EXIT TIME AND PRINCIPAL EIGENVALUE OF NON-REVERSIBLE ELLIPTIC DIFFUSIONS

DORIAN LE PEUTREC, LAURENT MICHEL, AND BORIS NECTOUX

Abstract. In this work, we analyse the metastability of non-reversible diffusion processes

$$dX_t = \boldsymbol{b}(X_t)dt + \sqrt{h}\,dB_t$$

on a bounded domain Ω when $\boldsymbol{b}$ admits the decomposition $\boldsymbol{b} = -(\nabla f + \boldsymbol{\ell})$ and $\nabla f \cdot \boldsymbol{\ell} = 0$. In this setting, we first show that, when $h \to 0$, the principal eigenvalue of the generator of $(X_t)_{t\geq 0}$ with Dirichlet boundary conditions on the boundary $\partial\Omega$ of Ω is exponentially close to the inverse of the mean exit time from Ω, uniformly in the initial conditions $X_0 = x$ within the compacts of Ω. The asymptotic behavior of the law of the exit time in this limit is also obtained. The main novelty of these first results follows from the consideration of non-reversible elliptic diffusions whose associated dynamical systems $\dot{X} = \boldsymbol{b}(X)$ admit equilibrium points on $\partial\Omega$. In a second time, when in addition $\operatorname{div}\boldsymbol{\ell} = 0$, we derive a new sharp asymptotic equivalent in the limit $h \to 0$ of the principal eigenvalue of the generator of the process and of its mean exit time from Ω. Our proofs combine tools from large deviations theory and from semiclassical analysis, and truly relies on the notion of quasi-stationary distribution.

Keywords. Metastability, Eyring-Kramers type formulas, mean exit time, principal eigenvalue, non-reversible processes.
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1. Introduction

1.1. **Purpose of this work.** Let $\mathrm{L} > 0$ and $M = (\mathrm{L}\mathbb{T})^d$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one dimensional torus. Let $(X_t)_{t\geq 0}$ be the solution on M of the stochastic differential equation

$$dX_t = \boldsymbol{b}(X_t)\,dt + \sqrt{h}\,dB_t, \tag{1.1}$$

where $h > 0$, $(B_t)_{t\geq 0}$ denotes the Brownian motion on M, and $\boldsymbol{b} : M \to \mathbb{R}^d$ is a vector field. Such an equation is one of the most important models in statistical physics. In all this work, $\Omega \subset M$ is a $\mathcal{C}^\infty$ domain and we denote by

$$\tau_{\Omega^c} = \inf\{t \geq 0, X_t \notin \Omega\}$$

the first exit time from Ω for the process (1.1).

When h is small, due to the existence of stable equilibrium points of the system $\dot{X} = \boldsymbol{b}(X)$, the process (1.1) remains trapped during a very long time in a neighborhood of such a point in M, called a metastable region, before going to another metastable region. For this reason, the process (1.1) is said to be metastable. This phenomenon of metastability has been widely studied through the asymptotic behavior in the zero white noise limit $h \to 0$ of the law of τ_{Ω^c} and of the principal eigenvalue $-\lambda^L_{1,h}$ of the infinitesimal generator of the diffusion (1.1) with Dirichlet boundary conditions on $\partial\Omega$. When the ω-limit set of each trajectory of the dynamical system $\dot{X} = \boldsymbol{b}(X)$ lying entirely in $\overline{\Omega}$ is contained in Ω, the limit of $h \ln \mathbb{E}[\tau_{\Omega^c}]$ when $h \to 0$ has been studied in [20] (see also [21, 41]). When in addition $\boldsymbol{b} \cdot n_\Omega < 0$ on $\partial\Omega$ (where n_Ω is the unit outward normal vector to $\partial\Omega$), it is proved in [10] that $\lambda_{1,h}^L \mathbb{E}[\tau_{\Omega^c}] \to 1$ when $h \to 0$ (see also [28, 29]). We also mention [43, 44] where formulas were obtained through formal computations.

When the process (1.1) is reversible, i.e. when there exists a function f such that $\boldsymbol{b} = -\nabla f$, we refer to [52, 26, 15, 47] for sharp asymptotics formulas on $\lambda_{1,h}^L$ or on $\mathbb{E}[\tau_{\Omega^c}]$ when the system does not have equilibrium points on $\partial\Omega$, and to [42, 35, 48] when it does (see also [38]). When $\boldsymbol{b}\cdot n_\Omega = 0$, the cycling effect of a two-dimensional randomly perturbed system has been studied in [12]. We refer to [1, 14, 13] for a comprehensive review of the literature on this topic.

Remark. *For asymptotic estimates of eigenvalues and transition times in the boundaryless case, we refer to* [27, 46, 6, 18, 5, 2, 22, 25, 45] *when elliptic reversible processes are considered, and to* [4, 31, 34, 36] *when the considered process is elliptic, non-reversible, and admits the Gibbs measure* (1.2) *as invariant measure.*

The purpose of this work is to investigate the asymptotic behaviors when $h \to 0$ of $\lambda_{1,h}^L$ and of the law and the expected time of τ_{Ω^c} for non-reversible processes of the form (1.1) when the smooth vector field $\boldsymbol{b} : M \to \mathbb{R}^d$ decomposes into the pointwise orthogonal sum of a smooth gradient field with a vector field (see (**Ortho**)).

First, we prove in this case the following: when Ω is roughly a single well (see (**One-Well**)) of the potential energy function f (see Theorem 1, which is the first main result of this work):

R1. In the limit $h \to 0$, $\lambda_{1,h}^L \mathbb{E}[\tau_{\Omega^c}]$ converges to 1 and the law of $\lambda_{1,h}^L \tau_{\Omega^c}$ converges to an exponential law of mean 1, both exponentially fast and uniformly w.r.t. the initial conditions x living in the (relevant) compacts of Ω. The asymptotic behavior of the spectral gap is also investigated.

When in addition the Gibbs measure

$$\mu_G(dx) = \frac{e^{-\frac{2}{h}f}}{\int_M e^{-\frac{2}{h}f}}\, dx \tag{1.2}$$

is invariant (see (**Div-free**)) and under an additional assumption on the shape of $\partial\Omega$ near its lowest energy points (see (**Normal**)), we prove that (see Theorem 2, which is the second main result of this work):

R2. In the limit $h \to 0$, $\lambda_{1,h}^L$, and thus $\mathbb{E}[\tau_{\Omega^c}]$, satisfy an Eyring-Kramers type formula.

Concerning item R1 above, the main novelty compared to the existing literature arises from the fact that these results are derived when, simultaneously, the process (1.1) is non-reversible and the dynamical system $\dot X = \boldsymbol{b}(X)$ is allowed to admit equilibrium points on $\partial\Omega$[1]. The latter situation, which is known to introduce several technical difficulties [11], is natural for applications [44]. For instance, this situation occurs when one is interested in the so-called state-to-state dynamics associated with (1.1). In this case, the set Ω, which is associated with a macroscopic state, is indeed typically defined as the basin of attraction of some asymptotically stable equilibrium point $x_0 \in M$ for the dynamical system $\dot X = \boldsymbol{b}(X)$, so that $\partial\Omega$ contains equilibrium points of $\dot X = \boldsymbol{b}(X)$. We refer for instance to [49, 33, 40, 13] for more material and references on state-to-state dynamics. Let us also mention that the condition (**Normal**)

[1]We mention that in our setting (more precisely under (**Ortho**)), every ω-limit set is composed of a single equilibrium point, see Section 1.3.

is automatically satisfied when Ω is a basin of attraction, see the discussion after (**Normal**) on this subject.

Finally, concerning item R2 above, the Eyring-Kramers type formula we derive for $\lambda^L_{1,h}$ in Theorem 2, which leads to the inverse formula for $\mathbb{E}[\tau_{\Omega^c}]$ according to item R1, is new when considering such non-reversible processes, whether or not there are equilibrium points of $\dot{X} = \boldsymbol{b}(X)$ on $\partial\Omega$. It exhibits the precise effect of the boundary $\partial\Omega$ on the sharp equivalent as $h \to 0$ of both $\lambda^L_{1,h}$ and $\mathbb{E}[\tau_{\Omega^c}]$.

1.2. **Assumptions.** For $\mu \in \mathbb{R}$, we use the notation

$\{f \le \mu\} := \{x \in M,\ f(x) \le \mu\}$, $\{f < \mu\} := \{x \in M,\ f(x) < \mu\}$, and $\{f = \mu\} := \{x \in M,\ f(x) = \mu\}$.

Moreover, for $r > 0$ and $y \in M$, $B(y, r)$ denotes the open ball of radius r centered at y in M:

$$B(y, r) := \{z \in M, |y - z| < r\}.$$

Throughout this work, we assume that there exist a smooth vector field $\boldsymbol{\ell} : M \to \mathbb{R}^d$ and a smooth Morse function $f : M \to \mathbb{R}$ such that the vector field $\boldsymbol{b} : M \to \mathbb{R}^d$ satisfies the following orthogonal decomposition:

(**Ortho**) $$\boldsymbol{b}(x) = -(\nabla f(x) + \boldsymbol{\ell}(x)) \quad \text{and} \quad \boldsymbol{\ell}(x)\cdot\nabla f(x) = 0 \qquad \text{for every } x \in M.$$

We recall that a smooth function is a Morse function if all its critical points are non degenerate.

Let us now define

$$\mathbf{C}_{\min} := \Omega \cap \{f < \min_{\partial\Omega} f\}. \tag{1.3}$$

Notice that $\mathbf{C}_{\min} = \overline{\Omega} \cap \{f < \min_{\partial\Omega} f\}$ and that, when $\mathbf{C}_{\min}$ is nonempty and connected, it is a connected component of $\{f < \min_{\partial\Omega} f\}$.

Our second main assumption roughly says that Ω looks like a single well of the potential f:

(**One-Well**) $f : M \to \mathbb{R}$ admits precisely one critical point x_0 in Ω and $\partial\mathbf{C}_{\min} \cap \partial\Omega \neq \emptyset$.

Note that when (**One-Well**) holds, $\mathbf{C}_{\min}$ is nonempty and connected, x_0 belongs to $\mathbf{C}_{\min}$, and

$$f(x_0) = \min_{x \in \overline{\Omega}} f(x). \tag{1.4}$$

We refer to Figure 1.1 for a schematic representation of $\mathbf{C}_{\min}$ when (**One-Well**) holds.

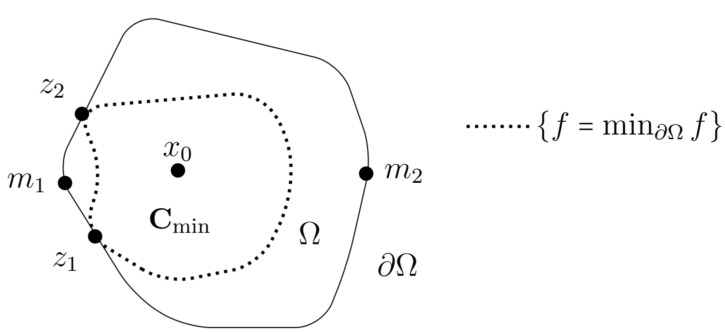

FIGURE 1.1. Schematic representation of $\mathbf{C}_{\min}$ when (**One-Well**) holds. On this figure, $\partial\mathbf{C}_{\min} \cap \partial\Omega = \{z_1, z_2\}$ and $m_1, m_2 \in \partial\Omega$ are the local maxima of f in M.

The first main result of this work, namely Theorem 1, only requires the assumptions (**Ortho**) and (**One-Well**). Our second main result, namely Theorem 2, requires two additional assumptions which are the topic of the rest of this section. The first one implies the invariance of the Gibbs measure $\mu_G(dx) = \frac{e^{-\frac{2}{h}f}}{\int_M e^{-\frac{2}{h}f}}\,dx$ defined in (1.2):

(**Div-free**) $$\text{For every } x \in M, \quad \operatorname{div} \boldsymbol{\ell}(x) = 0.$$

It is well-known that a process solution to an elliptic stochastic differential equation on M with sufficiently smooth coefficients admits a unique invariant probability measure. Furthermore, using the standard characterization[2] of an invariant probability measure with the adjoint of the operator $-\frac{h}{2}\Delta + \boldsymbol{b}\cdot\nabla$, the conditions (**Ortho**) and (**Div-free**) are necessary and sufficient to ensure that the measure μ_G is an (and thus the) invariant probability measure of the process (1.1) for all $h > 0$.

Throughout this work, we say that $z \in M$ is a saddle point of f when z is a critical point of f of index 1, i.e. when the matrix $\operatorname{Hess} f(z)$, which is invertible according to (**Ortho**), admits precisely one negative eigenvalue. Our last assumption (**Normal**) below deals with the points $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$. These points, which are global minima of $f|_{\partial\Omega}$, play a crucial role in the asymptotic equivalents of the mean exit time from Ω resulting from Theorems 1 and 2. Let us mention that, according to [35, Item (b) in Proposition 12], when such a z is a critical point of f, it is a saddle point.

For $x \in M$, we define the Jacobian matrix

$$\mathsf{L}(x) := \operatorname{Jac} \boldsymbol{\ell}(x).$$

In order to state our last assumption, we need some elements of the following proposition resulting from [34, Lemma 1.8] and [3, Lemma 1.4] (see also [32] for a similar result) on the Jacobian matrix of the vector field $\boldsymbol{b}$ at a saddle point of f.

Lemma 1. *Assume* (**Ortho**) *and let $z \in M$ be a critical point of f with index $p \in \{0, \dots, d\}$. Then, the matrix* $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ *admits precisely p eigenvalues in* $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re}\mathsf{z} < 0\}$ *and $d - p$ eigenvalues in* $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re}\mathsf{z} > 0\}$.

When z is a saddle point, we denote by $\mu(z)$ the eigenvalue of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ *in* $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re}\mathsf{z} < 0\}$ *and by $\lambda(z)$ the negative eigenvalue of* $\operatorname{Hess} f(z)$. *We have moreover in this case:*

(1) The eigenvalue $\mu(z)$ is real, and thus negative.

(2) Let $\xi(z)$ be a real unit eigenvector of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ *associated with $\mu(z)$. Then, the matrix* $\operatorname{Hess} f(z) + 2|\mu(z)|\,\xi(z)\xi(z)^t$ *is positive definite and of determinant* $-\det \operatorname{Hess} f(z)$.

(3) It holds $|\mu(z)| \geq |\lambda(z)|$, with equality if, and only if, ${}^t\mathsf{L}(z)\xi(z) = 0$.

Let us now formulate our last assumption, on the local shape of f near the points of $\partial\mathbf{C}_{\min} \cap \partial\Omega$ when (**Ortho**) holds. In the following, for any $z \in \partial\Omega$, $n_\Omega(z)$ denotes the unit outward normal vector to $\partial\Omega$ at z.

(**Normal**) $$\forall z \in \partial\mathbf{C}_{\min} \cap \partial\Omega, \text{ it holds: } \begin{cases} \text{when } \nabla f(z) = 0,\ \xi(z) \in \operatorname{Span}(n_\Omega(z)), \\ \text{when } \nabla f(z) \neq 0,\ \det \operatorname{Hess}(f|_{\partial\Omega})(z) \neq 0 \text{ and } \boldsymbol{\ell}(z) = 0, \end{cases}$$

where $\xi(z)$ is an eigenvector of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ associated with its unique negative eigenvalue, see Lemma 1.

[2]See for instance [53, page 259].

We end this section by discussing the geometric consequences of (**Normal**).

Let $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$ be such that $\nabla f(z) = 0$. When (**Normal**) holds, the tangent space $T_z\partial\Omega$ to $\partial\Omega$ at z satisfies $T_z\partial\Omega = z + \{\xi(z)\}^\perp$. Since $\xi(z)$ is an eigenvector of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ associated with its unique eigenvalue in $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} < 0\}$ and, according to Lemma 1, the $d-1$ remaining eigenvalues of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ belong to $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} > 0\}$, it follows that the (complexification of the) hyperplane $\{\xi(z)\}^\perp$ is the sum of the generalized eigenspaces of $-\operatorname{Jac} \boldsymbol{b}(z) = \operatorname{Hess} f(z) + \mathsf{L}(z)$ corresponding to its eigenvalues in $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} > 0\}$. Moreover, it follows from [34, Lemma 4.1] that, in a neighborhood $\mathcal{O}_z$ of z in M,

$$(1.5) \qquad (\partial\Omega \cap \mathcal{O}_z) \smallsetminus \{z\} \subset \{f > f(z)\}.$$

In particular, z is a strict global minimum of $f|_{\partial\Omega}$. We refer to Figure 1.2 for a schematic representation of $\xi(z)$ and $\mathbf{C}_{\min}$ near such a point z when (**Normal**) holds.

Let us also mention here that, as explained in Section 1.3 below, $\nabla f(z) = 0$ implies that z is an equilibrium point for the dynamical system $\dot{X} = \boldsymbol{b}(X)$, i.e. that $\boldsymbol{b}(z) = 0$. Hence, from a dynamical point of view, the above discussion simply says that, when (**Normal**) holds: at every $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$ such that $\nabla f(z) = 0$, the boundary $\partial\Omega$ of Ω is tangent to the stable manifold of z for the dynamical system $\dot{X} = \boldsymbol{b}(X)$, which has dimension $d-1$. We recall that the stable (resp. unstable) manifold of an equilibrium point z is defined as the set of the elements of M whose trajectories (for the dynamics $\dot{X} = \boldsymbol{b}(X)$) converge to z in the future (resp. in the past), and that (the complexification of) its tangent space at z is the sum of the generalized eigenspaces of $\operatorname{Jac} \boldsymbol{b}(z)$ corresponding to its eigenvalues in $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} < 0\}$ (resp. in $\{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} > 0\}$).

Let us now consider $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$ such that $\nabla f(z) \neq 0$. Since z is a global minimum of $f|_{\partial\Omega}$, the tangent space $T_z\partial\Omega$ satisfies $T_z\partial\Omega = z + \{\nabla f(z)\}^\perp$, $\partial_n f(z) > 0$, and $\boldsymbol{b}(z) = -\nabla f(z) - \boldsymbol{\ell}(z)$ is inward-pointing. Thus, according to (**Ortho**), the condition $\boldsymbol{\ell}(z) = 0$ in the second part of (**Normal**) is equivalent to $\boldsymbol{b}(z) \in \operatorname{Span}(n_\Omega(z))$. It is thus in a way the counterpart of the first assumption of (**Normal**) when z is not an equilibrium point for the dynamics $\dot{X} = \boldsymbol{b}(X)$, since it gives the condition for $\boldsymbol{b}(z)$ to be orthogonal to $T_z\partial\Omega$.

In particular, when (**Normal**) holds, any $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$ is a strict global minimum of $f|_{\partial\Omega}$, whether $\nabla f(z) \neq 0$ or $\nabla f(z) = 0$. Thus, since $\partial\Omega$ is compact:

$$(1.6) \qquad (\textbf{Normal}) \Rightarrow \ \operatorname{Card}(\partial\mathbf{C}_{\min} \cap \partial\Omega) < +\infty.$$

1.3. **The deterministic dynamical system.** We give here basic properties on the ω-limit sets of the deterministic dynamical system $\dot{X} = \boldsymbol{b}(X)$ associated with the stochastic differential equation (1.1) when (**Ortho**) holds.

For every $x \in M$, we denote by $\varphi_t(x)$ the solution on M to the ordinary differential equation

$$(1.7) \qquad \frac{d}{dt}\varphi_t(x) = \boldsymbol{b}(\varphi_t(x)) \text{ with initial condition } \varphi_0(x) = x.$$

Notice that, since $\boldsymbol{b}$ is (globally) Lipschitz continuous over M, such curves are defined globally.

Let us now describe the ω-limit set of some $x \in M$ for the dynamical system (1.7). This set, denoted by $\omega(x)$, is defined by (see e.g. [54, Definition 8.1.1])

$$\omega(x) := \{y \in M,\ \exists (s_n)_{n\in\mathbb{N}} \in (\mathbb{R}_+)^{\mathbb{N}},\ \lim_{n\to\infty} s_n = +\infty,\ \lim_{n\to\infty} \varphi_{s_n}(x) = y\}.$$

Let us recall that, for all $x \in M$, $\omega(x)$ is nonempty, connected, closed, and invariant under the flow of (1.7) (see e.g. [54, Proposition 8.1.3]). Moreover, since $\boldsymbol{\ell} \cdot \nabla f = 0$ according to (**Ortho**):

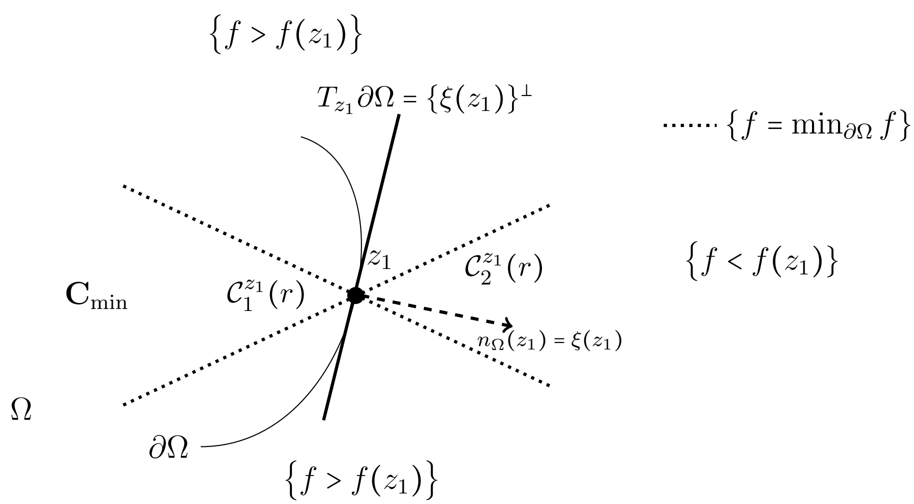

FIGURE 1.2. Schematic representation of $\partial\Omega$ near $z_1 \in \partial\mathbf{C}_{\min} \cap \partial\Omega$ when (**Normal**) holds and $\nabla f(z_1) = 0$ (recall that z_1 is then a saddle point of f).

for every $x \in M$ and $t \in \mathbb{R}$,

$$\frac{d}{dt} f(\varphi_t(x)) = -|\nabla f|^2(\varphi_t(x)). \tag{1.8}$$

Hence, following the proof of [54, Theorem 15.0.3], we have, as for gradient vector fields: for all $x \in M$, $\omega(x) \subset \{y \in M,\ \nabla f(y) = 0\}$. Since the Morse function $f : M \to \mathbb{R}$ has a finite number of critical points in M and $\omega(x)$ is nonempty and connected: for all $x \in M$, there exists a critical point $y \in M$ of f such that $\omega(x) = \{y\}$, so in particular $\lim_{t\to+\infty} \varphi_t(x) = y$.

