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Microlocalization of Resonant States and Estimates of the Residue of the Scattering Amplitude

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Abstract: We obtain some microlocal estimates of the resonant states associated to a resonance z_0 of an h-differential operator. More precisely, we show that the normalized resonant states are $\mathcal{O}(\sqrt{|\mathrm{Im}\, z_0|/h} + h^\infty)$ outside the set of trapped trajectories and are $\mathcal{O}(h^\infty)$ in the incoming area of the phase space.

As an application, we show that the residue of the scattering amplitude of a Schrödinger operator is small in some directions under an estimate of the norm of the spectral projector. Finally we prove such a bound in some examples.

1. Introduction

The original motivation of this paper is the study of the residue of the scattering amplitude associated to a Schrödinger operator $P(h) = -h^2\Delta + V(x)$ on $\mathbb{R}^n$. The first works treating this question are due to Lahmar-Benbernou [17] and Lahmar-Benbernou and Martinez [18]. In these papers, they consider the case where the potential $V(x)$ is a "well in an island" with non-degenerate local minimum. In this situation, the form of the resonances is given by the work of Helffer and Sjöstrand [13]. Near a resonance z_0 simple, isolated and close to the energy of this local minimum, the scattering amplitude can be written

$$f(\omega, \omega', z, h) = \frac{f^{res}(\omega, \omega', h)}{z - z_0} + f^{hol}(\omega, \omega', z, h),$$

with f^{hol} holomorphic near z_0. Using the form of the resonant states associated to z_0, Lahmar-Benbernou and Martinez proved that

$$|f^{res}(\omega, \omega', h)| = g(h)|\mathrm{Im}\, z_0|,$$

where $g(h)$ has an asymptotic expansion with respect to h. Moreover, they showed that for some directions (ω, ω'), determined by the Agmon distance to the well, the residue is $\mathcal{O}(h^\infty)$ while for some other ones, they obtained an explicit non-vanishing principal term for $g(h)$. Their proof is based on the knowledge of the resonant states given by [13].

In [30], Stefanov generalized some parts of this result and proved that for $V \in C_0^\infty(\mathbb{R}^n)$ and z_0 a resonance which is simple and isolated in a sense made precise in [30], one has

$$|f^{res}(\omega, \omega', h)| \leq Ch^{-\frac{n-1}{2}} |\mathrm{Im}\, z_0|. \tag{1.1}$$

This result was next improved by the second author in [23] where estimate (1.1) is established for general long-range potentials under a weaker separation condition on resonances. In [30] and [23], the method employed stands on the semiclassical maximum principle of Tang and Zworski [32] and a resolvent estimate of Burq [5].

In the case where $|\mathrm{Im}\, z_0| \leq Ch^M$ for $M \gg 1$ and $|\mathrm{Im}\, z_0| \neq \mathcal{O}(h^\infty)$, it shows only that $|f^{res}| = \mathcal{O}(h^N)$ for $N \in \mathbb{R}$, whereas it is proven in [18] that the decay of the residue may depend on the direction considered. In particular, one can think that there exists some couple of directions (ω, ω') such that the associated residue is $\mathcal{O}(h^\infty)$. One of our motivations is to show the existence of such directions for resonances "far" from the real axis.

In the case where the potential V is compactly supported, we have a nice representation formula for the scattering amplitude, so that one can easily see the link between the problem of the residue and the estimate of the resonant states announced in the title. Indeed, as is proven in [24], one has

$$f(\omega, \omega', z, h) = c(z; h) \int_{\mathbb{R}^n} e^{-i\sqrt{z}\langle x, \omega'\rangle/h} [h^2\Delta, \chi_1] R(z, h) [h^2\Delta, \chi_2] e^{i\sqrt{z}\langle x, \omega\rangle/h} dx, \tag{1.2}$$

where $R(z, h) : L^2_{comp} \to H^2_{loc}$ denotes the meromorphic continuation of the resolvent of P to a conic neighborhood of the real axis and

$$c(z; h) = \frac{1}{2} z^{\frac{n-3}{4}} (2\pi h)^{-\frac{n+1}{2}} e^{-i\frac{(n-3)\pi}{4}}. \tag{1.3}$$

Moreover, if one denotes by P_θ the operator obtained from P by analytic dilatation (see [28]), and if one assumes that the dilatation is performed sufficiently far, one gets

$$f(\omega, \omega', z, h) = c(z; h) \int_{\mathbb{R}^n} e^{-i\sqrt{z}\langle x, \omega'\rangle/h} [h^2\Delta, \chi_1] (P_\theta - z)^{-1} [h^2\Delta, \chi_2] e^{i\sqrt{z}\langle x, \omega\rangle/h} dx. \tag{1.4}$$

Assume that z_0 is a simple resonance and that there is no other resonance in a disk D centered in z_0, then the residue is given by the formula

$$f^{res}(\omega, \omega', h) = c(z_0; h) \langle [h^2\Delta, \chi_1] \Pi_\theta [h^2\Delta, \chi_2] e^{i\sqrt{z_0}\langle x, \omega\rangle/h}, e^{i\sqrt{z_0}\langle x, \omega'\rangle/h} \rangle, \tag{1.5}$$

where Π_θ is the spectral projector associated to z_0. Moreover, as Π_θ is a rank one operator, there exist $u_\theta, v_\theta \in L^2$ such that $\Pi_\theta = \langle ., v_\theta\rangle u_\theta$ and one can show that $(P_\theta - z_0) u_\theta = 0$ and $(P_{-\theta} - \overline{z}_0) v_\theta = 0$. It follows that

$$f^{res}(\omega, \omega', h) = -c(z_0; h) \langle u_\theta, [h^2\Delta, \chi_1] e^{i\sqrt{z_0}\langle x, \omega'\rangle/h} \rangle \langle [h^2\Delta, \chi_2] e^{i\sqrt{z_0}\langle x, \omega\rangle/h}, v_\theta \rangle.$$

On the other hand, it is easy to see that the functions $[h^2\Delta, \chi_*]e^{i\sqrt{z_0}\langle x,\omega^*\rangle/h}$ are microlocalized near $\{(x,\xi);\ R_1 < |x| < R_2, \xi/|\xi| \sim \omega^*\}$ for $\omega^* = \omega, \omega'$. Our approach consists to show that for suitable directions, the resonant state u_θ is microlocalized out of this set. In fact, the microlocal estimate that we will prove holds for more general operators than Schrödinger ones.

To state precisely our results, we need to introduce the following class of symbol (see the book of Dimassi and Sjöstrand [7] for more details). We say that $g \in C^\infty(\mathbb{R}^d; \mathbb{R}^*_+)$ is an order function if $\forall\alpha \in \mathbb{N}^d$, $\partial_x^\alpha g(x) = \mathcal{O}(g)$ uniformly on $\mathbb{R}^d$. A function $a(x;h)$ defined on $\mathbb{R}^d\times]0,h_0]$ for some $h_0 > 0$ is said to be a symbol in the class $S_d(g)$ if $a(x;h)$ depends smoothly on x and

$$\forall\alpha \in \mathbb{N}^d, \quad \partial_x^\alpha a(x;h) = \mathcal{O}(g),$$

uniformly with respect to $(x,h) \in \mathbb{R}^d\times]0,h_0]$. We will say that $a(x;h)$ belongs to $S_d^{cl}(g)$ if there exists a sequence $a_j(x) \in S_d(g)$ such that for all $N \in \mathbb{N}$,

$$a(x;h) - \sum_{j=0}^{N} a_j(x)h^j \in h^{N+1}S_d(g),$$

uniformly with respect to h. For $a(x,\xi;h) \in S_{2n}(g)$, one can define the h-pseudodifferential operator (in the Weyl quantization) $A = \mathrm{Op}_h^w(a) = a(x,hD_x)$ associated with a. For $f \in C_0^\infty(\mathbb{R}^n)$,

$$(\mathrm{Op}_h^w(a)f)(x) = \frac{1}{(2\pi h)^n}\iint e^{i\langle x-y.\xi\rangle/h}a\Big(\frac{x+y}{2},\xi;h\Big)f(y)\,d\xi\,dy.$$

In this case, we say that a is the Weyl symbol of A.

In this paper, we consider $P(h)$ an h-differential operator on $\mathbb{R}^n$, having the form

$$P(h) = \sum_{|\alpha|\le 2} a_\alpha(x;h)(hD_x)^\alpha, \tag{1.6}$$

where $a_\alpha(x;h) \in S_n^{cl}(1)$ and $a_\alpha(x;h)$ does not depend on h for $|\alpha| = 2$. We assume that P is formally self-adjoint on $L^2(\mathbb{R}^n)$, that is

$$\forall u, v \in C_0^\infty(\mathbb{R}^n) \qquad \int (Pu)\overline{v}\,dx = \int u\overline{(Pv)}\,dx. \tag{1.7}$$

We suppose also that P is elliptic, that is,

$$\sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge |\xi|^2/C. \tag{1.8}$$

To define the resonances, we assume that the coefficients $a_\alpha(x;h)$ extend holomorphically in x in the domain

$$\Upsilon = \{x \in \mathbb{C}^n;\ |\mathrm{Im}\,x| \le \delta_0\langle \mathrm{Re}\,x\rangle \text{ and } |x| \ge R_0\}, \tag{1.9}$$

$R_0 > 0$, $\delta_0 \in]0,1[$ and that P converge to $-h^2\Delta$ at infinity in the following sense:

$$\sum_{|\alpha|\le 2} a_\alpha(x;h)\xi^\alpha \longrightarrow \xi^2, \tag{1.10}$$

as $|x| \to +\infty$, $x \in \Gamma$, uniformly with respect to h. Under these assumptions, it is clear that P is a self-adjoint operator with domain $H^2(\mathbb{R}^n)$ and one can define the resonances associated to P by the method of analytic distortions (see Aguilar–Combes [1], Hunziker [14] and Sjöstrand–Zworski [28]).

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field such that $F(x) = 0$ if $|x| \leq R_0$ and $F(x) = |x|$ for $|x|$ large enough. For $\nu \in \mathbb{R}$ small enough, we consider the unitary operator U_ν on $L^2(\mathbb{R}^n)$ defined by:

$$U_\nu \varphi(x) = \det(1 + \nu dF(x))^{-\frac{1}{2}} \varphi(x + \nu F(x)).$$

Then, the operator $U_\nu P(h) U_\nu^{-1}$ has coefficients which are analytic with respect to ν near 0 and can be continued to complex values of ν. For $\nu = i\theta$, with $\theta > 0$ small enough, we get a differential operator denoted by P_θ. It is well-known that the spectrum of P_θ is discrete in the sector $S_\theta = \{z \in \mathbb{C};\ \operatorname{Re} z > 0 \text{ and } -2\theta < \arg z \leq 0\}$ (see [28] and [26]) and by definition, the resonances of P are the eigenvalues of P_θ.

We denote by $p(x, \xi; h) \in S^{cl}_{2n}(\langle \xi \rangle^2)$ the Weyl symbol of P and $p_0(x, \xi) = \sum_{|\alpha| \leq 2} a_{\alpha,0}(x) \xi^\alpha$ is its principal symbol. The Hamilton vector field associated with p_0 is $\mathrm{H}_{p_0} = \partial_\xi p_0 . \partial_x - \partial_x p_0 . \partial_\xi$ and $\exp(t\mathrm{H}_{p_0})$, $t \in \mathbb{R}$ is the corresponding Hamiltonian flow. We define the outgoing tail and the incoming tail at the energy E by

$$\Gamma_\pm(E) = \{(x, \xi) \in p_0^{-1}(E);\ \exp(t\mathrm{H}_{p_0})(x, \xi) \not\to \infty,\ t \to \mp\infty\}.$$

Hence, the set of trapped trajectories is

$$\mathcal{T}(E) = \Gamma_+(E) \cap \Gamma_-(E) = \{(x, \xi) \in p_0^{-1}(E);\ t \mapsto \exp(t\mathrm{H}_{p_0})(x, \xi) \text{ is bounded on } \mathbb{R}\}.$$

For $E > 0$, $\mathcal{T}(E)$ is a compact set (see the appendix of the paper of C. Gérard–Sjöstrand [11]). Setting $\mathcal{T}([a, b]) = \overline{\bigcup_{E \in [a,b]} \mathcal{T}(E)}$, we give another proof of a result of Stefanov on the localisation of the resonant states:

Theorem 1. *Let $E_0 > 0$ be a fixed energy level, $\epsilon > 0$ small enough, $\theta = h/C$ with $C > 0$, let $z \in \mathbb{C}$ be a resonance of P with $\operatorname{Re} z \in [E_0 - \epsilon, E_0 + \epsilon]$, $|\operatorname{Im} z| < \epsilon\theta$, and let $u_\theta \in L^2(\mathbb{R}^n)$ be a resonant state associated to z:*

$$(P_\theta - z) u_\theta = 0. \tag{1.11}$$

If $w(x, \xi) \in S_{2n}(1)$ with $\operatorname{supp} w \cap \mathcal{T}([E_0 - \epsilon, E_0 + \epsilon]) = \emptyset$, then

$$\mathrm{Op}_h^w(w) u_\theta = \mathcal{O}\left(\sqrt{\frac{|\operatorname{Im} z|}{h}} + h^\infty \right) \|u_\theta\|. \tag{1.12}$$

Remark 1.1. For compactly supported perturbations of the Laplacian, this is a straightforward consequence of the estimate given in Proposition 3 of [30] and propagation of singularities given in Lemma 4.1 of [31]. As remarked by Stefanov, the same arguments can be adapted for long-range perturbations of the Laplacian in view of Sect. 8 of [31].

