\|v\|_{\mathrm{H}^2(\Omega)} \leq C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\triangle v\|_{\mathrm{L}^2(\Omega)}\right). \tag{4.2a}$$

If v *also belongs to* $\mathrm{H}^3(\Omega)$ *then*

$$\|v\|_{\mathrm{H}^3(\Omega)} \leq C\left(\|v\|_{\mathrm{L}^2(\Omega)} + \|\nabla\triangle v\|_{\mathbb{L}^2(\Omega)}\right). \tag{4.2b}$$

We also use the

Lemma 12 (Aubin's lemma) *Let p, $1 \leq p < +\infty$. Let B be a Banach space. Let X, Y be Banach spaces such that $X \subset B \subseteq Y$, with compact injection from X to B. Let F be a set of functions included in $\mathrm{L}^p(0,T;B)$. Suppose F is bounded in $\mathrm{L}^p(0,T;X)$ and suppose the set $\{\partial_t f, f \in F\}$ is bounded in $\mathrm{L}^p(0,T;Y)$, then F is compact in $\mathrm{L}^p(0,T;B)$.*

PROOF. See [20].

5 Proof of theorem 3

The proof is based on the method found in Alouges-Soyeur [4], and Labbé [7]. We use a penalization method, replacing the boundary condition by a volume term on a thin layer. First, we introduce for any η belonging to $(0, \min(L^- - l, L^+ + l))$ the nonlinear operator $\mathcal{H}_s^\eta$:

$$\boldsymbol{m} \mapsto \frac{1}{2\eta}\begin{cases} 0 & \text{in } \mathbb{R}^3 \setminus (\, B \times (\mathcal{I}_l \setminus \mathcal{I}_{l+\eta})\,), \\ 2K_s((\boldsymbol{m}\cdot\boldsymbol{\nu})\boldsymbol{\nu} - \boldsymbol{m}) + 2J_1(\boldsymbol{m}^\sigma - \boldsymbol{m}) & \\ \quad +4J_2\Big((\boldsymbol{m}\cdot\boldsymbol{m}^\sigma)\boldsymbol{m}^\sigma - |\boldsymbol{m}^\sigma|^2\boldsymbol{m}\Big) & \text{in } B \times (\mathcal{I}_l \setminus \mathcal{I}_{l+\eta}), \end{cases} \tag{5.1}$$

where $\boldsymbol{m}^\sigma(\cdot,\cdot,z,\cdot) = \boldsymbol{m}(\cdot,\cdot,-z,\cdot)$. We keep the same notations throughout the rest of this section. We also introduce the corresponding energy

$$\begin{aligned}\mathrm{E}_s^\eta = & \frac{K_s}{2\eta}\int_{B\times(\mathcal{I}_l\setminus\mathcal{I}_{l+\eta})}\Big(|\boldsymbol{m}|^2 - (\boldsymbol{m}\cdot\boldsymbol{\nu})^2\Big)\,\mathrm{d}\boldsymbol{x} \\ & + \frac{J_1}{2\eta}\int_{B\times(\mathcal{I}_l\setminus\mathcal{I}_{l+\eta})}\left(\frac{|\boldsymbol{m}|^2 + |\boldsymbol{m}^\sigma|^2}{2} - (\boldsymbol{m}\cdot\boldsymbol{m}^\sigma)\right)\mathrm{d}\boldsymbol{x} \\ & + \frac{J_2}{2\eta}\int_{B\times\mathcal{I}_l\setminus\mathcal{I}_{l+\eta}}\Big(|\boldsymbol{m}^\sigma|^2|\boldsymbol{m}|^2 - (\boldsymbol{m}\cdot\boldsymbol{m}^\sigma)^2\Big)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

The general idea is to introduce the solution $\boldsymbol{m}^\eta$ of the Landau-Lifshitz system with excitation $\boldsymbol{h} = \mathcal{H}(\boldsymbol{m}) + \mathcal{H}_s^\eta(\boldsymbol{m})$ and to have η tend to zero. However, we must first prove the existence of solutions with such nonlinear excitation. Following Alouges and SoyeurSoyeur [4], and Labbé [7], we introduce, for all positive integers k and positive real η, the penalized problem

$$\begin{aligned}\alpha\frac{\partial\boldsymbol{m}^{k,(\eta)}}{\partial t} + \boldsymbol{m}^{k,(\eta)} \times \frac{\partial\boldsymbol{m}^{k,(\eta)}}{\partial t} &= (1+\alpha^2)(\mathcal{H}(\boldsymbol{m}^{k,(\eta)}) + \mathcal{H}_s^\eta(\boldsymbol{m}^{k,(\eta)})) \\ &\quad - k(1+\alpha^2)(|\boldsymbol{m}^{k,(\eta)}|^2 - 1)\boldsymbol{m}^{k,(\eta)}) \qquad \text{in } \Omega, \\ \frac{\partial\boldsymbol{m}^{k,(\eta)}}{\partial\boldsymbol{\nu}} &= 0 \qquad \text{on } \partial\Omega.\end{aligned}$$

To solve this system of equations, we use Galerkin's method.

Galerkin's method We introduce the orthonormal base of $\mathrm{L}^2(\Omega)$ whose elements w_i are the eigenvectors of the Laplacian operator with the homogenous Neumann boundary condition. The sequence $(w_1, \ldots, w_n, \ldots)$ is also an orthogonal basis of $\mathrm{H}^1(\Omega)$. This basis exists when the embedding of $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Omega)$ is compact, which is the case here since Ω satisfies the cone property and is bounded, see [12]. We define V_n as the subspace generated by $\{w_1, \ldots, w_n\}$. By classical results, each w_i belongs to $\mathcal{C}^\infty(\overline{\Omega})$. For each $n \geq 1$, we search $\boldsymbol{m}_n^{k,(\eta)}$ in $V_n \otimes \mathcal{C}^1([0, T_n^*); \mathbb{R}^3)$ that verifies the following weak formulation for all test function $\boldsymbol{\psi}$ in $V_n \otimes \mathcal{C}^1([0, +\infty); \mathbb{R}^3)$

$$\left(\alpha \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t} + \boldsymbol{m}_n^{k,(\eta)} \times \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t} - (1+\alpha^2)\mathcal{H}_v(\boldsymbol{m}_n^{k,(\eta)}), \boldsymbol{\psi}\right)_{\mathbb{L}^2(Q_T)} \\ +(1+\alpha^2)\left(k(|\boldsymbol{m}_n^{k,(\eta)}|^2 - 1)\boldsymbol{m}_n^{k,(\eta)}) - \mathcal{H}_s^\eta(\boldsymbol{m}_n^{k,(\eta)}), \boldsymbol{\psi}\right)_{\mathbb{L}^2(Q_T)} = 0, \tag{5.2}$$

and the following initial condition

$$\boldsymbol{m}_n^{k,(\eta)}(\cdot, 0) = \mathcal{P}_n(\boldsymbol{m}_0), \tag{5.3}$$

where $\mathcal{P}_n$ is the orthogonal projection on V_n, as a subspace of $\mathrm{L}^2(\Omega)$. We expand $\boldsymbol{m}_n^{k,(\eta)}$ on the $(w_i)_i$ basis as $\sum_{i=1}^n \boldsymbol{\varphi}_{i,n}(t) w_i$ where each $\boldsymbol{\varphi}_{i,n}$ belongs to $\mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^3)$. We define Φ_n as the finite sequences $(\boldsymbol{\varphi_{i,n}})_{i \in [\![1,n]\!]}$, and obtain an equivalent system expressed in terms of Φ_n:

$$\frac{\mathrm{d}\Phi_n}{\mathrm{d}t} - \mathbf{A}(\Phi_n(t))\frac{\mathrm{d}\Phi_n}{\mathrm{d}t} = F(\Phi_n(t)),$$

where F is a polynomial, thus of class $\mathcal{C}^\infty$ and $\Phi_n(t) \mapsto \mathbf{A}(\Phi_n(t))$ is linear continuous, thus smooth. Moreover, $\mathbf{A}(\Phi)$ is an antisymmetric matrix for all Φ. So the matrix $\mathbf{I} - \mathbf{A}(\Phi)$ is nonsingular and the function $\Phi \mapsto (\mathbf{I} - \mathbf{A}(\Phi))^{-1}$ is of class $\mathcal{C}^\infty$. By the Cauchy-Lipschitz theorem, for every n, there is a solution $\boldsymbol{m}_n^{k,(\eta)}$ on some time interval $(0, T^n)$ with Φ_n of class $\mathcal{C}^\infty$. Next, we derive some estimates on $\boldsymbol{m}_n^{k,(\eta)}$. In equation (5.2), we take $\psi = \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t}$ and after integration over $(0, T)$ we obtain for all time $T > 0$

$$\mathrm{E}_v(\boldsymbol{m}_n^{k,(\eta)}(T)) + \frac{\alpha}{1+\alpha^2}\int_O^T \left\|\frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t}\right\|_{\mathbb{L}^2(\Omega)}^2 \mathrm{d}t \\ + \mathrm{E}_s^\eta(\boldsymbol{m}_n^{k,(\eta)}(T)) + \frac{k}{4}\int_\Omega (|\boldsymbol{m}_n^{k,(\eta)}(T)|^2 - 1)^2 \,\mathrm{d}\boldsymbol{x} \\ \leq \mathrm{E}_v(\boldsymbol{m}_n^{k,(\eta)}(0)) + \mathrm{E}_s^\eta(\boldsymbol{m}_n^{k,(\eta)}(0)) \\ + \frac{k}{4}\int_\Omega (|\boldsymbol{m}_n^{k,(\eta)}(0)|^2 - 1)^2 \,\mathrm{d}\boldsymbol{x}. \tag{5.4}$$

For any positive integer n, $\boldsymbol{m}_n^{k,(\eta)}$ exists for any time $T > 0$ in $\mathcal{C}^\infty(0, T; \mathbb{H}^1(\Omega))$. Since $\mathbb{H}^1(\Omega)$ is embedded in $\mathbb{L}^4(\Omega)$, the right-hand side of equation (5.4) remains bounded independently of n. So for all $T > 0$,

- $\boldsymbol{m}_n^{k,(\eta)}$ is bounded in $\mathbb{H}^1(\Omega \times (0,T))$,
- $\boldsymbol{m}_n^{k,(\eta)}$ is bounded in $\mathrm{L}^\infty((0,T);\mathbb{H}^1(\Omega))$,
- $\frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t}$ is bounded in $\mathbb{L}^2(\Omega \times (0,T))$,
- $(|\boldsymbol{m}_n^{k,(\eta)}|^2 - 1)$ is bounded in $\mathbb{L}^2(\Omega \times (0,T))$.

There exists a subsequence $\boldsymbol{m}_{n_j}^{k,(\eta)}$ still written $\boldsymbol{m}_n^{k,(\eta)}$, and $\boldsymbol{m}^{k,(\eta)}$ in $\mathbb{H}^1(\Omega \times (0,T))$ such that for all $T > 0$

$$\begin{aligned} \lim_{n\to\infty} \boldsymbol{m}_n^{k,(\eta)} &= \boldsymbol{m}^{k,(\eta)} && \text{weakly in } \mathbb{H}^1(\Omega \times (0,T)), \\ \lim_{n\to\infty} (|\boldsymbol{m}_n^{k,(\eta)}|^2 - 1) &= (|\boldsymbol{m}^{k,(\eta)}|^2 - 1) && \text{weakly in } \mathbb{L}^2(\Omega \times (0,T)). \end{aligned}$$

Moreover, $\boldsymbol{m}^{k,(\eta)}$ belongs to $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$. According to Aubin's lemma, for all $1 \le p < 6$ and $1 \le q < +\infty$:

$$\lim_{n\to\infty} \boldsymbol{m}_n^{k,(\eta)} = \boldsymbol{m}^{k,(\eta)} \qquad \text{strongly in } \mathrm{L}^q((0,T);\mathbb{L}^p(\Omega)).$$

Hence, $\boldsymbol{m}^{k,(\eta)}$ verifies the following properties:

(1) $\boldsymbol{m}^{k,(\eta)}(\cdot,0) = \boldsymbol{m}_0$ since $\mathcal{P}_n(\boldsymbol{m}_0)$ tends strongly to $\boldsymbol{m}_0$ in $\mathbb{H}^1(\Omega \times (0,T))$ as n tends to infinity.