Now, recall that an equilibrium point for the dynamical system (1.7) is by definition a point $z \in M$ such that $\boldsymbol{b}(z) = 0$, that is such that $\omega(z) = \{z\}$. It follows that

$$\{z \in M, \boldsymbol{b}(z) = 0\} \subset \{z \in M, \nabla f(z) = 0\}.$$

Moreover, since $\operatorname{Hess} f$ is invertible at any critical point of f, a Taylor expansion of $\boldsymbol{\ell} \cdot \nabla f = 0$ around such a point shows that $\boldsymbol{\ell}(z) = 0$ whenever $\nabla f(z) = 0$. Thus, when (**Ortho**) holds, we have the equality $\{z \in M, \nabla f(z) = 0\} = \{z \in M, \boldsymbol{b}(z) = 0\}$ and, for all $x \in M$, there exists $y \in M$ such that

$$\omega(x) = \{y\} \subset \{z \in M, \nabla f(z) = 0\} = \{z \in M, \boldsymbol{b}(z) = 0\}. \tag{1.9}$$

With the same reasoning when $t \to -\infty$: for all $x \in M$, there exist two critical points $y_\pm$ of f such that

$$\lim_{t\to+\infty} \varphi_t(x) = y_+ \quad \text{and} \quad \lim_{t\to-\infty} \varphi_t(x) = y_-. \tag{1.10}$$

Definition 2. *For every $x \in \Omega$, we set $t_x := \inf\{t \geq 0,\ \varphi_t(x) \notin \Omega\} > 0$. The domain of attraction of $F \subset \Omega$ is defined by*

$$\mathcal{A}(F) := \big\{x \in \Omega,\ t_x = +\infty \text{ and } \omega(x) \subset F\big\}. \tag{1.11}$$

Notice that when (**Ortho**) and (**One-Well**) hold, (1.8) and (1.9) imply that

$$\mathbf{C}_{\min} \subset \mathcal{A}(\{x_0\}). \tag{1.12}$$

1.4. **Main results.** We denote by $L^2(\Omega)$ the space of functions which are square integrable on Ω for the Lebesgue measure on Ω. The associated Sobolev spaces of regularity $k \geq 1$ are denoted by $H^k(\Omega)$. The space $H^1_0(\Omega)$ denotes the spaces of functions $w \in H^1(\Omega)$ such that $w = 0$ on $\partial\Omega$. We also denote by $L^2_w(\Omega)$ the space of functions which are square integrable on Ω for the measure $e^{-\frac{2}{h}f}dx$ on Ω. The notation w indicates that the weight $e^{-\frac{2}{h}f}dx$ appears in the inner product. The associated weighted Sobolev spaces of regularity $k \geq 1$ are denoted by $H^k_w(\Omega)$.

According to (**Ortho**), it is natural to work in $L^2_w(\Omega)$ to study the spectral properties of (minus) the infinitesimal generator L_h of the process (1.1) with Dirichlet conditions on $\partial\Omega$:

$$L_h = -\frac{h}{2}\Delta + \nabla f \cdot \nabla + \boldsymbol{\ell} \cdot \nabla \quad \text{with domain } D(L_h) = H^2_w(\Omega) \cap \{w \in H^1_w(\Omega), w = 0 \text{ on } \partial\Omega\}.$$

Its adjoint L_h^* on $L^2_w(\Omega)$, whose domain is still $D(L_h)$, has indeed the rather nice form

$$L_h^* = -\frac{h}{2}\Delta + \nabla f \cdot \nabla - \boldsymbol{\ell} \cdot \nabla - \operatorname{div} \boldsymbol{\ell}.$$

In particular, when (**Div-free**) holds, L_h^* is L_h with $\boldsymbol{\ell}$ replaced by $-\boldsymbol{\ell}$, and the process (1.1) is reversible when $\boldsymbol{\ell} = 0$.

To study the spectral properties of L_h, we actually use a unitary transformation to work in the flat space $L^2(\Omega)$, where computations such as integrations by parts are easier to perform. We denote by $\nabla_{f,h} := h\, e^{-\frac{f}{h}} \nabla e^{\frac{f}{h}} = h\nabla + \nabla f$ the distorted gradient *à la* Witten and

$$\Delta_{f,h} := \nabla_{f,h}^* \nabla_{f,h} = -h^2\Delta + |\nabla f|^2 - h\Delta f \tag{1.13}$$

the Witten Laplacian associated with f, where adjoints are now taken on $L^2(\Omega)$. Let us then define

$$P_h := 2h\, e^{-\frac{f}{h}}\, L_h\, e^{\frac{f}{h}} = \Delta_{f,h} + 2\boldsymbol{\ell} \cdot \nabla_{f,h} = \Delta_{f,h} + 2h\,\boldsymbol{\ell} \cdot \nabla \tag{1.14}$$

with domain $D(P_h) = H^2(\Omega) \cap H^1_0(\Omega)$ on $L^2(\Omega)$. According to (1.14), the operators $2h\, L_h$ and P_h are unitarily equivalent, and thus have the same spectral properties. In particular, for all $h > 0$, $\lambda \in \sigma(L_h)$ if and only if $2h\,\lambda \in \sigma(P_h)$, and the algebraic and geometric multiplicities of λ are the same for both L_h and $(2h)^{-1}P_h$.

The following result describes general spectral properties of $(P_h, D(P_h))$, and thus of $(L_h, D(L_h))$, for every fixed $h > 0$.

Proposition 3. *Assume that* (**Ortho**) *holds. Then, for every $h > 0$:*

- *The operator $P_h : D(P_h) \to L^2(\Omega)$ is maximal quasi-accretive. More precisely, the operator $P_h + h\|\operatorname{div} \boldsymbol{\ell}\|_\infty : D(P_h) \to L^2(\Omega)$ is maximal accretive. Furthermore, P_h has a compact resolvent and is sectorial.*
- *The adjoint of $P_h : D(P_h) \to L^2(\Omega)$ is the operator*

$$P_h^* = \Delta_{f,h} - 2\boldsymbol{\ell} \cdot \nabla_{f,h} - 2h \operatorname{div} \boldsymbol{\ell} \quad \text{with domain } D(P_h).$$

  It is also maximal quasi-accretive, with a compact resolvent, and sectorial.
- *There exists $\Sigma \subset \mathbb{C}$ such that the spectra of P_h and of P_h^* satisfy*

$$\sigma(P_h) = \{\lambda^P_{1,h}\} \cup \Sigma \quad \text{and} \quad \sigma(P_h^*) = \{\lambda^P_{1,h}\} \cup \overline{\Sigma},$$

where $\lambda^P_{1,h} \in \mathbb{R}^*_+$ *is simple (i.e. has algebraic multiplicity* 1*) for both* P_h *and* P_h^* *and, for every* $\lambda \in \Sigma$, $\operatorname{Re}\lambda > \lambda^P_{1,h}$.
Moreover, P_h *(resp.* P_h^**) admits an eigenfunction* $u^P_{1,h}$ *(resp.* $u^{P^*}_{1,h}$*) associated with* $\lambda^P_{1,h}$ *which is positive within* Ω.

The proof of Proposition 3 uses standard arguments on elliptic operators with Dirichlet boundary conditions on a smooth bounded domain. It is proved in the appendix for the sake of completeness.

The eigenvalue $\lambda^P_{1,h}$ is the so-called principal eigenvalue of P_h. According to (1.14), the principal eigenvalue $\lambda^L_{1,h}$ of L_h acting on $L^2_w(\Omega)$ thus satisfies $2h\,\lambda^L_{1,h} = \lambda^P_{1,h}$. Moreover, by compacity of the resolvent of L_h, its spectrum is discrete and can only accumulate at infinity. Hence, the sectoriality of L_h and the last item of Proposition 3 imply the existence of a spectral gap for every $h > 0$, that is:

$$\forall h > 0\,,\ \exists c_h > 0\,,\quad \sigma(L_h) \cap \big\{\mathsf{z} \in \mathbb{C}\,,\ \operatorname{Re}\mathsf{z} \in (\lambda^L_{1,h}, \lambda^L_{1,h} + c_h)\big\} = \varnothing.$$

Furthermore, the analysis led in Section 3 (see Theorem 4) permits to specify the behaviour of $\lambda^L_{1,h}$ and of this spectral gap with respect to h: when f admits m_0 local minima in Ω, there exist $c_1, c_2 > 0$ and $h_0 > 0$ such that, for every $h \in (0, h_0]$, L_h admits m_0 eigenvalues (counted with multiplicity) in $\{\mathsf{z} \in \mathbb{C},\ |\mathsf{z}| \le e^{-\frac{c_1}{h}}\}$ and its remaining eigenvalues live in $\{\mathsf{z} \in \mathbb{C},\ \operatorname{Re}\mathsf{z} \ge c_2\}$. In particular, when (**One-Well**) is also satisfied:

$$\exists c, h_0 > 0\,,\ \forall h \in (0, h_0]\,,\quad \lambda^L_{1,h} \le e^{-\frac{c}{h}} \quad \text{and} \quad \sigma(L_h) \cap \big\{\mathsf{z} \in \mathbb{C}\,,\ \operatorname{Re}\mathsf{z} \in (\lambda^L_{1,h}, \lambda^L_{1,h} + c)\big\} = \varnothing.$$

We can now state the two main results of this work.

Theorem 1. *Assume* (**Ortho**) *and* (**One-Well**)*. Let* K *be a compact subset of* $\mathcal{A}(\{x_0\})$ *(see* (1.11)*). Then, there exist* $c > 0$ *and* $h_0 > 0$ *such that for all* $h \in (0, h_0]$ *and uniformly in* $x \in K$*:*

$$\mathbb{E}_x[\tau_{\Omega^c}] = \frac{(1 + O(e^{-\frac{c}{h}}))}{\lambda^L_{1,h}}. \tag{1.15}$$

In addition, there exist $c > 0$ *and* $h_0 > 0$ *such that, for all* $h \in (0, h_0]$*:*

$$\sup_{t \ge 0\,,\, x \in K} \Big| \mathbb{P}_x\Big[\tau_{\Omega^c} > \frac{t}{\lambda^L_{1,h}}\Big] - e^{-t} \Big| \le e^{-\frac{c}{h}}. \tag{1.16}$$

Furthermore, there exist $c > 0$ *and* $h_0 > 0$ *such that, for all* $h \in (0, h_0]$,

$$\sigma(L_h) \cap \big\{\mathsf{z} \in \mathbb{C}, \operatorname{Re}\mathsf{z} \le c\big\} = \{\lambda^L_{1,h}\} \quad \text{and} \quad \lim_{h \to 0} h \ln \lambda^L_{1,h} = -2\big(\min_{\partial\Omega} f - f(x_0)\big). \tag{1.17}$$

Let us make some comments with regard to Theorem 1:

- Equation (1.15) provides the following *leveling result* on the mean exit time from Ω: $\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_y[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}}))$, uniformly in x, y in the compacts of $\mathcal{A}(\mathbf{C}_{\min})$ (see (1.12)). As long as (**Ortho**) is satisfied, this leveling result extends the one obtained in [10, Corollary 1] when (1.7) admits equilibrium points on $\partial\Omega$. It also extends [48, Theorem 2] when the underlying process is non-reversible.
- Equation (1.16) implies that when $h \to 0$, the law of $\lambda^L_{1,h}\tau_{\Omega^c}$ converges exponentially fast to the exponential law of mean 1, uniformly in the compacts of $\mathcal{A}(\{x_0\})$. Notice that (1.15) is not a consequence of (1.16).

- Deriving Theorem 1 for all $x \in \mathcal{A}(\mathbf{C}_{\min})$ and not only for $x = x_0$ is of real interest for applications relying on the process (1.1). Indeed, ones wants in practice an estimate on the time this process remains trapped in the metastable domain Ω. Since it admits a density with respect to the Lebesgue measure dx on M, the probability that its trajectories pass through x_0 is zero.

Our second main result states that, under the additional assumptions (**Div-free**) and (**Normal**), the eigenvalue $\lambda_{1,h}^L$ satisfies an Eyring-Kramers type formula.

Theorem 2. *Assume* (**Ortho**), (**One-Well**), (**Div-free**), *and* (**Normal**). *Then, when $h \to 0$, the eigenvalue $\lambda_{1,h}^L$ satisfies the following Eyring-Kramers type formula:*

$$\lambda_{1,h}^L = \left(\kappa_1^L h^{-\frac{1}{2}} + \kappa_2^L + O(h^{\frac{1}{4}})\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}, \tag{1.18}$$

where

$$\begin{cases} \kappa_1^L &= \dfrac{\sqrt{\det \operatorname{Hess} f(x_0)}}{\sqrt{\pi}} \displaystyle\sum_{\substack{z \in \partial\mathbf{C}_{\min} \cap \partial\Omega \\ \nabla f(z) \neq 0}} \frac{\partial_{n_\Omega} f(z)}{\sqrt{\det \operatorname{Hess} f_{|\partial\Omega}(z)}} \\ \kappa_2^L &= \dfrac{\sqrt{\det \operatorname{Hess} f(x_0)}}{2\pi} \displaystyle\sum_{\substack{z \in \partial\mathbf{C}_{\min} \cap \partial\Omega \\ \nabla f(z) = 0}} \frac{2|\mu(z)|}{\sqrt{|\det \operatorname{Hess} f(z)|}} \end{cases}, \tag{1.19}$$

and $\mu(z)$ denotes the negative eigenvalue of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$ *at a saddle point z of f (see Lemma 1).*

Let us now comment the results of Theorem 2.

- Our analysis actually shows that the error term $O(h^{\frac{1}{4}})$ in (1.18) is of order $O(h^{\frac{1}{2}})$ when $\kappa_1^L = 0$ or $\kappa_2^L = 0$, see Theorem 5. It is moreover always of order $O(h^{\frac{1}{2}})$ when the process is reversible, i.e. when $\boldsymbol{\ell} = 0$ (see [35] or Proposition 19 below). In addition, whether or not the process is reversible, when the error term in (1.18) is $O(h^{\frac{1}{2}})$, it is in general optimal (see for instance [35, Remark 25] for a discussion).
- Let $\lambda_{1,h}^\Delta$ be the principal eigenvalue of $-\frac{h}{2}\Delta + \nabla f \cdot \nabla$. When $\kappa_1^L = 0$ (that is when $\nabla f(z) = 0$ for every $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$), we have:

$$\frac{\lambda_{1,h}^\Delta}{\lambda_{1,h}^L} \sim \frac{\sum_{z \in \partial\mathbf{C}_{\min} \cap \partial\Omega} |\lambda(z)| \, |\det \operatorname{Hess} f(z)|^{-\frac{1}{2}}}{\sum_{z \in \partial\mathbf{C}_{\min} \cap \partial\Omega} |\mu(z)| \, |\det \operatorname{Hess} f(z)|^{-\frac{1}{2}}},$$

  where, for $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$, $\lambda(z)$ is the negative eigenvalue of $\operatorname{Hess} f(z)$. According to Lemma 1, we have $|\mu(z)| \geq |\lambda(z)|$, with equality if and only if ${}^t\mathsf{L}(z)\xi(z) = 0$. Then, in view of (1.15) and of [48, Theorem 1], we accelerate the exit from Ω by adding, locally around $\partial\mathbf{C}_{\min} \cap \partial\Omega$, a generic drift term $\boldsymbol{\ell}(X_t)$ to the reversible process $dX_t = -\nabla f(X_t)dt + \sqrt{h}\, dB_t$. In the mathematical literature, this acceleration phenomenon has been studied for elliptic non-reversible diffusions on $\mathbb{R}^d$ through the analysis of different quantities: the rate of convergence to equilibrium at fixed $h > 0$ or as $h \to 0$, and the asymptotic equivalents of the transition times as $h \to 0$, see [37, 4, 32, 34, 36] and references therein.

- Let us finally mention that combining the analyses developed in this work and in [34, 35, 3], it is clearly possible to extend the results of Theorem 2 to the cases when f has several local minima in Ω and $\boldsymbol{\ell}$ admits a classical expansion $\sum_{k\geq 0} h^k \boldsymbol{\ell}_k$, where $\boldsymbol{\ell}_k$ are smooth vector fields over M such that the Gibbs measure (1.2) remains invariant for the process (1.1) for all $h>0$.

1.5. **Strategy of the proof and organization of the paper.** The proof of Theorem 1 relies crucially on the formula

$$\frac{1}{\lambda^L_{1,h}} = \mathbb{E}_{\nu_h}[\tau_{\Omega^c}] = \frac{\int_\Omega \mathbb{E}_x[\tau_{\Omega^c}] u^{P^*}_{1,h} e^{-\frac{f}{h}}}{\int_\Omega u^{P^*}_{1,h} e^{-\frac{f}{h}}}, \quad \text{where } \nu_h(dx) = \frac{u^{P^*}_{1,h} e^{-\frac{f}{h}}}{\int_\Omega u^{P^*}_{1,h} e^{-\frac{f}{h}}}\, dx \tag{1.20}$$

is a quasi-stationary distribution for the process (1.1) in Ω (actually it is the quasi-stationary distribution, see Section 4.2 for more details on ν_h).

To extract $\mathbb{E}_x[\tau_{\Omega^c}]$ from the integral in (1.20), in order to prove (1.15) for instance, we use a leveling result on $x\mapsto \mathbb{E}_x[\tau_{\Omega^c}]$. This is the purpose of Theorem 3, proved in Section 2 using large deviations techniques. Besides, we also need a priori estimates on the principal eigenvalue $\lambda^L_{1,h} = \lambda^P_{1,h}/2h$ of L_h, which is the purpose of Theorem 4 in Section 3, relying on the sole assumption (**Ortho**) and proved by semiclassical methods.

We derive in Section 4.1 from these a priori estimates information on the concentration of the principal eigenfunction $u^{P^*}_{1,h}$ of P^*_h, see Proposition 18. Afterwards, combining this information with the leveling results on $x\mapsto \mathbb{E}_x[\tau_{\Omega^c}]$ and the a priori estimates on $\lambda^L_{1,h}$, we prove Theorem 1 in Section 4.2.

Finally, when assuming in addition (**Div-free**) and (**Normal**), we prove the sharp asymptotic equivalents on $\lambda^L_{1,h}$ given in Theorem 2 by constructing a very precise quasi-mode for P_h. This is done in Section 5, see Theorem 5.

2. Leveling results on the mean exit time from Ω

The goal of this section is to prove Theorem 3 below which aims at giving, when (**Ortho**) and (**One-Well**) hold, sharp leveling results on $x\mapsto \mathbb{E}_x[\tau_{\Omega^c}]$ as well as the limit of $h\ln \mathbb{E}_x[\tau_{\Omega^c}]$ when $h\to 0$. To do so, we use techniques from the large deviations theory. This requires some care, since these techniques cannot be used directly on Ω due to the possible existence of equilibrium points of $\boldsymbol{b}$ on $\partial\Omega$ (recall indeed that $\boldsymbol{b}(z)=0$ if and only if $\nabla f(z)=0$, see (1.9)).

2.1. **Large deviations and mean exit time.** In this section we only assume (**Ortho**).

2.1.1. *The quasi-potential on a subset of M.* We now introduce the *quasi-potential* associated with the vector field $\boldsymbol{b}$ on $\overline{D}$, where D denotes a smooth bounded subdomain of M (which is possibly M), and recall some of its basic properties. For $x,y\in\overline{D}$ and $t_1<t_2\in\mathbb{R}$, let us denote by $\mathcal{C}^{x,y}([t_1,t_2],\overline{D})$ the set of continuous curves $\phi:[t_1,t_2]\to\overline{D}$ such that $\phi(t_1)=x$ and $\phi(t_2)=y$. For $\phi\in\mathcal{C}^{x,y}([t_1,t_2],\overline{D})$, define, if ϕ is absolutely continuous,

$$S_{t_1,t_2}(\phi) = \frac{1}{2}\int_{t_1}^{t_2} |\dot\phi_s - \boldsymbol{b}(\phi_s)|^2 ds \in \mathbb{R}^+,$$

where $\dot\phi_s = \frac{d}{ds}\phi_s$, and, otherwise, $S_{t_1,t_2}(\phi)=+\infty$. The function

$$V_D : (x,y)\in\overline{D}\times\overline{D}\mapsto \inf\left\{S_{0,T}(\phi),\ \ \phi\in\mathcal{C}^{x,y}([0,T],\overline{D}) \text{ and } T>0\right\}\in\mathbb{R}^+$$

is the so-called (Freidlin-Wentzell) quasi-potential of the process (1.1) on D. Notice that

$$V_D(x,x) = 0 \text{ for all } x \in \overline{D}. \tag{2.1}$$

For every $x, y \in \overline{D}$ and $S, S' \subset \overline{D}$, we also define

$$V_D(x,S') := \inf_{y\in S'} V_D(x,y)\,,\ V_D(S,y) := \inf_{x\in S} V_D(x,y)\,,\ \text{and } V_D(S,S') := \inf_{(x,y)\in S\times S'} V_D(x,y).$$

In the next lemma, we recall some basic and useful properties of the functional V_D.

Lemma 4. *One has the following:*

- $V_D : \overline{D}\times\overline{D} \to \mathbb{R}^+$ *is continuous.*
- *Assume that there exists a subset S of $\overline{D}$ such that, for any $T \geq 0$ and $\phi \in \mathcal{C}^{x,y}([0,T],\overline{D})$, there exists $t \in [0,T]$ such that $\phi_t \in S$. Then, it holds*

$$V_D(x,y) = \inf_{z\in S}[V_D(x,z) + V_D(z,y)].$$

- *For every $x \in \overline{D}$ and every $-\infty \leq t_- \leq t_+ \leq +\infty$ such that the solution $\varphi_t(x)$ of (1.7) satisfies $\{\varphi_t(x), t \in [t_-,t_+]\} \subset \overline{D}$, where $\varphi_{t_\pm}(x) := \lim_{t\to\pm\infty}\varphi_t(x)$ when $t_\pm = \pm\infty$ (see (1.10)), it holds*

$$V_D(\varphi_{t_-}(x), \varphi_{t_+}(x)) = 0.$$

- *Let $T > 0$ and G be a closed nonempty subset of $\mathcal{C}([0,T],M)$ (endowed with the uniform convergence topology). Then, the infimum*

$$\inf\left\{S_{0,T}(\phi),\ \ \phi \in G\right\}$$

is a minimum. In particular, this infimum is strictly positive as soon as G does not contain any trajectory of the dynamical system (1.7) defined on $[0,T]$.

The first item is a consequence of [20, Lemma 1.1 in Section 1 of Chapter 6] and implies the third one, while the second item can be proved by straightforward arguments. For the last one, we refer to the comments following the proof of [20, Theorem 1.1 in Chapter 4].

Lemma 5. *Assume* (**Ortho**)*. Then, for all $\phi \in \mathcal{C}^{x,y}([t_1,t_2],M)$, $S_{t_1,t_2}(\phi) \geq 2(f(y) - f(x))$.*

Proof. Using (**Ortho**), we have, for all $\phi \in \mathcal{C}^{x,y}([t_1,t_2],M)$,

$$S_{t_1,t_2}(\phi) = \frac{1}{2}\int_{t_1}^{t_2} |\dot\phi_s - (\nabla f(\phi_s) - \boldsymbol{\ell}(\phi_s))|^2 ds + 2\int_{t_1}^{t_2} \dot\phi_s\cdot\nabla f(\phi_s) ds \geq 2(f(\phi(t_2)) - f(\phi(t_1))),$$

which implies the result. □

Remark 6. *The proof of Lemma 5 also leads to the following: for every $x \in \overline{D}$ and every $-\infty \leq t_- \leq t_+ \leq +\infty$ such that the solution $\psi_t(x)$ of $\dot X = \nabla f(X) - \boldsymbol{\ell}(X)$ with initial condition $\psi_0(x) = x$ satisfies $\{\psi_t(x), t \in [t_-,t_+]\} \subset \overline{D}$, where $\psi_{t_\pm}(x) := \lim_{t\to\pm\infty}\psi_t(x)$ when $t_\pm = \pm\infty$, it holds*

$$V_D(\psi_{t_-}(x), \psi_{t_+}(x)) = 2(\psi_{t_+}(x) - \psi_{t_-}(x)).$$

2.1.2. *On the structure of the dynamical system.* To prove Theorem 3 we want to use [20, Theorem 5.3 in Chapter 6] with a suitable domain D such that

$$\nabla f \neq 0 \text{ on } \partial D. \tag{2.2}$$

The construction of D is the purpose of the next section. Before, we have to check that the conditions stated at the beginning of [20, Section 2 in Chapter 6] are satisfied. More precisely, we have to check that the exists a finite number of compact subsets $K_1, \dots, K_l$ of D such that:

- **(a)** For any $x \in \overline{D}$ such that $\varphi_t(x) \in \overline{D}$ for all $t \geq 0$, it holds $\omega(x) \subset K_q$ for some $q \in \{1, \dots, l\}$.
- **(b)** For all $i \in \{1, \dots, l\}$ and all $x, y \in K_i$, $V_D(x, y) = 0$.
- **(c)** If $x \in K_i$ and $y \notin K_i$ ($y \in \overline{D}$), either $V_D(x, y) > 0$ or $V_D(y, x) > 0$.