Remark 1.2. It seems also possible to obtain such type of results using the semi-classical measures introduced by P. Gérard [12] and Lions–Paul [19]. Assume that $\theta = o(h)$ and that $z \to E_0$ (as $h \to 0$) is a resonance with $|\operatorname{Im} z| \leq \epsilon\theta$. Let u_θ satisfying

$(P_\theta - z)u_\theta = 0$ and $\|u_\theta\| = 1$. Following the works of Burq [6] and Jecko [16], one can perhaps show that any semiclassical measure μ of the sequence $(u_\theta)_h$ verifies

$$\begin{cases} \operatorname{supp}\mu \subset \mathcal{T}(E_0), \\ \mathrm{H}_{p_0}\mu = 0. \end{cases} \tag{1.13}$$

Then it is enough to write $(P - z)u_\theta = (P - P_\theta)u_\theta$ with $\sigma\big(P - P_\theta\big) \in S_{2n}(\theta\langle\xi\rangle^2)$ and, as $\|u_\theta\|_{H^2} = \mathcal{O}(1)$, we deduce $(P - z)u_\theta = o(1)$ and one can apply the proof of Burq or Jecko.

Before we state our second result, let us introduce the following subspaces of the phase space. For $R > 0$, $\epsilon > 0$ and $\sigma \in [-1, 1]$, set

$$\Gamma_\pm(R, \epsilon, \sigma) = \{(x, \xi) \in \mathrm{T}^*(\mathbb{R}^n);\ |x| > R,\ |p_0(x, \xi) - E_0| < \epsilon \\ \text{and } \pm\langle x, \xi\rangle > \pm\sigma|x||\xi|\}.$$

We have the following theorem which says that a resonant state is outgoing.

Theorem 2. *Let $E_0 > 0$ and u_θ be a resonant state associated to a resonance z as in (1.11). We assume that* $\operatorname{Re} z \in [E_0 - \epsilon, E_0 + \epsilon]$, $|\operatorname{Im} z| < \epsilon\theta$ *and* $h/C < \theta < Ch\ln(1/h)$ *with* ϵ, $C > 0$. *Let $w(x, \xi) \in S_{2n}(1)$ and suppose that there exists $T > 0$ such that* $\exp(-T\mathrm{H}_{p_0})(\operatorname{supp}(w)) \subset \Gamma_-(R, \epsilon, \sigma)$ *with $R \gg 1$ and $\sigma < 0$. Then for $h > 0$ sufficiently small, one has*

$$\|w(x, hD_x)u_\theta\| = \mathcal{O}(h^\infty)\|u_\theta\|. \tag{1.14}$$

In particular $w \in C_0^\infty(\mathrm{T}^*(\mathbb{R}^n))$ such that $\operatorname{supp}(w) \subset \Gamma_+([E_0 - \epsilon, E_0 + \epsilon])^C = \overline{\bigcup_{E \in [E_0-\epsilon, E_0+\epsilon]} \Gamma_+(E)}^C$ satisfies the hypothesis of Theorem 2. Because, for each point $\rho \in \Gamma_+([E_0 - \epsilon, E_0 + \epsilon])^C$, $\exp(-t\mathrm{H}_{p_0})(\rho)$ is in a set $\Gamma_-(R, \epsilon, \sigma)$ if t is large enough.

Remark 1.3. It is possible to generalize this result to the black-box setting (see [28] and [27] for a precise formulation). Assume that the black-box is contained in $D(0, R_0)$, let $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi = 1$ near $D(0, R_0)$ and let w be supported in $\{|x| > R_0\}$ and satisfying the assumptions of the above theorem. If u_θ is a resonant state, then $\|w(x, hD_x)(1 - \chi(x))u_\theta\| = \mathcal{O}(h^\infty)\|u_\theta\|$.

Remark 1.4. Another possible generalization concerns the case of multiple resonances. Assume that z is a resonance whose multiplicity $N = N(h)$ is bounded uniformly with respect to h, then the conclusion of the theorem remains valid for all *generalized* resonant states (i.e. the functions $u_\theta \in L^2(\mathbb{R}^n)$ such that $(P_\theta - z)^N u_\theta = 0$). We will give the idea of the proof of this generalizations at the end of Sect. 3.

The plan of the paper is the following. In Sect. 2, we make precise the action of the FBI transform on pseudodifferential operators. In particular, we give the form of the term of order h in the expansion of the transformed symbol.

Section 3 is devoted to the proof of Theorem 2. The demonstration is based on the construction of a suitable escape function and an application of the result of Sect. 2. The main idea consists to choose a weight G which permits to gain ellipticity near 0, whereas the dilatation F gives ellipticity at infinity.

In Sect. 4, we prove Theorem 1 using again the results of Sect. 2.

Applying Theorem 2, we obtain in Sect. 5 an estimate of the residue of the scattering amplitude associated to a Schrödinger operator. We treat the case of resonances whose imaginary part is bounded by $\mathcal{O}(hln(1/h))$. This estimate involves the norm of the associated spectral projector on the space of resonant states.

In Sect. 6, we give some examples where the spectral projector above satisfies nice estimates. These bounds on the projector permit to show that the associated residue is $\mathcal{O}(h^{\infty})$ for some particular directions.

2. Microlocal Exponential Estimate

In this section, we give a microlocal exponential weighted estimate for C^{∞} symbols using a Fourier–Bros–Iagolnitzer (in short FBI) transform, widely studied by Sjöstrand [25]. The result is a slight modification of Proposition 3.1 of Martinez [21] (see also the book of Martinez [20] for a related presentation).

For $u \in \mathcal{S}'(\mathbb{R}^n)$, the FBI transform of u is given by

$$Tu(x,\xi;h) = \alpha_n(h) \int e^{i(x-y)\xi/h-(x-y)^2/2h} u(y)dy, \tag{2.1}$$

with $\alpha_n(h) = 2^{-n/2}(\pi h)^{-3n/4}$. As proved in [20], we know that $Tu \in \mathcal{C}^{\infty}(\mathbb{R}^{2n})$ and that $e^{\xi^2/2h}Tu(x,\xi;h)$ is an holomorphic function of $z = x - i\xi$. Moreover, if $u \in L^2(\mathbb{R}^n)$ then $\|Tu\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)}$.

Let A be a h-differential operator of Weyl symbol $a(x,\xi;h) \sim \sum_{j\geq 0} a_j(x,\xi)h^j \in S^{cl}_{2n}(\langle\xi\rangle^d)$. As a is polynomial with respect to ξ with coefficients in $S_n(1)$, one can find an almost analytic extension $\widetilde{a}(x,\xi;h) \in S^{cl}_{2n}(\langle\xi\rangle^d)$ of a in a $D_\epsilon \times \mathbb{C}^n$, where $D_\epsilon = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| < \epsilon\}$, which satisfies

$$\widetilde{a}_{|_{\mathbb{R}^{2n}}} = a, \tag{2.2}$$

$$\partial_{\overline{x}}\widetilde{a} = \mathcal{O}(|\operatorname{Im} x|^{\infty})\langle\xi\rangle^d. \tag{2.3}$$

Theorem 3 (Martinez). *Let $f(x,\xi) \in S_{2n}(1)$ and $G(x,\xi) \in C_0^{\infty}(\mathbb{R}^{2n})$. Then there exists a symbol $q(x,\xi;t,h) \sim \sum_{j\geq 0} q_j(x,\xi;t)h^j \in S^{cl}_{2n}(\langle\xi\rangle^d)$ uniformly with respect to t and an operator $R(t,h)$ such that for all $u,\ v \in C_0^{\infty}(\mathbb{R}^n)$, one has*

$$\begin{aligned}&\big\langle f e^{-tG/h} T \operatorname{Op}_h^w(a)u, e^{-tG/h}Tv\big\rangle_{L^2(\mathbb{R}^{2n})}\\ &\quad = \big\langle\big(q(x,\xi;t,h) + R(t,h)\big)e^{-tG/h}Tu, e^{-tG/h}Tv\big\rangle_{L^2(\mathbb{R}^{2n})},\end{aligned} \tag{2.4}$$

where $\operatorname{supp} q_j \subset \operatorname{supp} f$ *for all $j \in \mathbb{N}$. Here, we have with the notation $\partial_z = (\partial_x + i\partial_\xi)/2$,*

$$q_0(x,\xi;t) = f(x,\xi)\widetilde{a_0}\big(x + 2t\partial zG(x,\xi), \xi - 2it\partial zG(x,\xi)\big), \tag{2.5}$$

$$\begin{aligned}q_1(x,\xi;t) =& \Big(fa_1 - f\partial^2_{xx}a_0/4 - f\partial^2_{\xi\xi}a_0/4 - \partial_x f\partial_x a_0/2 - \partial_\xi f\partial_\xi a_0/2\Big)(x,\xi)\\ &+ \frac{i}{2}\Big(\partial_\xi a_0\partial_x f - \partial_x a_0\partial_\xi f\Big)(x,\xi) + \mathcal{O}(t),\end{aligned} \tag{2.6}$$

and

$$\big\|\langle\xi\rangle^{\sigma}R(t,h)\langle\xi\rangle^{-d-\sigma}\big\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \mathcal{O}(h^{\infty} + h^{-3n/2}|t|^{\infty}e^{2\sup|G||t|/h}), \tag{2.7}$$

for all $\sigma \in \mathbb{R}$, uniformly with respect to t and h small enough.

Remark 2.1. Theorem 3 holds also for $A = \mathrm{Op}_h^w(a)$ with $a(x,\xi;h) \in S^{cl}_{2n}(1)$, since one can find an almost analytic extension of a which satisfies (2.2) and (2.3) with $d = 0$.

Proof. This theorem is a slight adaptation of Proposition 3.1 of Martinez [20] and we follow his proof.

Let

$$\begin{aligned} T_0 &= \big\langle f e^{-tG/h} T A u, e^{-tG/h} T v \big\rangle \\ &= \frac{\alpha_n(h)^2}{(2\pi h)^n} \int e^{\Phi/h} f(x,\xi) a\Big(\frac{y+z}{2}, \eta; h\Big) u(z) \overline{v(y')}\, dx\, d\xi\, dy\, d\eta\, dy', \end{aligned} \tag{2.8}$$

where

$$\Phi = -2tG(x,\xi) + i(x-y)\xi - i(x-y')\xi - (x-y)^2/2 - (x-y')^2/2 + i(y-z)\eta. \tag{2.9}$$

We have, for $Y = (y,\eta) \in \mathbb{R}^{2n}$ and $X = \big(x + 2t\partial z G(x,\xi), \xi - 2i\partial z G(x,\xi)\big) \in \mathbb{C}^{2n}$,

$$\begin{aligned} & a(Y;h) - \widetilde{a}(X;h) \\ &\quad = \int_0^1 \Big((Y - \mathrm{Re}\, X) \frac{\partial \widetilde{a}}{\partial \mathrm{Re}\, X}\big(sY + (1-s)X\big) - \mathrm{Im}\, X \frac{\partial \widetilde{a}}{\partial \mathrm{Im}\, X}\big(sY + (1-s)X\big)\Big)\, ds \\ &\quad = \int_0^1 \Big((Y - X) \frac{\partial \widetilde{a}}{\partial \mathrm{Re}\, X}\big(sY + (1-s)X\big) + 2i\mathrm{Im}\, X \frac{\partial \widetilde{a}}{\partial \overline{X}}\big(sY + (1-s)X\big)\Big)\, ds \\ &\quad = (Y-X) b(x,\xi,y,\eta;t,h) + r(x,\xi,y,\eta;t,h), \end{aligned} \tag{2.10}$$

with $b \in S^{cl}_{4n}(\langle \xi,\eta\rangle^2)$ and $r \in S_{4n}(|t|^\infty \langle \xi,\eta\rangle^2)$ uniformly with respect to t. In addition

$$b_0(x,\xi,y,\eta;t) = \int_0^1 (\partial_x a_0, \partial_\xi a_0)\big(sy + (1-s)x, s\eta + (1-s)\xi\big)\, ds + \mathcal{O}(t\langle \xi,\eta\rangle^d). \tag{2.11}$$

So, we have

$$T_0 = \big\langle f(x,\xi)\widetilde{a}(X;h) e^{-tG/h} T u, e^{-tG/h} T v\big\rangle + T_1 + R_1$$

with

$$\begin{aligned} T_1 &= \frac{\alpha_n(h)^2}{(2\pi h)^n} \int e^{\Phi/h} \big(((y+z)/2, \eta) - X\big) f(x,\xi) b\Big(x,\xi,\frac{y+z}{2},\eta\Big) \\ &\quad \times u(z)\overline{v(y')}\, dx\, d\xi\, dy\, d\eta\, dy', \end{aligned} \tag{2.12}$$

$$R_1 = \frac{\alpha_n(h)^2}{(2\pi h)^n} \int e^{\Phi/h} f(x,\xi) r(x,\xi,(y+z)/2,\eta) u(z)\overline{v(y')}\, dx\, d\xi\, dy\, d\eta\, dy'. \tag{2.13}$$

We have

$$\begin{aligned} \big(\partial_x \Phi + i\partial_\xi \Phi + i\partial_\eta \Phi\big)/2 &= (y+z)/2 - x - 2t\partial z G(x,\xi), \\ -i\partial_y \Phi - i\big(\partial_x \Phi + i\partial_\xi \Phi\big)/2 &= \eta - \xi + 2it\partial z G(x,\xi). \end{aligned}$$