(2) We now take the limit in equation (5.4). Since $|\boldsymbol{m}_0| = 1$ and $\boldsymbol{m}_n^{k,(\eta)}(0)$ strongly converges to $\boldsymbol{m}_0$ in $\mathbb{H}^1(\Omega)$ and in $\mathbb{L}^6(\Omega)$, the right-hand side of equation (5.4) converges to $\mathrm{E}_s^\eta(\boldsymbol{m}_0) + \mathrm{E}_v(\boldsymbol{m}_0)$. By the lower semi-continuity of convex functions we obtain

$$\int_{Q_T} \left| \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t} \right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le \liminf_{n\to\infty} \int_{Q_T} \left| \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t} \right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Before considering the energy terms and the penalization term we take for each particular T a further subsequence such that

$$\lim_{n\to\infty} \boldsymbol{m}_n^{k,(\eta)}(T) = \boldsymbol{m}^{k,(\eta)}(T) \qquad \text{weakly in } \mathbb{H}^1(\Omega).$$

Then, by the lower semi-continuity of convex applications

$$\int_\Omega |\nabla \boldsymbol{m}^{k,(\eta)}(T|^2 \,\mathrm{d}\boldsymbol{x} \le \liminf_{n\to\infty} \int_\Omega |\nabla \boldsymbol{m}_n^{k,(\eta)}(T)|^2 \,\mathrm{d}\boldsymbol{x}.$$

Moreover $\boldsymbol{m}_n^{k,(\eta)}(T)$ converges strongly in $\mathbb{L}^2 \cap \mathbb{L}^p(\Omega)$ to $\boldsymbol{m}^{k,(\eta)}(T)$ for $1 \le p < 6$. Hence,

$$\lim_{n\to\infty} \mathrm{E}_v(\boldsymbol{m}_n^{k,(\eta)}(T)) + \mathrm{E}_s^\eta(\boldsymbol{m}_n^{k,(\eta)}(T)) = \mathrm{E}_v(\boldsymbol{m}^{k,(\eta)}(T)) + \mathrm{E}_s^\eta(\boldsymbol{m}^{k,(\eta)}(T)).$$

We derive from the previous relations that

$$
\begin{aligned}
&\mathrm{E}_v(\boldsymbol{m}^{k,(\eta)}(T)) + \frac{\alpha}{1+\alpha^2}\int_{Q_T}\left|\frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+\mathrm{E}_s^\eta(\boldsymbol{m}^{k,(\eta)}(T)) + \frac{k}{4}\int_\Omega (|\boldsymbol{m}^{k,(\eta)}(T)|^2-1)^2\,\mathrm{d}\boldsymbol{x} \leq \mathrm{E}_v(\boldsymbol{m}_0)+\mathrm{E}_s^\eta(\boldsymbol{m}_0).
\end{aligned}
\tag{5.5}
$$

(3) For all $T>0$, for all $\boldsymbol{\varphi}$ in $\mathcal{C}^\infty(0,T;\mathbb{R}^3)$, for all $1\geq i\geq n$, $\boldsymbol{m}_n^{k,(\eta)}$ verifies

$$
\begin{aligned}
\int_{Q_T} \boldsymbol{m}_n^{k,(\eta)} \times \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t} w_i\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \alpha\int_{Q_T}\frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial t} w_i\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
= (1+\alpha^2)\int_{Q_T}\mathcal{H}_{d,a}(\boldsymbol{m}_n^{k,(\eta)})w_i\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
-(1+\alpha^2)A\int_{Q_T}\sum_{j=1}^3 \frac{\partial \boldsymbol{m}_n^{k,(\eta)}}{\partial x_j}\cdot\frac{\partial \boldsymbol{\varphi} w_i}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
+(1+\alpha^2)\int_{Q_T}\mathcal{H}_s^\eta(\boldsymbol{m}_n^{k,(\eta)})\cdot\boldsymbol{\varphi} w_i\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
-(1+\alpha^2)k\int_{Q_T}(|\boldsymbol{m}_n^{k,(\eta)}|^2-1)\boldsymbol{m}_n^{k,(\eta)}w_i\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.&
\end{aligned}
$$

We recall that $\boldsymbol{\varphi} w_i$ belongs to $\mathcal{C}^\infty(\overline{Q_T})$. By Aubin's lemma, $\boldsymbol{m}_n^{k,(\eta)}$ tends strongly to $\boldsymbol{m}^{k,(\eta)}$ in $\mathbb{L}^4(Q_T)$. We take the limit in every term and obtain

$$
\begin{aligned}
\int_{Q_T} \boldsymbol{m}^{k,(\eta)} \times \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \alpha\int_{Q_T}\frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
= (1+\alpha^2)\int_{Q_T}\mathcal{H}_{d,a}(\boldsymbol{m}^{k,(\eta)})\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
-(1+\alpha^2)A\int_{Q_T}\sum_{j=1}^3 \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial x_j}\cdot\frac{\partial \boldsymbol{\psi}}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
+(1+\alpha^2)\int_{Q_T}\mathcal{H}_s^\eta(\boldsymbol{m}^{k,(\eta)})\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t& \\
-(1+\alpha^2)k\int_{Q_T}(|\boldsymbol{m}^{k,(\eta)}|^2-1)\boldsymbol{m}^{k,(\eta)}\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,&
\end{aligned}
\tag{5.6}
$$

for all $\boldsymbol{\psi}$ in $\left(\bigcup_{i=1}^{+\infty} V_n\right)\otimes\mathcal{C}^\infty([0,T];\mathbb{R}^3)$. Since this set is dense in $\mathbb{H}^1(\Omega\times(0,T))$, the equality (5.6) also holds for any $\boldsymbol{\psi}$ in $\mathbb{H}^1(\Omega\times(0,T))$.

Convergence of the penalized problem By estimate (5.5), there exists a subsequence of $(\boldsymbol{m}^{k,(\eta)})_k$, still denoted $(\boldsymbol{m}^{k,(\eta)})_k$, and $\boldsymbol{m}^{(\eta)}$ in $\mathbb{H}^1(Q_T)$ and in $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$ such that for all $T>0$:

$$
\begin{gathered}
\boldsymbol{m}^{k,(\eta)} \to \boldsymbol{m}^{(\eta)} \text{ weakly in } \mathbb{H}^1(Q_T),\\
|\boldsymbol{m}^{k,(\eta)}|^2-1 \to 0 \text{ strongly in } \mathbb{L}^2(Q_T).
\end{gathered}
$$

Moreover, by Aubin's lemma,

$$\boldsymbol{m}^{k,(\eta)} \to \boldsymbol{m}^{(\eta)} \qquad \text{strongly in } \mathrm{L}^q(0,T;\mathbb{L}^p(\Omega)),$$

for $1 \leq p < 6$, and $1 \leq q < +\infty$. Especially, the convergence is obtained in $\mathbb{L}^4(Q_T)$.

The properties of $\boldsymbol{m}^{(\eta)}$ are

(1) For all $k \geq 0$, $\boldsymbol{m}^{k,(\eta)}(\cdot,0) = \boldsymbol{m}_0$, thus $\boldsymbol{m}^{(\eta)}(\cdot,0) = \boldsymbol{m}_0$.

(2) $|\boldsymbol{m}^{k,(\eta)}|^2 - 1$ tends strongly in $\mathbb{L}^2(Q_T)$ to both 0 and $|\boldsymbol{m}^{(\eta)}|^2 - 1$, hence $|\boldsymbol{m}^{(\eta)}| = 1$ a.e. in $\Omega \times (0,+\infty)$.

(3) Since the penalization term in the energy estimate is positive, it can be omitted when passing to the limit in (5.5). The other terms are handled as previously done in the convergence of $\boldsymbol{m}_n^{k,(\eta)}$ to $\boldsymbol{m}^{k,(\eta)}$ and we obtain

$$\mathrm{E}_v(\boldsymbol{m}^{(\eta)}(T)) + \mathrm{E}_s^\eta(\boldsymbol{m}^{(\eta)}(T)) + \frac{\alpha}{1+\alpha^2}\int_{Q_T}\left|\frac{\partial \boldsymbol{m}^{(\eta)}}{\partial t}\right|^2 \mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ \leq \mathrm{E}_v(\boldsymbol{m}_0) + \mathrm{E}_s^\eta(\boldsymbol{m}_0). \quad (5.7)$$

(4) Let $\boldsymbol{\phi}$ be a vector field belonging to $(\mathcal{C}^\infty(\Omega \times (0,T))^3$. We take $\boldsymbol{\psi} = \boldsymbol{m}^{k,(\eta)} \times \boldsymbol{\phi}$ in relation (5.6) and we obtain

$$\begin{aligned}\int_{Q_T} |\boldsymbol{m}^{k,(\eta)}|^2 \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \alpha\int_{Q_T} \boldsymbol{m}^{k,(\eta)} \times \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ \int_{Q_T}\left(\boldsymbol{m}^{k,(\eta)}\cdot\frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial t}\right)\boldsymbol{m}^{k,(\eta)}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &- (1+\alpha^2)\int_{Q_T}(\boldsymbol{m}^{k,(\eta)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{k,(\eta)}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ (1+\alpha^2)A\int_{Q_T}\sum_{j=1}^{3}\left(\boldsymbol{m}^{k,(\eta)} \times \frac{\partial \boldsymbol{m}^{k,(\eta)}}{\partial x_j}\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &- (1+\alpha^2)\int_{Q_T}(\boldsymbol{m}^{k,(\eta)} \times \mathcal{H}_s^\eta(\boldsymbol{m}^{k,(\eta)}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \quad (5.8)\end{aligned}$$

By Aubin's lemma, we know that $\boldsymbol{m}^{k,(\eta)}$ tends strongly to $\boldsymbol{m}^{(\eta)}$ in $\mathbb{L}^4(Q_T)$. We can take the limit in every term in (5.8) and obtain

$$\begin{aligned}\int_{Q_T} \frac{\partial \boldsymbol{m}^{(\eta)}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &= \alpha\int_{Q_T} \boldsymbol{m}^{(\eta)} \times \frac{\partial \boldsymbol{m}^{(\eta)}}{\partial t}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &- (1+\alpha^2)\int_{Q_T}(\boldsymbol{m}^{(\eta)} \times \mathcal{H}_{d,a}(\boldsymbol{m}^{(\eta)}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ (1+\alpha^2)A\int_{Q_T}\sum_{j=1}^{3}\left(\boldsymbol{m}^{(\eta)} \times \frac{\partial \boldsymbol{m}^{(\eta)}}{\partial x_j}\right)\cdot\frac{\partial\boldsymbol{\phi}}{\partial x_j}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &- (1+\alpha^2)\int_{Q_T}(\boldsymbol{m}^{(\eta)} \times \mathcal{H}_s^\eta(\boldsymbol{m}^{(\eta)}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.\end{aligned} \quad (5.9)$$

Convergence to the weak solution We study the convergence of $\boldsymbol{m}^{(\eta)}$ as the thickness η tends to 0. $\boldsymbol{m}_0$ belongs [2] to $\mathcal{C}^0((-L^-,-l)\cup(l,L^+);\mathbb{L}^4(B))$.

$$\lim_{\eta\to 0} \mathrm{E}_v(\boldsymbol{m}_0) + \mathrm{E}_s^\eta(\boldsymbol{m}_0) = \mathrm{E}_v(\boldsymbol{m}_0) + \mathrm{E}_s(\boldsymbol{m}_0).$$

Hence the right-hand side of estimate (5.7) is bounded independently of η. Moreover $|\boldsymbol{m}^{(\eta)}| = 1$ locally, for all $\eta > 0$, hence:

- $\boldsymbol{m}^{(\eta)}$ is bounded in $\mathbb{H}^1(\Omega\times(0,T))$,
- $\boldsymbol{m}^{(\eta)}$ is bounded in $\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$.