In the following, we write $\{y \in D, \nabla f(y) = 0\} = \{y_1, \dots, y_l\}$ and we define

$$K_i = \{y_i\}, \ \forall i \in \{1, \dots, l\}. \tag{2.3}$$

Lemma 7. *Assume* (**Ortho**) *and* (2.2)*. When the compact sets K_i, $i = 1, \dots, l$, are defined by* (2.3)*, Conditions* **(a)** *and* **(b)** *above are satisfied.*

Proof. By (1.9) and (2.2), if $\{\varphi_t(z), t \geq 0\} \subset \overline{D}$, $\omega(x) = \{y\}$ for some critical point y of f in D. Thus, Condition **(a)** holds. In addition, according to (2.1), Condition **(b)** holds. □

Condition **(c)** is the purpose of the next proposition.

Proposition 8. *Assume* (**Ortho**) *and* (2.2)*. When the compact sets K_i, $i = 1, \dots, l$, are defined by* (2.3)*, Condition* **(c)** *holds.*

The following lemma will be useful to prove Proposition 8.

Lemma 9. *Assume* (**Ortho**)*. Let $z \in \overline{D}$ be such that $\nabla f(z) \neq 0$ and, for some $T > 0$, $\{\varphi_t(z), t \in [0, T]\} \subset \overline{D}$. Then, for all $y \in \overline{D} \setminus \{\varphi_t(z), t \in [0, T]\}$ satisfying $f(y) > f(\varphi_T(z))$, it holds $V_D(z, y) > 0$.*

Proof. Set $\rho_0 = \inf\{|y - \varphi_t(z)|, 0 \leq t \leq T\} > 0$. Let $T' \in (0, T]$. From the last item of Lemma 4:

$$d_{T'} := \inf \left\{ S_{0,T'}(\phi), \phi \in \mathcal{C}([0, T'], M) \text{ s.t. } \phi_0 = z \text{ and } \max_{t \in [0, T']} |\phi_t - \varphi_t(z)| \geq \rho_0/2 \right\} > 0.$$

We then have, for $T' \in (0, T]$ and $\phi \in \mathcal{C}^{z,y}([0, T'], \overline{D})$, $S_{0,T'}(\phi) \geq d_{T'} \geq d_T > 0$. Consequently,

$$\inf \left\{ S_{0,T'}(\phi), \ \phi \in \mathcal{C}^{z,y}([0, T'], \overline{D}) \text{ and } T' \in (0, T] \right\} > 0. \tag{2.4}$$

Let us now consider the infimum above when $T' \geq T$. Let $0 < T_1 < T_2 < T$ be such that $f(y) > f(\varphi_{T_1}(z))$. Notice that (1.8) and $\nabla f(z) \neq 0$ imply

$$f(z) > f(\varphi_{T_1}(z)) > f(\varphi_{T_2}(z)).$$

It follows that

$$\varphi(z)|_{[0,T_2]} \notin G^z_{T_2} := \left\{ \phi \in \mathcal{C}([0, T_2], \overline{D}), \ \phi_0 = z \text{ and, for all } t \in [0, T_2], f(\phi_t) \geq f(\varphi_{T_1}(z)) \right\}$$

and the last item of Lemma 4 then implies that

$$A := \inf \left\{ S_{0,T_2}(\phi), \phi \in G^z_{T_2} \right\} > 0.$$

Consider $T' \geq T$ and $\phi \in \mathcal{C}^{z,y}([0,T'],\overline{D})$. Assume that $\phi \in G^z_{T'}$. Then $\phi|_{[0,T_2]} \in G^z_{T_2}$, and thus $S_{0,T'}(\phi) \geq S_{0,T_2}(\phi) \geq A$. Assume now that $\phi \notin G^z_{T'}$, i.e. that $f(\phi_t) < f(\varphi_{T_1}(z))$ for some $t \in [0,T']$. Let $t_1 \in (0,T')$ be such that $f(\phi_{t_1}) = f(\varphi_{T_1}(z))$. Using Lemma 5, it holds

$$S_{0,T'}(\phi) \geq S_{t_1,T'}(\phi) \geq 2(f(\phi_{T'}) - f(\phi_{t_1})) = 2(f(y) - f(\varphi_{T_1}(z))) > 0.$$

In conclusion, for all $T' \geq T$ and $\phi \in \mathcal{C}^{z,y}([0,T'],\overline{D})$, $S_{0,T'}(\phi) \geq \min(f(y) - f(\varphi_{T_1}(z)), A) > 0$. Together with (2.4), this ends the proof of the lemma. □

We are now in position to prove Proposition 8.

Proof of Proposition 8. Let $x \in \overline{D}$ be such that $\nabla f(x) = 0$, so that $x \in D$ according to (2.2). Let us also consider $y \in \overline{D}$ such that $y \neq x$. According to Lemma 5, it suffices to consider the case when $f(x) = f(y)$. Since $x \in D$ and f admits a finite number of critical points in M, there exists a sphere $C(x,r) = \{w \in M, |w-x| = r\} \subset D$ of radius $0 < r < |x-y|$ such that $|\nabla f| > 0$ on $C(x,r)$. Then, using the two first items of Lemma 4, there exists $z \in C(x,r)$ such that

$$V_D(x,y) = \inf_{\xi \in C(x,r)} (V_D(x,\xi) + V_D(\xi,y)) = V_D(x,z) + V_D(z,y).$$

If $f(z) < f(x) = f(y)$, then Lemma 5 implies $V_D(x,y) \geq V_D(z,y) \geq 2(f(y) - f(z)) > 0$. Similarly, if $f(z) > f(x)$, then $V_D(x,y) \geq V_D(x,z) \geq 2(f(z) - f(x)) > 0$. Let us lastly consider the case when $f(z) = f(x)$. Since $z \in D$ and $\nabla f(z) \neq 0$, there exists $T > 0$ such that

$$\{\varphi_t(z), t \in [0,T]\} \subset \overline{D} \text{ and, according to (1.8), } f(z) > f(\varphi_t(z)) \text{ for all } t \in (0,T].$$

Using $f(z) = f(y)$ and $z \neq y$, it follows that $y \notin \{\varphi_t(z), t \in [0,T]\}$ and $f(y) > f(\varphi_T(z))$. Therefore, according to Lemma 9, $V_D(z,y) > 0$ and thus $V_D(x,y) > 0$, which completes the proof of Proposition 8. □

Following the terminology of [20], we say that a subset $N \subset M$ is *stable* if, for any $x \in N$ and $y \in M \setminus N$, $V_M(x,y) > 0$ (see the lines preceding [20, Lemma 4.2 in Chapter 6]). We then have:

Lemma 10. *Assume* (**Ortho**). *For any critical point x of f in M, the set $\{x\}$ is stable (in the sense defined above) if and only if x is a local minimum of f in M.*

Proof. Assume that x is a local minimum of the Morse function f in M, and take $y \in M \setminus \{x\}$. Since x is a strict minimum, there exists $0 < r < |x-y|$ such that $f > f(x)$ on $C(x,r) = \{w \in M, |w-x| = r\}$. Thus, according to Lemma 4, there exists $z^* \in C(x,r)$ such that

$$V_M(x,y) = \inf_{z \in C(x,r)} (V_M(x,z) + V_M(z,y)) = V_M(x,z^*) + V_M(z^*,y).$$

Using in addition Lemma 5, $V_M(x,z^*) \geq f(z^*) - f(x) > 0$ and thus $V_M(x,y) > 0$, which implies that $\{x\}$ is stable.

Let us now assume that x is not a local minimum of f in M. Then, according to Lemma 1, the dimension of the unstable manifold of x for the dynamical system $\dot{X} = \boldsymbol{b}(X)$ is at least one, and thus there exists $z^* \in M \setminus \{x\}$ such that $\varphi_t(z^*) \to x$ when $t \to -\infty$. It thus follows from the third item of Lemma 4 that $V_M(x,z^*) = 0$, showing that x is not stable. □

2.1.3. *Freidlin-Wentzell graphs and mean exit time.* Let us first introduce some notation. Let $\mathbf{L}$ be a finite set and $\mathbf{W} \subset \mathbf{L}$. A graph consisting of arrows $m \to n$ (for $m \in \mathbf{L} \setminus \mathbf{W}$, $n \in \mathbf{L}$, and $m \neq n$) is called a $\mathbf{W}$-graph over $\mathbf{L}$ (see the beginning of [20, Section 3 in Chapter 6]) if:

- every point $m \in \mathbf{L} \setminus \mathbf{W}$ is the initial point of exactly one arrow,
- there are no closed cycles in the graph.

The last condition can be replaced by the following one: for every point $m \in \mathbf{L} \setminus \mathbf{W}$, there exists a sequence of arrows leading from m to some $n \in \mathbf{W}$. The set of $\mathbf{W}$-graphs over $\mathbf{L}$ is denoted by $G^{\mathbf{L}}(\mathbf{W})$.

When Conditions **(a)**, **(b)**, and **(c)** hold, and when at least one of the compact subsets $K_1, \dots, K_l$ of D is stable, we label these sets so that $K_1, \dots, K_{p_s}$ are the stable compact sets among $K_1, \dots, K_l$, where $1 \leq p_s \leq l$. In this case, [20, Theorem 5.3 in Chapter 6] applies, and implies that, for every $x \in D$ and uniformly in x in the compact subsets of D,

$$\lim_{h\to 0} h \ln \mathbb{E}_x[\tau_{D^c}] \leq W_D, \quad \text{where } W_D := \min_{g \in G^{\{K_1,\dots,K_{p_s},\partial D\}}(\{\partial D\})} \sum_{(m\to n)\in g} V_D(m,n). \tag{2.5}$$

Corollary 11. *Assume* (**Ortho**), (2.2), *and that f admits $n+1$ local minima $x_0, x_1, \dots, x_n$ in D, with $n \geq 0$. Then, for all $x \in D$, and uniformly in x in the compact subsets of D,*

$$\lim_{h\to 0} h \ln \mathbb{E}_x[\tau_{D^c}] \leq \sum_{k=0}^{n} V_D(x_k, \partial D).$$

Proof. Let us define the compact sets K_i, $i = 1, \dots, l$, by (2.3). According to Lemma 7 and Proposition 8, Conditions **(a)**, **(b)**, and **(c)** are satisfied. Moreover, according to Lemma 10, the $\{x_k\}$, $0 \leq k \leq n$, are the stable compact sets among $K_1, \dots, K_l$, and thus $p_s = n+1$ and $\{K_1, \dots, K_{p_s}\} = \{\{x_k\}, 0 \leq k \leq n\}$. We conclude by applying (2.5) with the graph $(\{x_0\} \to \partial\Omega), \dots, (\{x_n\} \to \partial\Omega)$. □

2.2. Upper bound on the mean exit time when (Ortho) and (One-Well) hold.

Proposition 12. *Assume that* (**Ortho**) *and* (**One-Well**) *hold. Then, for every $\beta > 0$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0]$,*

$$\sup_{x \in \overline{\Omega}} \mathbb{E}_x[\tau_{\Omega^c}] \leq e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} e^{\frac{\beta}{h}}.$$

Proof. Let us assume that (**Ortho**) and (**One-Well**) hold. We set

$$D_\alpha := \{x \in M, \operatorname{dist}(x, \overline{\Omega}) < \alpha\}, \ \alpha > 0.$$

For every $\alpha > 0$, we have $\overline{\Omega} \subset D_\alpha$ and $\partial D_\alpha = \{x \in M, \operatorname{dist}(x, \overline{\Omega}) = \alpha\}$. In addition, there exists $\alpha_0 > 0$ such that, for every $\alpha \in (0, \alpha_0]$, D_α is a $\mathcal{C}^\infty$ subdomain of M and, since the critical points of f are isolated in M, $\{x \in \overline{D}_\alpha, \nabla f(x) = 0\} \subset \overline{\Omega}$. In particular, $|\nabla f| > 0$ on ∂D_α and the local minima of f in D_α are its local minimum x_0 in Ω and its local minima $x_1, \dots, x_n$ on $\partial\Omega$. Because $\overline{\Omega}$ is a compact subset of D_α, it follows from Corollary 11 that for every $\alpha \in (0, \alpha_0]$ and $\epsilon > 0$, we have for all h small enough:

$$\sup_{x \in \overline{\Omega}} \mathbb{E}_x[\tau_{\Omega^c}] \leq \sup_{x \in \overline{\Omega}} \mathbb{E}_x[\tau_{D_\alpha^c}] \leq e^{\frac{2}{h}\sum_{k=0}^{n} V_{D_\alpha}(x_0, \partial D_\alpha)} e^{\frac{\epsilon}{h}}.$$

In order to prove Proposition 12, it then enough to show that

$$V_{D_\alpha}(x_0,\partial D_\alpha)+\sum_{k=1}^n V_{D_\alpha}(x_k,\partial D_\alpha)\le 2(\min_{\partial\Omega} f-f(x_0))+o_\alpha(1). \tag{2.6}$$

Using the second item of Lemma 4, we have, for every $y\in\partial D_\alpha$ and $z\in\partial\Omega$,

$$V_{D_\alpha}(x_0,\partial D_\alpha)\le V_{D_\alpha}(x_0,y)\le V_{D_\alpha}(x_0,z)+V_{D_\alpha}(z,y).$$

Moreover, according to Lemma 5 and to Remark 6, for every $z\in\partial\mathbf{C}_{\min}\cap\partial\Omega$,

$$V_{D_\alpha}(x_0,z)=2(f(z)-f(x_0))=2(\min_{\partial\Omega} f-f(x_0)).$$

Consequently, for every $z\in\partial\mathbf{C}_{\min}\cap\partial\Omega$ and $\alpha>0$ small enough,

$$V_{D_\alpha}(x_0,\partial D_\alpha)\le 2(\min_{\partial\Omega} f-f(x_0))+V_{D_\alpha}(z,\partial D_\alpha)\le 2(\min_{\partial\Omega} f-f(x_0))+\frac12(1+\|\boldsymbol{b}\|_\infty)^2\alpha\,,$$

where we used the fact that for every $x\neq y\in M$, $\phi: t\in[0,|y-x|]\mapsto x+\frac{y-x}{|y-x|}t$ satisfies $S_{0,|y-x|}(\phi)\le\frac12(1+\|\boldsymbol{b}\|_\infty)^2|x-y|$. The same argument shows that $V_{D_\alpha}(x_k,\partial D_\alpha)\le\frac12(1+\|\boldsymbol{b}\|_\infty)^2\alpha$ for every $1\le k\le n$ (since $x_k\in\partial\Omega$). This implies (2.6) and thus completes the proof of Proposition 12. □

2.3. Leveling results for $x\mapsto\mathbb{E}_x[\tau_{\Omega^c}]$ and *commitor* functions.

The following result provides a local leveling result for $x\mapsto\mathbb{E}_x[\tau_{\Omega^c}]$.

Lemma 13. *Assume* (**Ortho**) *and* (**One-Well**)*. Let $\delta_1>0$ and $r_h=e^{-\delta_1/h}$. Then, there exist $h_0>0$ and $c>0$ such that, for all $h\in(0,h_0]$, $\sup_{x\in\bar B(x_0,r_h)}|\mathbb{E}_x[\tau_{\Omega^c}]-\mathbb{E}_{x_0}[\tau_{\Omega^c}]|\le e^{-\frac ch}\mathbb{E}_{x_0}[\tau_{\Omega^c}]$.*

Proof. Since (**Ortho**) holds, $\boldsymbol{b}(x_0)=0$ (see (1.9)). In addition, according to Lemma 1, the eigenvalues of the matrix $\mathrm{Jac}\,\boldsymbol{b}(x_0)=-(\mathrm{Hess}\,f(x_0)+\mathsf{L}(x_0))$ all belong to $\{\mathsf{z}\in\mathbb{C},\ \mathrm{Re}\,\mathsf{z}<0\}$ (in particular, x_0 is an asymptotically stable equilibrium point of the dynamical system (1.7)). The proof then follows the same lines as the one of [48, Lemma 3]. □

Denote by $\tau_{\bar B(x_0,r_h)}$ the first time the process (1.1) hits the closed ball $\bar B(x_0,r_h)$, where we recall that $r_h=e^{-\delta_1/h}$, $\delta_1>0$. The constant $\delta_1>0$ will be fixed in (2.9) below. We assume that h is small enough so that $\bar B(x_0,r_h)\subset\mathbf{C}_{\min}$. The function

$$x\mapsto\mathbb{P}_x[\tau_{\bar B(x_0,r_h)}<\tau_{\Omega^c}]$$

is called the commitor function (or the *equilibrium potential*) between Ω and $\bar B(x_0,r_h)$. The following result provides a (global) leveling result for $x\mapsto\mathbb{E}_x[\tau_{\Omega^c}]$ in $\mathcal{A}(\{x_0\})$.

Proposition 14. *Assume* (**Ortho**) *and* (**One-Well**)*. Then, there exists $\delta_1>0$ such that, for all compact subset K of $\mathcal{A}(\{x_0\})$ (see* (1.11) *and* (1.12)*), there exist $h_0>0$ and $c>0$ such that for all $h\in(0,h_0]$,*

$$\sup_{x\in K}\left|\mathbb{P}_x[\tau_{\bar B(x_0,r_h)}<\tau_{\Omega^c}]-1\right|\le e^{-\frac ch}.$$

Remark 15. *Applying* [9, Theorem 2] *with $\Omega=\mathcal{A}(\{x_0\})$ leads to a slightly weaker version of Proposition 14, where $\delta_1>0$ depends on K.*

Proof. For $\eta \in (0, \min_{\partial\Omega} f - f(x_0))$, set

$$\mathbf{C}_{\min}(\eta) := \mathbf{C}_{\min} \cap \{f < \min_{\partial\Omega} f - \eta\} = \{x \in \Omega, f(x) < \min_{\partial\Omega} f - \eta\}. \tag{2.7}$$

The set $\mathbf{C}_{\min}(\eta)$ is open, smooth (since $\nabla f \neq 0$ on $\partial\mathbf{C}_{\min}(\eta)$), and is the connected component of $\{f < \min_{\partial\Omega} f - \eta\}$ containing x_0 (see for instance [15, Proposition 18]). Recall also that x_0 is an asymptotically stable equilibrium point of the dynamical system (1.7). Moreover, (1.8) implies that $\varphi_t(x) \in \overline{\mathbf{C}}_{\min}(\eta)$ for all $x \in \overline{\mathbf{C}}_{\min}(\eta)$ and $t \in \mathbb{R}^+$, and thus that $\lim_{t\to+\infty} \varphi_t(x) = x_0$ since x_0 is the unique critical point of f in $\overline{\mathbf{C}}_{\min}(\eta)$ (see indeed (1.10)).

Fix now

$$\eta_0 \in (0, \min_{\partial\Omega} f - f(x_0)) \text{ and } \eta_* \in (\eta_0, \min_{\partial\Omega} f - f(x_0)). \tag{2.8}$$

It holds $\overline{\mathbf{C}}_{\min}(\eta_*) \subset \mathbf{C}_{\min}(\eta_0)$. In the following $h > 0$ is small enough so that $\bar{B}(x_0, r_h) \subset \mathbf{C}_{\min}(\eta_*)$, where we recall that $r_h = e^{-\delta_1/h}$. According to [9, Theorem 2], there exist $\delta_1 > 0$ (which is now kept fixed), $h_0 > 0$, and $c > 0$ such that for all $h \in (0, h_0]$:

$$\sup_{y\in\overline{\mathbf{C}}_{\min}(\eta_*)} \mathbb{P}_y\big[\tau_{\mathbf{C}^c_{\min}(\eta_0)} \le \tau_{\bar{B}(x_0,r_h)}\big] \le e^{-\frac{c}{h}}. \tag{2.9}$$

Since the trajectories of the process (1.1) are continuous, one has $\{\tau_{\Omega^c} < \tau_{\bar{B}(x_0,r_h)}\} \subset \{\tau_{\mathbf{C}^c_{\min}(\eta_0)} < \tau_{\bar{B}(x_0,r_h)}\}$ for all $y \in \overline{\mathbf{C}}_{\min}(\eta_*)$ when $X_0 = y$, so that (using also $\{\tau_{\Omega^c} = \tau_{\bar{B}(x_0,r_h)}\} = \varnothing$):

$$\sup_{y\in\overline{\mathbf{C}}_{\min}(\eta_*)} \mathbb{P}_y\big[\tau_{\Omega^c} \le \tau_{\bar{B}(x_0,r_h)}\big] \le e^{-\frac{c}{h}}, \tag{2.10}$$

which proves the proposition when $K = \overline{\mathbf{C}}_{\min}(\eta_*)$. Let us now consider the case when $K \subset \mathcal{A}(\{x_0\})$. In view of (2.10), it is enough to treat the case when $K \subset \Omega \setminus \mathbf{C}_{\min}(\eta_*)$. Pick $K \subset \Omega \setminus \mathbf{C}_{\min}(\eta_*)$ with $K \subset \mathcal{A}(\{x_0\})$. Recall that this implies that for all $x \in K$, $\varphi_t(x) \in \Omega$ for all $t \ge 0$ and $\lim_{t\to+\infty} \varphi_t(x) = x_0$. Then, there exists $T_K > 0$ such that $\varphi_{T_K}(x) \in \mathbf{C}_{\min}(\eta_*)$ for all $x \in K$. The set $\{\varphi_{T_K}(x), x \in K\}$ is a compact subset of the open set $\mathbf{C}_{\min}(\eta_*)$ and the compact subset $\{\varphi_t(x), (x,t) \in K \times [0, T_K]\}$ of Ω does not contain $x_0 \notin K$. We can thus consider $\delta > 0$ small enough such that:

- C1. $\{\varphi_{T_K}(x) + z, x \in K \text{ and } |z| \le \delta\} \subset \mathbf{C}_{\min}(\eta_*)$,
- C2. $x_0 \notin K_{T_K,\delta} := \{\varphi_t(x) + z, (x,t) \in K \times [0, T_K] \text{ and } |z| \le \delta\}$.

By item C2 above, for any h small enough, $\bar{B}(x_0, r_h) \cap K_{T_K,\delta} = \varnothing$. Then, for all $x \in K$, if $X_0 = x$ and $\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta$:

$$T_K < \tau_{\bar{B}(x_0,r_h)}. \tag{2.11}$$

Moreover, according to [10, Lemma 1] and its note, since M is compact, there exists $c' > 0$ such that for all h small enough:

$$\sup_{x\in M} \mathbb{P}_x\Big[\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| > \delta\Big] \le e^{-\frac{c'}{h}}. \tag{2.12}$$

On the other, by item C1 above, if $X_0 = x \in K$ and $\sup_{t\in[0,T_K]}|X_t - \varphi_t(x)| \le \delta$, it holds $X_{T_K} \in \mathbf{C}_{\min}(\eta_*)$. Then, for all $x \in K$, using the Markov property and (2.11), we have

$$\begin{aligned}\mathbb{P}_x\Big[\tau_{\bar B(x_0,r_h)} < \tau_{\Omega^c}, \sup_{t\in[0,T_K]}|X_t - \varphi_t(x)| \le \delta\Big] &= \mathbb{E}_x\Big[\mathbb{E}_{X_{T_K}}\big[\mathbf{1}_{\tau_{\bar B(x_0,r_h)}<\tau_{\Omega^c}}\big]\mathbf{1}_{\sup_{t\in[0,T_K]}|X_t-\varphi_t(x)|\le\delta}\Big]\\ &\ge (1-e^{-\frac{c}{h}})\mathbb{P}_x\Big[\sup_{t\in[0,T_K]}|X_t - \varphi_t(x)| \le \delta\Big]\\ &\ge (1-e^{-\frac{c}{h}})(1-e^{-\frac{c'}{h}}),\end{aligned}$$

where we used respectively (2.10) and (2.12) at the second and third equalities. In conclusion, we have proved that for some $c > 0$ and every h small enough, $\sup_{x\in K}|\mathbb{P}_x[\tau_{\bar B(x_0,r_h)} < \tau_{\Omega^c}]-1| \le e^{-\frac{c}{h}}$, which completes the proof of Proposition 14. ☐

Proposition 16. *Assume* (**Ortho**) *and* (**One-Well**). *Then, for every $\eta_* \in (0, \min_{\partial\Omega} f - f(x_0))$, there exist $h_0 > 0$ and $c > 0$ such that, for all $h \in (0, h_0]$,*

$$\sup_{x\in\overline{\mathbf{C}}_{\min}(\eta_*)} \mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] \le e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}e^{-\frac{c}{h}},$$

where $\mathbf{C}_{\min}(\eta_*)$ is defined in (2.7).