Thus there exists a constant vector-field $L(\partial_x, \partial_\xi, \partial_y, \partial_\eta)$ such that $L(\Phi) = \big(((y+z)/2, \eta) - X\big)$. Making an integration by part with L in (2.12) and using

$$a^1(x,\xi,(y+z)/2,\eta;t,h) = {}^t L\big(f(x,\xi)b(x,\xi,(y+z)/2,\eta;t,h)\big),$$

which satisfies $a^1 \in S^{cl}_{4n}(\langle \xi,\eta\rangle^2)$, $\operatorname{supp}_{(x,\xi)} a^1 \subset \operatorname{supp} f$ and

$$\begin{aligned} a^1_0(x,\xi,y,\eta;t) = \big(&-\partial_x/2 - i\partial_\xi/2 - i\partial_\eta/2, i\partial_y/2 + i\partial_x/2 - \partial_\xi/2\big) \\ &.\big(f(x,\xi)b_0(x,\xi,y,\eta;t)\big), \end{aligned} \tag{2.14}$$

we get

$$T_1 = h\big\langle e^{-tG/h}T_{a^1}u, e^{-tG/h}Tv\big\rangle, \tag{2.15}$$

with

$$T_{a^1}u(x,\xi) = \alpha_n(h)\int e^{i(x-y)\xi/h-(x-y)^2/2h}\operatorname{Op}^w_h(a^1(x,\xi,.,.;t,h))u(y)\,dy. \tag{2.16}$$

We repeat the same work for T_1 as this done for T_0 and, by induction, we can find, for $j=0,1,\dots,N$, symbols $q_j(x,\xi;t) \in S_{2n}(1)$ uniformly with respect to t. Moreover, $\operatorname{supp} q_j \subset \operatorname{supp} f$ and

$$q_0(x,\xi;t) = f(x,\xi)a_0\big(x+2t\partial zG(x,\xi), \xi - 2it\partial zG(x,\xi)\big), \tag{2.17}$$

$$\begin{aligned} q_1(x,\xi;t) &= f(x,\xi)a_1\big(x+2t\partial zG(x,\xi), \xi - 2it\partial zG(x,\xi)\big) \\ &\quad + a^1_0\big(x,\xi,x+2t\partial zG(x,\xi), \xi - 2it\partial zG(x,\xi);t\big) \\ &= \Big(fa_1 - f\partial^2_{xx}a_0/4 - f\partial^2_{\xi\xi}a_0/4 - \partial_x f\partial_x a_0/2 - \partial_\xi f\partial_\xi a_0/2\Big)(x,\xi) \\ &\quad + \frac{i}{2}\Big(\partial_\xi a_0\partial_x f - \partial_x a_0\partial_\xi f\Big)(x,\xi) + \mathcal{O}(t) \end{aligned} \tag{2.18}$$

such that, for each $N \in \mathbb{N}$,

$$T_0 = \Big\langle \sum_{j=0}^{N-1} q_j(x,\xi;t)h^j e^{tG/h}Tu, e^{tG/h}Tv\Big\rangle + h^N\big\langle e^{-tG/h}T_{a^N}u, e^{-tG/h}Tv\big\rangle + R_N, \tag{2.19}$$

where a^N and R_N satisfy the same properties as a^1 and R_1. Using what T is an isometry on $L^2(\mathbb{R}^{2n})$, we write

$$\begin{aligned} T_{a^N} &= \big\langle e^{-tG/h}T_{a^N}T^*Tu, e^{-tG/h}Tv\big\rangle, \\ R_N &= \big\langle e^{-tG/h}T_{r_N}T^*Tu, e^{-tG/h}Tv\big\rangle, \end{aligned} \tag{2.20}$$

where $a^N, r_N \in S_{4n}(\langle\xi,\eta\rangle^2)$ have their support inside $\operatorname{supp} f$. Applying Lemma 6.1 of [21] with $t=0$ and $l=0$, we get

$$\begin{aligned} \|\langle\xi\rangle^\sigma T_{p^N}T^*\langle\xi\rangle^{-d-\sigma}\| &= \mathcal{O}(1), \\ \|\langle\xi\rangle^\sigma T_{r_N}T^*\langle\xi\rangle^{-d-\sigma}\| &= \mathcal{O}(|t|^\infty), \end{aligned} \tag{2.21}$$

which give Theorem 3 since $|e^{-tG/h}| \le e^{\sup|G||t|/h}$. $\square$

3. Proof of Theorem 2

We begin the proof with some geometric results.

Lemma 3.1 (C. Gérard–Sjöstrand). *Assume that* $K \subset p_0^{-1}([E_0-\epsilon, E_0+\epsilon])$ *is compact and satisfies* $K \cap \mathcal{T}([E_0-\epsilon, E_0+\epsilon]) = \emptyset$*. Then, one can find a function* $f(x,\xi) \in \mathcal{C}_b^\infty(\mathrm{T}^*(\mathbb{R}^n))$ *such that* $\mathrm{H}_{p_0} f \geq 0$ *on* $p_0^{-1}([E_0-\epsilon, E_0+\epsilon])$ *and* $\mathrm{H}_{p_0} f > 1$ *on* K.

Proof. We follow the proof of Proposition A.6 of C. Gérard and Sjöstrand [11]. We give the proof for a reason of completeness and we use their notation. Let $H_T = \{(x,\xi) \in \mathrm{T}^*(\mathbb{R}^n);\ p_0(x,\xi) \in [E_0-\epsilon, E_0+\epsilon]$ and $x.\xi = T\}$ with T large enough. Let $\widetilde{T} > 0$ and $0 < f_+ \in C^\infty\big(p_0^{-1}([E_0-\epsilon, E_0+\epsilon]) \backslash \Gamma_-([E_0-\epsilon, E_0+\epsilon])\big)$ be equal to $\chi(x.\xi)\mathrm{H}_{p_0}(x.\xi)$ outside a compact with $\chi \in C_0^\infty([-T-\widetilde{T}-1, T+\widetilde{T}+1]; [0,1])$ equal to 1 near $[-T-\widetilde{T}, T+\widetilde{T}]$. As in [11], we can solve $\mathrm{H}_{p_0}(G_+) = f_+$ in $p_0^{-1}([E_0-\epsilon, E_0+\epsilon]) \backslash \Gamma_-([E_0-\epsilon, E_0+\epsilon])$ with $G_+ = T$ on H_T. We have $G_+ \leq T+\widetilde{T}+1$ and, if f_+ is large enough in a compact,

$$\limsup_{\Gamma_- \cap H_{-T}} G_+ \leq -T. \tag{3.1}$$

We construct G_- with analogous properties.

Let $\chi_\pm \in C^\infty(\mathbb{R}; \mathbb{R}^\pm)$ with $\operatorname{supp} \chi_\pm \subset [\mp T, \pm\infty[$ and with $\chi_+(t) + \chi_-(t) = t$. Put $\widetilde{G} = \chi_+(G_+) + \chi_-(G_-)$. By (3.1), we have, near $p_0^{-1}([E_0-\epsilon, E_0+\epsilon])$, $\widetilde{G} \in C_b^\infty$, $\mathrm{H}_{p_0}(\widetilde{G}) \geq 0$ and $\mathrm{H}_{p_0}(\widetilde{G}) > c > 0$ for $(x,\xi) \in \{-T-\widetilde{T} < x.\xi < -T\} \cup \{T < x.\xi < T+\widetilde{T}\}$.

As K is compact and $K \cap \mathcal{T}([E_0-\epsilon, E_0+\epsilon]) = \emptyset$, there is $s > 0$ and $\widetilde{T} > 0$ such that $(x,\xi) \in K$ implies $\exp(s\mathrm{H}_p)(x,\xi) \in \{T < x.\xi < T+\widetilde{T}\}$ or $\exp(-s\mathrm{H}_p)(x,\xi) \in \{-T-\widetilde{T} < x.\xi < -T\}$. Then, we can take $f \in C^\infty(\mathrm{T}^*(\mathbb{R}^n))$ with $f(x,\xi) = \widetilde{G}(\exp(s\mathrm{H}_p)(x,\xi))/c + \widetilde{G}(\exp(-s\mathrm{H}_p)(x,\xi))/c$ near $p^{-1}([E_0-\epsilon, E_0+\epsilon])$. □

Let $\sigma \in]-1, 0[$ and $\alpha \in C_0^\infty(\mathbb{R}; [0,1])$ be a decreasing function such that $\alpha(x) = 1$ if $x < \sigma$ and $\alpha(x) = 0$ if $x > \sigma/2$. We define

$$\widetilde{w}(x,\xi) = \rho(|x|) f(p_0(x,\xi)) \alpha\Big(\frac{x.\xi}{|x||\xi|}\Big), \tag{3.2}$$

where $f \in C_0^\infty([E_0-\epsilon, E_0+\epsilon])$ and $\rho \in C^\infty(\mathbb{R}; [0,1])$ is growing with $\rho(x) = 1$ for $x > R$ and $\rho(x) = 0$ for $x < R-1$. It is obvious that $\widetilde{w} \in S_{2n}(1)$. We have the following lemma for $\widetilde{w}$, that will be useful later.

Lemma 3.2. *For* $\sigma > 0$ *small enough and* R *large enough we have* $H_{p_0}\widetilde{w} \leq 0$.

Proof. Using (1.10), one can show that, for $p_0(x,\xi) \in [E_0-\epsilon, E_0+\epsilon]$,

$$\mathrm{H}_{p_0} = \begin{pmatrix} 2\xi + o(1) \\ o(1/x) \end{pmatrix}, \tag{3.3}$$

where $o(1)$ is a function which tends to 0 as $|x| \to +\infty$. Then

$$\mathrm{H}_{p_0}\widetilde{w}(x,\xi) = f(p_0(x,\xi))\rho(|x|)\mathrm{H}_{p_0}\alpha\Big(\frac{x.\xi}{|x||\xi|}\Big) + f(p(x,\xi))\alpha\Big(\frac{x.\xi}{|x||\xi|}\Big)\mathrm{H}_{p_0}\rho(|x|)$$

$$|\xi|^2 \geq E_0 - \epsilon + o(1).$$

For $(x, \xi) \in \operatorname{supp}(\mathrm{H}_{p_0} \widetilde{w})$, we have

$$\mathrm{H}_{p_0} \alpha\Big(\frac{x.\xi}{|x||\xi|}\Big) = \alpha'\Big(\frac{x.\xi}{|x|}\Big)\Big(\frac{2|\xi|^2}{|x|} - \frac{2(x.\xi)^2}{|x|^3} + \frac{o(1)}{|x|}\Big)$$
$$\leq \frac{1}{|x|}\alpha'\Big(\frac{x.\xi}{|x||\xi|}\Big)\Big(2E_0 - 2\epsilon + o(1) - 2\sigma^2\Big),$$

and, on the other hand,

$$\mathrm{H}_{p_0} \rho(|x|) = \rho'(|x|)\Big(\frac{x.\xi}{|x|} + o(1)\Big)$$
$$\leq \rho'(|x|)(\sigma/2 + o(1)).$$

If we fix $\sigma > 0$ small enough and after R large enough, we have $\mathrm{H}_{p_0} \widetilde{w} \leq 0$. □

Now, we can begin the proof of Theorem 2. Consider $z \in \mathbb{C}$, $u_\theta \in H^2(\mathbb{R}^n)$ and w as in Theorem 2 such that $(P_\theta - z)u_\theta = 0$. For $N \in \mathbb{N}$, let $\tilde{w}_j(x, \xi)$, $j = 1, \dots, N, \infty$ be of the form (3.2) with $w \prec w_1 \prec \cdots \prec w_N \prec w_\infty$, where the ω_j are defined by $\omega_j(x, \xi) = \tilde{\omega}_1(\exp(t\mathrm{H}_{p_0})(x, \xi))$. Here, the notation $g_1 \prec g_2$ means that $g_2 = 1$ near the support of g_1 and one can easily see that $\mathrm{H}_{p_0}\omega_j \leq 0, \forall j$. Denoting by $\langle ., . \rangle$ the scalar product on $L^2(\mathbb{R}^{2n})$, we have

$$0 = \big\langle w_1^2(x, \xi)e^{-tG/h}T(P_\theta - z)u_\theta, e^{-tG/h}Tu_\theta\big\rangle. \tag{3.4}$$

We can apply Theorem 3 with P_θ to get

$$\big\langle (q_\theta(x, \xi; t, h) - z)e^{-tG/h}Tu_\theta, e^{-tG/h}Tu_\theta\big\rangle$$
$$= \mathcal{O}\big((h^\infty + t^\infty h^{-3n/2})e^{\sup |G||t|/h}\big)\big\|e^{-tG/h}Tu_\theta\big\|^2, \tag{3.5}$$

with

$$q_\theta(x, \xi; t, h) = q_{\theta,0}(x, \xi; t) + hq_{\theta,1}(x, \xi; 0) + (h|t| + h^2)r_\theta(x, \xi; t, h), \tag{3.6}$$

where $q_{\theta,0}$ and $q_{\theta,1}$ are given by (2.5) and (2.6) and $r_\theta \in S_{2n}(1)$ uniformly with respect to t, θ and $\operatorname{supp} r_\theta \subset \operatorname{supp} w_1$.