Since the considered spaces are reflexive, there exist a subsequence of $\boldsymbol{m}^{(\eta)}$ and $\boldsymbol{m}$ in $\mathbb{H}^1(\Omega\times(0,T))\cap\mathrm{L}^\infty(0,T;\mathbb{H}^1(\Omega))$ such that

$$\boldsymbol{m}^{(\eta)} \to \boldsymbol{m} \qquad \text{weakly in } \mathbb{H}^1(\Omega\times(0,T)), \tag{5.10a}$$

$$\boldsymbol{m}^{(\eta)} \to \boldsymbol{m} \qquad \text{strongly in } \mathbb{L}^2(\Omega\times(0,T)), \tag{5.10b}$$

$$\boldsymbol{m}^{(\eta)} \to \boldsymbol{m} \qquad \text{strongly in } \mathbb{L}^\infty(\mathcal{I}_l,\mathbb{L}^2(B\times(0,T))). \tag{5.10c}$$

Since $|\boldsymbol{m}^{(\eta)}| = 1$, for all p, $2\le p<+\infty$, $\boldsymbol{m}^{(\eta)}$ tends strongly to $\boldsymbol{m}$ in $\mathbb{L}^\infty(\mathcal{I}_l,\mathbb{L}^p(B\times(0,T)))$.

We now prove that $\boldsymbol{m}$ is a weak solution to the Landau-Lifshitz equation:

(1) Clearly, $\boldsymbol{m}(\cdot,0)=\boldsymbol{m}_0$.

(2) By the strong $\mathbb{L}^2$ convergence, $|\boldsymbol{m}|=1$ almost everywhere in $\Omega\times(0,T)$.

(3) We take the limit in equality (5.9). All the volume terms converge to their intuitive limit as in the previous steps of the proof. $\boldsymbol{m}^{(\eta)}$ tends strongly to $\boldsymbol{m}$ in $\mathbb{L}^\infty(\mathcal{I}_l,\mathbb{L}^4(B\times(0,T)))$. Thus,

$$\limsup_{\eta\to 0}\left|\int_{Q_T}\left(\boldsymbol{m}^{(\eta)}\times\mathcal{H}_s^\eta(\boldsymbol{m}^{(\eta)})\right)\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \right.$$
$$\left. - \int_{Q_T}(\boldsymbol{m}\times\mathcal{H}_s^\eta(\boldsymbol{m}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t)\right| = 0.$$

Moreover $\boldsymbol{m}$ belongs [3] , so

$$\lim_{\eta\to 0}\int_{Q_T}(\boldsymbol{m}\times\mathcal{H}_s^\eta(\boldsymbol{m}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_{(\Gamma^\pm)\times(0,T)}(\boldsymbol{m}\times\mathcal{H}_s(\boldsymbol{m}))\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Hence, the boundary terms converge to their intuitive limits and we obtain relation (3.2).

(4) In order to take the limit in estimate (5.7), we extract for any $T>0$ a subsequence, depending on T, such that $\boldsymbol{m}^{(\eta)}(\cdot,T)$ tends to $\boldsymbol{m}(\cdot,T)$ weakly in $\mathbb{H}^1(\Omega)$. All the volume terms converge and are handled as in the precedent stage. It remains to calculate the limit as η tends to 0 of $\mathrm{E}_s^\eta(\boldsymbol{m}^{(\eta)}(T))$, which requires a little more work. First, for any

[2] This is proved in [12] or in [15].

[3] This a consequence of interpolation results found in [12] to $\mathrm{L}^\infty(0,T;\mathcal{C}^0(\mathcal{I}_l;\mathbb{L}^4(B)))$ or in [15].

$0 \le s < 1$ $\boldsymbol{m}^{(\eta)}(\cdot, T)$ tends to $\boldsymbol{m}(\cdot, T)$ strongly in $\mathbb{H}^s(\Omega)$. Hence, for any $1 \le p < 4$, the convergence holds in the normed spaces $\mathbb{L}^\infty(\mathcal{I}_l, \mathbb{L}^p(B))$. Since $|\boldsymbol{m}^{(\eta)}| = |\boldsymbol{m}| = 1$ almost everywhere, the convergence holds even for $1 \le p < +\infty$. Especially, for $p = 4$, we obtain

$$\limsup_{\eta \to 0} \left| \mathrm{E}_s^\eta(\boldsymbol{m}^{(\eta)}(T)) - \mathrm{E}_s^\eta(\boldsymbol{m}(T)) \right| = 0.$$

Since $\boldsymbol{m}(T)$ belongs to $\mathbb{H}^1(\Omega)$ embedded in $\mathcal{C}^0(\mathcal{I}_l, \mathbb{L}^4(B))$

$$\lim_{\eta \to 0} \mathrm{E}_s^\eta(\boldsymbol{m}(T)) = \mathrm{E}^s(\boldsymbol{m}(T)).$$

Thus, $\boldsymbol{m}$ verifies the energy estimate

$$\mathrm{E}_v(\boldsymbol{m}(T)) + \mathrm{E}_s(\boldsymbol{m}(T)) + \frac{\alpha}{1+\alpha^2} \int_{Q_T} \left| \frac{\partial \boldsymbol{m}}{\partial t} \right|^2 \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \le \mathrm{E}_v(\boldsymbol{m}_0) + \mathrm{E}_s(\boldsymbol{m}_0) \\ \le \mathrm{E}(\boldsymbol{m}_0) \quad (5.11)$$

Hence, $\boldsymbol{m}$ verifies all the required properties and is therefore a weak solution. □

6 Proof of theorem 5

The general idea is to introduce a sequence $(\boldsymbol{m}^n)_{n \in \mathbb{N}}$ whose elements satisfy equations (6.2), (3.1b), (3.1c) and a variation of equation (3.4)

$$\frac{\partial \boldsymbol{m}^n}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma, \\ Q_r^+(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}) - (Q_r^+(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}) \cdot \gamma \boldsymbol{m}^n) \gamma \boldsymbol{m}^n & \text{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}) - (Q_r^-(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}) \cdot \gamma \boldsymbol{m}^n) \gamma \boldsymbol{m}^n & \text{on } \Gamma^-, \end{cases} \quad (6.1)$$

for all $n \ge 0$. We call this sequence the outer converging sequence. The limit being the solution to theorem 5. Knowing $\boldsymbol{m}^n$, the proof of the existence of $\boldsymbol{m}^{n+1}$ require itself the convergence of yet another sequence which is denoted as the inner converging sequence. The proof of the existence and the convergence of such sequences is based on the method and inequalities found in [14].

The construction of the inner converging sequences uses a modification of the proof of Carbou and Fabrie [14]. First, we recall the reader about some inequalities.

Definition 13 *We call the developed equation of Landau-Lifshitz the following equation where the dissipating exchange term has been developed*

$$\begin{aligned}\frac{\partial \boldsymbol{m}}{\partial t} &= \alpha A \triangle \boldsymbol{m} + \alpha A |\nabla \boldsymbol{m}|^2 \boldsymbol{m} - A \boldsymbol{m} \times \triangle \boldsymbol{m} \\ &\quad - \boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{m}) - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \mathcal{H}_{d,a}(\boldsymbol{m})).\end{aligned} \tag{6.2}$$

It is easily verified that if $\boldsymbol{m}$ belongs to $\mathrm{L}^2(0,T;\mathbb{H}^2(\Omega))$ and if $|\boldsymbol{m}| = 1$ almost everywhere, then equation (3.1a) and equation (6.2) are equivalent.

Lemma 14 *Let* $\boldsymbol{m}$ *be in* $\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))$. *If* $\boldsymbol{m}$ *verifies either the Landau-Lifshitz equation* (3.1a) *or its developed version* (6.2), *then*

$$\begin{aligned}&\boldsymbol{m} \in \mathrm{H}^1(0,T;\mathbb{H}^1(\Omega)),\\ &\boldsymbol{m} \in \mathcal{C}^0(0,T;\mathbb{H}^2(\Omega)),\\ &\boldsymbol{m} \in \mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\Omega)).\end{aligned}$$

PROOF. Calculating the gradient of either equation (6.2) or equation (3.1a), $\boldsymbol{m}$ belongs to the space $\mathrm{H}^1(0,T;\mathbb{H}^1(\Omega))$. By interpolation, see [13], $\boldsymbol{m}$ belongs to $\mathcal{C}^0(0,T;\mathbb{H}^2(\Omega))$. For the last assertion, we use interpolations and the fact that if A is a continuous bilinear operator from spaces X, Y into space Z then A is bilinear continuous from the spaces $(\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(X), (\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(Y)$ into $(\mathrm{L}^\infty \cap \mathrm{H}^{\frac{1}{2}})(Z)$.

6.2 Landau-Lifshitz with nonzero affine Neumann boundary condition

To construct the needed sequences, we need to prove the existence of solutions of the Landau-Lifshitz system with some affine terms. From this result, we derive the existence of solutions with an nonzero affine Neumann boundary condition.

Proposition 15 *Let* T^* *be a positive real. Let* $\boldsymbol{v}$ *be in* $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ *for all* $T < T^*$ *with*

$$\frac{\partial(\boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0))}{\partial \boldsymbol{\nu}} = 0.$$

Then there exist a unique maximal $\widetilde{T^*} \leq T^*$ *and a unique* $\boldsymbol{u}$ *in* $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$, *for all $T < \widetilde{T^*}$, where $\boldsymbol{u}$ is the solution to*

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} &= -\frac{\partial \boldsymbol{v}}{\partial t} + \alpha A \triangle(\boldsymbol{u}+\boldsymbol{v}) + \alpha A|\nabla(\boldsymbol{u}+\boldsymbol{v})|^2(\boldsymbol{u}+\boldsymbol{v}) - A(\boldsymbol{u}+\boldsymbol{v}) \times \triangle(\boldsymbol{u}+\boldsymbol{v}) \\
&\quad - (\boldsymbol{u}+\boldsymbol{v}) \times \mathcal{H}_{d,a}(\boldsymbol{u}+\boldsymbol{v}) - \alpha(\boldsymbol{u}+\boldsymbol{v}) \times ((\boldsymbol{u}+\boldsymbol{v}) \times \mathcal{H}_{d,a}(\boldsymbol{u}+\boldsymbol{v})),
\end{aligned} \tag{6.3}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = 0, \tag{6.4}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0). \tag{6.5}$$

Moreover, if $\widetilde{T^*} < T^*$, then

$$\lim_{t \to \widetilde{T^*}} \|\boldsymbol{u}\|_{\mathbb{H}^2}(t) = +\infty.$$

PROOF. We use Galerkin's method. For the estimates, we use inequalities (4.2a) and (4.2b). Let $(w_i)_i$ be the scalar eigenfunctions of the Laplace operator with the homogenous Neumann boundary condition. The eigenfunctions $(w_i)_i$ are an orthonormal basis of $\mathrm{L}^2(\Omega)$ and also an orthogonal basis of $\mathrm{H}^1(\Omega)$ and of $\{\boldsymbol{f} \in \mathrm{H}^2(\Omega) | \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\nu}} = 0\}$.

We look for a local solution $\boldsymbol{u}^n$ in $V_n \otimes \mathcal{C}^\infty([0,T_n);\mathbb{R}^3)$ of the system

$$\boldsymbol{u}^n(\cdot,0) = \mathcal{P}^n(\boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0)), \tag{6.6}$$

$$\begin{aligned}
\frac{\partial \boldsymbol{u}^n}{\partial t} &= \mathcal{P}^n\Big(-\frac{\partial \boldsymbol{v}}{\partial t} + \alpha A \triangle(\boldsymbol{u}^n+\boldsymbol{v}) + \alpha A|\nabla(\boldsymbol{u}^n+\boldsymbol{v})|^2(\boldsymbol{u}^n+\boldsymbol{v}) - A(\boldsymbol{u}^n+\boldsymbol{v}) \times \triangle(\boldsymbol{u}^n+\boldsymbol{v}) \\
&\quad - (\boldsymbol{u}^n+\boldsymbol{v}) \times \mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v}) - \alpha(\boldsymbol{u}^n+\boldsymbol{v}) \times ((\boldsymbol{u}^n+\boldsymbol{v}) \times \mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v}))\Big).
\end{aligned} \tag{6.7}$$

where $\mathcal{P}^n$ is the orthogonal projector on the vector subspace V_n generated by $\{w_1,\dots,w_n\}$. We decompose $\boldsymbol{u}^n = \sum_{k=1}^n \boldsymbol{\phi}_k^n(t) w_k$ on the $(w_1,\dots,w_n)$ base.