Proof. The proof of Proposition 16 is inspired by the one of [10, Lemma 6]. Take $\eta_0 \in (0, \eta^*)$. For ease of notation, we set

$$K := \overline{\mathbf{C}}_{\min}(\eta_*) \quad \text{and} \quad D' := \mathbf{C}_{\min}(\eta_0).$$

Recall that $K \subset D' \subset \mathcal{A}(\{x_0\})$ and assume that $h > 0$ is small enough so that $\bar B(x_0, r_h) \subset \operatorname{int} K$ (see (2.8) and the lines below). According to [20, Theorems 3.1 and 4.1 in Chapter 4] (note that $n_{\mathbf{C}_{\min}(\eta)} = \nabla f/|\nabla f|$ and then, using (**Ortho**), $\boldsymbol{b}\cdot n_{\mathbf{C}_{\min}(\eta)} > 0$ on $\partial\mathbf{C}_{\min}(\eta)$), we have uniformly in y in the compacts of D':

$$\lim_{h\to0} h\ln\mathbb{E}_y[\tau_{D'^c}] = 2(\min_{\partial\Omega} f - f(x_0) - \eta_0). \tag{2.13}$$

In particular, for every $\beta > 0$ and every h small enough,

$$A_h^{D'} := \sup_{y\in K}\mathbb{E}_y[\tau_{D'^c}] \le e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0)-\eta_0)}e^{\frac{\beta}{h}}. \tag{2.14}$$

Similarly, according to Proposition 12, it holds for every $\beta > 0$ and every h small enough,

$$A_h^{\Omega} := \sup_{x\in\overline{\Omega}}\mathbb{E}_x[\tau_{\Omega^c}] \le e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}e^{\frac{\beta}{h}}. \tag{2.15}$$

Besides, using the strong Markov property, we have for all $x \in K$:

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] + \mathbb{E}_x\Big[\mathbf{1}_{\tau_{\Omega^c}>\tau_{\bar B(x_0,r_h)}}\mathbb{E}_{X_{\tau_{\bar B(x_0,r_h)}}}[\tau_{\Omega^c}]\Big]. \tag{2.16}$$

In addition, by continuity of the trajectories of the process (1.1), we have $\tau_{D'^c} < \tau_{\Omega^c}$ when $X_0 = y \in \bar B(x_0, r_h)$. Thus, using the strong Markov property,

$$\mathbb{E}_{X_{\tau_{\bar B(x_0,r_h)}}}[\tau_{\Omega^c}] \ge \mathbb{E}_{X_{\tau_{\bar B(x_0,r_h)}}}[\tau_{\Omega^c} - \tau_{D'^c}] = \mathbb{E}_{X_{\tau_{\bar B(x_0,r_h)}}}\Big[\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\Big]. \tag{2.17}$$

For $x \in D'$, let μ_x^h be the hitting distribution on $\partial D'$ for the process (1.1) when $X_0 = x$, i.e.:

$$\mu_x^h(B) = \mathbb{P}_x[X_{\tau_{D'^c}} \in B], \text{ for every Borel subset } B \text{ of } \partial D'. \tag{2.18}$$

The properties of D' listed just after (2.7) allow us to use [9, Theorem 1] (see also Eq. (5.1) there), leading to $\|\mu_x^h - \mu_y^h\| \le e^{-\frac{c}{h}}$ uniformly in $x, y \in K$ (where $\|\cdot\|$ is the total variation distance). Using this and (2.17) with $y = X_{\tau_{\bar B(x_0,r_h)}}$, we deduce from (2.16) that for all $x \in K$:

$$\begin{aligned}\mathbb{E}_x[\tau_{\Omega^c}] &\ge \mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] + \mathbb{E}_x\Big[\mathbf{1}_{\tau_{\Omega^c} > \tau_{\bar B(x_0,r_h)}} \mathbb{E}_{X_{\tau_{\bar B(x_0,r_h)}}}\big[\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\big]\Big]\\ &\ge \mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] + \mathbb{E}_x\Big[\mathbf{1}_{\tau_{\Omega^c} > \tau_{\bar B(x_0,r_h)}}\Big(\mathbb{E}_x\big[\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\big] - A_h^{\Omega} e^{-\frac{c}{h}}\Big)\Big]\\ &\ge \mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] + \mathbb{P}_x[\tau_{\Omega^c} > \tau_{\bar B(x_0,r_h)}]\,\mathbb{E}_x\big[\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\big] - A_h^{\Omega} e^{-\frac{c}{h}}.\end{aligned}$$

On the other hand, according to the strong Markov property, $\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_x[\tau_{D'^c}] + \mathbb{E}_x[\,\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\,]$ for all $x \in K$. It follows that for all $x \in K$,

$$\begin{aligned}\mathbb{E}_x[\tau_{\bar B(x_0,r_h)} \wedge \tau_{\Omega^c}] &\le (1 - \mathbb{P}_x[\tau_{\Omega^c} > \tau_{\bar B(x_0,r_h)}])\,\mathbb{E}_x[\,\mathbb{E}_{X_{\tau_{D'^c}}}[\tau_{\Omega^c}]\,] + A_h^{\Omega} e^{-\frac{c}{h}} + \mathbb{E}_x[\tau_{D'^c}]\\ &\le (1 - \mathbb{P}_x[\tau_{\Omega^c} > \tau_{\bar B(x_0,r_h)}])\,A_h^{\Omega} + A_h^{\Omega} e^{-\frac{c}{h}} + A_h^{D'},\end{aligned}$$

which implies Proposition 16, using (2.14), (2.15), and Proposition 14 (with $K = \overline{\mathbf{C}}_{\min}(\eta_*)$). □

Theorem 3. *Assume* (**Ortho**) *and* (**One-Well**)*. Let K a compact subset of $\mathcal{A}(\{x_0\})$ (see* (1.11) *and* (1.12)*). Then, there exist $h_0 > 0$ and $c > 0$ such that, for all $h \in (0, h_0]$ and uniformly in $x \in K$:*

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})) \text{ and } \lim_{h\to 0} h \ln \mathbb{E}_x[\tau_{\Omega^c}] = 2(\min_{\partial\Omega} f - f(x_0)).$$

Proof. First of all, according to Proposition 12, (2.13), and to the fact that $\mathbb{E}_y[\tau_{D'^c}] \le \mathbb{E}_y[\tau_{\Omega^c}]$ for all $y \in D'$, we have, uniformly in y in the compacts of D':

$$\lim_{h\to 0} h \ln \mathbb{E}_y[\tau_{\Omega^c}] = 2(\min_{\partial\Omega} f - f(x_0)). \tag{2.19}$$

Let K be a compact subset of $\mathcal{A}(\{x_0\})$. Assume first that $K = \overline{\mathbf{C}}_{\min}(\eta_*)$ (see (2.7) and (2.8)). Using (2.16), Lemma 13, and Propositions 16 and 14, we have uniformly in $x \in K$:

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})) + O(e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} e^{-\frac{c}{h}}).$$

Using in addition (2.19) with $y = x_0 \in D'$, we deduce that for some $c > 0$ and uniformly in $x \in K = \overline{\mathbf{C}}_{\min}(\eta_*)$, it holds for every h small enough:

$$\mathbb{E}_x[\tau_{\Omega^c}] = \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})). \tag{2.20}$$

This proves Theorem 3 when $K = \overline{\mathbf{C}}_{\min}(\eta_*)$. Let us now consider the general case $K \subset \mathcal{A}(\{x_0\})$. Let $T_K \ge 0$ be such that $\varphi_{T_K}(x) \in \mathbf{C}_{\min}(\eta_*)$ for all $x \in K$, and take $\delta > 0$ small enough so that:

- $\{\varphi_t(x) + z, (x,t) \in K \times [0, T_K] \text{ and } |z| \le \delta\} \subset \Omega$,
- $\{\varphi_{T_K}(x) + z, x \in K \text{ and } |z| \le \delta\} \subset \mathbf{C}_{\min}(\eta_*)$.

These two conditions imply that for all $x \in K$, if $X_0 = x$ and $\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta$:

$$T_K < \tau_{\Omega^c} \text{ and } X_{T_K} \in \mathbf{C}_{\min}(\eta_*). \tag{2.21}$$

From the Markov property, (2.21), (2.12), and (2.20), we have uniformly in $x \in K$:

$$\begin{aligned}\mathbb{E}_x[\tau_{\Omega^c}\, \mathbf{1}_{\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta}] &= T_K \mathbb{P}_x\big[\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta\big]\\ &\quad + \mathbb{E}_x\big[\mathbb{E}_{X_{T_K}}[\tau_{\Omega^c}]\, \mathbf{1}_{\sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta}\big]\\ &= T_K(1 + O(e^{-\frac{c}{h}})) + \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}}))\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_x\big[\tau_{\Omega^c}\,\mathbf{1}_{\sup_{t\in[0,T_K]}|X_t-\varphi_t(x)|>\delta}\,\mathbf{1}_{T_K<\tau_{\Omega^c}}\big] &= T_K\mathbb{P}_x\big[\sup_{t\in[0,T_K]}|X_t-\varphi_t(x)|>\delta, T_K<\tau_{\Omega^c}\big]\\ &\quad+\mathbb{E}_x\Big[\mathbb{E}_{X_{T_K}}\big[\tau_{\Omega^c}\big]\,\mathbf{1}_{\sup_{t\in[0,T_K]}|X_t-\varphi_t(x)|>\delta}\,\mathbf{1}_{T_K<\tau_{\Omega^c}}\Big]\\ &= T_K O(e^{-\frac{c}{h}})+\mathbb{E}_{x_0}\big[\tau_{\Omega^c}\big]O(e^{-\frac{c}{h}}).\end{aligned}$$

On the other hand, using (2.12), it holds for every $x\in K$:

$$\mathbb{E}_x\big[\tau_{\Omega^c}\,\mathbf{1}_{\sup_{t\in[0,T_K]}|X_t-\varphi_t(x)|>\delta}\,\mathbf{1}_{\tau_{\Omega^c}\le T_K}\big]\le T_K e^{-\frac{c}{h}}.$$

Combining the three previous estimates leads to $\mathbb{E}_x[\tau_{\Omega^c}]=\mathbb{E}_{x_0}[\tau_{\Omega^c}](1+O(e^{-\frac{c}{h}}))$ for all h small enough, uniformly in $x\in K$. This ends the proof of Theorem 3. ☐

3. Spectral analysis of $\mathrm{Re}\,(P_h)$ and of P_h

Recall that we assume (**Ortho**) throughout this work.

3.1. Analysis of the real part of P_h.

This section is devoted to a preliminary spectral analysis of the operator (see Proposition 3)

$$\mathrm{Re}\,(P_h):=\frac{1}{2}(P_h+P_h^*)=\Delta_{f,h}+2\,\boldsymbol{\ell}\cdot\nabla f-h\,\mathrm{div}\,\boldsymbol{\ell}=\Delta_{f,h}-h\,\mathrm{div}\,\boldsymbol{\ell}$$

with domain $D(\mathrm{Re}\,(P_h))=H^2(\Omega)\cap H^1_0(\Omega)=D(P_h)=D(P_h^*)$. This operator is self-adjoint with a compact resolvent and is the Friedrichs extension of the closed quadratic form

$$u\in H^1_0(\Omega)\mapsto\int_\Omega|\nabla_{f,h}u|^2-h\int_\Omega(\mathrm{div}\,\boldsymbol{\ell})\,|u|^2\,. \tag{3.1}$$

It is consequently bounded from below by $-h\|\,\mathrm{div}\,\boldsymbol{\ell}\|_{L^\infty(\Omega)}$, and hence

$$\sigma(\mathrm{Re}\,(P_h))\subset[-h\|\,\mathrm{div}\,\boldsymbol{\ell}\|_{L^\infty(\Omega)},+\infty).$$

When $\mathrm{div}\,\boldsymbol{\ell}=0$, the operator $\mathrm{Re}\,(P_h)$ is nothing but the Witten Laplacian $\Delta_{f,h}$ (see (1.13)) with domain $D(\Delta_{f,h})=H^2(\Omega)\cap H^1_0(\Omega)$ and is in particular positive. Let us now define

$$\mathsf{U}_0=\{x\in\Omega,\ x\text{ is a local minimum of } f\ \}\ \text{ and }\ \mathsf{m}_0:=\mathrm{Card}(\mathsf{U}_0)<+\infty^{3}. \tag{3.2}$$

Then, according to [35, Theorem 1], there exist $c_0>0$ and $h_0>0$ such that for all $h\in(0,h_0]$:

$$\dim\ \mathrm{Ran}\ \pi_{[0,c_0h]}\big(\Delta_{f,h}\big)=\ \mathsf{m}_0, \tag{3.3}$$

where, for a Borel set $I\subset\mathbb{R}$, $\pi_I\big(\Delta_{f,h}\big)$ denotes the spectral projector associated with $\Delta_{f,h}$ and I. For ease of notation, we set

$$\pi_h^\Delta:=\pi_{[0,c_0h]}\big(\Delta_{f,h}\big). \tag{3.4}$$

Moreover, the m_0 eigenvalues of $\Delta_{f,h}$ in $[0,c_0h]$ are exponentially small in the limit $h\to0$, i.e. there exists $c>0$ such that for every $h>0$ small enough,

$$\sigma(\Delta_{f,h})\cap[0,c_0h]\subset[0,e^{-\frac{c}{h}}]. \tag{3.5}$$

[3]We recall that f has a finite number of critical point in M by (**Ortho**).

Additionally, we can apply [34, Lemma 3.1] since (**Ortho**) holds: for every critical point $\mathbf{u} \in M$ of f, there exists a smooth map J defined around $\mathbf{u}$ and with values in $\mathcal{M}_d(\mathbb{R})$ such that $J(\mathbf{u})$ is antisymmetric and $\boldsymbol{\ell}(x) = J(x)\nabla f(x)$ around $\mathbf{u}$. It follows that

$$\begin{aligned}\operatorname{div} \boldsymbol{\ell}(\mathbf{u}) = \operatorname{Tr}\big(J(\mathbf{u}) \operatorname{Hess} f(\mathbf{u})\big) &= \operatorname{Tr}\big(\operatorname{Hess} f(\mathbf{u}) J(\mathbf{u})\big)\\ &= \operatorname{Tr}\big({}^t\big(\operatorname{Hess} f(\mathbf{u}) J(\mathbf{u})\big)\big) = -\operatorname{Tr}\big(J(\mathbf{u}) \operatorname{Hess} f(\mathbf{u})\big),\end{aligned}$$

and hence:

$$\text{for every critical point } \mathbf{u} \in M \text{ of } f, \quad \operatorname{div} \boldsymbol{\ell}(\mathbf{u}) \;=\; 0. \tag{3.6}$$

The above analysis together with standard tools of spectral theory and semiclassical analysis for Schrödinger operators (see e.g. [8, 16]) lead to the following proposition. The proof basically relies on the fact that (3.6) implies that $\operatorname{Re}(P_h)$ is a perturbation of $\Delta_{f,h}$ of order $O(h^{\frac{3}{2}})$.

Proposition 17. *Let us assume that (**Ortho**) holds. Then, there exist $C, c > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0]$, one has, counting the eigenvalues with multiplicity,*

$$\sigma(\operatorname{Re}(P_h)) \cap (-\infty, ch] \subset [-Ch^{\frac{3}{2}}, e^{-\frac{c}{h}}] \quad \text{and} \quad \operatorname{Card}\big(\sigma(\operatorname{Re}(P_h)) \cap (-\infty, ch]\big) = \mathsf{m}_0,$$

where m_0 is defined in (3.2).

Moreover, there exists $c_1 > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0]$:

$$\forall u \in H^2(\Omega) \cap H^1_0(\Omega), \quad \langle \operatorname{Re}(P_h)(1-\pi_h^{\Delta})u, (1-\pi_h^{\Delta})u\rangle_{L^2} \geq c_1 h \|(1-\pi_h^{\Delta})u\|^2_{L^2},$$

where π_h^{Δ} is the spectral projector associated with $\Delta_{f,h}$ and the interval $[0, c_0 h]$ (see (3.4)).

Note that the spectrum of the operator $\operatorname{Re}(P_h)$ is a priori not included in $[0, +\infty)$.

Proof. Let us define $\mathsf{m} := \operatorname{Card}(\{x \in \overline{\Omega},\ \nabla f(x) = 0\})$ and, when $\mathsf{m}_0 > 0$, let us order the elements $x_1, \dots, x_{\mathsf{m}}$ of $\{x \in \overline{\Omega},\ \nabla f(x) = 0\}$ so that (see (3.2))

$$\big\{x_1, \dots, x_{\mathsf{m}_0}\big\} \;=\; \mathsf{U}_0.$$

We consider, for every $x_j \in \Omega$, a smooth open connected neighborhood O_j of x_j in Ω such that $\overline{O_j} \subset \Omega$. When moreover $j \in \{1, \dots, \mathsf{m}_0\}$, we also assume that x_j is the only point where f attains its minimal value in $\overline{O_j}$. Similarly, when $x_j \in \partial\Omega$, we consider a smooth open set $O_j \subset \Omega$ such that $\overline{O_j}$ is a neighborhood of x_j in $\overline{\Omega}$. In addition, we assume that $\overline{O_i} \cap \overline{O_j} = \varnothing$ when $i \neq j$, so that each $\overline{O_i}$ contains precisely one critical point of f, x_i, which is in its interior.

Step 1. Let us first prove that there exists $c > 0$ such that, for every $h > 0$ small enough,

$$\dim \operatorname{Ran} \pi_{(-\infty, e^{-\frac{c}{h}}]}\big(\operatorname{Re}(P_h)\big) \;\geq\; \mathsf{m}_0. \tag{3.7}$$

This is obvious when $\mathsf{m}_0 = 0$. When $\mathsf{m}_0 > 0$, let us introduce, for every $j \in \{1, \dots, \mathsf{m}_0\}$, a cut-off function $\chi_j \in \mathcal{C}^\infty_c(O_j)$ such that $\chi_j = 1$ in a neighborhood of x_j and

$$\psi_j := \frac{\chi_j e^{-\frac{f}{h}}}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}}. \tag{3.8}$$

Since x_j is the only point where f attains its minimal value on $\operatorname{supp} \chi_j \subset O_j$, standard Laplace asymptotics give, in the limit $h \to 0$,

$$\|\chi_j e^{-\frac{f}{h}}\|^2_{L^2} = \frac{(\pi h)^{\frac{d}{2}}}{(\det \operatorname{Hess} f(x_j))^{\frac{1}{2}}} e^{-2\frac{f(x_j)}{h}}\big(1 + O(h)\big).$$

Using in addition the fact that $\chi_j = 1$ near x_j and thus that $f - f(x_j) > 2c_j$ on $\operatorname{supp}\nabla\chi_j$ for some $c_j > 0$, we have when $h \to 0$:

$$\|\Delta_{f,h}\psi_j\|_{L^2} = \frac{h}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}}\|(-h\operatorname{div} + \nabla f\cdot)(e^{-\frac{f}{h}}\nabla\chi_j)\|_{L^2} \le e^{-\frac{c_j}{h}}. \tag{3.9}$$

Since moreover $\operatorname{div}\boldsymbol{\ell}(x_j) = 0$ according to (3.6), Laplace asymptotics give, when $h \to 0$,

$$\|(\operatorname{div}\boldsymbol{\ell})\psi_j\|_{L^2} = O(h^{\frac{1}{2}}). \tag{3.10}$$

The two above relations imply the following one which will be useful in the sequel:

$$\|\operatorname{Re}(P_h)\psi_j\|_{L^2} = \|(\Delta_{f,h} - h\operatorname{div}\boldsymbol{\ell})\psi_j\|_{L^2} = O(h^{\frac{3}{2}}). \tag{3.11}$$

Besides, using (3.1), an integration by parts, (**Ortho**), and $f - f(x_j) > 2c_j$ on $\operatorname{supp}\nabla\chi_j$, it holds when $h \to 0$:

$$\begin{aligned}\langle \operatorname{Re}(P_h)\psi_j, \psi_j\rangle_{L^2} &= \frac{h^2}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}^2}\int_{O_j}|\nabla\chi_j|^2 e^{-2\frac{f}{h}} - \frac{h}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}^2}\int_{O_j}\operatorname{div}\boldsymbol{\ell}\,\chi_j^2\, e^{-2\frac{f}{h}} \\ &= \frac{h^2}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}^2}\int_{O_j}|\nabla\chi_j|^2 e^{-2\frac{f}{h}} + 2\frac{h}{\|\chi_j e^{-\frac{f}{h}}\|_{L^2}^2}\int_{O_j}\chi_j\ \boldsymbol{\ell}\cdot\nabla\chi_j\ e^{-2\frac{f}{h}}\ \le\ e^{-\frac{c_j}{h}}.\end{aligned}$$

Since the ψ_j, $j \in \{1,\dots,\mathsf{m}_0\}$, are normalized in $L^2(\Omega)$ with disjoint supports, it follows from the Min-Max principle that $\operatorname{Re}(P_h)$ admits, for $c := \min(c_1,\dots,c_{\mathsf{m}_0})$, at least m_0 eigenvalues less that $e^{-\frac{c}{h}}$ when $h \to 0$, which proves (3.7).

Step 2. Let us now prove that there exists $c_1 > 0$ such that, for every $h > 0$ small enough:

$$\forall u \in H^2(\Omega)\cap H_0^1(\Omega),\quad \langle \operatorname{Re}(P_h)(1-\pi_h^{\Delta})u, (1-\pi_h^{\Delta})u\rangle_{L^2} \ge c_1 h\|(1-\pi_h^{\Delta})u\|_{L^2}^2. \tag{3.12}$$

To this end, we first define a cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d, [0,1])$ such that $\chi = 1$ in $\{|x| \le 1\}$, $\chi = 0$ in $\{|x| \ge 2\}$, and $\sqrt{1-\chi^2} \in \mathcal{C}^\infty(\mathbb{R}^d)$. Then, for every $j \in \{1,\dots,\mathsf{m}\}$, we define the following smooth function on $\overline{\Omega}$:

$$\chi_{j,h} : x \in \overline{\Omega} \longmapsto \chi(h^{-\varepsilon}(x - x_j)) \in \mathbb{R}^+,$$

where $\varepsilon \in (0, \frac{1}{2})$ is arbitrary but fixed. In particular, for every $h > 0$ small enough, $\operatorname{supp}\chi_{j,h} \subset O_j$ when $x_j \in \Omega$ and, when $x_j \in \partial\Omega$, $\overline{O_j}$ is a neighborhood of $\operatorname{supp}\chi_{j,h}$ in $\overline{\Omega}$. Lastly, we define the smooth function

$$\chi_{0,h} : x \in \overline{\Omega} \longmapsto \big(1 - \sum_{j=1}^{\mathsf{m}}\chi_{j,h}^2\big)^{\frac{1}{2}},$$

so that $\sum_{j=0}^{\mathsf{m}}\chi_{j,h}^2 = 1$ on $\overline{\Omega}$.