Lemma 3.3. *We have the following expansions*

$$\operatorname{Im} q_{\theta,0}(x, \xi; t) = -w_1^2(x, \xi)\mathrm{H}_{p_0}(tG(x, \xi) + \theta F(x)\xi) + w_1^2(x, \xi)\mathcal{O}(\theta^2 + t^2), \tag{3.7}$$

and

$$\operatorname{Im} q_{\theta,1}(x, \xi; 0) = \frac{1}{2}\mathrm{H}_{p_0} w_1^2(x, \xi) + w_2^2(x, \xi)\mathcal{O}(\theta). \tag{3.8}$$

Proof. First, we recall that P being formally self-adjoint, the symbols p_0 and p_1 are real valued. We will denote by $p_\theta(x, \xi; h)$ the symbol of P_θ, and by definition, we have

$$p_{\theta,0}(x, \xi) = p_0\big(x + i\theta F(x), (1 + i\theta\partial_x F(x))^{-1}\xi\big). \tag{3.9}$$

Notice that one has

$$\begin{aligned} p_{\theta,0}(x,\xi) &= p_0(x+i\theta F(x), \xi - i\theta\partial_x F(x)\xi) + \mathcal{O}(\theta^2)\langle\xi\rangle^2 \\ &= p_0(x,\xi) + i\theta\big(F(x)\partial_x p_0(x,\xi) - \partial_x F(x)\xi\partial_\xi p_0(x,\xi)\big) + \mathcal{O}(\theta^2)\langle\xi\rangle^2 \\ &= p_0(x,\xi) - i\theta \mathrm{H}_{p_0}(F(x)\xi) + \mathcal{O}(\theta^2)\langle\xi\rangle^2. \end{aligned} \tag{3.10}$$

Combining Eqs. (3.10) and (2.5) one gets

$$\begin{aligned} q_{\theta,0}(x,\xi;t) =& w_1^2(x,\xi)\widetilde{p_{\theta,0}}\big(x+2t\partial_z G(x,\xi), \xi - 2it\partial_z G(x,\xi)\big) \\ =& w_1^2(x,\xi)\Big(\widetilde{p_0}\big(x+2t\partial_z G(x,\xi), \xi - 2it\partial_z G(x,\xi)\big) \\ & - i\theta \mathrm{H}_{p_0}(F(x)\xi)\big(x+2t\partial_z G(x,\xi), \xi - 2it\partial_z G(x,\xi)\big) + \mathcal{O}(\theta^2)\Big). \end{aligned}$$

By Taylor expansion, we obtain

$$\begin{aligned} q_{\theta,0}(x,\xi;t) = w_1^2(x,\xi)\Big(& p_0(x,\xi) + t\big(\partial_x p_0 \partial_x G + \partial_\xi p_0 \partial_\xi G\big)(x,\xi) \\ & - it\mathrm{H}_{p_0}G(x,\xi) - i\theta\mathrm{H}_{p_0}(F(x)\xi)(x,\xi) + \mathcal{O}(\theta^2+t^2)\Big). \end{aligned} \tag{3.11}$$

Taking the imaginary part, we obtain the announced expansion for $q_{\theta,0}$. Now, let us prove the formula on $q_{\theta,1}$. From formula (2.6), we know that

$$\begin{aligned} q_{\theta,1}(x,\xi;0) = \big(& w_1^2 p_{\theta,1} - w_1^2\partial_{xx}^2 p_{\theta,0}/4 - w_1^2\partial_{\xi\xi}^2 p_{\theta,0}/4 - \partial_x w_1^2 \partial_x p_{\theta,0}/2 \\ & - \partial_\xi w_1^2 \partial_\xi p_{\theta,0}/2\big)(x,\xi) + \frac{i}{2}\mathrm{H}_{p_{\theta,0}} w_1^2(x,\xi). \end{aligned} \tag{3.12}$$

By Taylor expansion, we get

$$p_{\theta,1}(x,\xi) = p_1(x,\xi) + \mathcal{O}(\theta\langle\xi\rangle^2),$$

and

$$\mathrm{H}_{p_{\theta,0}} w_1^2(x,\xi) = \mathrm{H}_{p_0} w_1^2(x,\xi) + \mathcal{O}(\theta) w_2^2(x,\xi).$$

The symbols p_1, p_0 and w_1 being real valued, the result comes directly by taking the imaginary part of (3.12). □

As $x\xi$ is an escape function and $F(x) = x$ for x large enough, we have, on $p_0^{-1}([E_0-\epsilon, E_0+\epsilon])$,

$$\mathrm{H}_{p_0}(F(x)\xi) \geq \begin{cases} c > 0 & \text{for } |x| \geq R, \\ -M & \text{for } |x| \leq R, \end{cases} \tag{3.13}$$

with $R > 0$ large enough. We fix $K = \operatorname{supp} w_\infty \cap \overline{B(0,R_0)} \subset p_0^{-1}([E_0-\epsilon, E_0+\epsilon]) \cap \mathcal{T}([E_0-\epsilon, E_0+\epsilon])^c$ and we denote $f(x,\xi)$ the function given by Lemma 3.1. Let $\chi_1 \in C_0^\infty(\mathbb{R}^n, [0;1])$ such that $\chi_2(x) = 1$ for $|x| \leq R+1$ and $\chi_2 \in C_0^\infty(\mathbb{R};[0,1])$ with $\chi_2(E) = 1$ on $[E_0-\epsilon, E_0+\epsilon]$). As in [21], we set

$$G(x,\xi) = \chi_1(x)\chi_2(p_0(x,\xi)) f(x,\xi) \in C_0^\infty(\mathbb{R}^{2n}). \tag{3.14}$$

Since χ_1 can be chosen arbitrarily flat, the quantities

$$\mu = \sup |\chi_2(p_0(x,\xi)) f(x,\xi) H_{p_0}\chi_1(x)|, \tag{3.15}$$

can be chosen arbitrarily small. We take $t = L\theta$, and we have, on the support on w_∞,

$$\mathrm{H}_{p_0}(tG(x,\xi) + \theta F(x)\xi) \geq \theta \begin{cases} -L\mu + c & \text{for } |x| \geq R_0, \\ L - L\mu - M & \text{for } |x| \leq R_0. \end{cases} \tag{3.16}$$

We fix $L \geq 2M$ and μ small enough so that (3.16) becomes

$$\mathrm{H}_{p_0}(tG(x,\xi) + \theta F(x)\xi) \geq \theta c/2, \tag{3.17}$$

on supp w_∞.

Equations (3.6), (3.17) and Lemma 3.3 imply

$$\begin{aligned} -\operatorname{Im} \big\langle (q_\theta(x,\xi;t,h) - z) e^{-tG/h} T u_\theta, e^{-tG/h} T u_\theta \big\rangle \\ \geq \theta c/4 \big\| w_1 e^{-tG/h} T u_\theta \big\|^2 - h \big\langle w_1 \mathrm{H}_{p_0} w_1 e^{-tG/h} T u_\theta, e^{-tG/h} T u_\theta \big\rangle \\ + \mathcal{O}(\theta^2 + h^2) \big\| w_2 e^{-tG/h} T u_\theta \big\|^2. \end{aligned} \tag{3.18}$$

Since $w_1 \mathrm{H}_{p_0} w_1 \leq 0$, by Lemma 3.2,

$$\begin{aligned} -\operatorname{Im} \big\langle (q_\theta(x,\xi;t,h) - z) e^{-tG/h} T u_\theta, e^{-tG/h} T u_\theta \big\rangle \\ \geq \theta c/4 \big\| w_1 e^{-tG/h} T u_\theta \big\|^2 + \mathcal{O}(\theta^2 + h^2) \big\| w_2 e^{-tG/h} T u_\theta \big\|^2. \end{aligned} \tag{3.19}$$

Using (3.5), we get

$$\big\| w_1 e^{-tG/h} T u_\theta \big\|^2 \leq \mathcal{O}(\theta) \big\| w_2 e^{-tG/h} T u_\theta \big\|^2 + \mathcal{O}(h^\infty) \big\| T u_\theta \big\|^2,$$

and by induction,

$$\begin{aligned} \big\| w_1 e^{-tG/h} T u_\theta \big\|^2 &\leq \mathcal{O}(\theta^{N-1}) \big\| w_N e^{-tG/h} T u_\theta \big\|^2 + \mathcal{O}(h^\infty) \big\| T u_\theta \big\|^2 \\ &\leq \mathcal{O}\big(\theta^{N-1} h^{-2CL \sup |G|}\big) \big\| u_\theta \big\|^2, \end{aligned} \tag{3.20}$$

which implies

$$\big\| w_1 T u_\theta \big\| = \mathcal{O}(h^\infty) \big\| u_\theta \big\|. \tag{3.21}$$

Now, choose $w_0 \in S_{2n}(1)$ such that $w \prec w_0 \prec w_1$. One can write

$$\| T \operatorname{Op}_h^w(w) u_\theta \| \leq \| w_0 T \operatorname{Op}_h^w(w) u_\theta \| + \| (1 - w_0) T \operatorname{Op}_h^w(w) u_\theta \|.$$

Using two times Theorem 3 with $A = \operatorname{Op}_h^w(w)$ and inequality (3.21), we have

$$\begin{aligned} \| w_0 T \operatorname{Op}_h^w(w) u_\theta \|^2 &= \langle w_0 T \operatorname{Op}_h^w(w) u_\theta, w_0 T \operatorname{Op}_h^w(w) u_\theta \rangle \\ &= \langle q(x,\xi;h) T u_\theta, T u_\theta \rangle + \mathcal{O}(h^\infty) \| u_\theta \|^2 \\ &= \mathcal{O}(h^\infty) \| u_\theta \|^2, \end{aligned} \tag{3.22}$$

since $q(x,\xi;h)\in S_{2n}(1)$ satisfy $q\prec w_1$. On the other hand,

$$\begin{aligned}(1-w_0)T\operatorname{Op}_h^w(w)u_\theta(x,\xi)\\ =\frac{\alpha_n(h)}{(2\pi h)^n}\int e^{\Phi/h}(1-w_0)(x,\xi)w\big((y+z)/2,\eta;h\big)u_\theta(z)\,dz\,dy\,d\eta,\end{aligned}\tag{3.23}$$

with

$$\Phi(x,\xi,y,\eta,z)=i(x-y)\xi-(x-y)^2/2+i(y-z)\eta.\tag{3.24}$$

We notice that $\partial_y\Phi-i/2\partial_\eta\Phi=i(\eta-\xi)+(x-(y+z)/2)$. So, making integrations by parts in (3.23) with

$$L=\frac{(x-(y+z)/2)-i(\eta-\xi)}{(x-(y+z)/2)^2+(\eta-\xi)^2}(\partial_y-i/2\partial_\eta),\tag{3.25}$$

and using the fact that $\operatorname{supp}w\cap\operatorname{supp}(1-w_0)=\emptyset$, we find, for each $N\in\mathbb{N}$,

$$\begin{aligned}(1-w_0)T\operatorname{Op}_h^w(w)u_\theta&=h^N\frac{\alpha_n(h)}{(2\pi h)^n}\int e^{\Phi/h}s_N\big(x,\xi,(y+z)/2,\eta;h\big)u_\theta(z)\,dx\,d\xi\,dy\,d\eta\\&=h^NT_{s_N}T^*Tu_\theta,\end{aligned}$$

with $s_N\in S_{4n}(1)$. So (2.21) implies $(1-w_0)T\operatorname{Op}_h^w(w)u_\theta=\mathcal{O}(h^\infty)\|u_\theta\|$ and we get

$$\operatorname{Op}_h^w(w)u_\theta=\mathcal{O}(h^\infty)\|u_\theta\|,\tag{3.26}$$

which gives Theorem 2. □

Let us explain briefly how to generalize Theorem 2 to the black-box setting and to multiple resonances. Assume that χ, w and u_θ are as in Remark 1.3. We have $(P_\theta-z)(1-\chi)=(Q_\theta-z)(1-\chi)$, where Q_θ is a differential operator satisfying the assumptions of Theorem 2. Following the proof above we get

$$\begin{aligned}&\big|\operatorname{Im}\big\langle w_1(x,\xi)^2e^{-tG/h}T(Q_\theta-z)(1-\chi)u_\theta,e^{-tG/h}T(1-\chi)u_\theta\big\rangle\big|\\&\quad\geq C\theta\big\|w_1e^{-tG/h}T(1-\chi)u_\theta\big\|^2-\mathcal{O}(\theta^2)\big\|w_2e^{-tG/h}T(1-\chi)u_\theta\big\|^2.\end{aligned}$$

On the other hand, we can always assume that $\operatorname{supp}\chi\cap\operatorname{supp}_x w_1=\emptyset$ and one deduces from Theorem 3 that

$$\begin{aligned}&\big\langle w_1(x,\xi)^2e^{-tG/h}T(Q_\theta-z)(1-\chi)u_\theta,e^{-tG/h}T(1-\chi)u_\theta\big\rangle\\&\quad=\big\langle w_1(x,\xi)^2e^{-tG/h}T[Q_\theta,\chi]u_\theta,e^{-tG/h}T(1-\chi)u_\theta\big\rangle=\mathcal{O}(h^\infty)\|u_\theta\|.\end{aligned}$$

It follows that $\|w_1(x,\xi)T(1-\chi)u_\theta\|=\mathcal{O}(h^\infty)\|u_\theta\|+\mathcal{O}(h)\|w_2(x,\xi)T(1-\chi)u_\theta\|$ and working as in the proof of Theorem 2 we show that $\|w(x,hD_x)(1-\chi)u_\theta\|=\mathcal{O}(h^\infty)\|u_\theta\|$. □

Now, as in Remark 1.4, assume that z_0 is a resonance whose multiplicity $N=N(h)$ is bounded uniformly with respect to h and that u_θ is a generalized resonant state associated to z_0. By definition, $(P_\theta-z_0)^Nu_\theta=0$ and we deduce from Theorem 2, that

$$\|w_{N-1}(x,hD_x)(P_\theta-z)^{N-1}u_\theta\|=\mathcal{O}(h^\infty)\|(P_\theta-z)^{N-1}u_\theta\|=\mathcal{O}(h^\infty)\|u_\theta\|.\tag{3.27}$$

Using this estimate and the proof of Remark 1.3, we obtain

$$\| w_{N-2}(x, hD_x)(P_\theta - z)^{N-2} u_\theta \| = \mathcal{O}(h^\infty) \| u_\theta \|,$$

and repeating this argument $N-2$ times (here we use that N is bounded with respect to h), we deduce that if w satisfies the assumptions of Theorem 2 and u_θ is a generalized resonant state associated to z_0, then $\| w(x, hD_x) u_\theta \| = \mathcal{O}(h^\infty) \| u_\theta \|$. □

4. Proof of Theorem 1

The proof uses essentially the same arguments as in the proof of Theorem 2.