By the Cauchy-Lipschitz theorem, $\boldsymbol{u}^n$ exists at least locally in time. We now give some estimates on $\boldsymbol{u}^n$. We recall that estimates on $\nabla\triangle\boldsymbol{u}^n$ give an estimate on $\boldsymbol{u}^n$ in $\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))$ by corollary 11. For the estimates, we define $\boldsymbol{w}^n = \boldsymbol{u}^n + \boldsymbol{v}$. In all the subsequent estimates, η is a positive real that can be chosen arbitrarily small but independently of n. C is a constant that only depends on the domain Ω. $C(\eta)$ is a constant depending also on η.

First estimate Multiplying equation (6.7) by $\boldsymbol{u}^n$ and integrating over Ω gives

$$\frac{1}{2}\frac{\mathrm{d}\|\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A\|\nabla \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} = -\underbrace{\int_\Omega \frac{\partial \boldsymbol{v}}{\partial t}\cdot \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{I} + \alpha A\underbrace{\int_\Omega \triangle \boldsymbol{v}\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}}_{II}$$
$$+\alpha A\underbrace{\int_\Omega |\nabla \boldsymbol{w}^n|^2 \boldsymbol{w}^n\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}}_{III} - A\underbrace{\int_\Omega (\boldsymbol{v}\times\triangle \boldsymbol{w}^n)\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}}_{IV} - \underbrace{\int_\Omega (\boldsymbol{v}\times \mathcal{H}_{d,a}(\boldsymbol{w}^n))\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}}_{V}$$
$$-\alpha\underbrace{\int_\Omega (\boldsymbol{v}\times(\boldsymbol{w}^n\times \mathcal{H}_{d,a}(\boldsymbol{w}^n)))\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}}_{VI}. \quad (6.8)$$

First, we estimate $I = \int_\Omega \frac{\partial \boldsymbol{v}}{\partial t}\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$. By Cauchy-Schwartz inequality, we obtain

$$|I| \le \frac{1}{2}\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\|\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \quad (6.9a)$$

Then, we estimate $II = \int_\Omega \triangle \boldsymbol{v}\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and we obtain

$$|II| \le \frac{1}{2}\|\triangle \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\|\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \quad (6.9b)$$

Let's estimate $III = \int_\Omega |\nabla(\boldsymbol{u}^n+\boldsymbol{v})|^2(\boldsymbol{u}^n+\boldsymbol{v})\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$. By Hölder inequality, we obtain

$$|III| \le 2(\|\nabla \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})(\|\boldsymbol{u}^n\|_{\mathbb{L}^\infty(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)})\|\boldsymbol{u}^n\|_{\mathbb{L}^\infty(\Omega)}. \quad (6.9c)$$

If we estimate $IV = \int_\Omega (\boldsymbol{v}\times\triangle(\boldsymbol{u}^n+\boldsymbol{v}))\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, we obtain by Hölder inequality

$$|IV| \le \left(\|\triangle \boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)}\right)\|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)}\|\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)}. \quad (6.9d)$$

We estimate $V = \int_\Omega (\boldsymbol{v}\times \mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v}))\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and obtain using theorem 1

$$|V| \le C\|\boldsymbol{v}\|_{\mathbb{L}^4(\Omega)}(\|\boldsymbol{u}^n\|_{\mathbb{L}^4(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{L}^4(\Omega)})\|\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)}. \quad (6.9e)$$

Estimating $VI = \int_\Omega (\boldsymbol{v}\times((\boldsymbol{u}^n+\boldsymbol{v})\times \mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v})))\cdot \boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$ yields

$$|VI| \le C\|\boldsymbol{v}\|_{\mathbb{L}^6(\Omega)}(\|\boldsymbol{u}^n\|_{\mathbb{L}^6(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{L}^6(\Omega)})^2\|\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)}. \quad (6.9f)$$

Combining equations (6.9), and using classical Sobolev embeddings, we obtain the whole first estimate

$$\frac{1}{2}\frac{\mathrm{d}\|\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)^2}}{\mathrm{d}t} + \alpha A\|\nabla \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} \le P_1(\|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)}) + P_2(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)}) + \frac{1}{2}\left\|\frac{\partial \boldsymbol{v}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)}, \quad (6.10)$$

where P_1 and P_2 are polynomials that do not depend on n.

Second estimate Multiplying equation (6.7) by $\triangle^2 \boldsymbol{u}^n$ and integrating over Ω gives us:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}\|\triangle \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A \|\nabla\triangle \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} = & \underbrace{\int_\Omega \frac{\partial \nabla \boldsymbol{v}}{\partial t}\cdot \nabla\triangle \boldsymbol{u}^n}_{I} - \alpha A \underbrace{\int_\Omega \nabla\triangle \boldsymbol{v}\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{II} \\
& - 2\alpha A \underbrace{\int_\Omega (\mathrm{D}^2 \boldsymbol{w}^n \nabla \boldsymbol{w}^n)(\boldsymbol{w}^n\cdot\nabla\triangle \boldsymbol{u}^n)\,\mathrm{d}\boldsymbol{x}}_{III} - \alpha A \underbrace{\int_\Omega |\nabla \boldsymbol{w}^n|^2 \nabla \boldsymbol{w}^n \cdot \nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{IV} \\
& + A \underbrace{\int_\Omega (\nabla \boldsymbol{w}^n \times \triangle \boldsymbol{w}^n)\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{V} + A \underbrace{\int_\Omega (\boldsymbol{w}^n \times \nabla\triangle \boldsymbol{v})\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{VI} \\
& - \underbrace{\int_\Omega (\nabla \boldsymbol{w}^n \times \mathcal{H}_{d,a}(\boldsymbol{w}^n))\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{VII} - \underbrace{\int_\Omega (\boldsymbol{w}^n \times \nabla\mathcal{H}_{d,a}(\boldsymbol{w}^n))\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{VIII} \\
& - \alpha \underbrace{\int_\Omega (\boldsymbol{w}^n \times (\boldsymbol{w}^n \times \nabla\mathcal{H}_{d,a}(\boldsymbol{w}^n)))\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{IX} \\
& - \alpha \underbrace{\int_\Omega (\boldsymbol{w}^n \times (\nabla \boldsymbol{w}^n \times \mathcal{H}_{d,a}(\boldsymbol{w}^n)))\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{X} \\
& - \alpha \underbrace{\int_\Omega (\nabla \boldsymbol{w}^n \times (\boldsymbol{w}^n \times \mathcal{H}_{d,a}(\boldsymbol{w}^n)))\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}}_{XI}. \quad (6.11)
\end{aligned}
$$

We estimate $I = \int_\Omega \frac{\partial \nabla \boldsymbol{v}}{\partial t}\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}$, and we obtain for any positive η

$$
|I| \le \frac{1}{4\eta}\left\|\frac{\partial \nabla \boldsymbol{v}}{\partial t}\right\|^2_{\mathbb{L}^2(\Omega)} + \eta\|\nabla\triangle \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \quad (6.12a)
$$

Using Hölder inequality in $II = \int_\Omega \nabla\triangle \boldsymbol{v}\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}$, we obtain

$$
|II| \le \frac{1}{4\eta}\|\nabla\triangle \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \eta\|\nabla\triangle \boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \quad (6.12b)
$$

Let's estimate $III = \int_\Omega \mathrm{D}^2(\boldsymbol{u}^n + \boldsymbol{v})\nabla(\boldsymbol{u}^n + \boldsymbol{v})(\boldsymbol{u}^n + \boldsymbol{v})\cdot\nabla\triangle \boldsymbol{u}^n \,\mathrm{d}\boldsymbol{x}$, using Hölder inequality, interpolation inequality for L^p spaces and embedding theorems for Sobolev spaces.

$$
\begin{aligned}
|III| \leq & (\|\boldsymbol{u}^n\|_{\mathbb{L}^\infty(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)})(\|\mathrm{D}^2\boldsymbol{u}^n\|_{\mathbb{L}^3(\Omega)} + \|\mathrm{D}^2\boldsymbol{v}\|_{\mathbb{L}^3(\Omega)}) \\
& \quad (\|\nabla\boldsymbol{u}^n\|_{\mathbb{L}^6(\Omega)} + \|\nabla\boldsymbol{v}\|_{\mathbb{L}^6(\Omega)})\|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} \\
\leq & C(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})^{\frac{5}{2}} \\
& \quad (\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)} + \|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} + \|\nabla\triangle\boldsymbol{v}\|_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}}\|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} \\
\leq & C''\left(1+\frac{1}{\eta^3}\right)\Big(P_4(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)}) + P_4(\|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})\Big) \\
& + \frac{\eta}{6}\|\nabla\triangle\boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)},
\end{aligned}
\tag{6.12c}
$$

where P_4 is a polynomial independent of η and n.
We estimate $IV = \int_\Omega |\nabla(\boldsymbol{u}^n+\boldsymbol{v})|^2\nabla(\boldsymbol{u}^n+\boldsymbol{v})\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and we obtain

$$
|IV| \leq \frac{8}{\eta}\|\nabla\boldsymbol{u}^n\|^6_{\mathbb{L}^6(\Omega)} + \frac{8}{\eta}\|\nabla\boldsymbol{v}\|^6_{\mathbb{L}^6(\Omega)} + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{6.12d}
$$

If we estimate $V = \int_\Omega(\nabla(\boldsymbol{u}^n+\boldsymbol{v})\times\triangle(\boldsymbol{u}^n+\boldsymbol{v}))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$ using Hölder inequality and embedding properties of Sobolev spaces, we obtain

$$
\begin{aligned}
|V| \leq & (\|\nabla\boldsymbol{u}^n\|_{\mathbb{L}^6(\Omega)} + \|\nabla\boldsymbol{v}\|_{\mathbb{L}^6(\Omega)})\left(\|\triangle\boldsymbol{u}^n\|_{\mathbb{L}^3(\Omega)} + \|\triangle\boldsymbol{v}\|_{\mathbb{L}^3(\Omega)}\right)\|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} \\
\leq & C(\|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)})^{\frac{3}{2}} \\
& \quad (\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)} + \|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} + \|\nabla\triangle\boldsymbol{v}\|_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}}\|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} \\
\leq & C''\left(1+\frac{1}{\eta^3}\right)\Big(P_5(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)}) + P_5(\|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})\Big) + \frac{\eta}{6}\|\nabla\triangle\boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)},
\end{aligned}
\tag{6.12e}
$$

where P_5 is a polynomial that do not depend on η.
Let's estimate $VI = \int_\Omega((\boldsymbol{u}^n+\boldsymbol{v})\times\nabla\triangle\boldsymbol{v})\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$

$$
\begin{aligned}
|VI| \leq & (\|\boldsymbol{u}^n\|_{\mathbb{L}^\infty(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{L}^\infty(\Omega)})\|\nabla\triangle\boldsymbol{v}\|_{\mathbb{L}^2(\Omega)}\|\nabla\triangle\boldsymbol{u}^n\|_{\mathbb{L}^2(\Omega)} \\
\leq & \frac{C}{\eta}(\|\boldsymbol{u}^n\|^2_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})^2\|\nabla\triangle\boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}.
\end{aligned}
\tag{6.12f}
$$

Since $\mathcal{H}_{d,a}$ is continuous from $\mathbb{L}^4$ to $\mathbb{L}^4$, the estimation of $VII = \int_\Omega(\nabla(\boldsymbol{u}^n+\boldsymbol{v})\times\mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v}))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and $VIII = \int_\Omega((\boldsymbol{u}^n+\boldsymbol{v})\times\nabla\mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v}))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$ yields

$$
|VII| + |VIII| \leq \frac{C}{\eta}(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})^4 + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \tag{6.12g}
$$

And we can also estimate $IX = \int_\Omega((\boldsymbol{u}^n+\boldsymbol{v})\times((\boldsymbol{u}^n+\boldsymbol{v})\times\nabla\mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v})))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and $X = \int_\Omega((\boldsymbol{u}^n+\boldsymbol{v})\times(\nabla(\boldsymbol{u}^n+\boldsymbol{v})\times\mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v})))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$, and $XI = \int_\Omega(\nabla(\boldsymbol{u}^n+\boldsymbol{v})\times((\boldsymbol{u}^n+\boldsymbol{v})\times\mathcal{H}_{d,a}(\boldsymbol{u}^n+\boldsymbol{v})))\cdot\nabla\triangle\boldsymbol{u}^n\,\mathrm{d}\boldsymbol{x}$. Since

$\mathcal{H}_{d,a}$ is continuous from $\mathbb{L}^6$ to $\mathbb{L}^6$,

$$|IX| + |X| + |XI| \leq \frac{C}{\eta}(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)} + \|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})^6 + \eta\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}. \quad (6.12\text{h})$$

Then, using embedding theorems in Sobolev spaces and choosing η small enough, we derive from inequalities (6.12) the whole second estimate

$$\frac{1}{2}\frac{\mathrm{d}\|\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \frac{\alpha A}{2}\|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} \leq g_1 P_3(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)}) + g_2, \quad (6.13)$$

where P_3 is a polynomial independent of n, and g_1 and g_2 two elements of $\mathrm{L}^1(0,T;\mathbb{R})$ only depending on the $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ norm of $\boldsymbol{v}$.