Step 2a. Analysis on $\operatorname{supp}\chi_{0,h}$**.** Since $\operatorname{supp}\chi_{0,h}$ is at a distance greater than h^ε from the set of the critical points of the Morse function f in $\overline{\Omega}$, there exists $c > 0$ such that, for every $h > 0$ small enough, $|\nabla f(x)|^2 \ge 3ch^{2\varepsilon}$ on $\operatorname{supp}\chi_{0,h}$. Since $2\varepsilon < 1$, it follows that for every $h > 0$ small enough and every $u \in H^2(\Omega)\cap H_0^1(\Omega)$:

$$\begin{aligned}\langle \operatorname{Re}(P_h)\chi_{0,h}u, \chi_{0,h}u\rangle_{L^2} &= \langle(-h^2\Delta + |\nabla f|^2 - h\Delta f - h\operatorname{div}\boldsymbol{\ell})\chi_{0,h}u, \chi_{0,h}u\rangle_{L^2} \\ &\ge \langle(|\nabla f|^2 - h\Delta f - h\operatorname{div}\boldsymbol{\ell})\chi_{0,h}u, \chi_{0,h}u\rangle_{L^2} \\ &\ge 2ch^{2\varepsilon}\|\chi_{0,h}u\|_{L^2}^2. \end{aligned} \tag{3.13}$$

Step 2b. Analysis on $\mathrm{supp}\,\chi_{j,h}$ **when** $x_j \notin \mathsf{U}_0$**.** In this case, it holds $O_j \cap \mathsf{U}_0 = \varnothing$. Applying [35, Theorem 1] to the Witten Laplacian $\Delta^{O_j}_{f,h}$ with domain $D(\Delta^{O_j}_{f,h}) = H^2(O_j) \cap H^1_0(O_j)$ then implies the existence of $c > 0$ such that, for every $h > 0$ small enough,

$$\dim \text{ Ran } \pi_{[0,3ch]}\big(\Delta^{O_j}_{f,h}\big) = 0.$$

It follows that for every $h > 0$ small enough and every $u \in H^2(\Omega) \cap H^1_0(\Omega)$:

$$\begin{aligned} \langle \text{ Re } (P_h)\chi_{j,h}u, \chi_{j,h}u\rangle_{L^2} &= \langle (\Delta^{O_j}_{f,h} - h \,\mathrm{div}\,\boldsymbol{\ell})\chi_{j,h}u, \chi_{j,h}u\rangle_{L^2} \\ &\geq \langle (3ch - h \,\mathrm{div}\,\boldsymbol{\ell})\chi_{j,h}u, \chi_{j,h}u\rangle_{L^2} \\ &= \big(3ch + O(h^{1+\varepsilon})\big)\|\chi_{j,h}u\|^2_{L^2} \;\geq\; 2ch\|\chi_{j,h}u\|^2_{L^2}\,, \end{aligned} \tag{3.14}$$

where, to obtain the last inequality, we have used that $\mathrm{div}\,\boldsymbol{\ell}(x_j) = 0$ (see (3.6)) and $\mathrm{supp}\,\chi_{j,h} \subset \{|x - x_j| \leq 2h^\varepsilon\}$ imply that, for every $h > 0$ small enough, $\|\,\mathrm{div}\,\boldsymbol{\ell}\|_{L^\infty} = O(h^\varepsilon)$ on $\mathrm{supp}\,\chi_{j,h}$.

Step 2c. Analysis on $\mathrm{supp}\,\chi_{j,h}$ **when** $x_j \in \mathsf{U}_0$**.** In this case, it holds $O_j \cap \mathsf{U}_0 = \{x_j\}$ and, applying again [35, Theorem 1] to the Witten Laplacian $\Delta^{O_j}_{f,h}$ with domain $D(\Delta^{O_j}_{f,h}) = H^2(O_j) \cap H^1_0(O_j)$ then implies the existence of $c > 0$ such that, for every $h > 0$ small enough,

$$\dim \text{ Ran } \pi_{[0,3ch]}\big(\Delta^{O_j}_{f,h}\big) = 1. \tag{3.15}$$

Let us define

$$\psi_{j,h} := \frac{\chi_{j,h} e^{-\frac{f}{h}}}{\|\chi_{j,h} e^{-\frac{f}{h}}\|_{L^2}} \in \mathcal{C}^\infty_c(O_j, \mathbb{R}^+)$$

and note that $\psi_{j,h}$ both belongs to $D(\Delta^{O_j}_{f,h})$ and to $D(\Delta_{f,h})$. Moreover, using $\mathrm{supp}\,\chi_{j,h} \subset \{|x - x_j| \leq 2h^\varepsilon\}$, tail estimates and Laplace aymptotics, there exists $c' > 0$ such that, for every $h > 0$ small enough,

$$\langle \Delta_{f,h}\psi_{j,h}, \psi_{j,h}\rangle_{L^2} = \frac{h^2}{\|\chi_{j,h} e^{-\frac{f}{h}}\|^2_{L^2}} \int_{O_j} |\nabla \chi_{j,h}|^2 e^{-2\frac{f}{h}} \leq e^{-2c'\frac{h^{2\varepsilon}}{h}}\,.$$

Hence, using the spectral estimate

$$\forall b > 0\,,\ \forall u \in Q\,(T)\ ,\ \big\|\pi_{[b,+\infty)}(T)\,u\big\|^2 \leq \frac{q_T(u)}{b}\,, \tag{3.16}$$

with $b = 3ch$ and $T = \Delta^{O_j}_{f,h}$, valid for any nonnegative self-adjoint operator $(T, D(T))$ on a Hilbert space $(\mathcal{H}, \|\cdot\|)$ with associated quadratic form $(q_T, Q(T))$, we obtain (since $2\varepsilon < 1$)

$$\pi^{\Delta,O_j}_h \psi_{j,h} = \psi_{j,h} + O(e^{-c'\frac{h^{2\varepsilon}}{h}}) \ \text{ in } L^2(O_j), \tag{3.17}$$

where for conciseness we have set $\pi^{\Delta,O_j}_h := \pi_{[0,3ch]}\big(\Delta^{O_j}_{f,h}\big)$. In particular, according to (3.15), π^{Δ,O_j}_h is the orthogonal projector on $\mathrm{Span}(\Psi_j)$, where, using also (3.17),

$$\Psi_j \ := \ \frac{\pi^{\Delta,O_j}_h \psi_{j,h}}{\|\pi^{\Delta,O_j}_h \psi_{j,h}\|} = \psi_{j,h} + O(e^{-c'\frac{h^{2\varepsilon}}{h}}) \ \text{ in } L^2(O_j). \tag{3.18}$$

Note lastly that the same analysis with $\chi_{j,h}\psi_{j,h}$, $b = c_0 h$, and $T = \Delta_{f,h}$, shows that

$$\pi^{\Delta}_h(\chi_{j,h}\psi_{j,h}) = \chi_{j,h}\psi_{j,h} + O(e^{-c'\frac{h^{2\varepsilon}}{h}}) \ \text{ in } L^2(\Omega). \tag{3.19}$$

We can now finish this step. Let us recall that $\operatorname{div} \boldsymbol{\ell}(x_j) = 0$ and $\operatorname{supp} \chi_{j,h} \subset \{|x - x_j| \le 2h^\varepsilon\}$ imply that, for every $h > 0$ small enough, $\|\operatorname{div} \boldsymbol{\ell}\|_{L^\infty} = O(h^\varepsilon)$ on $\operatorname{supp} \chi_{j,h}$. Thus, for every $h > 0$ small enough and every $u \in H^2(\Omega) \cap H_0^1(\Omega)$, setting $w := (1 - \pi_h^\Delta)u \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\langle \operatorname{Re}(P_h)\chi_{j,h}w, \chi_{j,h}w\rangle_{L^2} = \langle \Delta_{f,h}^{O_j}\chi_{j,h}w, \chi_{j,h}w\rangle_{L^2} + O(h^{1+\varepsilon})\|\chi_{j,h}w\|_{L^2}^2.$$

Therefore, using in addition (3.15),

$$\langle \operatorname{Re}(P_h)\chi_{j,h}w, \chi_{j,h}w\rangle_{L^2} \;\ge\; 3ch\|(1 - \pi_h^{\Delta,O_j})\chi_{j,h}w\|^2 + O(h^{1+\varepsilon})\|\chi_{j,h}w\|^2. \tag{3.20}$$

Besides, using (3.18) and then (3.19) together with $w \in \big(\operatorname{Ran} \pi_h^\Delta\big)^\perp$,

$$\begin{aligned}\chi_{j,h}w &= (1 - \pi_h^{\Delta,O_j})\chi_{j,h}w + \langle \chi_{j,h}w, \Psi_j\rangle_{L^2(O_j)}\Psi_j\\ &= (1 - \pi_h^{\Delta,O_j})\chi_{j,h}w + \langle \chi_{j,h}w, \psi_{j,h}\rangle_{L^2(O_j)}\Psi_j + O(e^{-c'\frac{h^{2\varepsilon}}{h}})\|\chi_{j,h}w\|\\ &= (1 - \pi_h^{\Delta,O_j})\chi_{j,h}w + O(e^{-c'\frac{h^{2\varepsilon}}{h}})\|w\|_{L^2}.\end{aligned}$$

Injecting this estimate in (3.20), we obtain that for every $h > 0$ small enough and every $u \in H^2(\Omega) \cap H_0^1(\Omega)$, setting $w := (1 - \pi_h^\Delta)u$,

$$\langle \operatorname{Re}(P_h)\chi_{j,h}w, \chi_{j,h}w\rangle_{L^2} \ge 2ch\|\chi_{j,h}w\|_{L^2}^2 + O(e^{-c'\frac{h^{2\varepsilon}}{h}})\|w\|_{L^2}^2. \tag{3.21}$$

Step 2d. Proof of (3.12). Let us recall the so-called IMS localization formula (see for example [8]):

$$\begin{aligned}\forall w \in H^2(\Omega) \cap H_0^1(\Omega)\,,\quad \langle \operatorname{Re}(P_h)w, w\rangle &= \sum_{j=0}^{\mathsf{m}}\langle \operatorname{Re}(P_h)\chi_{j,h}w, \chi_{j,h}w\rangle - \sum_{j=0}^{\mathsf{m}} h^2\,\big\||\nabla\chi_{j,h}|\,w\big\|_{L^2(\Omega)}^2\\ &= \sum_{j=0}^{n}\langle \operatorname{Re}(P_h)\chi_{j,h}w, \chi_{j,h}w\rangle + O(h^{2-2\varepsilon})\|w\|_{L^2(\Omega)}^2.\end{aligned}$$

Using in addition the estimates (3.13), (3.14), and (3.21), we obtain the existence of $c > 0$ such that, for every $h > 0$ small enough and every $u \in H^2(\Omega) \cap H_0^1(\Omega)$, setting $w := (1 - \pi_h^\Delta)u \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\langle \operatorname{Re}(P_h)w, w\rangle \ge 2ch\sum_{j=0}^{\mathsf{m}}\|\chi_{j,h}w\|_{L^2}^2 + O(h^{2-2\varepsilon} + e^{-c\frac{h^{2\varepsilon}}{h}})\|w\|_{L^2(\Omega)}^2 \ge ch\|w\|_{L^2(\Omega)}^2\,.$$

This proves (3.12).

Step 3. End of the proof of Proposition 17. Let us first recall from (3.7) the existence of $c > 0$ such that, for every $h > 0$ small enough, the dimension of $\operatorname{Ran} \pi_{(-\infty, e^{-\frac{c}{h}}]}\big(\operatorname{Re}(P_h)\big)$ is at least m_0. Moreover, since $\dim \operatorname{Ran} \pi_h^\Delta = \mathsf{m}_0$ (see (3.3)), it follows from (3.12) and from the Min-Max principle that the $(\mathsf{m}_0 + 1)$-th eigenvalue of $\operatorname{Re}(P_h)$ is bounded from below by $c_1 h$ when $h \to 0$. The dimension of $\operatorname{Ran} \pi_{(-\infty, e^{-\frac{c}{h}}]}\big(\operatorname{Re}(P_h)\big)$ is thus precisely m_0 for every $h > 0$ small enough. To conclude, it just remains to show that the m_0 eigenvalues of $\operatorname{Re}(P_h)$ in $(-\infty, e^{-\frac{c}{h}}]$ are of the order $O(h^{\frac{3}{2}})$ in the limit $h \to 0$.

To this end, note that it is possible to construct, for every $h > 0$ sufficiently small, a simple closed loop $\gamma \subset \{\mathsf{z} \in \mathbb{C}, \operatorname{Re} \mathsf{z} \le \frac{c_1}{2}h\}$ such that:

- γ contains $[-h\|\operatorname{div} \boldsymbol{\ell}\|_{L^\infty}, \frac{c_1}{2}h]$, and thus $\sigma(\operatorname{Re}(P_h)) \cap (-\infty, \frac{c_1}{2}h]$, in its interior,
- for some $c, c' > 0$ independent of h, $|\gamma| \le ch$ and $\operatorname{dist}(\gamma, \sigma(\operatorname{Re}(P_h))) \ge c'h$.

The rank-m_0 orthogonal spectral projector π_h associated with $\mathrm{Re}\,(P_h)$ and $\sigma(\mathrm{Re}\,(P_h))\cap]-\infty, e^{-\frac{c}{h}}]$ then satisfies, for every $h>0$ small enough,

$$\pi_h = \frac{1}{2i\pi}\int_\gamma (\mathsf{z} - \mathrm{Re}\,(P_h))^{-1} d\mathsf{z}.$$

For $j\in\{1,\dots,\mathsf{m}_0\}$, let ψ_j be the function defined in (3.8) and recall the relation (3.11) which has not yet been used in this proof:

$$\|\,\mathrm{Re}\,(P_h)\psi_j\|_{L^2} = O(h^{\frac32}).$$

Using $\|(\mathsf{z} - \mathrm{Re}\,(P_h))^{-1}\| \le \frac{1}{c'h}$ for every $\mathsf{z}\in\gamma$, it follows that for every $h>0$ small enough,

$$\begin{aligned}(1-\pi_h)\psi_j &= \frac{1}{2\pi i}\int_\gamma \big(\mathsf{z}^{-1} - (\mathsf{z} - \mathrm{Re}\,(P_h))^{-1}\big)\psi_j\, d\mathsf{z}\\ &= \frac{-1}{2\pi i}\int_\gamma \mathsf{z}^{-1}\big(\mathsf{z} - \mathrm{Re}\,(P_h)\big)^{-1}\,\mathrm{Re}\,(P_h)\psi_j\, d\mathsf{z} = O(h^{\frac12}).\end{aligned} \tag{3.22}$$

Since the family $\big(\psi_j\big)_{j\in\{1,\dots,\mathsf{m}_0\}}$ is orthonormal, the family $\big(\pi_h\psi_j = \psi_j + O(h^{\frac12})\big)_{j\in\{1,\dots,\mathsf{m}_0\}}$ is linearly independent, and hence a basis of $\mathrm{Ran}\,\pi_h$, when $h\to 0$. In addition, any normalized vector $\Psi\in\mathrm{Ran}\,\pi_h$ writes $\Psi = \sum_{k=1}^{\mathsf{m}_0}\mu_k\pi_h\psi_j$, where the complex numbers $\mu_1,\dots,\mu_k$ satisfy $\sum_{k=1}^{\mathsf{m}_0}|\mu_k|^2 = 1+O(h^{\frac12})$. It thus follows from (3.11) that, when $h\to 0$:

$$\|\,\mathrm{Re}\,(P_h)\Psi\|_{L^2(\Omega)} = \|\sum_{k=1}^{\mathsf{m}_0}\mu_k\pi_h\,\mathrm{Re}\,(P_h)\psi_j\|_{L^2(\Omega)} \le \sum_{k=1}^{\mathsf{m}_0}|\mu_k|\|\,\mathrm{Re}\,(P_h)\psi_j\|_{L^2(\Omega)} = O(h^{\frac32}),$$

which implies that the m_0 eigenvalues of $\mathrm{Re}\,(P_h)$ in $(-\infty, e^{-\frac{c}{h}}]$ are of the order $O(h^{\frac32})$.

□

3.2. **Small eigenvalues of P_h and resolvent estimates.** The aim of this section is to prove Theorem 4 on the number of small eigenvalues of P_h (or equivalently of L_h, see (1.14)).

Theorem 4. *Let us assume that* (**Ortho**) *holds. Then, there exists $c_2>0$ such that, for all $c_3\in(0,c_2)$, there exist $h_0>0$ and $C>0$ such that, for all $\mathsf{z}\in\{\mathsf{z}\in\mathbb{C}, \mathrm{Re}\,\mathsf{z}\le c_2 h, |\mathsf{z}|\ge c_3 h\}$ and $h\in(0,h_0]$,*

$$P_h - \mathsf{z} \text{ is invertible and } \quad \|(P_h-\mathsf{z})^{-1}\|\le Ch^{-1}.$$

In addition, there exists $h_0>0$ such that for all $h\in(0,h_0]$, $\sigma(P_h)\cap\{\mathsf{z}\in\mathbb{C}, \mathrm{Re}\,\mathsf{z}\le c_2 h\}$ is composed of exactly m_0 eigenvalues $\lambda_{1,h},\lambda_{2,h},\dots,\lambda_{\mathsf{m}_0,h}$ (counted with algebraic multiplicity), where m_0 is defined in (3.2). Finally, there exists $c>0$ such that for all $j\in\{1,\dots,\mathsf{m}_0\}$ and h small enough, $|\lambda_{j,h}|\le e^{-\frac{c}{h}}$. All these results also hold for P_h^*.

Proof. Note first that the last sentence in the statement of Theorem 4 concerning P_h^* is an immediate consequence of the part concerning P_h since $\sigma(P_h^*) = \overline{\sigma(P_h)}$ (with multiplicity) and, for all $\mathsf{z}\in\mathbb{C}\smallsetminus\sigma(P_h)$, $\|(P_h-\mathsf{z})^{-1}\| = \|(P_h^*-\bar{\mathsf{z}})^{-1}\|$ (see indeed [30, Section 6.6 in Chapter 3]).

Let us also recall the relations (3.3), (3.4), and (3.5) stated in the beginning of Section 3.1. Let us consider, for $j\in\{1,\dots,\mathsf{m}_0\}$, a $L^2(\Omega)$-normalized eigenfunction $u_{j,h}^{\Delta}$ of $\Delta_{f,h}$ associated with its j-th eigenvalue. Since $P_h = \Delta_{f,h} + 2\boldsymbol{\ell}\cdot\nabla_{f,h}$ has domain $D(P_h) = D(\Delta_{f,h})$ and the quadratic form associated with $\Delta_{f,h}$ is given by (3.1) with $\boldsymbol{\ell}=0$:

$$\exists c>0 \text{ such that, when } h\to 0, \quad \|P_h u_{j,h}^{\Delta}\|_{L^2(\Omega)}\le e^{-\frac{c}{h}}. \tag{3.23}$$

Similarly, since $P_h^* = \Delta_{f,h} - 2\boldsymbol{\ell}\cdot\nabla_{f,h} - 2h\operatorname{div}\boldsymbol{\ell}$ has domain $D(P_h^*) = D(\Delta_{f,h})$, there exists $c>0$ such that, for every $h>0$ small enough,

$$\|P_h^* u_{j,h}^{\Delta}\|_{L^2(\Omega)} \le e^{-\frac{c}{h}} + 2h\|(\operatorname{div}\boldsymbol{\ell})u_{j,h}^{\Delta}\|_{L^2(\Omega)}.$$

Considering now the orthonormal family $(\psi_j)_{j\in\{1,\dots,\mathsf{m}_0\}}$ defined in the previous section in (3.8) and using the spectral estimate (3.16) with $b = c_0 h$, $T = \Delta_{f,h}$, and (3.9), there exists $c'>0$ such that, for every $j\in\{1,\dots,\mathsf{m}_0\}$ and $h>0$ small enough,

$$\pi_h^{\Delta}\psi_j = \psi_j + O(e^{-\frac{c'}{h}}) \ \text{ in } L^2(\Omega). \tag{3.24}$$

Using in addition (3.10), it thus follows that, for every $h>0$ small enough,

$$\|(\operatorname{div}\boldsymbol{\ell})\pi_h^{\Delta}\psi_j\|_{L^2(\Omega)} = O(h^{\frac12}).$$

Hence, since (3.24) implies that each $u_{j,h}^{\Delta}$ writes $u_{j,h}^{\Delta} = \sum_{k=1}^{\mathsf{m}_0}\mu_k\pi_h^{\Delta}\psi_j$ for some complex numbers $\mu_1,\dots,\mu_k$ satisfying $\sum_{k=1}^{\mathsf{m}_0}|\mu_k|^2 = 1 + O(e^{-\frac{c'}{h}})$, we obtain that for every $h>0$ small enough,

$$\|P_h^* u_{j,h}^{\Delta}\|_{L^2(\Omega)} = O(h^{\frac32}). \tag{3.25}$$

Let us now define the operator $\hat P_h$ by

$$\hat P_h := (1-\pi_h^{\Delta})P_h(1-\pi_h^{\Delta}) \text{ with domain } (1-\pi_h^{\Delta})D(P_h) \text{ on } \hat{\mathsf{E}} := (1-\pi_h^{\Delta})L^2(\Omega),$$

where we recall that $D(P_h) = D(\Delta_{f,h}) = H^2(\Omega)\cap H_0^1(\Omega)$. Note that the space $\hat{\mathsf{E}}$ (equipped with the restricted $L^2(\Omega)$-Hermitian inner product) is a Hilbert space and that the operator $\hat P_h : D(\hat P_h)\to\hat{\mathsf{E}}$ is well defined, since $(1-\pi_h^{\Delta})D(P_h) = \hat{\mathsf{E}}\cap D(P_h)\subset D(P_h)$, with dense domain in $\hat{\mathsf{E}}$.

The rest of the proof is reminiscent of the analysis led in [34, Section 2B.] and is divided into two steps.

Step 1. Resolvent estimates for $\hat P_h : D(\hat P_h)\to\hat{\mathsf{E}}$. First, the operator $\hat P_h$ is closed. This follows from the fact that $P_h : D(P_h)\to L^2(\Omega)$ is closed and from the relation $\hat P_h = P_h + \pi_h^{\Delta}P_h\pi_h^{\Delta} - \pi_h^{\Delta}P_h - P_h\pi_h^{\Delta}$ on $D(\hat P_h)$, since $\pi_h^{\Delta}P_h\pi_h^{\Delta} - \pi_h^{\Delta}P_h - P_h\pi_h^{\Delta}$ extends into a bounded operator T_h on $L^2(\Omega)$. Indeed, $P_h\pi_h^{\Delta}$ and then $\pi_h^{\Delta}P_h\pi_h^{\Delta}$ extend into bounded operators on $L^2(\Omega)$ since π_h^{Δ} is continuous with finite rank, and it is also the case for $\pi_h^{\Delta}P_h$ since for all $u\in D(P_h) = D(P_h^*)$,

$$\pi_h^{\Delta}P_h u = \sum_{j=1}^{\mathsf{m}_0}\langle u_{j,h}^{\Delta}, P_h u\rangle_{L^2(\Omega)}u_{j,h}^{\Delta} = \sum_{j=1}^{\mathsf{m}_0}\langle P_h^* u_{j,h}^{\Delta}, u\rangle_{L^2(\Omega)}u_{j,h}^{\Delta}.$$

The above considerations also imply that the adjoint of $\hat P_h$ is the operator

$$\hat P_h^* = (1-\pi_h^{\Delta})P_h^*(1-\pi_h^{\Delta}) \text{ with domain } (1-\pi_h^{\Delta})D(P_h).$$

Let us now prove the following resolvent estimates for $\hat P_h$: there exist $C>0$ and $c_2>0$ such that, for all $h>0$ small enough and $\mathsf{z}\in\mathbb{C}$ such that $\operatorname{Re}\mathsf{z}\le c_2 h$,

$$\hat P_h - \mathsf{z} \text{ is invertible and } \|(\hat P_h - \mathsf{z})^{-1}\| \le Ch^{-1}. \tag{3.26}$$

To prove this claim, let us consider $w \in D(\hat P_h) = (1-\pi_h^\Delta)D(P_h)$ and $\mathsf{z} \in \mathbb{C}$. Then, according to Proposition 17, it holds, for every $h > 0$ small enough,

$$\begin{aligned}\text{Re}\,\langle(\hat P_h - \mathsf{z})w, w\rangle_{L^2(\Omega)} &= \text{Re}\,\langle P_h(1-\pi_h^\Delta)w,(1-\pi_h^\Delta)w\rangle_{L^2(\Omega)} - (\text{Re}\,\mathsf{z})\,\|(1-\pi_h^\Delta)w\|^2_{L^2(\Omega)}\\ &= \langle\,\text{Re}\,(P_h)(1-\pi_h^\Delta)w,(1-\pi_h^\Delta)w\rangle_{L^2(\Omega)} - (\text{Re}\,\mathsf{z})\|(1-\pi_h^\Delta)w\|^2_{L^2(\Omega)}\\ &\geq [c_1h - \text{Re}\,\mathsf{z}]\|(1-\pi_h^\Delta)w\|^2_{L^2(\Omega)} = [c_1h - \text{Re}\,\mathsf{z}]\|w\|^2_{L^2(\Omega)}.\end{aligned}$$

The same inequality also holds for $\hat P_h^* - \mathsf{z}$ since $\text{Re}\,(P_h) = \text{Re}\,(P_h^*)$. Let us now fix $c_2 \in (0, c_1)$. When $\text{Re}\,\mathsf{z} \leq c_2h$ and $h > 0$ is small enough, the previous inequality implies

$$\|(\hat P_h - \mathsf{z})w\|_{L^2(\Omega)} \geq (c_1 - c_2)h\|w\|_{L^2(\Omega)}. \tag{3.27}$$

Consequently, when $\text{Re}\,\mathsf{z} \leq c_2h$ and $h > 0$ is small enough, $\hat P_h - \mathsf{z}$ is injective and its range is closed. Since the same inequality also holds for its adjoint $\hat P_h^* - \bar{\mathsf{z}}$, the range of $\hat P_h - \mathsf{z}$ is dense in $\hat{\mathsf{E}}$. Thus, $\hat P_h - \mathsf{z} : D(\hat P_h) \to \hat{\mathsf{E}}$ is invertible and the relation (3.26) follows from (3.27).