For $N \in \mathbb{N}$, let $w \prec w_0 \prec \cdots \prec w_N \prec w_\infty \in S_{2n}(1)$ with $\operatorname{supp} w_\infty \cap \Gamma = \emptyset$ and let $g_0 \prec \cdots \prec g_N \prec g_\infty \in C_0^\infty([E_0 - 3\epsilon, E_0 + 3\epsilon])$ with $g_0 = 1$ near $[E_0 - 2\epsilon + E_0 + 2\epsilon]$. Applying Theorem 3 with $f = g_0^2(p_0(x,\xi))$ and $t = \widetilde{C}h$, we get

$$\begin{aligned} 0 &= \operatorname{Im} \big\langle g_0(p_0(x,\xi)) e^{-tG/h} T(P_\theta - z) u_\theta, e^{-tG/h} T u_\theta \big\rangle \\ &= \operatorname{Im} \big\langle \big(q_\theta(x,\xi;\widetilde{C}h, h) + \mathcal{O}(h^\infty) - z \big) e^{-tG/h} T u_\theta, e^{-tG/h} T u_\theta \big\rangle, \end{aligned} \tag{4.1}$$

with

$$q_\theta(x,\xi;\widetilde{C}h, h) = \sum_{j=0}^{\infty} q_{\theta,j}(x,\xi;\widetilde{C}h) h^j.$$

Following Lemma 3.3, one can choose $G(x,\xi) \in C_0^\infty(\operatorname{supp} w_2)$ as in (3.14) such that:

$$\operatorname{Im} q_{\theta,0}(x,\xi;\widetilde{C}h) \leq -g_0^2(p_0) w_0^2 h + \mathcal{O}(h^2) g_1^2(p_0) w_1^2. \tag{4.2}$$

Using Remark 2.1 and the fact that $\operatorname{Im} q_j(x,\xi;0) = 0$, we get that

$$\operatorname{Im} q_{\theta,j}(x,\xi;\widetilde{C}h) = \mathcal{O}(h^2) g_1^2(p_0) w_1^2 \tag{4.3}$$

for $j \geq 1$. So (4.1) implies

$$\begin{aligned} \| g_0(p_0) w_0 e^{-tG/h} T u_\theta \|^2 &= \mathcal{O}(h) \| g_1(p_0) w_1 e^{-tG/h} T u_\theta \|^2 \\ &\quad + \mathcal{O}\left(\frac{|\operatorname{Im} z|}{h} + h^\infty \right) \| e^{-tG/h} T u_\theta \|^2. \end{aligned} \tag{4.4}$$

By induction,

$$\begin{aligned} \| g_0(p_0) w_0 e^{-tG/h} T u_\theta \|^2 &= \mathcal{O}(h^N) \| g_N(p_0) w_N e^{-tG/h} T u_\theta \|^2 \\ &\quad + \mathcal{O}\left(\frac{|\operatorname{Im} z|}{h} + h^\infty \right) \| e^{-tG/h} T u_\theta \|^2, \end{aligned}$$

and since $e^{-tG/h} = \mathcal{O}(1)$ and $e^{tG/h} = \mathcal{O}(1)$,

$$\| g_0(p_0) w_0 T u_\theta \|^2 = \mathcal{O}\left(\frac{|\operatorname{Im} z|}{h} + h^\infty \right) \| T u_\theta \|^2. \tag{4.5}$$

Now, let $g_{-\infty} \prec g_{-N} \prec \cdots \prec g_{-1} \prec g_0 \in C_0^\infty(\mathbb{R})$ with $g_{-\infty} = 1$ on $[E_0 - \epsilon, E_0 + \epsilon]$. Applying Theorem 3 with $t = 0$ and $f = (1 - g_0(p_0))^2 \text{sign}(p_0 - E_0) \in S_{2n}(1)$, we have

$$\begin{aligned} 0 &= \operatorname{Re} \big\langle (1 - g_0(p_0))^2 T(P_\theta - z) u_\theta, T u_\theta \big\rangle \\ &= \operatorname{Re} \big\langle \big(q_\theta(x, \xi; 0, h) - z + \mathcal{O}(h^\infty) \langle \xi \rangle^2 \big) T u_\theta, T u_\theta \big\rangle. \end{aligned} \tag{4.6}$$

We have

$$\begin{aligned} \operatorname{Re}\ (q_\theta(x, \xi; 0) - z) &= (1 - g_0(p_0))^2 \text{sign}(p_0 - E_0)(p_0 - \operatorname{Re} z) + \mathcal{O}(h)(1 - g_{-1}(p_0))^2 \\ &\geq (1 - g_0(p_0))^2 \langle \xi \rangle^2 / C + \mathcal{O}(h)(1 - g_{-1}(p_0))^2, \end{aligned} \tag{4.7}$$

with $C > 0$ large enough. On the other hand, we know that, for $j \geq 1$, $q_{\theta,j}(x, \xi; 0) \in S_{2n}(\langle \xi \rangle^2)$ satisfies $\operatorname{supp} q_{\theta,j} \subset \operatorname{supp}(1 - g_0(p_0))$. So (4.6) proves that

$$\|(1 - g_0(p_0)) \langle \xi \rangle^2 T u_\theta \|^2 = \mathcal{O}(h) \|(1 - g_{-1}(p_0)) \langle \xi \rangle^2 T u_\theta \|^2 + \mathcal{O}(h^\infty) \| \langle \xi \rangle^2 T u_\theta \|^2, \tag{4.8}$$

and by induction

$$\begin{aligned} \|(1 - g_0(p_0)) \langle \xi \rangle^2 T u_\theta \|^2 &= \mathcal{O}(h^N) \|(1 - g_{-N}(p_0)) \langle \xi \rangle^2 T u_\theta \|^2 + \mathcal{O}(h^\infty) \| \langle \xi \rangle^2 T u_\theta \|^2 \\ &= \mathcal{O}(h^N) \| \langle \xi \rangle^2 T u_\theta \|^2. \end{aligned} \tag{4.9}$$

As P_θ is elliptic in the classical sense, (1.11) implies,

$$\|u_\theta\|_{H^2(\mathbb{R}^n)} = \mathcal{O}(1) \|u_\theta\|_{L^2(\mathbb{R}^n)},$$

which, in view of (2.21), implies

$$\| \langle \xi \rangle^2 T u_\theta \| = \mathcal{O}(1) \|u_\theta\|. \tag{4.10}$$

So (4.9) becomes

$$\|(1 - g_0(p_0)) T u_\theta \|^2 = \mathcal{O}(h^\infty) \|u_\theta\|, \tag{4.11}$$

which gives, with (4.5),

$$\|w_0 T u_\theta\|^2 = \mathcal{O}\left(\frac{|\operatorname{Im} z|}{h} + h^\infty \right) \|u_\theta\|^2. \tag{4.12}$$

We conclude as at the end of the proof of Theorem 2. □

5. Residue Estimate of the Scattering Amplitude

In this section, we assume that P is a Schrödinger operator

$$P = -h^2 \Delta + V(x), \tag{5.1}$$

where $V(x) \in S_n(1)$ extends holomorphically to the domain Υ defined in (1.9). To define the scattering amplitude, we make a long-range assumption on $V(x)$:

$$\exists \rho > 0 \ \exists C > 0 \ \forall x \in \Gamma, \qquad |V(x)| \leq C|x|^{-\rho}. \tag{5.2}$$

In particular, P satisfies the assumptions of Sect. 1. We can define the scattering matrix $S(z;h)$, $z \in \mathbb{R}_+^*$, related to $P_0 = -h^2\Delta$ and P, as a unitary operator:

$$S(z;h) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1}).$$

Next, introduce the operator $T(z;h)$ defined by $S(z;h) = Id - 2i\pi T(z;h)$. It is well-known (see [15]) that $T(z;h)$ has a kernel $T(\omega, \omega', z; h)$, smooth in $(\omega, \omega') \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \{\omega = \omega'\}$ and the scattering amplitude is given by

$$f(\omega, \omega', z; h) = c_1(z;h) T(\omega, \omega', z; h),$$

with

$$c_1(z;h) = -2\pi (2z)^{-\frac{n-1}{4}} (2\pi h)^{\frac{n-1}{2}} e^{-i\frac{(n-3)\pi}{4}}. \tag{5.3}$$

In [10], C. Gérard and Martinez have shown that for $\omega \neq \omega'$ fixed, the scattering amplitude has a meromorphic continuation to a conic neighborhood of $\mathbb{R}_+^*$, whose poles are the resonances of P. Moreover, the multiplicity of each pole is exactly the multiplicity of the resonance.

In this section, we still assume that $z_0(h)$ is a *simple* resonance of P such that $\operatorname{Re} z_0 \in [E_0 - \epsilon, E_0 + \epsilon]$ and $0 < -\operatorname{Im} z_0 < Ch \ln(1/h)$. Under this condition the scattering amplitude takes the form

$$f(\omega, \omega', z; h) = \frac{f^{res}(\omega, \omega'; h)}{z - z_0} + f^{hol}(\omega, \omega', z; h), \tag{5.4}$$

where $f^{hol}(\omega, \omega', z; h)$ is holomorphic near z_0. Our aim is to give an estimate of the residue f^{res} in some special directions:

Definition 5.1. *We say that $\omega \in \mathbb{S}^{n-1}$ is an* **incoming direction** *(resp.* **outgoing direction***) for the energy E_0 iff there is ϵ, $R > 0$ and $W \subset \mathbb{S}^{n-1}$, a neighborhood of ω, such that, for all $(x, \xi) \in p^{-1}([E_0 - \epsilon, E_0 + \epsilon])$*

$$|x| \geq R \text{ and } \frac{\xi}{|\xi|} \in W \Longrightarrow \lim_{t \to -\infty} \exp(t\mathrm{H}_{p_0})(x, \xi) = \infty. \tag{5.5}$$

(resp. $\lim \exp(t\mathrm{H}_{p_0})(x, \xi) = \infty$ *as $t \to +\infty$).*

Remark 5.2. If $\rho > 1$, ω is an incoming direction iff there is $R > 0$ such that

$$p(x, \xi) = E_0, \ |x| \geq R \text{ and } \frac{\xi}{|\xi|} = \omega \Longrightarrow \lim_{t \to -\infty} \exp(t\mathrm{H}_{p_0})(x, \xi) = \infty.$$

This is a consequence of Proposition 6.1 of [22].

For $\theta \geq C|\operatorname{Im} z|$ with $C > 0$ sufficiently large, we denote by Π_θ the spectral projector associated to the resonance z_0:

$$\Pi_\theta = \frac{1}{2i\pi} \int_{\partial D} (z - P_\theta)^{-1} dz, \tag{5.6}$$

where $D = D(z_0, r(h)) \subset \mathbb{C}$ is a small disk such that z_0 is the only resonance in $\overline{D}$.

Theorem 4. *Let $E_0 > 0$ and $\omega, \omega' \in S^{n-1}$ with $\omega \neq \omega'$. If ω is an outgoing direction or if ω' is an incoming direction, then there exists ϵ, $C' > 0$ such that for all simple resonance $z_0 \in [E_0 - \epsilon, E_0 + \epsilon] - i[0, \theta/C']$ with $h/C < \theta < Ch\ln(1/h)$, $C > 0$ one has*

$$f^{res}(\omega, \omega', h) = \mathcal{O}(h^\infty)\|\Pi_\theta\|. \tag{5.7}$$

Remark 5.3. As for Theorem 2, the assumption that z_0 is simple is not necessary to estimate the corresponding residue f^{res}. If we suppose only that z_0 is a resonance whose multiplicity N is bounded with respect to h, then it is possible to show that

$$f^{res}(\omega, \omega', h) = \mathcal{O}(h^\infty)\sum_{j=0}^{N}\|A_j\|, \tag{5.8}$$

where $A_j = \displaystyle\int_{\partial D}(z - z_0)^j(z - P_\theta)^{-1}dz$ is a finite rank operator. We will give the proof of this result at the end of Sect. 5.2.

For the proof of Theorem 4, we need a representation formula of the scattering amplitude. This is the object of the next section.

5.1. Representation formula. In this section, we recall some results due to C. Gérard and Martinez [10]. We just have to be careful with the fact that in our case, the dilatation angle θ may depend on h. Moreover, we recall only how to continue the meromorphic part of the scattering amplitude. The main idea consists to extend Isozaki-Kitada's formula to complex energies. For this purpose, C. Gérard and Martinez show that the symbols and the phases involved in that formula can be chosen to be analytic in a complex neighborhood of $\mathbb{R}^{2n}$.