If we combine the two estimates, using inequalities (4.2a) and (4.2b), we obtain

$$\frac{\mathrm{d}\|\boldsymbol{u}^n\|^2_{\mathbb{H}^2(\Omega)}}{\mathrm{d}t} + \|\nabla\triangle\boldsymbol{u}^n\|^2_{\mathbb{L}^2(\Omega)} \leq gP'(\|\boldsymbol{u}^n\|_{\mathbb{H}^2(\Omega)}) + g',$$

where P' is a polynomial independent of n, and g and g' two elements of $\mathrm{L}^1(0,T;\mathbb{R})$ depending only on the $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ norm of $\boldsymbol{v}$.

Applying Gronwall's lemma, we deduce a minimum time of existence $\widetilde{T^*}$ for all n. Moreover, for any $T < \widetilde{T^*}$, there exists a constant C_T such that for any $n \geq 0$

$$\|\boldsymbol{u}^n\|_{\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))} \leq C_T, \qquad \|\boldsymbol{u}^n\|_{\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))} \leq C_T.$$

Since $\boldsymbol{u}^n$ verifies equation (6.7), we can apply, up to a minor modification, lemma 14

$$\|\boldsymbol{u}^n\|_{\mathrm{H}^1(0,T;\mathbb{H}^1(\Omega))} \leq C_T, \qquad \|\boldsymbol{u}^n\|_{\mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\Omega))} \leq C_T.$$

We can therefore extract a subsequence $\boldsymbol{u}^{n_k}$, such that

$$\begin{aligned}
\lim_{k\to+\infty} \boldsymbol{u}^{n_k} &= \boldsymbol{u} && \text{weakly in } \mathrm{H}^1(0,T;\mathbb{H}^1(\Omega)),\\
\lim_{k\to+\infty} \boldsymbol{u}^{n_k} &= \boldsymbol{u} && \text{weakly in } \mathrm{L}^2(0,T;\mathbb{H}^3(\Omega)),\\
\lim_{k\to+\infty} \boldsymbol{u}^{n_k} &= \boldsymbol{u} && \text{weakly in } \mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\Omega)).
\end{aligned}$$

We now verify that $\boldsymbol{u}$ is a solution to system (6.3), (6.4) and (6.5). First, $\boldsymbol{u}^n(\cdot,0)$ tends to $\boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0)$ in $\mathbb{H}^2(\Omega)$. Thus, $\boldsymbol{u}(\cdot,0) = \boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0)$. The boundary condition is $\frac{\partial\boldsymbol{u}^n}{\partial\boldsymbol{\nu}} = 0$. Since $\frac{\partial\boldsymbol{u}^{n_k}}{\partial\boldsymbol{\nu}}$ tends weakly to $\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}}$ in $\mathrm{L}^2(0,T;\mathbb{H}^{\frac{3}{2}}(\partial\Omega))$, $\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{\nu}} = 0$. It remains to prove that equation (6.3) is verified by $\boldsymbol{u}$. By compactness results [4], the subsequence also converges strongly in

[4] See [15].

- $\mathbb{H}^{s_1,s_2}(\Omega \times (0,T))$ for all $0 \leq s_1 < 3$, $0 \leq s_2 < \frac{3}{2}$,
- $\mathcal{C}^0(0,T;\mathbb{H}^s(\Omega))$ for all $0 \leq s < 2$,
- $\mathbb{L}^\infty(\Omega \times (0,T))$.

By Aubin's lemma, the subsequence also converges strongly in $\mathrm{L}^p(0,T;\mathbb{H}^2(\Omega))$ for all $1 \leq p < +\infty$. Then, we take the limit in equation (6.7). $\boldsymbol{u}$ verifies equation (6.3). *A posteriori*, equation (6.3) and lemma 14 imply that $\boldsymbol{u}+\boldsymbol{v}$ also belongs to $\mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\Omega))$ for any $T < \widetilde{T^*}$. Hence $\boldsymbol{u}$ also belongs to $\mathrm{H}^{\frac{3}{2}}(0,T;\mathbb{L}^2(\Omega))$.

Suppose now there exist two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ with the same initial condition and homogenous Neumann boundary condition. Multiplying equation (6.3) by $\delta\boldsymbol{u} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ and integrating over Ω yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}\|\delta\boldsymbol{u}\|\mathbb{L}^2(\Omega)^2}{\mathrm{d}t} + \alpha A\|\nabla\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \leq \\
&\alpha A\int_\Omega |\nabla(\boldsymbol{u}^1+\boldsymbol{v})|^2|\delta\boldsymbol{u}|^2 \quad +\alpha A\int_\Omega (\nabla\delta\boldsymbol{u}\cdot\nabla(\boldsymbol{u}_2+\boldsymbol{u}_1+2\boldsymbol{v}))\,((\boldsymbol{u}_2+\boldsymbol{v})\cdot\delta\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} \\
&\quad + A\sum_{i=1}^3\int_\Omega\left(\frac{\partial(\boldsymbol{u}_1+\boldsymbol{v})}{\partial x_i}\times\frac{\partial\delta\boldsymbol{u}}{\partial x_i}\right)\cdot\delta\boldsymbol{u} - \int_\Omega((\boldsymbol{u}_1+\boldsymbol{v})\times\mathcal{H}_{d,a}(\delta\boldsymbol{u}))\cdot\delta\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\
&\qquad - \alpha\int_\Omega((\boldsymbol{u}_1+\boldsymbol{v})\times(\delta\boldsymbol{u}\times\mathcal{H}_{d,a}(\boldsymbol{u}_2+\boldsymbol{v})))\cdot\delta\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \\
&\qquad\quad - \alpha\int_\Omega((\boldsymbol{u}_1+\boldsymbol{v})\times((\boldsymbol{u}_1+\boldsymbol{v})\times\mathcal{H}_{d,a}(\delta\boldsymbol{u})))\cdot\delta\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}. \quad (6.14)
\end{aligned}$$

We estimate the right-hand side of the precedent inequality. We obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}\|\delta\boldsymbol{u}\|^2}{\mathrm{d}t} + \alpha A\|\nabla\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} &\leq \alpha A\|\nabla(\boldsymbol{u}^1+\boldsymbol{v})\|^2_{\mathbb{L}^\infty(\Omega)}\|\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad + \frac{(\alpha A)^2}{4\eta}\|\nabla(\boldsymbol{u}^1+\boldsymbol{u}^2+2\boldsymbol{v})\|^2_{\mathbb{L}^\infty(\Omega)}\|\boldsymbol{u}^2+\boldsymbol{v}\|^2_{\mathbb{L}^\infty(\Omega)}\|\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad + \frac{A^2}{4\eta}\|\nabla(\boldsymbol{u}_1+\boldsymbol{v})\|^2_{\mathbb{L}^\infty(\Omega)}\|\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + 3\eta\|\nabla\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad + C\left(1+\frac{1}{\eta}\right)(1+\|\boldsymbol{u}_2+\boldsymbol{v}\|^2_{\mathbb{H}^1(\Omega)}+\|\boldsymbol{u}_1+\boldsymbol{v}\|^2_{\mathbb{L}^\infty(\Omega)})^2\|\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} \\
&\leq g(t)\|\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)} + 3\eta\|\nabla\delta\boldsymbol{u}\|^2_{\mathbb{L}^2(\Omega)},
\end{aligned}$$

where g belongs to $\mathrm{L}^1(0,T;\mathbb{R}^+)$. Choosing η sufficiently small, we can apply Gronwall's lemma. Since $\|\boldsymbol{u}_2(\cdot,0)-\boldsymbol{u}_1(\cdot,0)\|^2_{\mathbb{L}^2(\Omega)} = 0$, we obtain $\boldsymbol{u}_2 = \boldsymbol{u}_1$.

We now prove the explosion at the end of time of existence. Suppose $\widetilde{T^*} < T^*$, choose $\delta = \min(T^* - \widetilde{T^*}, \widetilde{T^*})/2$. The $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(\widetilde{T^*}/2,\widetilde{T^*}+\delta))$ norm of $\boldsymbol{v}$ is bounded so there exists a constant C such that

$$\|\boldsymbol{v}\|_{\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(\widetilde{T^*}-t,\widetilde{T^*}-t+\delta))} < C,$$

for any $t < \widetilde{T^*}/2$. Suppose also that $\|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)}$ is bounded on $(0, \widetilde{T^*})$. Hence, there exists a Δt, such that the equation with initial condition $\boldsymbol{m}(\cdot, t)$ and $\boldsymbol{v}(\cdot, t + \cdot)$ for any t in $(\widetilde{T^*}/2, \widetilde{T^*})$ has a solution which exists over $(0,\Delta t)$. Choosing $t > \widetilde{T^*} - \Delta t$, we construct a solution that extends the original solution beyond $\widetilde{T^*}$, hence a contradiction.

The previous proposition leads to the following corollary.

Corollary 16 *Let $T^* > 0$ and $\boldsymbol{m}_0$ in $\mathbb{H}^2(\Omega)$ such that*

$$\frac{\partial \boldsymbol{m}_0}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus (\Gamma^{\pm}), \\ Q_r^+(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) - (Q_r^+(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) \cdot \gamma \boldsymbol{m}_0)\gamma \boldsymbol{m}_0 & \text{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) - (Q_r^-(\gamma \boldsymbol{m}_0, \gamma' \boldsymbol{m}_0) \cdot \gamma \boldsymbol{m}_0)\gamma \boldsymbol{m}_0 & \text{on } \Gamma^-. \end{cases}$$

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0, T))$ for all $T < T^*$ be such that

$$\begin{aligned} \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\nu}} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma, & \frac{\partial \boldsymbol{b}}{\partial \boldsymbol{\nu}} &= 0 \quad \text{on } \partial\Omega \setminus \Gamma, \\ \boldsymbol{a}(\cdot, 0) &= \boldsymbol{m}_0, & \boldsymbol{b}(\cdot, 0) &= \boldsymbol{m}_0. \end{aligned}$$

Then there exists a unique maximal $\widetilde{T^*}$ and a unique $\boldsymbol{m}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0, T))$ for all $T < T^*$ satisfying the Landau-Lifshitz developed equation (6.2) and such that

$$\begin{aligned} &\boldsymbol{m}(\cdot, 0) = \boldsymbol{m}_0, \\ &\frac{\partial \boldsymbol{m}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma, \\ Q_r^+(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) - (Q_r^+(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) \cdot \gamma \boldsymbol{b})\gamma \boldsymbol{b} & \text{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) - (Q_r^-(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) \cdot \gamma \boldsymbol{b})\gamma \boldsymbol{b} & \text{on } \Gamma^-. \end{cases} \end{aligned}$$

Moreover, if $\widetilde{T^*} < T^*$, then

$$\lim_{t \to \widetilde{T^*}} \|\boldsymbol{m}\|_{\mathbb{H}^2}(t) = +\infty$$

PROOF. It relies on an extension result. Suppose there exists $\boldsymbol{v}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0, T))$ for each $T < T^*$ such that

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma, \\ Q_r^+(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) - (Q_r^+(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) \cdot \gamma \boldsymbol{b})\gamma \boldsymbol{b} & \text{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) - (Q_r^-(\gamma \boldsymbol{a}, \gamma' \boldsymbol{a}) \cdot \gamma \boldsymbol{b})\gamma \boldsymbol{b} & \text{on } \Gamma^-. \end{cases}$$

Since $\frac{\partial(\boldsymbol{m}_0 - \boldsymbol{v}(\cdot,0))}{\partial \boldsymbol{\nu}} = 0$, we apply proposition 15 to construct $\boldsymbol{u}$. Choosing $\boldsymbol{m} = \boldsymbol{u} + \boldsymbol{v}$ gives the solution. The uniqueness of $\boldsymbol{m}$ is a direct consequence of the uniqueness of $\boldsymbol{u}$.