Step 2. Grushin problem and end of the proof of Theorem 4. Define the operators:

$$R_- : \mathbb{C}^{\mathsf{m}_0} \to L^2(\Omega),\ (\mu_k)_{j=1}^{\mathsf{m}_0} \mapsto \sum_{j=1}^{\mathsf{m}_0} \mu_j\, u_{j,h}^\Delta,\ \text{ and } R_+ : L^2(\Omega) \to \mathbb{C}^{\mathsf{m}_0},\ \ u \mapsto (\langle u, u_{j,h}^\Delta\rangle_{L^2(\Omega)})_{j=1}^{\mathsf{m}_0}.$$

We equip $\mathbb{C}^{\mathsf{m}_0}$ with the ℓ^2 norm. Note the relations

$$R_+^* = R_-\,,\quad R_-R_+ = \pi_h^\Delta\,,\ \text{ and } R_+R_- = \mathsf{I}_{\mathbb{C}^{\mathsf{m}_0}}\,, \tag{3.28}$$

and that, for all $h > 0$,

$$\|R_+\| \leq 1 \quad\text{and}\quad \|R_-\| \leq 1. \tag{3.29}$$

Moreover, according to (3.23) and (3.25), there exists $c > 0$ such that for every $h > 0$ small enough, it holds:

$$\|R_+P_h\| = O(h^{\frac32}) \quad\text{and}\quad \|P_hR_-\| \leq e^{-\frac{c}{h}}. \tag{3.30}$$

For $\mathsf{z} \in \mathbb{C}$, let us denote by $\mathcal{P}_h(\mathsf{z})$ the linear operator defined by

$$(u, u_-) \in D(P_h) \times \mathbb{C}^{\mathsf{m}_0} \mapsto \begin{pmatrix}(P_h - \mathsf{z})u + R_-u_-\\ R_+u\end{pmatrix} \in L^2(\Omega) \times \mathbb{C}^{\mathsf{m}_0}.$$

Using (3.26) and the same analysis as the one made to prove [34, Lemma 2.2], we deduce that, when $\text{Re}\,\mathsf{z} \leq c_2h$ and $h > 0$ is small enough, $\mathcal{P}_h(\mathsf{z})$ is invertible (i.e. the Grushin problem $\mathcal{P}_h(\mathsf{z})$ is well posed) and its inverse writes

$$(f, g) \in L^2(\Omega) \times \mathbb{C}^{\mathsf{m}_0} \mapsto \begin{pmatrix}\mathcal{E}(\mathsf{z}) & \mathcal{E}_+(\mathsf{z})\\ \mathcal{E}_-(\mathsf{z}) & \mathcal{E}_{-+}(\mathsf{z})\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix} \in D(P_h) \times \mathbb{C}^{\mathsf{m}_0},$$

where the operators $\mathcal{E}$, $\mathcal{E}_+$, $\mathcal{E}_-$, and $\mathcal{E}_{-+}$ are holomorphic on $\{\text{Re}\,\mathsf{z} \leq c_2h\}$ and satisfy:

(1) $\mathcal{E}(\mathsf{z}) = (\hat P_h - \mathsf{z})^{-1}(1-\pi_h^\Delta)$ and thus, according to (3.26):

$$\text{for every } \mathsf{z} \in \{\text{Re}\,\mathsf{z} \leq c_2h\},\quad \|\mathcal{E}(\mathsf{z})\| \leq Ch^{-1}, \tag{3.31}$$

(2) $\mathcal{E}_{-+}(\mathsf{z}) = -R_+(P_h - \mathsf{z})R_- + R_+P_h(\hat P_h - \mathsf{z})^{-1}(1-\pi_h^\Delta)P_hR_-$,
(3) $\mathcal{E}_+(\mathsf{z}) = R_- - (\hat P_h - \mathsf{z})^{-1}(1-\pi_h^\Delta)P_hR_-$,
(4) $\mathcal{E}_-(\mathsf{z}) = R_+ - R_+P_h(\hat P_h - \mathsf{z})^{-1}(1-\pi_h^\Delta)$.

Moreover, $P_h - \mathsf{z}$ is invertible if and only if $\mathcal{E}_{-+}(\mathsf{z})$ is invertible, and in this case,

$$(P_h - \mathsf{z})^{-1} = \mathcal{E}(\mathsf{z}) - \mathcal{E}_+(\mathsf{z})\mathcal{E}_{-+}(\mathsf{z})^{-1}\mathcal{E}_-(\mathsf{z}). \tag{3.32}$$

We refer to [51] for more details on so-called Grushin problems.

Using (3.26), (3.28), (3.29), and (3.30), one deduces that there exists $c > 0$ such that, for every $h > 0$ small enough and uniformly with respect to $\mathsf{z} \in \{\operatorname{Re} \mathsf{z} \le c_2 h\}$,

$$\mathcal{E}_-(\mathsf{z}) = R_+ + O(h^{\frac12}),\ \mathcal{E}_+(\mathsf{z}) = R_- + O(e^{-\frac{c}{h}}),\ \text{and } \mathcal{E}_{-+}(\mathsf{z}) = \mathsf{z}\mathsf{I}_{\mathbb{C}^{\mathsf{m}_0}} + O(e^{-\frac{c}{h}}).$$

In particular, when in addition $|\mathsf{z}| \ge e^{-\frac{c}{2h}}$, $\mathcal{E}_{-+}(\mathsf{z})$ is invertible and thus so is $P_h - \mathsf{z}$ (see the line above (3.32)). Therefore, for every $h > 0$ small enough:

$$\sigma(P_h) \cap \left\{\mathsf{z} \in \mathbb{C},\ \operatorname{Re} \mathsf{z} \le c_2 h\right\} \subset \left\{|\mathsf{z}| \le e^{-\frac{c}{2h}}\right\}. \tag{3.33}$$

Let us now fix $c_3 \in (0, c_2)$. The operator $\mathcal{E}_{-+}(\mathsf{z})$ is then invertible for every $h > 0$ small enough and every $\mathsf{z} \in \{\operatorname{Re} \mathsf{z} \le c_2 h, |\mathsf{z}| \ge c_3 h\}$, and satisfies $\mathcal{E}_{-+}(\mathsf{z})^{-1} = \mathsf{z}^{-1}(\mathsf{I}_{\mathbb{C}^{\mathsf{m}_0}} + O(e^{-\frac{c}{2h}}))$. Hence, according to (3.32), since $R_-R_+ = \pi_h^\Delta$, $\|\pi_h^\Delta\| \le 1$, $\|R_+\| \le 1$, and $\|R_-\| \le 1$, the previous estimates on $\mathcal{E}_+(\mathsf{z})$, $\mathcal{E}_-(\mathsf{z})$, and $\mathcal{E}_{-+}(\mathsf{z})$ imply that for all h small enough and uniformly with respect to $\mathsf{z} \in \{\operatorname{Re} \mathsf{z} \le c_2 h, |\mathsf{z}| \ge c_3 h\}$:

$$(P_h - \mathsf{z})^{-1} = \mathcal{E}(\mathsf{z}) - \mathsf{z}^{-1}(\pi_h^\Delta + O(h^{\frac12})) = \mathcal{E}(\mathsf{z}) - \mathsf{z}^{-1}\pi_h^\Delta + O(h^{-\frac12}). \tag{3.34}$$

Using in addition (3.31), there exists $K > 0$ such that for all for h small enough and $\mathsf{z} \in \{\operatorname{Re} \mathsf{z} \le c_2 h, |\mathsf{z}| \ge c_3 h\}$:

$$\|(P_h - \mathsf{z})^{-1}\| \le C h^{-1} + |\mathsf{z}|^{-1} + O(h^{-\frac12}) \le K h^{-1}.$$

Lastly, take $\beta \in (c_3, c_2)$. According to (3.33), the spectral Riesz projector

$$\pi_h^P := \frac{1}{2i\pi} \int_{\{|\mathsf{z}| = \beta h\}} (\mathsf{z} - P_h)^{-1} d\mathsf{z} \tag{3.35}$$

is well defined for every $h > 0$ small enough and its rank is the number of eigenvalues of P_h in $\left\{\operatorname{Re} \mathsf{z} \le c_2 h\right\}$, counted with algebraic multiplicity. Moreover, Equation (3.34) implies that for every $h > 0$ small enough,

$$\pi_h^P = \pi_h^\Delta + O(h^{\frac12}) \tag{3.36}$$

and thus, $\dim \operatorname{Ran}(\pi_h^P) = \dim \operatorname{Ran}(\pi_h^\Delta) = \mathsf{m}_0$ (see (3.3)). Therefore, for every h small enough, $\sigma(P_h) \cap \left\{\operatorname{Re} \mathsf{z} \le c_2 h\right\}$ is composed of m_0 eigenvalues, counted with algebraic multiplicity, which are exponentially small. This concludes the proof of Theorem 4. □

4. Proof of Theorem 1

4.1. **Rough asymptotic estimates on $u_{1,h}^P$ and on $u_{1,h}^{P^*}$.** We assume from now on, without loss of generality, that the principal eigenmodes $u_{1,h}^P$ of P_h and $u_{1,h}^{P^*}$ of P_h^* defined in Proposition 3 are normalized in $L^2(\Omega)$. We derive in the following proposition a priori estimates on these eigenmodes which will be used in Section 4.2 to prove Theorem 1.

Proposition 18. *Assume* (**Ortho**) *and* (**One-Well**). *For any $\eta \in (0, \min_{\partial\Omega} f - f(x_0))$, let $\chi_\eta : \overline{\Omega} \to [0,1]$ be a smooth function such that $\chi_\eta = 1$ on $\mathbf{C}_{min}(\eta)$ (see (2.7)) and $\chi_\eta = 0$ on $\overline{\Omega} \setminus \mathbf{C}_{min}(\eta/2)$. Set*

$$\mathsf{u}_\eta = \frac{\chi_\eta e^{-\frac{f}{h}}}{\|\chi_\eta e^{-\frac{f}{h}}\|_{L^2(\Omega)}}.$$

Then, there exists $c>0$ such that for all h small enough, $u_{1,h}^P$ and $u_{1,h}^{P^*}$ satisfy

$$u_{1,h}^P = \mathsf{u}_\eta + O(e^{-\frac{c}{h}}) \quad \text{and} \quad u_{1,h}^{P^*} = \mathsf{u}_\eta + O(h^{\frac12}) \quad \text{in } L^2(\Omega), \tag{4.1}$$

as well as

$$\frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{min}(\eta)} u_{1,h}^P\, e^{-\frac{f}{h}}}{\int_\Omega u_{1,h}^P e^{-\frac{f}{h}}} = O(e^{-\frac{c}{h}}) \quad \text{and} \quad \frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{min}(\eta)} u_{1,h}^{P^*}\, e^{-\frac{f}{h}}}{\int_\Omega u_{1,h}^{P^*} e^{-\frac{f}{h}}} = O(e^{-\frac{c}{h}}). \tag{4.2}$$

Proof. Assume (**Ortho**). According to Theorem 4, P_h admits precisely m_0 eigenvalues in $\{\operatorname{Re} \mathsf{z} \le c_2 h\}$, where we recall that m_0 is the number of local minima of f in Ω (see (3.2)), and these m_0 eigenvalues are exponentially small. When in addition (**One-Well**) holds, $\mathsf{U}_0 = \{x_0\}$ and then $\mathsf{m}_0 = 1$. Thus, $\lambda_{1,h}^P$ is the unique eigenvalue of P_h in $\{\operatorname{Re} \mathsf{z} \le c_2 h\}$ and π_h^P (see (3.35)) has rank 1. Notice that the same holds for $\pi_h^{P^*}$.

In what follows, we assume (**Ortho**) and (**One-Well**).

Step 1. Proof of (4.1). Laplace's method provides (since $\chi_\eta = 1$ in a neighborhood of x_0 which is, according to (**One-Well**), the unique global minimum point of f in $\overline{\Omega}$):

$$\|\chi_\eta e^{-\frac{f}{h}}\|_{L^2(\Omega)} = (\pi\, h)^{\frac{d}{4}} \left(\det \operatorname{Hess} f(x_0)\right)^{-\frac14}\, e^{-\frac{f(x_0)}{h}}(1+O(h)). \tag{4.3}$$

Since $P_h = \Delta_{f,h} + 2\boldsymbol{\ell}\cdot\nabla_{f,h} = (-h\operatorname{div} + \nabla f\cdot)\nabla_{f,h} + 2\boldsymbol{\ell}\cdot\nabla_{f,h}$ with $\nabla_{f,h} = h\, e^{-\frac{f}{h}}\nabla e^{\frac{f}{h}}$, the function $P_h\mathsf{u}_\eta$ is supported in $\operatorname{supp}\nabla\chi_\eta$, where $f - f(x_0)$ is larger than $\min_{\partial\Omega} f - f(x_0) - \eta > 0$. Hence, following the reasoning used to prove (3.9), there exists $c>0$ such that, for every $h>0$ small enough:

$$\|P_h \mathsf{u}_\eta\|_{L^2(\Omega)} \le e^{-\frac{c}{h}}. \tag{4.4}$$

Since moreover $P_h^* = 2\operatorname{Re}(P_h) - P_h$, (4.4) and (3.11) imply that, in the limit $h\to 0$:

$$\|P_h^* \mathsf{u}_\eta\|_{L^2(\Omega)} = O(h^{\frac32}). \tag{4.5}$$

On the other hand, since $\mathsf{u}_\eta \in D(P_h)$, following the argument leading to (3.22), the relation (3.35) and the resolvent estimate of Theorem 4 imply the existence of $C>0$ such that, when $h\to 0$,

$$\|(1-\pi_h^P)\mathsf{u}_\eta\|_{L^2(\Omega)} \le C h^{-1} \|P_h\mathsf{u}_\eta\|_{L^2(\Omega)}. \tag{4.6}$$

Consequently, using also (4.4), there exists $c>0$ such that, for every $h>0$ small enough:

$$\pi_h^P \mathsf{u}_\eta = \mathsf{u}_\eta + O(e^{-\frac{c}{h}}) \text{ in } L^2(\Omega).$$

In particular, $\|\pi_h^P\mathsf{u}_\eta\|_{L^2(\Omega)} = 1 + O(e^{-\frac{c}{h}})$ for all h small enough and, since π_h^P has rank 1, $\mathsf{u}_\eta \ge 0$, and $u_{1,h}^P, u_{1,h}^{P^*} > 0$ in Ω, it holds:

$$u_{1,h}^P = +\frac{\pi_h^P \mathsf{u}_\eta}{\|\pi_h^P\mathsf{u}_\eta\|_{L^2(\Omega)}} = \frac{\pi_h^P\mathsf{u}_\eta}{1+O(e^{-\frac{c}{h}})} = \mathsf{u}_\eta + O(e^{-\frac{c}{h}}) \quad \text{in } L^2(\Omega). \tag{4.7}$$

Similarly, using the resolvent estimate of Theorem 4 for P_h^* together with (4.5), we deduce that, when $h\to 0$, $\pi_h^{P^*}\mathsf{u}_\eta = \mathsf{u}_\eta + O(h^{\frac12})$ and

$$u_{1,h}^{P^*} = +\frac{\pi_h^{P^*} \mathsf{u}_\eta}{\|\pi_h^{P^*}\mathsf{u}_\eta\|_{L^2(\Omega)}} = \frac{\pi_h^{P^*}\mathsf{u}_\eta}{1+O(h^{\frac12})} = \mathsf{u}_\eta + O(h^{\frac12}) \quad \text{in } L^2(\Omega). \tag{4.8}$$

This ends the proof of (4.1).

Step 2. Proof of (4.2)**.** According to (4.1), we have:

$$\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}} = \int_\Omega \mathsf{u}_\eta\, e^{-\frac{f}{h}} + O(h^{\frac12})\|e^{-\frac{f}{h}}\|_{L^2(\Omega)} = \frac{\int_\Omega \chi_\eta\, e^{-\frac{2}{h}f}}{\|\chi_\eta e^{-\frac{f}{h}}\|_{L^2(\Omega)}} + O(h^{\frac12})\|e^{-\frac{f}{h}}\|_{L^2(\Omega)}.$$

Hence, using Laplace's method as we did to get (4.3), we have when $h\to 0$:

$$\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}} = (\pi\, h)^{\frac{d}{4}}\big(\det \mathrm{Hess}\, f(x_0)\big)^{-\frac14}\ e^{-\frac{f(x_0)}{h}}\big(1+O(h^{\frac12})\big).$$

Thus, since $f - f(x_0) \geq \min_{\partial\Omega} f - f(x_0) - \eta > 0$ on $\Omega\setminus \overline{\mathbf{C}}_{\min}(\eta)$, there exists $c>0$ such that, for every h small enough:

$$\frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta)} u_{1,h}^{P*}\, e^{-\frac{f}{h}}}{\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}}} = \frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta)} \chi_\eta\, e^{-\frac{2}{h}f}}{\|\chi_\eta e^{-\frac{f}{h}}\|_{L^2(\Omega)} \int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}}} + O(h^{\frac12})\frac{\|e^{-\frac{f}{h}}\|_{L^2(\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta))}}{\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}}} = O(e^{-\frac{c}{h}}),$$

which proves (4.2) for $u_{1,h}^{P*}$. The proof for $u_{1,h}^{P}$ is analogous. □

4.2. **Proof of Theorem 1.** Assume (**Ortho**) and (**One-Well**). We recall that a quasi-stationary distribution for the process (1.1) in Ω is a probability measure μ_h on Ω such that, for any time $t\geq 0$ and any Borel set $A\subset\Omega$, $\mathbb{P}_{\mu_h}(X_t\in A\,|\, t<\tau_{\Omega^c}) = \mu_h(A)$. Let us now introduce the following probability distribution on Ω (see Proposition 3):

$$\nu_h(dx) = \frac{u_{1,h}^{P*} e^{-\frac{f}{h}}}{\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}}}\ dx.$$

Using the smoothness of the killed semigroup $P_t f(x) = \mathbb{E}_x[f(X_t)1_{t<\tau_{\Omega^c}}]$ (summarized e.g. in [39, Section 2.1]) and similar computations as those used in the proof of [33, Proposition 2.2], one deduces that ν_h is a quasi-stationary distribution[4] for the process (1.1) in Ω and that, when X_0 is initially distributed according to the measure ν_h, it holds:

$$\tau_{\Omega^c} \sim \mathcal{E}(\lambda_{1,h}^L),\ \text{where we recall that } \lambda_{1,h}^L = \frac{\lambda_{1,h}^P}{2h}, \tag{4.9}$$

and where $\mathcal{E}(\lambda_{1,h}^L)$ stands for the exponential law of parameter $\lambda_{1,h}^L$.

Step 1. Proof of (1.17)**.** Note that the first statement of (1.17) has already been proved at the very beginning of the proof of Proposition 18. Moreover, according to (4.9), it holds,

$$\frac{1}{\lambda_{1,h}^L} = \mathbb{E}_{\nu_h}[\tau_{\Omega^c}] = \int_\Omega \mathbb{E}_x[\tau_{\Omega^c}]\,\nu_h(dx) = \frac{\int_\Omega \mathbb{E}_x[\tau_{\Omega^c}] u_{1,h}^{P*}\, e^{-\frac{f}{h}}}{\int_\Omega u_{1,h}^{P*} e^{-\frac{f}{h}}}. \tag{4.10}$$

[4]Even if the uniqueness of ν_h is not required here, we mention that for elliptic processes with smooth coefficients and when Ω is a smooth bounded domain, it is well-known that the quasi-stationary distribution in Ω is unique, see e.g. [23, 7, 50, 24].

Take now $\eta_0 \in (0, \min_{\partial\Omega} f - f(x_0))$ and recall that $\mathbf{C}_{\min}(\eta_0) = \mathbf{C}_{\min} \cap \{f < \min_{\partial\Omega} f - \eta_0\}$ (see (2.7)). One then has:

$$\frac{1}{\lambda^L_{1,h}} = \frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}(x)\, e^{-\frac{f(x)}{h}}\, dx}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} + \frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}(x)\, e^{-\frac{f(x)}{h}}\, dx}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}}$$
$$\geq \frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}\, e^{-\frac{f}{h}}}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}}. \tag{4.11}$$

Moreover, Theorem 3 with $K = \overline{\mathbf{C}}_{\min}(\eta_0)$ ($\subset \mathcal{A}(\{x_0\})$) implies that for some $c > 0$ and every $h > 0$ small enough:

$$\frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}\, e^{-\frac{f}{h}}}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} = \frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_0)} u^{P*}_{1,h}\, e^{-\frac{f}{h}}}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} \times \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})).$$

Then, using in addition (4.2) and taking $c > 0$ smaller if necessary, we have when $h \to 0$:

$$\frac{1}{\lambda^L_{1,h}} \geq \frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}\, e^{-\frac{f}{h}}}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} = \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})), \tag{4.12}$$

which leads, applying again Theorem 3, to

$$\limsup_{h\to 0} h \ln \lambda^L_{1,h} \ \leq\ -\lim_{h\to 0} h \ln \mathbb{E}_{x_0}[\tau_{\Omega^c}] \ =\ -2(\min_{\partial\Omega} f - f(x_0)).$$

Finally, the fact that

$$\liminf_{h\to 0} h \ln \lambda^L_{1,h} \ \geq\ -2(\min_{\partial\Omega} f - f(x_0))$$

is a direct consequence Proposition 12 together with the inequality $\lambda^L_{1,h} \sup_{x\in\overline{\Omega}} \mathbb{E}_x[\tau_{\Omega^c}] \geq 1$. This standard inequality can be derived as follows. Define the smooth function $g : x \in \overline{\Omega} \mapsto v_h - \lambda^L_{1,h} \mathbb{E}_x[\tau_{\Omega^c}]$, where v_h is the principal eigenvalue of L_h satisfying $v_h > 0$ in Ω and $\sup_{\overline{\Omega}} v_h = 1$. It then holds $L_h g = \lambda^L_{1,h}(v_h - 1) \leq 0$. Hence, according to the weak maximum principle [19, Theorem 1 in Section 6.4.1], we have $g \leq 0$ on $\overline{\Omega}$ and thus the announced inequality.