For $R > 0$ large enough, $d > 0$, $\epsilon > 0$ and $\sigma \in]0, 1[$, we denote

$$\Gamma^{\pm}_{\mathbb{C}}(R, d, \nu, \sigma) = \Big\{(x, \xi) \in \mathbb{C}^{2n};\ |\mathrm{Re}\, x| > R,\ d^{-1} < |\mathrm{Re}\,\xi| < d,\ \ |\mathrm{Im}\, x| \leq \epsilon\langle \mathrm{Re}\, x\rangle,$$
$$|\mathrm{Im}\,\xi| \leq \epsilon\langle \mathrm{Re}\,\xi\rangle \text{ and } \pm\langle \mathrm{Re}\, x, \mathrm{Re}\,\xi\rangle \geq \pm\sigma|x||\xi|\Big\}.$$

Let $\epsilon > 0$, $d > 1$, $-1 < \sigma_a^- < \sigma_a^+ < 0 < \sigma_b^- < \sigma_b^+ < 1$ and $R_0 > 0$ be sufficiently large. For $* = a, b$, we denote $\Gamma^* = \Gamma^+_{\mathbb{C}}(R_0, d, \epsilon, \sigma_*^+) \cup \Gamma^-_{\mathbb{C}}(R_0, d, \epsilon, \sigma_*^-)$. C. Gérard and Martinez construct some phases $\Phi_* \in C^\infty(\mathbb{C}^{2n})$ and some symbols $k_* \in C^\infty(\mathbb{C}^{2n}) \cap S_{2n}(1)$ satisfying the general assumptions of Isozaki-Kitada [15], and such that the following properties hold:

The phases Φ_* have an holomorphic extension to Γ^* and satisfy

$$\begin{cases} (\nabla_x\Phi_*(x, \xi))^2 + V(x) = \xi^2, \\ \partial_x^\alpha\partial_\xi^\beta\big(\Phi_*(x, \xi) - \langle x, \xi\rangle\big) = \mathcal{O}\big(\langle x\rangle^{1-\rho-|\alpha|}\big), \end{cases} \tag{5.9}$$

uniformly in Γ^*.

There exists $0 < \delta \ll 1$ and $\epsilon_1 > 0$ such that k_* are supported in Γ^*, extend holomorphically in the variables $|x|, |\xi|$ to $\Gamma^+_{\mathbb{C}}(2R_0, d/2, \epsilon, \sigma_*^+ + \delta) \cup \Gamma^-_{\mathbb{C}}(2R_0, d/2, \epsilon, \sigma_*^- - \delta)$ and

$$k_*(x, \xi; h) = \mathcal{O}\big(e^{-\epsilon_1\langle x\rangle/h}\big), \tag{5.10}$$

uniformly with respect to $h \in]0,1]$ and $(x,\xi) \in \Gamma^+_{\mathbb{C}}(2R_0, d/2, \epsilon, \sigma^+_* + \delta) \cup \Gamma^-_{\mathbb{C}}(2R_0, d/2, \epsilon, \sigma^-_* - \delta)$.

With this construction, one can show that for real energies, the scattering amplitude takes the form

$$f(\omega,\omega',z;h) = f_1(\omega,\omega',z;h) + f_2(\omega,\omega',z;h), \tag{5.11}$$

where $f_1(\omega,\omega',z;h)$ has an holomorphic continuation with respect to $z \in \{|\operatorname{Im} z| \leq \epsilon_0 |\operatorname{Re} z|\}$ for ϵ_0 sufficiently small independent on h, and

$$\begin{aligned} &f_2(\omega,\omega',z;h) \\ &\quad = c_2(z;h)\big\langle \big(P-(z+i0)\big)^{-1} k_b(.,\sqrt{z}\omega)e^{i\Phi_b(.,\sqrt{z}\omega)/h}, k_a(.,\sqrt{z}\omega')e^{i\Phi_a(.,\sqrt{z}\omega')/h}\big\rangle, \end{aligned} \tag{5.12}$$

with

$$c_2(z;h) = -2\pi z^{\frac{n-3}{4}} (2\pi h)^{-\frac{n+1}{2}} e^{-i\frac{(n-3)\pi}{4}}.$$

The function f_2 can be continued meromorphically by the following process. For $\mu > 0$ small enough, let U_μ be defined as in Sect. 1 with $F(x) = x\chi(|x|)$, $\chi \in C^\infty(\mathbb{R})$, $\chi = 1$ outside a big interval and $\chi = 0$ near 0, then one has

$$f_2(\omega,\omega',z;h) = c_2(z;h)\big\langle U_\mu\big(P-(z+i0)\big)^{-1} U_\mu^{-1} U_\mu\big(k_b e^{i\Phi_b/h}\big), U_{\overline{\mu}}\big(k_a e^{i\Phi_a/h}\big)\big\rangle. \tag{5.13}$$

Using the above properties on k_* and Φ_* it is easy to see that $U_\mu(k_* e^{ih^{-1}\Phi_*})$ is well defined for μ complex and $|\operatorname{Im} z| \ll |\operatorname{Im} \mu| \ll 1$. A simple calculus shows that, for $|\operatorname{Im} z| \leq \epsilon_0 |\operatorname{Re} z|$ and $\mu = i\theta$, one has

$$U_{i\theta}\big(k_b(.,\sqrt{z}\omega)e^{ih^{-1}\Phi_b(.,\sqrt{z}\omega)}\big) = \widetilde{k}_b(x,\omega,z;\theta,h)e^{i\widetilde{\Phi}_b(x,\omega,z;\theta,h)/h}, \tag{5.14}$$

with

$$\begin{aligned} &\widetilde{k}_b(x,\omega,z;\theta,h) \\ &\quad = J_{i\theta}(x) k_b\big(x+i\theta F(x), \sqrt{z}\omega\big) e^{-\langle \operatorname{Re}\sqrt{z}\theta F(x) + \operatorname{Im}\sqrt{z}x, \omega\rangle/h + \mathcal{O}((\theta + \operatorname{Im}\sqrt{z})\langle x\rangle^{1-\rho})/h}, \end{aligned}$$

and

$$\widetilde{\Phi}_b(x,\omega,z;\theta,h) = \Phi_b(x, \operatorname{Re}\sqrt{z}\omega) - \theta \operatorname{Im}\sqrt{z}\langle F(x),\omega\rangle.$$

From estimate (5.10), one deduces that there exists $\epsilon_2 > 0$ such that $\widetilde{k}_b(x,\omega,z;\theta,h) = \mathcal{O}\big(e^{\theta(C-\langle x\rangle)/h}\big)$ uniformly on $\mathbb{C}^n$ and

$$\widetilde{k}_b(x,\omega,z;\theta,h) = \mathcal{O}\big(e^{-\epsilon_2\langle x\rangle/h}\big), \tag{5.15}$$

uniformly with respect to $x \in \{|x| \geq 2R_0\} \cap \big(\{\langle x,\omega\rangle \geq (\sigma_b^+ + \delta)|x|\} \cup \{\langle x,\omega\rangle \leq (\sigma_b^- - \delta)|x|\}\big)$ and $h \in]0,1]$. Similarly, one can write

$$U_{-i\theta}\big(k_a(.,\sqrt{z}\omega')e^{i\Phi_a(.,\sqrt{z}\omega')/h}\big) = \widetilde{k}_a(x,\omega',z;\theta,h)e^{i\widetilde{\Phi}_a(x,\omega',z;\theta,h)/h},$$

with

$$\widetilde{\Phi}_a(x,\omega',z;\theta,h)=\Phi_a(x,\operatorname{Re}\sqrt{z}\omega')+\theta\operatorname{Im}\sqrt{z}\langle F(x),\omega'\rangle,$$

$\widetilde{k}_a(x,\omega,z;\theta,h)=\mathcal{O}(e^{\theta(C-\langle x\rangle)/h})$ uniformly on $\mathbb{C}^n$ and

$$\widetilde{k}_a(x,\omega,z;\theta,h)=\mathcal{O}\big(e^{-\epsilon_2\langle x\rangle/h}\big),$$

uniformly with respect to $x\in\{|x|\geq 2R_0\}\cap\big(\{\langle x,\omega'\rangle\geq(\sigma_a^+ +\delta)|x|\}\cup\{\langle x,\omega'\rangle\leq(\sigma_a^- -\delta)|x|\}\big)$ and $h\in]0,1]$. It follows that f_2 can be written

$$f_2(\omega,\omega',z;h)=c_2(z;h)\big\langle(P_\theta-z)^{-1}\widetilde{k}_b e^{i\widetilde{\Phi}_b/h},\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle,\tag{5.16}$$

for $\theta>0$ and $|\operatorname{Im}z|\ll\theta|\operatorname{Re}z|$. From this formula, one deduces easily the form of the residue of f at a simple pole z_0:

$$f^{res}(\omega,\omega',h)=c_2(z_0;h)\big\langle\Pi_\theta\widetilde{k}_b e^{i\widetilde{\Phi}_b/h},\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle,\tag{5.17}$$

where the functions $\widetilde{k}_*$, $\widetilde{\Phi}_*$ are evaluated in $z=z_0(h)$.

5.2. Proof of Theorem 4. Before going further, let us discuss the properties of Π_θ when z_0 is simple. If one denotes by u_θ a resonant state associated to z_0, the rank one operator Π_θ can be written

$$\Pi_\theta=\langle .,u_{-\theta}\rangle u_\theta,\tag{5.18}$$

where $u_{-\theta}$ satisfies $(P_{-\theta}-\bar{z}_0)u_{-\theta}=0$. In particular, one has $\|\Pi_\theta\|=\|u_\theta\|\|u_{-\theta}\|$. Let $R>0$, $d>0$, $\sigma>0$ and $w_\pm\in S_{2n}(1)$ such that $exp(\pm T\mathrm{H_p})(\operatorname{supp}w_\pm)\subset\Gamma_\pm(R,\epsilon,\pm\sigma)$ for some $T>0$. It follows from (5.18) and Theorem 2 that

$$\begin{aligned}w_-(x,hD_x)\Pi_\theta&=\mathcal{O}(h^\infty)\|\Pi_\theta\|,\\ \Pi_\theta w_+(x,hD_x)&=\mathcal{O}(h^\infty)\|\Pi_\theta\|.\end{aligned}\tag{5.19}$$

The inequality (4.11) implies that for $\rho\in C_0^\infty(\mathbb{R})$, $\rho=1$ near $[E_0-\epsilon,E_0+\epsilon]$),

$$\Pi_\theta\rho(P)=\mathcal{O}(h^\infty)\|\Pi_\theta\|.\tag{5.20}$$

Now, we consider $\chi\in C_0^\infty(\mathbb{R}^n)$ such that $1_{|x|\leq 2R_0}\prec\chi\prec 1_{|x|\leq 3R_0}$. Then, one has the following

Lemma 5.4. *For $R_0>0$ large enough and $|\operatorname{Im}z|\ll\theta$, one has*

$$f^{res}(\omega,\omega';h)=c(z_0;h)\big\langle\Pi_\theta\chi\widetilde{k}_b e^{i\widetilde{\Phi}_b/h},\chi\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle+\mathcal{O}(h^\infty)\|\Pi_\theta\|.\tag{5.21}$$

Proof. First we prove that

$$\big\langle\Pi_\theta(1-\chi)\widetilde{k}_b e^{i\widetilde{\Phi}_b/h},\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle=\mathcal{O}(h^\infty)\|\Pi_\theta\|.\tag{5.22}$$

Let us denote $g_1=\Pi_\theta(1-\chi)\widetilde{k}_b e^{i\widetilde{\Phi}_b/h}$. Using (5.20), we have, for $\rho\in C_0^\infty([E_0-\epsilon,E_0+\epsilon])$ with $\rho=1$ near E_0,

$$g_1=\Pi_\theta(1-\chi)\rho(P)\widetilde{k}_b e^{i\widetilde{\Phi}_b}+\mathcal{O}(h^\infty)\|\Pi_\theta\|.$$

Next, we introduce $w_+ \in S_{2n}(1)$ such that $\operatorname{supp} w_+ \subset \Gamma_+(R_0, \epsilon, \sigma_b^- - 2\delta)$ and that $w_+ = 1$ on $\Gamma_+(2R_0, \epsilon/2, \sigma_b^- - \delta)$. It follows immediately from (5.19) that

$$\|\Pi_\theta (1-\chi)\rho(P) w_+(x, hD_x)\| = \mathcal{O}(h^\infty)\|\Pi_\theta\|,$$

so that

$$g_1 = \Pi_\theta w_-(x, hD_x)\widetilde{k}_b e^{i\widetilde{\Phi}_b/h} + \mathcal{O}(h^\infty)\|\Pi_\theta\|,$$

with $w_-(x, hD_x) = (1-\chi)\rho(P)(1-w_+)(x, hD_x)$. In particular, $\operatorname{supp} w_- \subset \Gamma_-(2R_0, \epsilon, \sigma_b^- - \delta)$. Moreover, the stationary phase method gives

$$w_-(x, hD_x)\big(\widetilde{k}_b e^{i\widetilde{\Phi}_b/h}\big) \sim \sum_\alpha h^{|\alpha|} C_\alpha \widetilde{k}_{b,\alpha} e^{i\widetilde{\Phi}_b/h} \partial_\xi^\alpha w_-(x, \nabla_x \widetilde{\Phi}_b), \tag{5.23}$$

with $\widetilde{k}_{b,\alpha} = \sum_{\beta\leq\alpha} \mathcal{O}(\partial_x^\beta \widetilde{k}_b)$. Now, if we assume that $\operatorname{Re}(x, \nabla_x \widetilde{\Phi}_b) \in \operatorname{supp} w_-$, we have $(x, \nabla_x\widetilde{\Phi}_b) \in \Gamma_{\mathbb{C}}^-(2R_0, \epsilon, \sigma_b^- - \delta)$. By (5.9), $\nabla_x \widetilde{\Phi}_b = \omega + \mathcal{O}(\langle x\rangle^{-\rho})$ and assuming that R_0 is sufficiently large, we get $\langle x, \omega\rangle \leq (\sigma_b^- - \delta)|x|$. By (5.15), $w_-(x, hD_x)\big(\widetilde{k}_b e^{i\widetilde{\Phi}_b/h}\big) = \mathcal{O}(h^\infty)$ comes and (5.22) follows. Using the same arguments, one proves that $(1-\chi)\widetilde{k}_a e^{i\widetilde{\Phi}_a} = \mathcal{O}(h^\infty)\|\Pi_\theta\|$. It follows that

$$\big\langle \Pi_\theta \chi \widetilde{k}_b e^{i\widetilde{\Phi}_b/h}, (1-\chi)\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle = \mathcal{O}(h^\infty)\|\Pi_\theta\|,$$

and the proof is complete. □

Now, we are in position to prove Theorem 4 and we assume that ω is outgoing. Let $w \in S_{2n}(1)$ with $\operatorname{supp} w \subset p^{-1}([E_0-\epsilon, E_0+\epsilon]) \cap \{\frac{\xi}{|\xi|} \in W\} \cap \{R_0 < |x| < 4R_0\}$ and $w = 1$ on $p^{-1}([E_0 - \epsilon/2, E_0 + \epsilon/2]) \cap \{\frac{\xi}{|\xi|} \in W'\} \cap \{2R_0 < |x| < 3R_0\}$, where W is given by Definition 5.1 and $\omega \in W' \subset\subset W$. Using the fact that $\nabla_x \Phi_b = \omega + \mathcal{O}(\langle x\rangle^{-\rho})$ and an argument similar to (5.23), one can prove that

$$f^{res}(\omega, \omega'; h) = c_2(z_0; h)\big\langle \Pi_\theta \chi w(x, hD_x)\widetilde{k}_b e^{i\widetilde{\Phi}_b/h}, \chi\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle + \mathcal{O}(h^\infty)\|\Pi_\theta\|.$$