We now construct $\boldsymbol{v}$. First, we recall that $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ is an algebra. Given χ in $\mathcal{C}_c^\infty(-\infty,+\infty;\mathbb{R}^+)$ such that

$$\chi(t)=\begin{cases}1 & \text{if } |x|<\frac{\min(L^+,L^-)}{2},\\ 0 & \text{if } |x|>\frac{3}{4}\min(L^+,L^-).\end{cases}$$

We define

$$\boldsymbol{g}=\begin{cases}Q_r^+(\boldsymbol{a},\boldsymbol{a}\circ\sigma)-(Q_r^+(\boldsymbol{a},\boldsymbol{a}\circ\sigma)\cdot\boldsymbol{b})\boldsymbol{b} & \text{in } \Omega^+,\\ Q_r^-(\boldsymbol{a},\boldsymbol{a}\circ\sigma)-(Q_r^+(\boldsymbol{a},\boldsymbol{a}\circ\sigma)\cdot\boldsymbol{b})\boldsymbol{b} & \text{in } \Omega^-.\end{cases}$$

where σ is the application that maps (x,y,z,t) to $(x,y,-z,t)$. Then,

$$\boldsymbol{v}(\cdot,\cdot,z,\cdot)=\int_0^z \chi(s)\boldsymbol{g}(\cdot,\cdot,s,\cdot)\,\mathrm{d}s,$$

is in $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ and has the required properties.

6.3 The converging sequences

For the construction of sequence (6.1), we need proposition 18 whose proof involves itself a converging sequence. It would be difficult to merge the two sequences in one because $\mathbb{H}^1(\Omega)$ is not an algebra. Since the elements of the sequence do not verify the Neumann homogenous condition, we cannot use corollary 11. We must use the more general propositions 9 and 10.

To build both sequences, we need an initial guess. It is provided by the following lemma.

Lemma 17 *Let $\boldsymbol{m}_0$ be in $\mathbb{H}^2(\Omega)$, with $\frac{\partial \boldsymbol{m}_0}{\partial \boldsymbol{\nu}}=0$ on $\partial\Omega\setminus\Gamma$. Then there exists $\boldsymbol{u}$ belonging to $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$ such that*

$$\boldsymbol{u}(\cdot,0)=\boldsymbol{m}_0 \qquad \textit{in } \Omega, \tag{6.15a}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}=0 \qquad \textit{on } \partial\Omega\setminus\Gamma\times(0,T). \tag{6.15b}$$

PROOF. To construct such a function, we need in the spirit of Lions and Magenes [15] to introduce the corresponding spaces $\mathrm{H}^{r,s,t}(B\times(0,L)\times(0,T))$ and to study the compatibility conditions between the traces. This construction is found in [16, App. A]. In this case, all the compatibility conditions hold: $\boldsymbol{u}$ does exist.

6.3.1 The inner converging sequence

Proposition 18 *Let $T^* > 0$. Let $\boldsymbol{m}_0$ in $\mathbb{H}^2(\Omega)$ verify the condition* (3.5) *and $\boldsymbol{u}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$ satisfy equations* (6.15a) *and* (6.15b)*. Then, there exists a unique maximal $\widetilde{T^*}$ and a unique $\boldsymbol{v}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < \widetilde{T^*}$ satisfying equations* (3.1c) *and* (6.2)*, and such that*

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \textit{on } \partial\Omega \setminus \Gamma \times (0,T), \\ Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) - (Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) \cdot \gamma \boldsymbol{v})\gamma \boldsymbol{v} & \textit{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) - (Q_r^-(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) \cdot \gamma \boldsymbol{v})\gamma \boldsymbol{v} & \textit{on } \Gamma^-. \end{cases} \tag{6.16}$$

Moreover, if $\widetilde{T^*} < T^*$, then

$$\lim_{t \to \widetilde{T^*}} \|\boldsymbol{v}\|_{\mathbb{H}^2}(t) = +\infty.$$

If $|\boldsymbol{m}_0| = 1$ a.e. in Ω then $|\boldsymbol{v}| = 1$ a.e. in $\Omega \times (0,T)$.

PROOF. The proof is divided in three steps:

(1) We construct a sequence $\boldsymbol{v}^n$ whose elements satisfies equations (3.1c) and (6.2), and

$$\frac{\partial \boldsymbol{v}^{n+1}}{\partial \boldsymbol{\nu}} = \begin{cases} 0 & \text{on } \partial\Omega \setminus \Gamma \times (0,T), \\ Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}), -(Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) \cdot \gamma \boldsymbol{v}^n)\gamma \boldsymbol{v}^n & \text{on } \Gamma^+, \\ Q_r^-(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}), -(Q_r^-(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) \cdot \gamma \boldsymbol{v}^n)\gamma \boldsymbol{v}^n & \text{on } \Gamma^-. \end{cases} \tag{6.17}$$

(2) We make estimates on size of the elements of sequence $(\boldsymbol{v}_n)_n$.
(3) The limit has the required properties and is the solution..

First step We need to construct a decreasing sequence of maximal T_n^* and $\boldsymbol{v}^n$ belonging to $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ such that for every $n \geq 0$, $\boldsymbol{v}^n$ satisfies equations (3.1c) , (6.2), and (6.17). To construct such a sequence, we define

- $\boldsymbol{v}^{-1} = \boldsymbol{u}$.
- Given $\boldsymbol{v}^{n-1}$ in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$, we construct $\boldsymbol{v}^n$ using corollary 16 with $\boldsymbol{a} = \boldsymbol{u}$ and $\boldsymbol{b} = \boldsymbol{v}^{n-1}$.

We initialize the sequence with $\boldsymbol{u}$ at $n = -1$ instead of $n = 0$ so that $\boldsymbol{v}^0$ satisfy equation (6.2).

Before making estimates, we note that compared to the proof of 15, the new Neumann boundary condition forces us to modify the proof. First, an integral over the boundary may appear in each of the estimate. Second, since inequalities (4.1a) and (4.1b) contain a boundary term on the right-hand side, an upper bound of $\|\boldsymbol{v}\|_{\mathbb{L}^2}$, $\|\triangle \boldsymbol{v}\|_{\mathbb{L}^2}$ and $\|\nabla\triangle \boldsymbol{v}\|_{\mathbb{L}^2}$ do not yield an upper bound of $\|\boldsymbol{v}\|_{\mathbb{H}^2}$ or $\|\boldsymbol{v}\|_{\mathbb{H}^3}$. We infer from Proposition 9 and the boundary condition verified by $\boldsymbol{v}^n$ for all positive integer n that

$$\begin{aligned}
\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{H}^2(\Omega)} \le C\Bigg(&\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} \\
&+ \left\|Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u}) - (Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v}^n)\gamma\boldsymbol{v}^n\right\|^2_{\mathbb{H}^{\frac12}(\Gamma^+)} \\
&+ \left\|Q_r^-(\gamma\boldsymbol{u},\gamma'\boldsymbol{u}) - (Q_r^-(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v}^n)\gamma\boldsymbol{v}^n\right\|^2_{\mathbb{H}^{\frac12}(\Gamma^-)} \Bigg) \\
\le C\left(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right) &+ P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})(1 + \||\boldsymbol{v}^n|^2\|^2_{\mathbb{H}^1(\Omega)}) \\
\le C\left(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right) &+ 3P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})(1 + \|\boldsymbol{v}^n\|^2_{\mathbb{H}^1(\Omega)}\|\boldsymbol{v}^n\|^2_{\mathbb{L}^\infty(\Omega)}) \\
\le C\left(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right) &+ C'P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})(1 + \|\boldsymbol{v}^n\|^3_{\mathbb{H}^1(\Omega)})\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}
\end{aligned} \tag{6.18}$$

Thus,

$$\begin{aligned}
\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{H}^2(\Omega)} \le C\left(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right) + C''(1 + P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})^2)(1 + \|\boldsymbol{v}^n\|^6_{\mathbb{H}^1(\Omega)}) \\
+ \frac{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)}}{2},
\end{aligned} \tag{6.19}$$

where P_1 is a given polynomial. The inequalities above are justified because $\mathbb{H}^2(\Omega)$ is an algebra and because of lemma 7. Thus, if $N \ge 1$

$$\begin{aligned}
\sup_{1\le n\le N}\left\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)}\right\} \le 2C\left(\sup_{1\le n\le N}\{\|\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\} + \sup_{1\le n\le N}\{\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\}\right) \\
+ 2C''P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})^2(1 + \sup_{1\le n\le N}\{\|\boldsymbol{v}^n\|^6_{\mathbb{H}^1(\Omega)}\}) \\
+ \|\boldsymbol{v}^0\|^2_{\mathbb{H}^2(\Omega)} + 2C''P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})^2(1 + \|\boldsymbol{v}^0\|^6_{\mathbb{H}^1(\Omega)}).
\end{aligned} \tag{6.20}$$

By the same method and proposition 10, we get

$$\begin{aligned}
\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{H}^3(\Omega)} \le C\Bigg(&\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} \\
&+ \left\|Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u}) - (Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v}^n)\gamma\boldsymbol{v}^n\right\|^2_{\mathbb{H}^{\frac32}(\Gamma^+)} \\
&+ \left\|Q_r^-(\gamma\boldsymbol{u},\gamma'\boldsymbol{u}) - (Q_r^-(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v}^n)\gamma\boldsymbol{v}^n\right\|^2_{\mathbb{H}^{\frac32}(\Gamma^-)} \Bigg) \\
\le C'\left(\|\boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla\triangle \boldsymbol{v}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right) &+ P_2(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)},
\end{aligned} \tag{6.21}$$

where P_2 is a given polynomial. Thus if $N \geq 1$,

$$\sup_{1\leq n\leq N}\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^3(\Omega)}\} \leq \sup_{1\leq n\leq N}\{\|\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\} + \sup_{1\leq n\leq N}\{\|\nabla\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\}$$
$$P_2(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)}) \sup_{1\leq n\leq N}\{\|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)}\} + P_2(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^0\|^2_{\mathbb{H}^2(\Omega)}. \tag{6.22}$$

Second step We now make three estimates on the norm of $\boldsymbol{v}^n$ for $n \geq 1$.

(1) We multiply equation (6.2) by $\boldsymbol{v}^n$ and integrate over Ω.
(2) We take the gradient of equation (6.2), multiply it by $\nabla\triangle\boldsymbol{v}^n$ and integrate over Ω.
(3) We take the gradient of equation (6.2), multiply it by $\nabla\boldsymbol{v}^n$ and integrate over Ω.

The first two estimates have mostly been made in the proof of proposition 15 or by Carbou-Fabrie in [14], except for the nonzero Neumann boundary condition. The third estimate is a simplification of the second estimate.

First estimate Using trace theorems, we obtain

$$\frac{1}{2}\frac{\mathrm{d}\|\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A\|\nabla\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} \leq \alpha A\|\nabla\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\|\boldsymbol{v}^n\|^2_{\mathbb{L}^\infty(\Omega)} + \alpha A\int_\Gamma\left|\frac{\partial\boldsymbol{v}^n}{\partial\boldsymbol{\nu}}\cdot\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x})$$
$$\leq P_1(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}), \tag{6.23}$$

where P_1 is a given polynomial.