Step 2. Proof of (1.15). Injecting the equality in (4.12) into the relation (4.11) leads to the existence of $c > 0$ such that, for every $h > 0$ small enough,

$$\frac{1}{\lambda^L_{1,h}} = \frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}] u^{P*}_{1,h}(x)\, e^{-\frac{f(x)}{h}}\, dx}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} + \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}})). \tag{4.13}$$

Moreover, it follows from (1.17), Proposition 12, and (4.2) that for some $c > 0$ and every $h > 0$ small enough,

$$\lambda^L_{1,h} \frac{\int_{\Omega\setminus\overline{\mathbf{C}}_{\min}(\eta_0)} \mathbb{E}_x[\tau_{\Omega^c}]\, u^{P*}_{1,h}(x)\, e^{-\frac{f(x)}{h}}\, dx}{\int_\Omega u^{P*}_{1,h} e^{-\frac{f}{h}}} \leq e^{-\frac{c}{h}},$$

Plugging this estimate into (4.13) leads to $1 + O(e^{-\frac{c}{h}}) = \lambda^L_{1,h} \mathbb{E}_{x_0}[\tau_{\Omega^c}](1 + O(e^{-\frac{c}{h}}))$ when $h \to 0$. Together with Theorem 3, this proves (1.15).

Step 3. Proof of (1.16). Set $m_h = e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) - \frac{\eta_0}{2})}$. Consider a compact subset K of $\mathcal{A}(\{x_0\})$ and $\eta_* \in (\eta_0, \min_{\partial\Omega} f - f(x_0))$, so that $\overline{\mathbf{C}}_{\min}(\eta_*) \subset \mathbf{C}_{\min}(\eta_0)$ and $\overline{\mathbf{C}}_{\min}(\eta_*) \subset \mathcal{A}(\{x_0\})$ (see (2.8)). We claim that, for all $x \in K$, $y \in \overline{\mathbf{C}}_{\min}(\eta_*)$, and all $u > 0$:

$$\mathbb{P}_x[\tau_{\Omega^c} > u] \le \mathbb{P}_y[\tau_{\Omega^c} > u - 2m_h] + R_1 \text{ and } \mathbb{P}_x[\tau_{\Omega^c} > u] \ge \mathbb{P}_y[\tau_{\Omega^c} > u + m_h] + R_2, \tag{4.14}$$

where, for $j \in \{1,2\}$, R_j is independent of $u > 0$ and of x, y, and satisfies, for some $c > 0$ and every h small enough: $|R_j| \le e^{-\frac{c}{h}}$.

To prove (4.14), we first consider the case when $K = \overline{\mathbf{C}}_{\min}(\eta_*)$. Using (2.13) and the Markov inequality, there exists $c > 0$ such that for every h small enough:

$$\sup_{x \in \overline{\mathbf{C}}_{\min}(\eta_*)} \mathbb{P}_x[\tau_{\mathbf{C}^c_{\min}(\eta_0)} > m_h] \le e^{-\frac{c}{h}}. \tag{4.15}$$

Recall that for $x' \in \mathbf{C}_{\min}(\eta_0)$, $\mu^h_{x'}$ denotes the hitting distribution on $\partial\mathbf{C}_{\min}(\eta_0)$ for the process (1.1) when $X_0 = x'$ (see (2.18)) and $\|\mu^h_{x'} - \mu^h_{y'}\| \le e^{-\frac{c}{h}}$ uniformly in $x', y' \in \overline{\mathbf{C}}_{\min}(\eta_*)$. We then have for all $u' > 0$, $v > 0$, and $x', y' \in \overline{\mathbf{C}}_{\min}(\eta_*)$, using the strong Markov property,

$$\begin{aligned}
\mathbb{P}_{x'}[\tau_{\Omega^c} > u'] \ge \mathbb{P}_{x'}[\tau_{\Omega^c} > u' + \tau_{\mathbf{C}^c_{\min}(\eta_0)}] &= \int \mathbb{P}_z[\tau_{\Omega^c} > u']\mu^h_{x'}(dz) \\
&\ge \int \mathbb{P}_z[\tau_{\Omega^c} > u']\mu^h_{y'}(dz) - \|\mu^h_{x'} - \mu^h_{y'}\| \\
&= \mathbb{P}_{y'}[\tau_{\Omega^c} > u' + \tau_{\mathbf{C}^c_{\min}(\eta_0)}] - \|\mu^h_{x'} - \mu^h_{y'}\| \\
&\ge \mathbb{P}_{y'}[\tau_{\Omega^c} > u' + v] - \mathbb{P}_{y'}[\tau_{\mathbf{C}^c_{\min}(\eta_0)} > v] - \|\mu^h_{x'} - \mu^h_{y'}\|.
\end{aligned} \tag{4.16}$$

Let $u, v > 0$ and $x, y \in \overline{\mathbf{C}}_{\min}(\eta_*)$. If $u - m_h \le 0$, $\mathbb{P}_x[\tau_{\Omega^c} > u] \le 1 = \mathbb{P}_y[\tau_{\Omega^c} > u - m_h]$. In addition, using (4.15) and (4.16) with $(x', y', u', v) = (x, y, u, m_h)$ and also with $(x', y', u', v) = (y, x, u - m_h, m_h)$ (when $u - m_h > 0$), we deduce that for all $x, y \in \overline{\mathbf{C}}_{\min}(\eta_*)$ and all $u > 0$:

$$\mathbb{P}_x[\tau_{\Omega^c} > u] \le \mathbb{P}_y[\tau_{\Omega^c} > u - m_h] + r_1 \text{ and } \mathbb{P}_x[\tau_{\Omega^c} > u] \ge \mathbb{P}_y[\tau_{\Omega^c} > u + m_h] + r_2, \tag{4.17}$$

where, for $j \in \{1,2\}$, r_j is independent of $u > 0$ and $x, y \in \overline{\mathbf{C}}_{\min}(\eta_*)$, and satisfies $|r_j| \le e^{-\frac{c}{h}}$ for some $c > 0$ independent of h. Notice that (4.17) implies (4.14) when $K = \overline{\mathbf{C}}_{\min}(\eta_*)$. Let us mention that the proof of (4.17) is inspired by the one of [22, Lemma 3].

Let us now prove (4.14) for an arbitrary $K \subset \mathcal{A}(\{x_0\})$. Take such a K and consider $T_K \ge 0$ as in the proof of Theorem 3. We have for every $x \in K$ and $y \in \overline{\mathbf{C}}_{\min}(\eta_*)$, using the Markov property, (2.12), (2.21), and the second inequality in (4.17),

$$\begin{aligned}
\mathbb{P}_x[\tau_{\Omega^c} > u] &\ge \mathbb{P}_x[\tau_{\Omega^c} > u + T_K] \\
&\ge \mathbb{P}_x\Big[\tau_{\Omega^c} > u + T_K, \sup_{t \in [0, T_K]} |X_t - \varphi_t(x)| \le \delta\Big] \\
&= \mathbb{E}_x\Big[\mathbb{P}_{X_{T_K}}[\tau_{\Omega^c} > u]\,\mathbf{1}_{\sup_{t \in [0,T_K]} |X_t - \varphi_t(x)| \le \delta}\Big] \\
&\ge \big(\mathbb{P}_y[\tau_{\Omega^c} > u + m_h] + r_2\big)(1 + O(e^{-\frac{c}{h}})).
\end{aligned}$$

This proves the second inequality in (4.14). Now let $h > 0$ be small enough so that $m_h > T_K$. Then, using the Markov property, (2.12), (2.21), and the first inequality in (4.17), it holds for all $u' > 0$, $x \in K$, and $y \in \overline{\mathbf{C}}_{\min}(\eta_*)$:

$$\begin{aligned}\mathbb{P}_x[\tau_{\Omega^c} > u' + m_h] &\le \mathbb{P}_x[\tau_{\Omega^c} > u' + T_K]\\ &= \mathbb{P}_x\Big[\tau_{\Omega^c} > u' + T_K, \sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta\Big]\\ &\quad + \mathbb{P}_x\Big[\tau_{\Omega^c} > u' + T_K, \sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| > \delta\Big]\\ &= \mathbb{P}_x\Big[\tau_{\Omega^c} > u' + T_K, \sup_{t\in[0,T_K]} |X_t - \varphi_t(x)| \le \delta\Big] + O(e^{-\frac{c}{h}})\\ &= \mathbb{E}_x\Big[\mathbb{P}_{X_{T_K}}[\tau_{\Omega^c} > u']\, \mathbf{1}_{\sup_{t\in[0,T_K]} |X_t-\varphi_t(x)|\le\delta}\Big] + O(e^{-\frac{c}{h}})\\ &\le \big(\mathbb{P}_y[\tau_{\Omega^c} > u' - m_h] + r_1\big)(1 + O(e^{-\frac{c}{h}})) + O(e^{-\frac{c}{h}}).\end{aligned}$$

Pick $u > 0$. Then, the first inequality in (4.14) is a consequence of the previous inequality when $u - 2m_h > 0$ (use it with $u' = u - m_h > 0$) and of the fact that when $u - 2m_h \le 0$, $\mathbb{P}_x[\tau_{\Omega^c} > u] \le 1 = \mathbb{P}_y[\tau_{\Omega^c} > u - 2m_h]$. This concludes the proof of (4.14).

We are now in position to prove Equation (1.16). According to (4.9), it holds for all $s \in \mathbb{R}$,

$$\mathbb{P}_{\nu_h}[\tau_{\Omega^c} > s] = e^{-\lambda^L_{1,h}\max(s,0)}, \tag{4.18}$$

and, according to (4.2), there exists $c > 0$ such that for all h small enough and for all $s \in \mathbb{R}$:

$$\begin{aligned}\mathbb{P}_{\nu_h}[\tau_{\Omega^c} > s] &= \frac{\int_\Omega \mathbb{P}_y[\tau_{\Omega^c} > s]\, u^{P^*}_{1,h}(y)\, e^{-\frac{1}{h}f(y)}dy}{\int_\Omega u^{P^*}_{1,h} e^{-\frac{f}{h}}}\\ &= \frac{\int_{\overline{\mathbf{C}}_{\min}(\eta_*)} \mathbb{P}_y[\tau_{\Omega^c} > s]\, u^{P^*}_{1,h}(y)\, e^{-\frac{1}{h}f(y)}dy}{\int_\Omega u^{P^*}_{1,h} e^{-\frac{f}{h}}} + O(e^{-\frac{c}{h}}).\end{aligned} \tag{4.19}$$

Moreover, from (1.17), there exists $c > 0$ such that for every h small enough:

$$\lambda^L_{1,h} m_h \le e^{-\frac{c}{h}}.$$

Consider $t > 0$ and $x \in K \subset \mathcal{A}(\{x_0\})$. Taking $s = t/\lambda^L_{1,h} + m_h > 0$ in (4.18) and using (4.14) and (4.19), there exists $h_0 > 0$ which does not depend on $t > 0$ and on $x \in K$ such that, taking $c > 0$ smaller if necessary (but not depending on $t > 0$ and on $x \in K$), it holds for every $h \in (0, h_0]$:

$$\mathbb{P}_x[\tau_{\Omega^c} > t/\lambda^L_{1,h}] \ge e^{-\lambda^L_{1,h}(t/\lambda^L_{1,h}+m_h)} - e^{-\frac{c}{h}} \text{ and then } \mathbb{P}_x[\tau_{\Omega^c} > t/\lambda^L_{1,h}] - e^{-t} \ge -\lambda^L_{1,h} m_h - e^{-\frac{c}{h}} \ge -2e^{-\frac{c}{h}}.$$

Similarly, taking now $s = t/\lambda^L_{1,h} - 2m_h$ and $h_0 > 0$ smaller if necessary (but not depending on $t > 0$ and on $x \in K$), it holds for every $h \in (0, h_0]$:

$$\begin{aligned}\mathbb{P}_x[\tau_{\Omega^c} > t/\lambda^L_{1,h}] - e^{-t} \le e^{-\lambda^L_{1,h}\max(t/\lambda^L_{1,h}-2m_h,0)} + e^{-\frac{c}{h}} - e^{-t} &\le \begin{cases} 3\lambda^L_{1,h} m_h + e^{-\frac{c}{h}} & \text{if } t > 2\lambda^L_{1,h} m_h\\ t + e^{-\frac{c}{h}} & \text{if } t \le 2\lambda^L_{1,h} m_h\end{cases}\\ &\le 4e^{-\frac{c}{h}}.\end{aligned}$$

Hence, for every compact $K \subset \mathcal{A}(\{x_0\})$, there exists $c > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$:

$$\sup_{x\in K, t\in\mathbb{R}^+} |\mathbb{P}_x[\tau_{\Omega^c} > t/\lambda^L_{1,h}] - e^{-t}| \le e^{-\frac{c}{h}},$$

which completes the proof of (1.16).

5. Proof of Theorem 2

In this last section, we prove Theorem 2. More precisely, we prove the following equivalent result on the principal eigenvalue $\lambda^P_{1,h}$ of P_h (see (1.14) and the lines below, and Proposition 3).

Theorem 5. *Assume* (**Ortho**), (**One-Well**), (**Div-free**), *and* (**Normal**). *Then, the principal eigenvalue $\lambda^P_{1,h}$ of P_h satisfies, when $h\to 0$:*

$$\lambda^P_{1,h} = \left(\kappa^P_1\, h^{\frac12} + \kappa^P_2\, h + O(h^{\frac54})\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))},$$

where $\kappa^P_1 = 2\kappa^L_1$ and $\kappa^P_2 = 2\kappa^L_2$ (see (1.19)), and the error term $O(h^{\frac54})$ is actually of order $O(h^{\frac32})$ when $\kappa^P_1 = 0$ or $\kappa^P_2 = 0$, i.e. when $\nabla f(z) = 0$ for every $z\in\partial\mathbf{C}_{\min}\cap\partial\Omega$ or $\nabla f(z)\neq 0$ for every $z\in\partial\mathbf{C}_{\min}\cap\partial\Omega$.

5.1. **General strategy.** In order to prove Theorem 5, we want to construct, for every h small enough, a very accurate approximation $\mathsf{f}_{1,h}$ of the eigenmode $u^P_{1,h}$ of P_h. The next proposition gives conditions ensuring that such an approximation is sufficiently accurate.

Proposition 19. *Assume* (**Ortho**) *and* (**One-Well**). *Assume moreover that, for all h small enough, there exists a $L^2(\Omega)$-normalized function $\mathsf{f}_{1,h}\in D(P_h)$ such that the following properties hold:*

(E1) $$\langle P_h\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)} = \left(\kappa^P_1 h^{\frac12} + \kappa^P_2 h + O(h^{\frac32})\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))},$$

(E2) $$\|P_h\mathsf{f}_{1,h}\|^2_{L^2(\Omega)} = O(h^2)\,\langle P_h\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)},$$

(E3) $$\|P^*_h\mathsf{f}_{1,h}\|^2_{L^2(\Omega)} = \left(\kappa^P_1 h^{\frac12}\, O(h^2) + \kappa^P_2\, h\, O(h)\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}.$$

Then, the asymptotic equivalent of Theorem 5 holds, i.e.

$$\lambda^P_{1,h} = \left(\kappa^P_1\, h^{\frac12} + \kappa^P_2\, h + O(h^{\frac54})\right) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} \quad \textit{when } h\to 0,$$

where the error term $O(h^{\frac54})$ is actually of order $O(h^{\frac32})$ when $\kappa^P_1 = 0$ or $\kappa^P_2 = 0$.

Proof. According to the argument leading to (4.6) and to (E1), (E2), we have, for some $C, c>0$ and every $h>0$ small enough:

(5.1) $$\|(1-\pi^P_h)\mathsf{f}_{1,h}\|_{L^2(\Omega)} \le Ch^{-1}\|P_h\mathsf{f}_{1,h}\|_{L^2(\Omega)} \quad\text{and thus}\quad \pi^P_h\mathsf{f}_{1,h} = \mathsf{f}_{1,h} + O(e^{-\frac{c}{h}}) \ \text{ in } L^2(\Omega).$$

Since $P_h\pi^P_h\mathsf{f}_{1,h} = \lambda^P_{1,h}\pi^P_h\mathsf{f}_{1,h}$, it follows from the second estimate of (5.1) that

$$\begin{aligned}\lambda^P_{1,h} &= \frac{\langle P_h\pi^P_h\mathsf{f}_{1,h},\pi^P_h\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}}{\|\pi^P_h\mathsf{f}_{1,h}\|^2_{L^2(\Omega)}}\\ &= (1+O(e^{-\frac{c}{h}}))\Big[\langle P_h\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)} + \langle P_h(\pi^P_h-1)\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)} + \langle P_h\pi^P_h\mathsf{f}_{1,h},(\pi^P_h-1)\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}\Big].\end{aligned}$$

Moreover (5.1), $\pi^P_h = O(1)$ (see (3.36)), and the Cauchy-Schwarz inequality imply:

$$|\langle P_h\pi^P_h\mathsf{f}_{1,h},(\pi^P_h-1)\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}| = |\langle \pi^P_h P_h\mathsf{f}_{1,h},(\pi^P_h-1)\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}| = \|P_h\mathsf{f}_{1,h}\|^2_{L^2(\Omega)}\,O(h^{-1})$$

and

$$|\langle P_h(\pi^P_h-1)\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}| = |\langle (\pi^P_h-1)\mathsf{f}_{1,h},P^*_h\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}| = \|P_h\mathsf{f}_{1,h}\|_{L^2(\Omega)}\|P^*_h\mathsf{f}_{1,h}\|_{L^2(\Omega)}\,O(h^{-1}).$$

Using in addition (E1), (E2), and (E3), it follows that

$$\lambda^P_{1,h} = (1+O(e^{-\frac{c}{h}}))\langle P_h\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}(1+O(h^{\ell})) = \langle P_h\mathsf{f}_{1,h},\mathsf{f}_{1,h}\rangle_{L^2(\Omega)}(1+O(h^{\ell})),$$

where $\ell = \frac{1}{2}$ when $\kappa_1^P = 0$ (and thus $\kappa_2^P \neq 0$), $\ell = 1$ when $\kappa_2^P = 0$ (and thus $\kappa_1^P \neq 0$), and $\ell = \frac{3}{4}$ when $\kappa_1^P \kappa_2^P \neq 0$. This leads to the statement of Proposition 19. □

5.2. **Proof of Theorem 5.** From now on, we assume (**Ortho**), (**One-Well**), (**Div-free**), and (**Normal**). According to Proposition 19, it is sufficient to construct a quasi-mode $\mathsf{f}_{1,h}$ satisfying (E1), (E2), and (E3) (see Proposition 22 below). The construction below is strongly inspired by to the ones made in [35, 34].

5.2.1. *System of coordinates near the points of $\partial \mathbf{C}_{\min} \cap \partial\Omega$.* Recall that $\partial \mathbf{C}_{\min} \cap \partial\Omega \neq \emptyset$ (see (**One-Well**)) and that $\partial \mathbf{C}_{\min} \cap \partial\Omega$ has a finite cardinality (see (1.6)). Take $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$. There exists a neighborhood V_z of z in $\overline{\Omega}$ and a coordinate system

$$p \in \mathsf{V}_z \mapsto v = (v', v_d) = (v_1, \dots, v_{d-1}, v_d) \in \mathbb{R}^{d-1} \times \mathbb{R}_- \tag{5.2}$$

such that

$$v(z) = 0, \quad \{p \in \mathsf{V}_z,\, v_d(p) < 0\} = \Omega \cap \mathsf{V}_z, \quad \{p \in \mathsf{V}_z,\, v_d(p) = 0\} = \partial\Omega \cap \mathsf{V}_z, \tag{5.3}$$

and

$$\forall i, j \in \{1, \dots, d\}, \quad g_z\Big(\frac{\partial}{\partial v_i}(z), \frac{\partial}{\partial v_j}(z)\Big) = \delta_{ij} \quad \text{and} \quad \frac{\partial}{\partial v_d}(z) = n_\Omega(z),$$

where g_z is the metric tensor in the new coordinates. We denote by $G = (G_{ij})_{1 \le i,j \le d}$ its matrix, by $G^{-1} = (G^{ij})_{1 \le i,j \le d}$ its inverse, and by $(\mathsf{e}_1, \dots, \mathsf{e}_d) = ({}^t(1, 0, \dots, 0), \dots, {}^t(0, \dots, 0, 1))$ the canonical basis of $\mathbb{R}^d$ so that, defining $J := \operatorname{Jac} v^{-1}$, we have

$$G = {}^tJJ\,, \quad G(0) = (\delta_{ij}) \;\text{ i.e. }\; {}^tJ(0) = J^{-1}(0)\,, \quad \text{and} \quad n_\Omega(z) = J(0)\mathsf{e}_d\,. \tag{5.4}$$

In addition, defining $\hat f := f \circ v^{-1}$ the function f in the new coordinates:

Case 1, when $\nabla f(z) \neq 0$: According for example to [26, Section 3.4], the v-coordinates can be chosen such that

$$\hat f(v', v_d) = f(z) + \mu(z) v_d + \frac{1}{2} v' \operatorname{Hess} \hat f|_{\{v_d = 0\}}(0)\, {}^t v', \tag{5.5}$$

where we recall that $\mu(z) := \partial_{n_\Omega} f(z) > 0$ and that, thanks to (**Normal**), 0 is a non degenerate (global) minimum of $\hat f|_{\{v_d=0\}}$.

Case 2, when $\nabla f(z) = 0$: We have $\nabla(\hat f + |\mu(z)| v_d^2)(0) = 0$ and, according to (5.4):

$$\operatorname{Hess}(\hat f + |\mu(z)| v_d^2)(0) = {}^tJ(0)\big(\operatorname{Hess} f(z) + 2|\mu(z)| n_\Omega(z) n_\Omega(z)^*\big) J(0), \tag{5.6}$$

where we recall that, from (**Normal**), $n_\Omega(z)$ is an eigenvector associated with the negative eigenvalue $\mu(z)$ of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$. Note also that the matrix in (5.6) is positive definite according to Lemma 1.

In particular, up to choosing V_z smaller, one can assume that when $\nabla f(z) \neq 0$,

$$\operatorname{argmin}_{v(\overline{\mathsf{V}_z})} \big(\hat f(v) - 2\mu(z) v_d\big) = \{0\}, \tag{5.7}$$

and when $\nabla f(z) = 0$,

$$\operatorname{argmin}_{v(\overline{\mathsf{V}_z})} \big(\hat f(v) + |\mu(z)| v_d^2\big) = \{0\}. \tag{5.8}$$

For $\delta_1 > 0$ and $\delta_2 > 0$ small enough, one finally defines the following neighborhood of z in $\partial\Omega$,

$$\mathsf{V}_{\partial\Omega}^{\delta_2}(z) := \big\{p \in \mathsf{V}_z,\, v_d(p) = 0 \text{ and } |v'(p)| \le \delta_2\big\} \;\; \text{(see (5.2)-(5.3))},$$

and the following neighborhood of z in $\overline{\Omega}$,

$$\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z) = \big\{p \in \mathsf{V}_z, |v'(p)| \le \delta_2 \text{ and } v_d(p) \in [-2\delta_1, 0]\big\}. \tag{5.9}$$

The set defined in (5.9) is a cylinder centered at z in the v-coordinates. Up to choosing $\delta_1 > 0$ and $\delta_2 > 0$ smaller, we can assume the cylinders $\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)$, $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$, pairwise disjoint. Since $f(z) = \min_{\partial\Omega} f > f(x_0)$, we can also assume that

$$\min_{\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)} f > f(x_0) \quad \text{(so in particular } x_0 \notin \mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)\text{)}, \tag{5.10}$$

and, in view of (1.5),

$$\operatorname{argmin}_{\mathsf{V}^{\delta_2}_{\partial\Omega}(z)} f = \{z\}. \tag{5.11}$$

The parameter $\delta_2 > 0$ is now kept fixed. Finally, according to (5.11) and up to choosing $\delta_1 > 0$ smaller, there exists $r > 0$ such that:

$$\big\{p \in \mathsf{V}_z, |v'(p)| = \delta_2 \text{ and } v_d(p) \in [-2\delta_1, 0]\big\} \subset \{f \ge f(z) + r\}. \tag{5.12}$$

We end this section by defining locally near each $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$ a function φ_z in the above v-coordinates, and used in the next section to define the quasi-mode $\mathsf{f}_{1,h}$ near z. Let $\chi \in C^\infty(\mathbb{R}^-, [0,1])$ be a cut-off function such that

$$\operatorname{supp} \chi \subset [-\delta_1, 0] \text{ and } \chi = 1 \text{ on } \Big[-\frac{\delta_1}{2}, 0\Big]. \tag{5.13}$$

For every $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$, the function φ_z is defined as follows (see (5.2), (5.3), and (5.9)):

Case 1, when $\nabla f(z) \neq 0$:

$$\forall v = (v', v_d) \in v\big(\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)\big), \quad \varphi_z(v', v_d) := \frac{\int_{v_d}^0 \chi(t) e^{\frac{2}{h}\mu(z)\, t} dt}{\int_{-2\delta_1}^0 \chi(t)\, e^{\frac{2}{h}\mu(z)\, t} dt}, \tag{5.14}$$

where we recall that $\mu(z) = \partial_{n_\Omega} f(z) > 0$, see (5.5).