As $\chi \in C_0^\infty(\mathbb{R}^n)$ and ω is outgoing, there exists $T > 0$ such that $\exp(T\mathrm{H}_{p_0})(\operatorname{supp} \chi w) \subset \Gamma_+(R, \epsilon, \sigma)$, with $R \gg 1$ and $0 < \sigma < 1$. It follows from (5.19) and from the estimates on $\widetilde{k}_*$ and $\widetilde{\Phi}_*$ that

$$\begin{aligned} f^{res}(\omega, \omega'; h) = c_2(z_0; h)\big\langle \Pi_\theta \chi w(x, hD_x)\widetilde{k}_b e^{i\widetilde{\Phi}_b/h}, \chi\widetilde{k}_a e^{i\widetilde{\Phi}_a/h}\big\rangle \\ + \mathcal{O}(h^\infty)\|\Pi_\theta\| = \mathcal{O}(h^\infty)\|\Pi_\theta\|, \end{aligned}$$

and the proof of Theorem 4 is complete. □

Here, we give the arguments to show Remark 5.3 concerning multiple resonances. In the situation that we deal with, we have

$$(P_\theta - z_0)^{-1} = \sum_{j=1}^N \frac{A_j}{(z-z_0)^j} + A^{hol}(z),$$

where $A_0 = \Pi_\theta$, $A^{hol}(z)$ is holomorphic near z_0 and the A_j are finite rank operators with $\operatorname{Im} A_j \subset \operatorname{Im} \Pi_\theta$. For $j = 0, \dots, N$ fixed, one can write

$$A_j = \sum_{k=1}^{N_j} \langle ., v^j_{\theta,k} \rangle u^j_{\theta,k}$$

with $(P_\theta - z_0)^N u^j_{\theta,k} = 0$ and $N_j \le N$. In particular, one can choose the sequence $(u^j_{\theta,k})_k$ or $(v^j_{\theta,k})_k$ orthogonal and one has $\|v^j_{\theta,k}\| \|u^j_{\theta,k}\| \le \|A_j\|$ for all k. Moreover, it follows from Remark 1.3 and the discussion following (5.18), that for supp $\omega_\pm \subset \Gamma_\pm(R, \epsilon, \pm\sigma)$, one gets

$$\begin{aligned} \omega_-(x, hD_x)A_j &= \mathcal{O}(h^\infty)\|A_j\|, \\ A_j \omega_+(x, hD_x) &= \mathcal{O}(h^\infty)\|A_j\|. \end{aligned} \tag{5.24}$$

Following the proof of Theorem 4, one can show that

$$f^{res}(\omega, \omega', h) = \sum_{j=1}^{N} \sum_{\alpha+\beta=j} \left\langle A_j k_{b,\beta}(z_0, h) e^{ih^{-1}\Phi_b}, k_{a,\alpha}(z_0, h) e^{ih^{-1}\Phi_a} \right\rangle,$$

where the functions $k_{b,\beta}$, $k_{a,\alpha}$ have the same properties as k_a, k_b. Hence, one can work as in the proof of Theorem 4, to get

$$|f^{res}(\omega, \omega', h)| = \mathcal{O}(h^\infty) \sum_j \|A_j\|,$$

which proves Remark 5.3. $\square$

6. Estimate on the Spectral Projector

In this section, we give some examples where the spectral projector Π_θ is bounded by $\mathcal{O}(h^{-M})$.

6.1. Case of resonances at distance h^M. In this section, we consider the case where the resonance z_0 satisfies $|\operatorname{Im} z_0| = \mathcal{O}(h^M)$ with $M >> 1$. In that case, it is possible to obtain some a priori estimates of the spectral projector by using the semiclassical maximum principle [32, 33, 29]. For this purpose, we need some exponential estimate of the modified resolvent $(P_\theta - z)^{-1}$in a suitable complex neighborhood of E_0. This was done by Tang and Zworski in [32, 33] in the case where θ is fixed. Here θ depends on h so that we have to check that this estimate is still available in our case.

Lemma 6.1 (Tang–Zworski). *Assume that $Ch < \theta < Mh \log(1/h)$, with $C > 0$ large enough, and let $\Omega_\theta = E + \theta\Omega$, where $E \in [E_0 - \epsilon, E_0 + \epsilon]$ and $\Omega \subset \mathbb{C}$ is a fixed simply connected and relatively compact domain. Let $g(h)$ be a strictly positive function such that $g(h) \ll \theta$, then there exists $C = C(\Omega)$ such that*

$$\forall z \in \Omega_\theta \setminus \bigcup_{z_j \in \operatorname{Res}(P) \cap \Omega_\theta} D(z_j, g(h)), \quad \|(P_\theta - z)^{-1}\| \le C e^{Ch^{-n} \log \frac{\theta}{g(h)}}.$$

Proof. The demonstration follows closely [32] and we just sketch it. The only difference is that θ (and so Ω_θ) depends on h, so that we have to be careful with the constants appearing in the proof. The main steps are the following.

As in [26] one can find $K \in \mathcal{L}(L^2, L^2)$ with $\|K\| = \mathcal{O}(1)$ and $rank(K) = \mathcal{O}(h^{-n})$ such that $(P_\theta + \theta K - z)$ is invertible for $z \in \Omega_\theta$ and

$$\|(P_\theta + \theta K - z)^{-1}\|_{\mathcal{L}(L^2, H^2)} = \mathcal{O}(1/\theta).$$

Using K as in [26, 32], we can construct, for $z \in \Omega_\theta$, an invertible operator

$$\mathcal{P}(z) = \begin{pmatrix} P_\theta - z & R_- \\ R_+ & 0 \end{pmatrix} : H^2 \oplus \mathbb{C}^N \to L^2 \oplus \mathbb{C}^N, \tag{6.1}$$

where $N = rank(K) = \mathcal{O}(h^{-n})$. Using the fact that $(P_\theta + \theta K - z)^{-1}(P_\theta - z) = \mathcal{O}(1)$, one shows that its inverse

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : L^2 \oplus \mathbb{C}^N \to H^2 \oplus \mathbb{C}^N, \tag{6.2}$$

satisfies $\|E(z)\|, \|E_-(z)\| = \mathcal{O}(\theta^{-1})$ and $\|E_+(z)\|, \|E_{-+}(z)\| = \mathcal{O}(1)$.

As $\|(P_\theta - z)^{-1}\| = \mathcal{O}(\theta^{-2})\big(1 + \|E_{-+}^{-1}(z)\|_{\mathbb{C}^N, \mathbb{C}^N}\big)$, for $z \in \Omega_\theta \setminus \operatorname{Res} P$, we obtain

$$\|(P_\theta - z)^{-1}\|_{\mathbb{C}^N, \mathbb{C}^N} = \mathcal{O}\big(\theta^{-2} h^{-n} e^{Ch^{-n}}\big) |\det(E_{-+}(z))|^{-1}, \tag{6.3}$$

and it remains to estimate $|\det(E_{-+}(z))|$ from below. For this purpose, we set

$$D_\theta(z, h) = \prod_{z_j \in \Omega_\theta \cap \operatorname{Res} P} \left(\frac{z - z_j}{\theta} \right)$$

and $\det(E_{-+}(z)) = G_\theta(z, h) D_\theta(z, h)$. Using the change of variable $\Omega_\theta \ni z \mapsto (z - E)/\theta \in \Omega$ we work on a domain independent of θ. Following the arguments of [26], one can show that $|G_\theta(z, h)| \geq e^{-Ch^{-n}}$ uniformly with $z \in \Omega_\theta$, which implies

$$\begin{aligned} \forall z \in \Omega_\theta \setminus \bigcup_{z_j \in \operatorname{Res}(P) \cap \Omega_\theta} D(z_j, g(h)), \qquad |\det(E_{-+}(z))| &\geq e^{-Ch^{-n}} \left(\frac{g(h)}{\theta} \right)^{\mathcal{O}(h^{-n})} \\ &\geq C e^{-C \log(\frac{\theta}{g(h)}) h^{-n}}. \end{aligned} \tag{6.4}$$

Combining estimates (6.3) and (6.4), one gets the announced result. □

Proposition 6.2. *Assume that V is compactly supported and let $E_0 > 0$. Let z_0 be a simple resonance of P such that $Res(P) \cap D(z_0, h^{M_1}) = \{z_0\}$, for M_1 sufficiently large and $|\operatorname{Im} z_0| \leq Ch^{M_2}$ with $M_2 \geq M_1 + 2n + 2$. Then*

$$\|\Pi_\theta\| = \mathcal{O}(1)$$

uniformly with respect to $h/C < \theta < Ch \log(1/h)$.

Proof. We can copy the proof of Proposition 3.1 of Stefanov [31] with $\theta_0 = h\ln(1/h)$ to get

$$\|(P_\theta - z)^{-1}\| \leq \frac{2}{\operatorname{Im} z}, \tag{6.5}$$

for all z satisfying $\operatorname{Im} z > 2e^{-h^{-1/3}}$. Let us denote $\tilde{z}_0 = \overline{z}_0 + 2ih^{M_2}$. Following [31], we want to apply the semiclassical maximum principle as it is presented in Stefanov [29] to the function $F(z,h) = \frac{z-z_0}{z-\tilde{z}_0}(P_\theta - z)^{-1}$ which is holomorphic on

$$\Omega(h) = \{z \in \mathbb{C};\ |\operatorname{Re} z - \operatorname{Re} z_0| < 2h^{M_1}, -h^{M_1-n-2} < \operatorname{Im} z < h^{M_1}\}.$$

From (6.5), it follows that $\|F(z,h)\| \leq Ch^{-M_1}$ on $\operatorname{Im} z = h^{M_1}$. On the other hand, we deduce from the exponential estimate of the resolvent proved in Lemma 6.1 below, that $\|F(z,h)\| \leq Ce^{Ch^{-n\ln(1/h)}}$ on $\Omega(h)$. By the semiclassical maximum principle, it follows that

$$\|F(z,h)\| \leq Ch^{-M_1} \text{ on } \tilde{\Omega}(h),$$

with $\tilde{\Omega}(h) = \{z \in \mathbb{C};\ |\operatorname{Re} z - \operatorname{Re} z_0| < h^{M_1}, -2h^{M_1} < \operatorname{Im} z < h^{M_1}\}$. In particular, $(2h)^{-M_1}\|\Pi_\theta\| \leq \frac{1}{|z_0-\tilde{z}_0|}\|\Pi_\theta\| = \|F(z_0,h)\| \leq Ch^{-M_1}$ and the proof is complete. □

From Theorem 4 and Proposition 6.2, one deduces immediately the following.

Corollary 6.3. *Assume that V is compactly supported and let $E_0 > 0$. Suppose that ω is outgoing or ω' is incoming and that $\omega \neq \omega'$. Let z_0 be a simple resonance of P such that* $\operatorname{Res}(P) \cap D(z_0, h^{M_1}) = \{z_0\}$, *for M_1 large enough,* $\operatorname{Re} z_0 \in [E_0 - \epsilon, E_0 + \epsilon]$ *and* $|\operatorname{Im} z_0| \leq Ch^{M_2}$ *with $M_2 \geq M_1 + 2n + 2$. Then*

$$f^{res}(\omega, \omega', h) = \mathcal{O}(h^\infty). \tag{6.6}$$

Remark 6.4. Let us notice that this result is not a consequence of the works [30] and [23]. Indeed, if one applies the theorems of [30] and [23] to this situation, one can only show that $f^{res}(\omega,\omega',h) = \mathcal{O}\big(h^{M_2-\frac{n-1}{2}}\big)$.