Second estimate See [14] for the estimates of the volume terms for the second estimate. We obtain

$$\frac{1}{2}\frac{\mathrm{d}\|\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A\|\nabla\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} \leq \left(1+\frac{1}{\eta^3}\right)P_2(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}) + \eta\|\mathrm{D}^3\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}$$
$$+\alpha A\left|\int_{\Gamma^\pm}\frac{\partial^2\boldsymbol{v}^n}{\partial\boldsymbol{\nu}\partial t}\cdot\triangle\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x})$$
$$\leq \left(1+\frac{1}{\eta^3}\right)P_2(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}) + \eta\|\nabla\triangle\boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}$$
$$+\eta P_3(\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)}) + \int_{\Gamma^\pm}\left|\frac{\partial^2\boldsymbol{v}^n}{\partial\boldsymbol{\nu}\partial t}\cdot\triangle\boldsymbol{v}^n\right|\mathrm{d}\sigma(\boldsymbol{x}). \tag{6.24}$$

The boundary term has a meaning even before using the boundary condition. If $\boldsymbol{v}$ belongs to $\mathbb{H}^{3,\frac{3}{2}}(\Omega\times(0,T))$, then $\frac{\partial^2\boldsymbol{v}}{\partial t\partial\boldsymbol{\nu}}$ belongs to $\mathrm{H}^{-\frac{1}{4}}(0,T;\mathbb{L}^2(\Omega))$ and $\gamma\triangle\boldsymbol{v}$ belongs to $\mathrm{H}^{\frac{1}{4}}(0,T;\mathbb{L}^2(\Omega))$. The evaluation of the integral on Γ^+

for $n \geq 1$ gives

$$\int_{\Gamma^+} \left| \frac{\partial^2 \boldsymbol{v}^n}{\partial \boldsymbol{\nu} \partial t} \cdot \triangle \boldsymbol{v}^n \right| \, \mathrm{d}\sigma(\boldsymbol{x}) \leq \frac{1}{4} \|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)} + \left\| \frac{\partial^2 \boldsymbol{v}^n}{\partial \boldsymbol{\nu} \partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)}.$$

But,

$$\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)} \leq C \|\triangle \boldsymbol{v}^n\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)} (\|\triangle \boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)} + \|\nabla \triangle \boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)})^{\frac{3}{2}} \tag{6.25}$$

$$\begin{aligned} \left\| \frac{\partial^2 \boldsymbol{v}^n}{\partial \boldsymbol{\nu} \partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)} &\leq \left\| \frac{\partial((Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}) \cdot \boldsymbol{v}^{n-1}) \boldsymbol{v}^{n-1} - Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u}))}{\partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)} \\ &\leq 4 \left\| \frac{\partial Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u})}{\partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)} (1 + \|\boldsymbol{v}^{n-1}\|^4_{\mathbb{L}^\infty(\Omega)}) \\ &\quad + 8 \|Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u})\|_{\mathbb{L}^\infty(\Gamma^+)} \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{L}^\infty(\Omega)} \left\| \frac{\partial \boldsymbol{v}^{n-1}}{\partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)}. \end{aligned} \tag{6.26}$$

There exists a polynomial P_3 such that

$$\left\| \frac{\partial Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u})}{\partial t} \right\|_{\mathbb{L}^2(\Gamma^+)} \leq P_3(\|\boldsymbol{u}\|_{\mathbb{L}^\infty(\Omega)}) \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^{\frac{1}{4}}_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^{\frac{3}{4}}_{\mathbb{H}^1(\Omega)} \tag{6.27a}$$

$$\|Q_r^+(\gamma \boldsymbol{u}, \gamma' \boldsymbol{u})\|_{\mathbb{L}^\infty(\Gamma^+)} \leq P_3(\|\boldsymbol{u}\|_{\mathbb{L}^\infty(\Omega)}) \tag{6.27b}$$

$$\begin{aligned} \left\| \frac{\partial \boldsymbol{v}^{n-1}}{\partial t} \right\|^2_{\mathbb{L}^2(\Gamma^+)} &\leq C \left\| \frac{\partial \boldsymbol{v}^{n-1}}{\partial t} \right\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)} \left\| \frac{\partial \boldsymbol{v}^{n-1}}{\partial t} \right\|^{\frac{3}{2}}_{\mathbb{H}^1(\Omega)} \\ &\leq P_3(\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)})^{\frac{1}{2}} \|\boldsymbol{v}^{n-1}\|^{\frac{3}{2}}_{\mathbb{H}^3(\Omega)} \\ &\leq \frac{1}{\eta^3} P_3(\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)})^2 + \eta \|\boldsymbol{v}^{n-1}\|^3_{\mathbb{H}^3(\Omega)}. \end{aligned} \tag{6.27c}$$

Thus,

$$\begin{aligned} \int_{\Gamma^+} \left| \frac{\partial^2 \boldsymbol{v}^n}{\partial \boldsymbol{\nu} \partial t} \cdot \triangle \boldsymbol{v}^n \right| \, \mathrm{d}\sigma(\boldsymbol{x}) &\leq C' \left(1 + \frac{1}{\eta^3}\right) \|\boldsymbol{v}^n\|^2_{\mathbb{H}^2(\Omega)} + \eta \|\boldsymbol{v}^n\|^2_{\mathbb{H}^3(\Omega)} \\ &\quad + C'' \left(1 + \frac{1}{\eta^3}\right) P_4(\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)}) + \eta \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^3(\Omega)} \\ &\quad + C \left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^2_{\mathbb{H}^1(\Omega)} \|\boldsymbol{v}^{n-1}\|^2_{\mathbb{L}^\infty(\Omega)}, \end{aligned} \tag{6.28}$$

where P_4, P_5 are polynomials and $\left\| \frac{\partial \boldsymbol{u}}{\partial t} \right\|^2_{\mathbb{H}^1(\Omega)}$ a $\mathrm{L}^1(0,T)$ function.

Third estimate This is a simplification of the previous estimate and we obtain for $n \geq 1$

$$\frac{1}{2}\frac{\mathrm{d}\|\nabla \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}}{\mathrm{d}t} + \alpha A\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)} \leq \left(1 + \frac{1}{\eta^{1/3}}\right) P_6(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}) + \eta\|\nabla\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}$$
$$+ \eta P_7(\|\boldsymbol{v}^{n-1}\|_{\mathbb{H}^2(\Omega)}) + \alpha A \int_{\Gamma^\pm} \left|\frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}} \cdot \triangle \boldsymbol{v}^n\right| \mathrm{d}\sigma(\boldsymbol{x}). \tag{6.29}$$

The evaluation of the boundary term on Γ^+ gives

$$\begin{aligned}\int_{\Gamma^+} \left|\frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}} \cdot \triangle \boldsymbol{v}^n\right| \mathrm{d}\sigma(\boldsymbol{x}) &\leq \frac{1}{2}\|\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Gamma^+)} + \frac{1}{2}\left\|\frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}}\right\|^2_{\mathbb{L}^2(\Gamma^+)} \\ &\leq C\|\triangle \boldsymbol{v}^n\|^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)}\|\nabla\triangle \boldsymbol{v}^n\|^{\frac{3}{2}}_{\mathbb{L}^2(\Omega)} + P_3(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)})\|\boldsymbol{v}^{n-1}\|^4_{\mathbb{L}^\infty(\Omega)} \\ &\leq \frac{1}{\eta^3} P_4(\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}) + C\|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^1(\Omega)}\|\boldsymbol{v}^{n-1}\|^2_{\mathbb{H}^2(\Omega)} \\ &\quad + \eta\|\nabla\triangle \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)},\end{aligned}$$

where the P_i are polynomials and C a constant only depending on the $\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))$ norm of $\boldsymbol{u}$ which is bounded for all $T < T^*$.

We choose η small enough, then combine all three estimates. Taking the maximum over $1 \leq n \leq N$, we obtain, for all t in $(0, \min_{0\leq n\leq N} T^*_n)$,

$$\begin{aligned}&\sup_{1\leq n\leq N}\{\|\boldsymbol{v}^n\|_{\mathbb{H}^1(\Omega)} + \|\triangle \boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)}\} \\ &\quad + \sup_{1\leq n\leq N}\left\{\int_0^t \|\nabla\triangle \boldsymbol{v}^n\|_{\mathbb{L}^2(\Omega)}\,\mathrm{d}s\right\} \leq \int_0^t P_5^\eta(\sup_{1\leq n\leq N}\|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)})\,\mathrm{d}s \\ &\qquad + \int_0^t P_5^\eta\|\boldsymbol{v}^0\|_{\mathbb{H}^2(\Omega)})\,\mathrm{d}s + \eta\int_0^t \|\nabla\triangle \boldsymbol{v}^0\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}s \\ &\qquad + \eta\int_0^t \|\nabla\triangle \boldsymbol{v}^{-1}\|^2_{\mathbb{L}^2(\Omega)}\,\mathrm{d}s.\end{aligned} \tag{6.30}$$

We now replace the $\mathbb{H}^2(\Omega)$ norm of $\boldsymbol{v}^n$ using inequality (6.20). Gronwall's lemma implies that the $\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))$ norms of $\boldsymbol{v}^n$, $\triangle \boldsymbol{v}^n$ and $\nabla \boldsymbol{v}^n$, as well as the $\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega))$ norm of $\nabla\triangle \boldsymbol{v}^n$ cannot explode before a given time $\widetilde{T^*}$ independent of n. Thus, using again inequality (6.20), the $\mathbb{H}^2$ norm of $\boldsymbol{v}^n$ remains bounded independently of n. Hence, there exists a common time of existence $\widetilde{T^*}$ for all $n \geq 0$. By lemma 14, for every $T < T^*$, the norm $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ is bounded independently of n. Thus, there exists a weakly converging subsequence $\boldsymbol{v}^{n_k}$ in $\mathrm{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*$.

Third step We need to prove that the limit $\boldsymbol{v}$ has the required properties (3.1c), (6.2), and (6.16). As in proposition 15, we can take the limit in equation (6.2) and $\boldsymbol{v}$ verifies this equation. Since for all positive integers n,

$\boldsymbol{v}^n(\cdot,0)=\boldsymbol{m}_0$, then $\boldsymbol{v}(\cdot,0)=\boldsymbol{m}_0$. Since $\frac{\partial \boldsymbol{v}^n}{\partial \boldsymbol{\nu}}=0$ on $\partial\Omega\setminus\Gamma$, $\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}=0$ on $\partial\Omega\setminus\Gamma$. We only need to verify the Neumann nonhomogenous boundary condition on Γ.

Suppose [5] that if $\boldsymbol{v}^{n_k}$ and $\boldsymbol{v}^{n_k+1}$ both converge weakly, then they converge to the same limit. Extract a further subsequence such that $\boldsymbol{v}^{n_k+1}$ also converges. According to our supposition, the limit is the same. On Γ^+,

$$\frac{\partial \boldsymbol{v}^{n_k+1}}{\partial \boldsymbol{\nu}}=Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})-(Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v}^{n_k})\gamma\boldsymbol{v}^{n_k}).$$

We take the limit,

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}}=Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})-(Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\cdot\gamma\boldsymbol{v})\gamma\boldsymbol{v}).$$

The corresponding result holds on Γ^-.