6.2. Estimate in dimension one.

Lemma 6.5. *We assume that $n = 1$ and that the critical points of $p_0(x,\xi)$ on the energy level are non-degenerate (i.e. the points $(x,\xi) \in p_0^{-1}(\{E_0\})$ such that $\nabla p_0(x,\xi) = 0$ satisfy* $\operatorname{Hess} p_0(x,\xi)$ *is invertible). Then there exists $M, \epsilon > 0$ such that, for $E \in [E_0 - \epsilon, E_0 + \epsilon]$ and $\theta = Nh$ with $N > 0$ large enough,*

$$\|(P_\theta - z)^{-1}\| = \mathcal{O}(h^{-M}) \prod_{z_j \in \operatorname{Res}(P)\cap\Omega_{E,\epsilon\theta}} \frac{\theta}{|z - z_j|}, \tag{6.7}$$

where $z \in \Omega_{E,\epsilon\theta/2}$, $\Omega_{E,\delta} = E + D(0,\delta)$ and h is small enough.

Proof. As for Lemma 6.1, the proof is a slight modification of Lemma 1 of Tang–Zworski [32]. It is shown in [2], that for a

$$K = \chi(x)g(A)f\Big(\frac{A-E}{\theta}\Big)g(A)\chi(x),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$, $f \in \mathcal{S}(\mathbb{R}; \mathbb{R}_+)$ with $\widehat{f} \in C_0^\infty(\mathbb{R})$ ($\widehat{f}$ is the Fourier transform of f), $g \in C_0^\infty(\mathbb{R})$ with $g = 1$ near E_0 and $A = \mathrm{Op}_h^w(a)$ with $a \in S_{2n}^{cl}(\langle \xi \rangle^2)$ elliptic in the sense of (1.8), we have

$$\|(P_\theta - i\theta K - z)^{-1}\| \leq \mathcal{O}(\theta^{-1}), \tag{6.8}$$

for $|\mathrm{Re}\, z - E| \leq \epsilon\theta$, $\mathrm{Im}\, z \geq -\epsilon\theta$ and

$$\|(P_\theta - z)^{-1}\| \leq \mathcal{O}(\theta^{-1}), \tag{6.9}$$

for $|\mathrm{Re}\, z - E| \leq \epsilon\theta$, $\mathrm{Im}\, z \geq C\theta$. In addition, the critical points of $a_0(x, \xi)$ in $[E_0 - \epsilon, E_0 + \epsilon]$ are non degenerate.

Let $b(x, \xi; h) \in S_{2n}^{cl}(\langle x \rangle^2 + \langle \xi \rangle^2; \mathbb{R})$ be such that $b = a$ for $|x| \leq R$, $b(x, \xi; h) \geq (\langle x \rangle^2 + \langle \xi \rangle^2)/C$ for $|x| \geq 2R$ and the critical points of $b_0(x, \xi)$ in $[E_0 - \epsilon, E_0 + \epsilon]$ are non-degenerate. We note $B = \mathrm{Op}_h^w(b)$ which is self-adjoint and has only pure spectrum near $[E_0 - \epsilon, E_0 + \epsilon]$. Since the symbol of $g(A)$ and $g(B)$ coincide modulo $\mathcal{O}(h^\infty)$ near the support of $\chi(x)$, we get

$$K = \chi(x) g(B) f\Big(\frac{A - E}{\theta}\Big) g(B) \chi(x) + \mathcal{O}(h^\infty), \tag{6.10}$$

and we have

$$\begin{aligned} &\chi(x) g(B) \Big(f\Big(\frac{A - E}{\theta}\Big) - f\Big(\frac{B - E}{\theta}\Big)\Big) g(B) \chi(x) \\ &\quad = \frac{1}{2\pi} \int \widehat{f}(t) \chi(x) g(B) \big(e^{itA/\theta} - e^{itB/\theta}\big) g(B) \chi(x)\, dt \\ &\quad = \frac{1}{2\pi} \iint_0^1 \widehat{f}(t) \chi(x) g(B) e^{itsA/\theta} \frac{it(A - B)}{\theta} g(B) e^{it(1-s)B/\theta} \chi(x)\, ds\, dt. \end{aligned} \tag{6.11}$$

Here $(A - B) g(B)$ is a h-pseudodifferential operator whose symbol, in $S_{2n}^{cl}(1)$, vanishes for $|x| \leq R$. On the other hand, we have $|st| \leq C$ since $\widehat{f} \in C_0^\infty(\mathbb{R})$ and the symbol of $\chi(x) g(B)$, in $S_{2n}(1)$, has compact support independent of R (modulo h^∞). If we fix R large enough, the theorem of Egorov implies

$$\chi(x) g(B) e^{itsA/\theta} (A - B) g(B) = \mathcal{O}(h^\infty). \tag{6.12}$$

So (6.10), (6.11) and (6.12) imply

$$K = \chi(x) g(B) f\Big(\frac{B - E}{\theta}\Big) g(B) \chi(x) + \mathcal{O}(h^\infty). \tag{6.13}$$

Let $k \in C_0^\infty(\mathbb{R}; [0, 1])$ with $k = 1$ near 0. As $f \in \mathcal{S}(\mathbb{R})$, the functional calculus implies

$$\begin{aligned} &\Big\| \chi(x) g(B) \Big(1 - k\Big(\frac{B - E}{\Lambda\theta}\Big)\Big) f\Big(\frac{B - E}{\theta}\Big) g(B) \chi(x) \Big\| \\ &\quad \leq \mathcal{O}(1) \sup_{t \in \mathbb{R}} \Big| g(t) \Big(1 - k\Big(\frac{t - E}{\Lambda\theta}\Big)\Big) f\Big(\frac{t - E}{\theta}\Big) g(t) \Big| \\ &\quad \leq \mathcal{O}(\Lambda^{-\infty}). \end{aligned} \tag{6.14}$$

So (6.13) shows that

$$\begin{aligned} K =&\chi(x)g(B)k\Big(\frac{B-E}{\Lambda\theta}\Big)f\Big(\frac{B-E}{\theta}\Big)g(B)\chi(x)+\mathcal{O}(\Lambda^{-\infty}+h^{\infty})\\ =&\widetilde{K}+\mathcal{O}(\Lambda^{-\infty}+h^{\infty}). \end{aligned} \tag{6.15}$$

Using (6.8), we get for $|\operatorname{Re} z - E| \leq \epsilon\theta$ and $\operatorname{Im} z \geq -\epsilon\theta$,

$$\begin{aligned} P_\theta - i\theta\widetilde{K} - z =& P_\theta - i\theta K - z + \theta\mathcal{O}(\Lambda^{-\infty}+h^{\infty})\\ =&(P_\theta - i\theta K - z)(1+\mathcal{O}(\Lambda^{-\infty}+h^{\infty})). \end{aligned} \tag{6.16}$$

Now we fix Λ large enough and we have

$$\big\|\big(P_\theta - i\theta\widetilde{K} - z\big)^{-1}\big\| \leq \mathcal{O}(\theta^{-1}), \tag{6.17}$$

for $|\operatorname{Re} z - E| \leq \epsilon\theta$, $\operatorname{Im} z \geq -\epsilon\theta$ and h small enough. As $b_0(x,\xi)$ has only non-degenerate critical point in the energy level E_0, the work of Brummelhuis–Paul–Uribe [4] or [3] shows that the number of eigenvalues of B in $[E-C\theta, E+C\theta]$ is $\mathcal{O}(\ln(1/\theta)$, so

$$\operatorname{rank}\widetilde{K} \leq \operatorname{rank} k\Big(\frac{B-E}{\Lambda\theta}\Big) \leq \#\operatorname{sp}(B)\cap[E-C\theta, E+C\theta] \leq \mathcal{O}(\ln(1/\theta)), \tag{6.18}$$

and

$$\|\widetilde{K}\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(1). \tag{6.19}$$

Now the end of the proof is a repetition of the proof of Lemma 6.1: we put a Grushin problem like (6.1) which is well posed and we note $\mathcal{E}(z)$ is inverse as in 6.2. We have $\|(P_\theta - z)^{-1}\| = \mathcal{O}(\theta^{-2})(1+\|E_{-+}^{-1}(z)\|)$ with $\|E_{-+}(z)\| = \mathcal{O}(1)$. As the minor $\widetilde{E}_{-+} = \mathcal{O}(h^{-C})$, we get

$$\|(P_\theta - z)^{-1}\| = \mathcal{O}(h^{-C})|\det(E_{-+}(z))|^{-1}. \tag{6.20}$$

As usual, we set

$$D_\theta(z,h) = \prod_{z_j\in\Omega_\theta\cap\operatorname{Res}(P)}\Big(\frac{z-z_j}{\theta}\Big)$$

and $\det(E_{-+}(z)) = G_\theta(z,h)D_\theta(z,h)$. Using the change of variable $\Omega_{E,\epsilon\theta} \ni z \mapsto (z-E)/\theta \in \Omega_{0,\epsilon}$ we work on a domain independent on θ. The majoration of the number of resonances

$$\#\operatorname{Res}(P)\cap\Omega_{E,\epsilon\theta} = \mathcal{O}(\ln(1/h)), \tag{6.21}$$

proved in [3] and the arguments of Sjöstrand [26] show that $|G_\theta(z,h)| \geq h^C/\widetilde{C}$ uniformly with $z \in \Omega_{E,\epsilon\theta/2}$. The lemma follows from (6.20). □

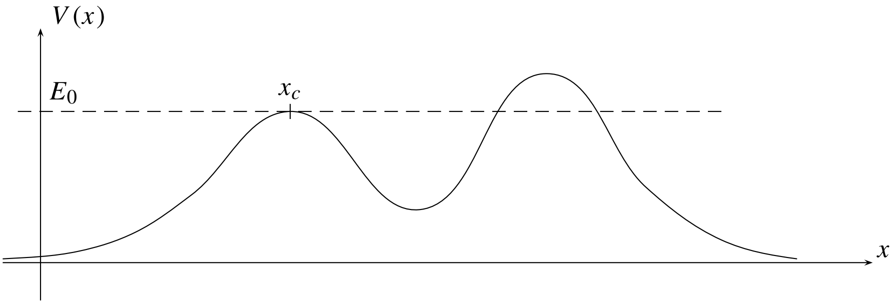
Corollary 6.6. *Under the hypotheses of Lemma 6.5, if* $\#\operatorname{Res}(P)\cap D(E_0,\theta) = \mathcal{O}(1)$ *and* $z_0 \in \operatorname{Res}(P)$ *is separated by* h^C *from the other resonances of* P*, then*

$$\Pi_\theta = \mathcal{O}(h^{-C'}). \tag{6.22}$$

Now we give an example in the 1 dimensional case where we can bound the projector Π_θ. Consider a short range potential $V(x)$ which is holomorphic in

$$\{x \in \mathbb{C};\ |\mathrm{Im}\, z| \leq \langle \mathrm{Re}\, z\rangle / C\},$$

and has the following form:



At x_c, $V(x)$ has a non-degenerate maximum. Such type of potential have been studied by Fujjié and Ramond [8] and [9]. In particular, the formula (41) of [9] implies that the resonances in $\Omega_{E_0,\epsilon\theta}$ are of the form

$$z_j = E_0 + \frac{S_0 - (2j+1)\pi h + ih\ln(2)}{K\ln(h)} + \mathcal{O}(h/\ln(h)^2), \tag{6.23}$$

with $j \in \mathbb{Z}$ and S_0, K are some fixed constants. Let $j_0 \in \mathbb{Z}$ fixed and $z \in D(z_{j_0}, h/\ln(1/h)C)$ with $C > 0$ large enough. Using (6.23), we get $|z - z_j| \geq (|j_0 - j|)h/\ln(1/h)C$ for $j \neq j_0$ and we have

$$\prod_{z_k \in \Omega_{E_0,\epsilon\theta} \text{ and } j \geq j_0} \frac{\theta}{|z - z_j|} \leq \prod_{z_j \in \Omega_{E_0,\epsilon\theta} \text{ and } j \geq j_0} \frac{C\ln(1/h)}{|j - j_0|} \leq \frac{(C\ln(1/h))^{N_+}}{N_+!},$$

where $N_+ = \#\{j \geq j_0;\ z_j \in \Omega_{E_0,\epsilon\theta}\}$. A similar formula can be obtained for the product over $j \leq j_0$ and we get

$$\prod_{z_k \in \Omega_{E_0,\epsilon\theta}} \frac{\theta}{|z - z_j|} \leq \frac{(C\ln(1/h))^{N_+}(C\ln(1/h))^{N_-}}{N_+!N_-!}. \tag{6.24}$$

Equation (6.23) implies that the number of resonances in $\Omega_{E_0,\epsilon\theta}$, noted $N = N_+ + N_-$, satisfy $N \sim \alpha\ln(1/h)$, with $\alpha > 0$ and, as j_0 is fixed,

$$|N_+ - N/2| \leq C \text{ and } |N_- - N/2| \leq C, \tag{6.25}$$

so

$$\prod_{z_k \in \Omega_{E_0,\epsilon\theta}} \frac{\theta}{|z - z_j|} \leq N^C \frac{(C\ln(1/h))^N}{((N/2)!)^2} \leq N^C \frac{(CN)^N}{((N/2)!)^2}, \tag{6.26}$$

The Stirling formula $N! \sim N^N e^{-N}\sqrt{2\pi N}$ implies

$$\prod_{z_k \in \Omega_{E_0,\epsilon\theta}} \frac{\theta}{|z - z_j|} \leq C^N \frac{N^N}{((N/2)^{N/2})^2} \leq C^N = \mathcal{O}(h^{-C}). \tag{6.27}$$

Using Lemma 6.5, we have proved

Corollary 6.7. *Under the previous hypotheses, the projector associated to a resonance* z_j *satisfies, for h small enough,*

$$\Pi_\theta = \mathcal{O}(h^{-C}). \tag{6.28}$$

In this case $+1 \in \mathbb{S}^0$ *is an incoming direction and* -1 *is an outgoing direction.*

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