We now prove that if $|\boldsymbol{m}_0|=1$ a.e., then $|\boldsymbol{v}|=1$ a.e. We work by estimates, multiplying equation (6.2) by $(|\boldsymbol{v}|^2-1)\boldsymbol{v}$ and integrating over Ω

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\||\boldsymbol{v}|^2-1\|^2_{\mathrm{L}^2(\Omega)}\right)+\frac{\alpha A}{2}\left\|\nabla(|\boldsymbol{v}|^2-1)\right\|^2_{\mathrm{L}^2(\Omega)}&\leq \alpha A\int_\Omega|\nabla\boldsymbol{v}|^2(|\boldsymbol{v}|^2-1)^2\,\mathrm{d}\boldsymbol{x}\\&+\frac{\alpha A}{2}\int_\Gamma^{\pm}\frac{\partial|\boldsymbol{v}|^2}{\partial\boldsymbol{\nu}}\left(|\boldsymbol{v}|^2-1\right)\,\mathrm{d}\sigma(\boldsymbol{x}).\end{aligned}$$

We estimate the boundary integral with (6.16).

$$\begin{aligned}\int_\Gamma^{\pm}\frac{\partial|\boldsymbol{v}|^2}{\partial\boldsymbol{\nu}}\left(|\boldsymbol{v}|^2-1\right)\,\mathrm{d}\sigma(\boldsymbol{x})&=2\int_{\Gamma^+}Q_r^+(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\left(|\boldsymbol{v}|^2-1\right)^2\,\mathrm{d}\sigma(\boldsymbol{x})\\&+2\int_{\Gamma^-}Q_r^-(\gamma\boldsymbol{u},\gamma'\boldsymbol{u})\left(|\boldsymbol{v}|^2-1\right)^2\,\mathrm{d}\sigma(\boldsymbol{x}).\end{aligned}$$

We finally obtain

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\||\boldsymbol{v}|^2-1\|^2_{\mathrm{L}^2(\Omega)}\right)+\frac{\alpha A}{2}\left\|\nabla(|\boldsymbol{v}|^2-1)\right\|^2_{\mathrm{L}^2(\Omega)}\\&\leq \alpha A\|\nabla\boldsymbol{v}\|^2_{\mathrm{L}^\infty(\Omega)}\||\boldsymbol{v}|^2-1\|^2_{\mathrm{L}^2(\Omega)}+C\alpha AP_1(\|\boldsymbol{v}\|_{\mathrm{L}^\infty})\||\boldsymbol{v}|^2-1\|^{\frac{1}{2}}_{\mathrm{L}^2(\Omega)}\||\boldsymbol{v}|^2-1\|^{\frac{3}{2}}_{\mathrm{H}^1(\Omega)}\\&\qquad\leq C(\eta)P_2(\|\boldsymbol{v}\|_{\mathbb{H}^2(\Omega)})\||\boldsymbol{v}|^2-1\|^2_{\mathrm{L}^2(\Omega)}+\eta\|\nabla(|\boldsymbol{v}|^2-1)\|^2_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

We choose η small enough, then we absorb $\|\nabla(|\boldsymbol{v}|^2-1)\|^2_{\mathrm{L}^2(\Omega)}$ in the left-hand side. We apply Gronwall's lemma and since $\||\boldsymbol{m}_0|^2-1\|^2_{\mathrm{L}^2(\Omega)}=0$, $|\boldsymbol{v}|=1$ a.e. in $\Omega\times(0,T)$.

It remains to prove our precedent assumption.

[5] We prove this supposition later in lemma 19.

Lemma 19 *For each $T < T^*$, the $\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))$ norm and the $\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))$ norm of $\boldsymbol{v}^{n+1} - \boldsymbol{v}^n$ tends to 0.*

PROOF. First, using inequality (4.1a), we evaluate the $\mathbb{H}^2$ norm of $\delta^n \boldsymbol{v} = \boldsymbol{v}^{n+1} - \boldsymbol{v}^n$ for $n \geq 1$

$$\begin{aligned}\|\delta^n \boldsymbol{v}\|^2_{\mathbb{H}^2(\Omega)} \leq C\left(\|\delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}\right)& \\ + P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)}) \sup_j \{\|\boldsymbol{v}^j\|^2_{\mathbb{H}^2(\Omega)}\} \|\delta^{n-1}\boldsymbol{v}\|^2_{\mathbb{H}^1(\Omega)},&\end{aligned} \tag{6.31}$$

and for the $\mathbb{H}^3$ norm using inequality (4.1b)

$$\begin{aligned}\|\delta^n \boldsymbol{v}\|^2_{\mathbb{H}^3(\Omega)} \leq C\left(\|\delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla\triangle \delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}\right)& \\ + P_2(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)}) \sup_j \{\|\boldsymbol{v}^j\|^2_{\mathbb{H}^2(\Omega)}\} \|\delta^{n-1}\boldsymbol{v}\|^2_{\mathbb{H}^2(\Omega)}.&\end{aligned} \tag{6.32}$$

Next, we make some estimates. For the first estimate, we multiply equation (6.2) by $\delta^n \boldsymbol{v} = \boldsymbol{v}^{n+1} - \boldsymbol{v}^n$ and integrate over Ω. For the second estimate, we take the gradient of equation (6.2), multiply the result by $\nabla\triangle(\boldsymbol{v}^{n+1} - \boldsymbol{v}^n)$ and integrate over Ω. For the third estimate, we take the gradient of equation (6.2), multiply by $\nabla(\boldsymbol{v}^{n+1} - \boldsymbol{v}^n)$ and integrate over Ω. We add the three estimates, combine them with the regularity inequalities (6.31) and (6.32), and integrate over $(0,t)$. Hence,

$$\begin{aligned}\|\delta^n \boldsymbol{v}\|^2_{\mathbb{H}^1(\Omega)} + \|\triangle \delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)} + \int_0^t \|\nabla\triangle \delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}\, \mathrm{d}s& \\ \leq \int_0^t P_1\left(\|\boldsymbol{v}^{n+1}\|_{\mathbb{H}^2(\Omega)}, \|\boldsymbol{v}^n\|_{\mathbb{H}^2(\Omega)}\right) \|\delta^n \boldsymbol{v}\|^2_{\mathbb{H}^2(\Omega)}\, \mathrm{d}s& \\ + \int_0^t P_1(\|\boldsymbol{u}\|_{\mathbb{H}^2(\Omega)}) \sup_j \{\|\boldsymbol{v}^j\|_{\mathbb{H}^2(\Omega)}\}^2 \|\delta^{n-1}\boldsymbol{v}\|^2_{\mathbb{H}^2(\Omega)}\, \mathrm{d}s& \\ + \eta \int_0^t \|\nabla\triangle \delta^n \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}\, \mathrm{d}s + \eta \int_0^t \|\nabla\triangle \delta^{n-1} \boldsymbol{v}^n\|^2_{\mathbb{L}^2(\Omega)}\, \mathrm{d}s,&\end{aligned}$$

where η may be chosen arbitrarily small. If we sum over $0 \leq k \leq n$, then

$$\begin{aligned}\sum_{k=1}^n \left(\|\delta^k \boldsymbol{v}\|^2_{\mathbb{H}^1(\Omega)} + \|\triangle \delta^k \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}\right) + \sum_{k=1}^n \int_0^t \|\nabla\triangle(\delta^k \boldsymbol{v})\|^2_{\mathbb{L}^2(\Omega)}\, \mathrm{d}s& \\ \leq \psi_1(t)\left(\sum_{k=1}^n \|\delta^k \boldsymbol{v}\|^2_{\mathbb{H}^2(\Omega)}\right) + 2\eta \sum_{k=1}^n \|\delta^k \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)}& \\ + \psi_1(t)\|\boldsymbol{v}^1 - \boldsymbol{v}^0\|^2_{\mathbb{H}^2(\Omega)} + \eta\|\nabla\triangle(\boldsymbol{v}^1 - \boldsymbol{v}^0)\|^2_{\mathbb{L}^2(\Omega)},&\end{aligned}$$

where Ψ_1 is a $\mathrm{L}^1(0,T)$ function. Since $\sum_{k=1}^n \delta^k \boldsymbol{v}(\cdot, 0) = 0$, the $\mathbb{H}^2(\Omega)$ norm of the initial condition remains bounded independently of n. After the use of inequality (6.31), we can apply Gronwall's lemma. Hence, for any $T < T^*$, there exists a constant C_T such that

$$\sum_{k=0}^{n} \|\boldsymbol{v}^{k+1} - \boldsymbol{v}^k\|^2_{\mathrm{L}^\infty(0,T;\mathbb{H}^2(\Omega))} \leq C_T,$$
$$\sum_{k=0}^{n} \|\boldsymbol{v}^{k+1} - \boldsymbol{v}^k\|^2_{\mathrm{L}^2(0,T;\mathbb{H}^3(\Omega))} \leq C_T.$$

Hence, the term in the sum tends to 0.

The uniqueness is proven by the same kind of estimates as in the proof of lemma 19. As in the proof of proposition 15, there is explosion of the $\mathbb{H}^2$ norm of $\boldsymbol{v}$ at the end of time of existence if $\widetilde{T^*} < T^*$. If that were not the case, we could extend the lifetime of the solution beyond T^*, hence a contradiction.

6.3.2 Convergence of sequence (6.1)

The existence of a sequence satisfying equations (3.1b), (3.1c), (6.1), and (6.2) is a direct consequence of lemma 17 and proposition 18. Each element of the sequence is in $\mathbb{H}^{3,\frac{3}{2}}(\Omega \times (0,T))$ for all $T < T^*_n$, where T^*_n is the lifetime of $\boldsymbol{m}^n$. We adapt the proof of proposition 18. The only differences are the regularity estimate (6.19) and inequality (6.28). The latter is replaced for $n \geq 1$ by

$$\begin{aligned}
\left\|\frac{\partial^2 \boldsymbol{m}^n}{\partial \boldsymbol{\nu} \partial t}\right\|^2_{\mathbb{L}^2(\Gamma^+)} &\leq \left\|\frac{\partial((Q_r^+(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}) \cdot \boldsymbol{m}^n)\boldsymbol{m}^n - Q_r^+(\gamma \boldsymbol{m}^{n-1}, \gamma' \boldsymbol{m}^{n-1}))}{\partial t}\right\|^2_{\mathbb{L}^2(\Gamma^+)} \\
&\leq \frac{C}{\eta}(\left\|\boldsymbol{m}^{n-1}\right\|^2_{\mathbb{H}^2(\Gamma^+)} + \|\boldsymbol{m}^n\|^2_{\mathbb{H}^2(\Gamma^+)}) \\
&\quad + \eta(\|\boldsymbol{m}^n\|^2_{\mathbb{L}^2(\Gamma^+)} \left\|\boldsymbol{m}^{n-1}\right\|^2_{\mathbb{L}^2(\Gamma^+)} + \|\boldsymbol{m}^n\|^2_{\mathbb{L}^2(\Gamma^+)}).
\end{aligned}$$

The former is replaced by

$$\begin{aligned}
\|\boldsymbol{m}^{n+1}\|^2_{\mathbb{H}^2(\Omega)} \leq C\Bigg(&\|\boldsymbol{m}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \|\triangle \boldsymbol{m}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} \\
&+ \left\|Q_r^+(\gamma \boldsymbol{m}^n, \gamma' \boldsymbol{m}^n) - (Q_r^+(\gamma \boldsymbol{m}^n, \gamma' \boldsymbol{m}^n) \cdot \gamma \boldsymbol{m}^{n+1})\gamma \boldsymbol{m}^{n+1}\right\|^2_{\mathbb{H}^{\frac{1}{2}}(\Gamma^+)} \\
&+ \left\|Q_r^-(\gamma \boldsymbol{m}^n, \gamma' \boldsymbol{m}^n) - (Q_r^-(\gamma \boldsymbol{m}^n, \gamma' \boldsymbol{m}^n) \cdot \gamma \boldsymbol{m}^{n+1})\gamma \boldsymbol{m}^{n+1}\right\|^2_{\mathbb{H}^{\frac{1}{2}}(\Gamma^-)}\Bigg) \\
\leq C\left(\|\boldsymbol{m}^{n+1}\|^2_{\mathbb{L}^2(\Omega)}\right.&\left. + \|\triangle \boldsymbol{m}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + C'(\|\boldsymbol{m}^n\|^2_{\mathbb{H}^1(\Omega)} + \|\boldsymbol{m}^{n+1}\|^2_{\mathbb{H}^1(\Omega)}\right).
\end{aligned}$$

This inequality holds because the local norm of $\boldsymbol{m}^n$ is equal to 1 for all $n \geq 1$. The rest of the proof of proposition 18 can be applied with no differences.

It is possible to find a common time of existence T^* and a converging subsequence $\boldsymbol{m}^{n_k}$, such that $\boldsymbol{m}^{n_k+1}$ converges to the same limit.

The limit is thus the strong solution to theorem 5. There is only one solution, and if $T^* < +\infty$, then

$$\lim_{t \to T^*} \|\boldsymbol{m}\|_{\mathbb{H}^2(\Omega)} = +\infty$$

If that were not the case, we could extend the solution past T^*, hence a contradiction.

7 Conclusion

We have proved for a particular geometry the existence of finite time strong solutions and infinite time weak solutions to Landau-Lifshitz system. The existence of such solutions is certainly valid with more complicated geometries. The proof of such existence would only require some technical modifications.

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