Case 2, when $\nabla f(z) = 0$:

$$\forall v = (v', v_d) \in v\big(\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)\big), \quad \varphi_z(v', v_d) := \frac{\int_{v_d}^0 \chi(t) e^{-\frac{1}{h}|\mu(z)|\, t^2} dt}{\int_{-2\delta_1}^0 \chi(t)\, e^{-\frac{1}{h}|\mu(z)|\, t^2} dt}, \tag{5.15}$$

where we recall that $\mu(z)$ is the negative eigenvalue of $\operatorname{Hess} f(z) + {}^t\mathsf{L}(z)$, see (5.6).

In both cases:

$$\begin{cases} \varphi_z \in C^\infty\big(v\big(\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)\big), [0,1]\big) \text{ only depends on } v_d,\ \varphi_z(v', 0) = 0, \text{ and} \\ \forall (v', v_d) \in v\big(\mathsf{V}^{\delta_1,\delta_2}_{\overline{\Omega}}(z)\big),\ \varphi_z(v', v_d) = 1 \text{ when } v_d \in [-2\delta_1, -\delta_1]. \end{cases} \tag{5.16}$$

5.2.2. *Definition of the quasi-mode* $\mathsf{f}_{1,h}$. We now define $\mathsf{f}_{1,h}$, using the v-coordinates and the above φ_z, $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$. Before, we recall that we defined in (5.9) pairwise disjoint cylinders around the $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$ which satisfy (5.10), (5.11), and (5.12). On the other hand, for every $p \in \partial \mathbf{C}_{\min} \setminus \partial\Omega$: $p \in \Omega$ and thus $\nabla f(p) \neq 0$, which implies that $\{f < f(p)\} \cap B(p, r)$ is connected for every $r > 0$ small enough and thus included in $\mathbf{C}_{\min}$.

These considerations imply the existence of the following subsets $\mathbf{C}_{\text{low}}$ and $\mathbf{C}_{\text{up}}$ of Ω.

Proposition 20. *Assume* (**Ortho**), (**One-Well**), (**Div-free**), *and* (**Normal**). *Then, there exist two* $\mathcal{C}^\infty$ *connected open sets* $\mathbf{C}_{\rm low}$ *and* $\mathbf{C}_{\rm up}$ *of* Ω *satisfying the following properties:*

(1) It holds $\overline{\mathbf{C}}_{\min} \subset \mathbf{C}_{\rm up} \cup \partial\Omega$ *and* $\operatorname{argmin}_{\overline{\mathbf{C}}_{\rm up}} f = \{x_0\}$.

(2) The set $\overline{\mathbf{C}}_{\rm up}$ *is a neighborhood in* $\overline{\Omega}$ *of each* $\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z)$, $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$.

(3) It holds $\overline{\mathbf{C}}_{\rm low} \subset \mathbf{C}_{\rm up}$ *and the strip* $\overline{\mathbf{C}}_{\rm up} \setminus \mathbf{C}_{\rm low}$ *satisfies*

$$\exists c > 0\,,\ f \geq f(x_0) + c \ \text{on}\ \overline{\mathbf{C}}_{\rm up} \setminus \mathbf{C}_{\rm low} \quad \text{and} \quad \overline{\mathbf{C}}_{\rm up} \setminus \mathbf{C}_{\rm low} = \bigcup_{z \in \partial\mathbf{C}_{\min} \cap \partial\Omega} \mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z) \ \bigcup\ \mathsf{O}, \tag{5.17}$$

where the subset O *of* $\overline{\Omega}$ *is such that:*

$$\exists c > 0\,,\ f \geq \min_{\partial\Omega} f + c \ \text{on}\ \mathsf{O}.$$

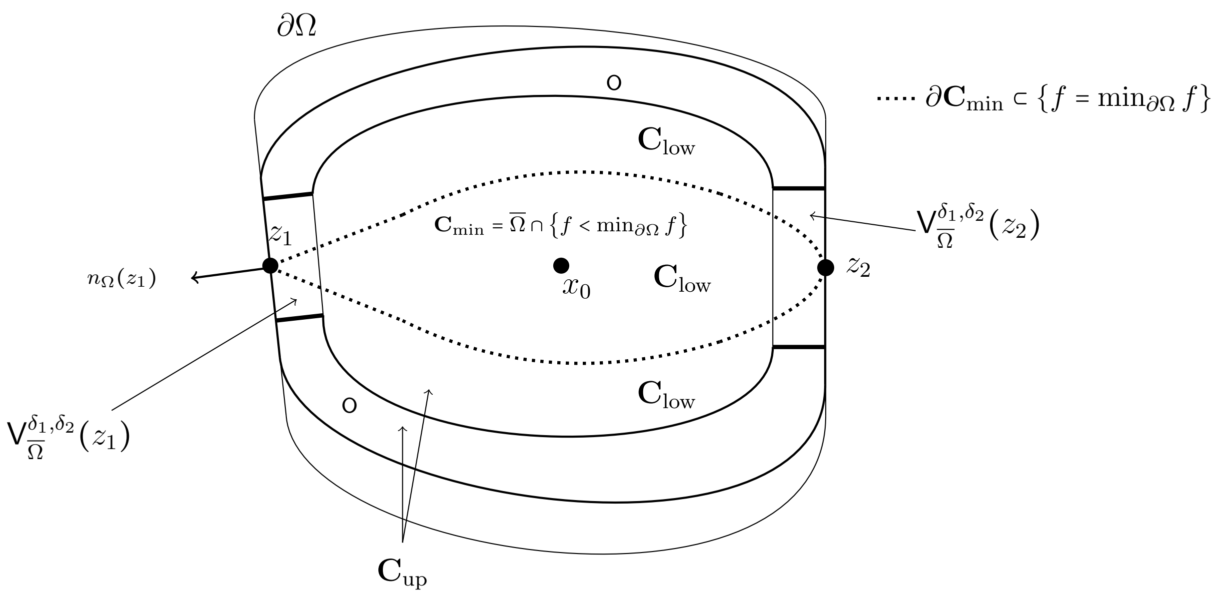

FIGURE 5.1. Schematic representation of $\mathbf{C}_{\rm low}$, $\mathbf{C}_{\rm up}$, and O (see Proposition 20). On the figure, $\partial\mathbf{C}_{\min} \cap \partial\Omega = \{z_1, z_2\}$ with $\nabla f(z_1) = 0$ and $|\nabla f(z_2)| \neq 0$.

We refer to Figure 5.1 for a schematic representation of $\mathbf{C}_{\rm low}$, $\mathbf{C}_{\rm up}$, and O. Notice that Proposition 20 implies

$$\operatorname{argmin}_{\overline{\mathbf{C}}_{\rm up}} f = \operatorname{argmin}_{\overline{\mathbf{C}}_{\rm low}} f = \{x_0\}. \tag{5.18}$$

Using the above sets $\mathbf{C}_{\rm up}$ and $\mathbf{C}_{\rm low}$, we define a function $\phi_{1,h} : \overline{\Omega} \to [0,1]$ as follows.

(i) For every $z \in \partial\mathbf{C}_{\min} \cap \partial\Omega$, $\phi_{1,h}$ is defined on the cylinder $\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z)$ (see (5.9)) by

$$\forall p \in \mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z),\ \phi_{1,h}(p) := \varphi_z(v(p)), \quad \text{see (5.14) and (5.15)}. \tag{5.19}$$

(ii) From (5.16), (5.17), and the fact that $\overline{\mathbf{C}}_{\rm low} \subset \mathbf{C}_{\rm up}$ (see Proposition 20), the above function $\phi_{1,h}$ satisfying (5.19) can be extended to $\overline{\Omega}$ so that

$$\phi_{1,h} = 0 \text{ on } \overline{\Omega} \setminus \mathbf{C}_{\rm up}, \quad \phi_{1,h} = 1 \text{ on } \mathbf{C}_{\rm low}, \text{ and } \phi_{1,h} \in C^\infty(\overline{\Omega}, [0,1]). \tag{5.20}$$

Moreover, in view of (5.14), (5.15), and (5.17), $\phi_{1,h}$ can be chosen on O such that, for some $C > 0$ and for every h small enough,

$$\forall \alpha \in \mathbb{N}^d,\ |\alpha| \in \{1,2\},\ \|\partial^\alpha \phi_{1,h}\|_{L^\infty(\mathsf{O})} \le C h^{-2}. \tag{5.21}$$

Notice that (5.20) implies

$$\operatorname{supp} \nabla \phi_{1,h} \subset \overline{\mathbf{C}}_{\mathrm{up}} \setminus \mathbf{C}_{\mathrm{low}}. \tag{5.22}$$

We are now in position to define the quasi-mode $\mathsf{f}_{1,h}$ for P_h.

Definition 21. *Assume* (**Ortho**), (**One-Well**), (**Div-free**), *and* (**Normal**). *Let $\phi_{1,h}$ be the above function satisfying* (5.19)–(5.21). *We define:*

$$\mathsf{f}_{1,h} := \frac{\phi_{1,h}\, e^{-\frac{f}{h}}}{Z_{1,h}}, \quad \textit{where} \quad Z_{1,h} := \|\phi_{1,h}\, e^{-\frac{f}{h}}\|_{L^2(\Omega)}.$$

5.2.3. *Quasi-modal estimates.*

Proposition 22. *Assume* (**Ortho**), (**One-Well**), (**Div-free**), *and* (**Normal**). *Let $\mathsf{f}_{1,h}$ be as introduced in Definition 21. Then, $\mathsf{f}_{1,h}$ belongs to $D(P_h)$ and satisfies* (E1), (E2), *and* (E3) *of Proposition 19. In particular, Theorem 5 holds true.*

Proof. First of all, the relation (5.20) implies $\mathsf{f}_{1,h} \in C^\infty(\overline{\Omega}, \mathbb{R}^+)$ and $\mathsf{f}_{1,h} = 0$ on $\partial\Omega$, and thus, since $\mathbf{C}_{\mathrm{up}} \subset \Omega$ (see Proposition 20),

$$\mathsf{f}_{1,h} \in D(P_h) = H^2(\Omega) \cap H^1_0(\Omega), \tag{5.23}$$

In the following, $c > 0$ is a constant independent of $h > 0$ which can change from one occurrence to another. The proof is divided into three steps.

Step 1. The function $\mathsf{f}_{1,h}$ satisfies (E1).

Asymptotic equivalent of $Z_{1,h}$. From Definition 21 and (5.20), we have

$$Z_{1,h}^2 = \int_\Omega \phi_{1,h}^2\, e^{-\frac{2}{h} f} = \int_{\mathbf{C}_{\mathrm{low}}} \phi_{1,h}^2\, e^{-\frac{2}{h} f} + \int_{\mathbf{C}_{\mathrm{up}} \setminus \mathbf{C}_{\mathrm{low}}} \phi_{1,h}^2\, e^{-\frac{2}{h} f} = \int_{\mathbf{C}_{\mathrm{low}}} \phi_{1,h}^2\, e^{-\frac{2}{h} f} + O\big(e^{-\frac{2}{h}(f(x_0)+c)}\big),$$

where we used $\operatorname{Ran} \phi_{1,h} \subset [0,1]$ and $f \ge f(x_0) + c$ on $\mathbf{C}_{\mathrm{up}} \setminus \mathbf{C}_{\mathrm{low}}$ (see (5.17)). Moreover, using $\phi_{1,h} = 1$ on $\mathbf{C}_{\mathrm{low}}$ and (5.18), the standard Laplace method implies that when $h \to 0$,

$$Z_{1,h}^2 = (\pi h)^{\frac{d}{2}} \big(\det \operatorname{Hess} f(x_0)\big)^{-\frac{1}{2}}\ e^{-\frac{2}{h} f(x_0)} \big(1 + O(h)\big). \tag{5.24}$$

Asymptotic equivalent of $\langle P_h \mathsf{f}_{1,h}, \mathsf{f}_{1,h}\rangle_{L^2(\Omega)}$. First, using (5.23) and (3.1),

$$\langle P_h \mathsf{f}_{1,h}, \mathsf{f}_{1,h}\rangle_{L^2(\Omega)} = \int_\Omega |\nabla_{f,h} \mathsf{f}_{1,h}|^2.$$

In addition, from Definition 21 and (5.22), $\nabla_{f,h} \mathsf{f}_{1,h} = Z_{1,h}^{-1} h e^{-\frac{f}{h}} \nabla \phi_{1,h}$ is supported in $\overline{\mathbf{C}}_{\mathrm{up}} \setminus \mathbf{C}_{\mathrm{low}}$. Hence, from (3) in Proposition 20, (5.21), and (5.24), we have for every h small enough:

$$\begin{aligned} \langle P_h \mathsf{f}_{1,h}, \mathsf{f}_{1,h}\rangle_{L^2(\Omega)} &= \sum_{z \in \partial \mathbf{C}_{\min} \cap \partial\Omega} \int_{\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z)} |\nabla_{f,h} \mathsf{f}_{1,h}|^2 + O\big(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}\big) \\ &= \sum_{z \in \partial \mathbf{C}_{\min} \cap \partial\Omega} Z_{1,h}^{-2} h^2 \int_{\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z)} \big|\nabla \phi_{1,h}\big|^2 e^{-\frac{2}{h} f} + O\big(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}\big). \end{aligned} \tag{5.25}$$

Let now z belong to $\partial \mathbf{C}_{\min} \cap \partial\Omega$ and recall the coordinates $p \mapsto v(p)$ defined in Section 5.2.1, see (5.2)–(5.4). We also define $\hat{\boldsymbol{\ell}} := \boldsymbol{\ell} \circ v^{-1}$. With these coordinates, we have on $\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)$:

$$(\nabla f)(v^{-1}) = {}^tJ^{-1}\nabla \hat f,\ (\nabla \phi_{1,h})(v^{-1}) = {}^tJ^{-1}\,\nabla \varphi_z, \text{ and } (\operatorname{Jac}\boldsymbol{\ell})(v^{-1}) = \operatorname{Jac}\hat{\boldsymbol{\ell}}\, J^{-1}. \tag{5.26}$$

Case 1, when $\nabla f(z) \neq 0$: Using (5.4), (5.26), and (5.14), we have

$$\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} \big|\nabla \phi_{1,h}\big|^2 e^{-\frac{2}{h}f} = \frac{\int_{|v'|\le \delta_2}\int_{-2\delta_1}^0 G^{dd}(v)\,\chi^2(v_d)\sqrt{|G|}(v)\, e^{-\frac{2}{h}(\hat f(v)-2\mu(z)v_d)}dv}{\Big(\int_{-2\delta_1}^0 \chi(t)\, e^{\frac{2}{h}\mu(z)\,t}dt\Big)^2} \tag{5.27}$$

and a straightforward computation shows that, when $h\to 0$ (see (5.13)),

$$N_z := \int_{-2\delta_1}^0 \chi(t)\, e^{\frac{2}{h}\mu(z)\,t}dt = \frac{h}{2\mu(z)}\big(1+O(e^{-\frac{c}{h}})\big). \tag{5.28}$$

On the other hand, using $G(0) = (\delta_{ij})$, (5.13), (5.5), and (5.7), the Laplace method leads to

$$\begin{aligned}\int_{|v'|\le\delta_2}\int_{-2\delta_1}^0 G^{dd}\chi^2\sqrt{|G|}\,e^{-\frac{2}{h}(\hat f-2\mu(z)v_d)}dv &= \big(1+O(h)\big)\int_{\mathbb{R}^{d-1}}\int_{-\infty}^0 e^{-\frac{2}{h}(\hat f-2\mu(z)v_d)}dv\\ &= \big(1+O(h)\big)\frac{h}{2\mu(z)}\frac{(\pi h)^{\frac{d-1}{2}}e^{-\frac{2}{h}\hat f(0)}}{\big(\det\operatorname{Hess}\hat f|_{\{v_d=0\}}(0)\big)^{\frac12}}.\end{aligned}$$

Combining this equation with (5.24), (5.27), and (5.28) (recall that $f(z) = \min_{\partial\Omega} f$), we get

$$\frac{h^2}{Z_{1,h}^2}\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)}|\nabla\phi_{1,h}|^2 e^{-\frac{2}{h}f} = \frac{2\partial_{n_\Omega}f(z)}{\sqrt{\pi}}\frac{\sqrt{\det\operatorname{Hess} f(x_0)}}{\sqrt{\det\operatorname{Hess} f|_{\partial\Omega}(z)}}\sqrt{h}\,e^{-\frac{2}{h}(\min_{\partial\Omega}f-f(x_0))}\big(1+O(h)\big). \tag{5.29}$$

Case 2, when $\nabla f(z) = 0$: Thanks to (5.4), (5.26), and (5.15), we have

$$\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)}|\nabla\phi_{1,h}|^2 e^{-\frac{2}{h}f} = \frac{\int_{|v'|\le\delta_2}\int_{-2\delta_1}^0 G^{dd}(v)\,\chi^2(v_d)\sqrt{|G|}(v)\,e^{-\frac{2}{h}(\hat f(v)+|\mu(z)|v_d^2)}dv}{\Big(\int_{-2\delta_1}^0\chi(t)\,e^{-\frac1h|\mu(z)|\,t^2}dt\Big)^2},$$

where the denominator of the r.h.s. satisfies in the limit $h\to 0$ (see (5.13)),

$$N_z := \int_{-2\delta_1}^0 \chi(t)\,e^{-\frac1h|\mu(z)|\,t^2}dt = \frac{\sqrt{\pi h}}{2\sqrt{|\mu(z)|}}\big(1+O(e^{-\frac{c}{h}})\big). \tag{5.30}$$

Furthermore, using $G(0) = (\delta_{ij})$, (5.13), (5.6), and (5.8), the Laplace method gives, when $h\to 0$,

$$\int_{|v'|\le\delta_2}\int_{-2\delta_1}^0 G^{dd}\chi^2\sqrt{|G|}\,e^{-\frac{2}{h}(\hat f+|\mu(z)|v_d^2)}dv = \frac{(\pi h)^{\frac d2}e^{-\frac2h\hat f(0)}}{\sqrt{\det\operatorname{Hess}(\hat f+|\mu(z)|v_d^2)(0)}}\Big(\frac12+O(\sqrt h)\Big),$$

where, from the second item in Lemma 1 and (5.6), $\det\operatorname{Hess}(\hat f+|\mu(z)|v_d^2)(0) = -\det\operatorname{Hess} f(z)$. We refer to [35, Remark 25] for an explanation on the optimality of the remainder term $O(\sqrt h)$ in the previous equality. Using in addition (5.30), we obtain

$$\frac{h^2}{Z_{1,h}^2}\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)}|\nabla\phi_{1,h}|^2 e^{-\frac2h f} = \frac{2|\mu(z)|}{\pi}\frac{\sqrt{\det\operatorname{Hess} f(x_0)}}{\sqrt{|\det\operatorname{Hess} f(z)|}}\,h\,e^{-\frac2h(\min_{\partial\Omega}f-f(x_0))}\big(1+O(\sqrt h)\big). \tag{5.31}$$

Finally, (5.25), (5.29), and (5.31) imply that $\mathsf{f}_{1,h}$ satisfies (E1).

Step 2. The function $\mathsf{f}_{1,h}$ satisfies (E2).

Recall that $\nabla_{f,h}\mathsf{f}_{1,h} = Z_{1,h}^{-1}\, he^{-\frac{f}{h}}\nabla\phi_{1,h}$ is supported in $\overline{\mathbf{C}}_{\rm up}\setminus\mathbf{C}_{\rm low}$, so the same holds for $P_h\mathsf{f}_{1,h} = (\nabla^*_{f,h}+2\boldsymbol{\ell}\cdot)\nabla_{f,h}\mathsf{f}_{1,h}$. Thus, Proposition 20, (5.21), and (5.24) imply that for h small enough,

$$\int_\Omega |P_h\,\mathsf{f}_{1,h}|^2 = \sum_{z\in\partial\mathbf{C}_{\rm min}\cap\partial\Omega}\ \int_{\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z)}|P_h\,\mathsf{f}_{1,h}|^2 + O\big(e^{-\frac{2}{h}(\min_{\partial\Omega}f-f(x_0)+c)}\big). \tag{5.32}$$

Since $\operatorname{div}\boldsymbol{\ell}=0$, the same relation holds when replacing $P_h\,\mathsf{f}_{1,h}$ by $P_h^*\mathsf{f}_{1,h} = (\Delta_{f,h}-2\boldsymbol{\ell}\cdot\nabla_{f,h})\mathsf{f}_{1,h}$.

Let now z belong to $\partial\mathbf{C}_{\rm min}\cap\partial\Omega$. Using the relations $\Delta_{f,h} = 2he^{-\frac{f}{h}}(-\frac{h}{2}\Delta+\nabla f\cdot\nabla)e^{\frac{f}{h}}$ and (5.26) with φ_z only depending on the variable v_d, we get in the v-coordinates on $v(\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z))$:

$$\begin{aligned}(\Delta_{f,h}\mathsf{f}_{1,h})\circ v^{-1} &= \frac{2h\,e^{-\hat f/h}}{Z_{1,h}}\Big[\frac{-h}{2\sqrt{|G|}}\operatorname{div}\big(\sqrt{|G|}\,G^{-1}\nabla\varphi_z\big)+\sum_{i,j}G^{ij}\partial_{v_j}\varphi_z\partial_{v_i}\hat f\Big]\\ &= \frac{h\,e^{-\hat f/h}}{Z_{1,h}}\Big[\frac{-h}{\sqrt{|G|}}\sum_i\partial_{v_i}(\sqrt{|G|}G^{id}\partial_{v_d}\varphi_z)+2\partial_{v_d}\varphi_z\sum_i G^{id}\partial_{v_i}\hat f\Big].\end{aligned}$$

Moreover, recall that $\varphi_z(v) = \int_{v_d}^0\chi(t)e^{-\frac{1}{h}\theta(t)}dt/N_z$, where $\theta(t) = -2\mu(z)t$ when $\nabla f(z)\neq 0$ and $\theta(t) = |\mu(z)|t^2$ when $\nabla f(z)=0$ (see (5.14) and (5.15)), so that

$$\partial_{v_d}\varphi_z(v) = -\frac{1}{N_z}\chi(v_d)e^{-\frac{\theta(v_d)}{h}}\ \text{ and }\ \partial^2_{v_d}\varphi_z(v) = -\frac{1}{N_z}\chi'(v_d)e^{-\frac{\theta(v_d)}{h}}+\frac{1}{h\,N_z}\chi(v_d)\theta'(v_d)e^{-\frac{\theta(v_d)}{h}}. \tag{5.33}$$

Hence, we have on $v(\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z))$:

$$\begin{aligned}(\Delta_{f,h}\mathsf{f}_{1,h})\circ v^{-1} = \frac{he^{-\frac{1}{h}(\hat f+\theta)}}{N_zZ_{1,h}}\Big[&\frac{h}{\sqrt{|G|}}\sum_i\partial_{v_i}(\sqrt{|G|}G^{id})\chi(v_d)\\ &-\chi(v_d)\big(G^{dd}\theta'(v_d)+2\sum_j G^{jd}\partial_{v_j}\hat f\big)+hG^{dd}\chi'(v_d)\Big].\end{aligned} \tag{5.34}$$

Besides, we deduce from $\boldsymbol{\ell}\cdot\nabla_{f,h}\mathsf{f}_{1,h} = \frac{he^{-\frac{f}{h}}}{Z_{1,h}}\boldsymbol{\ell}\cdot\nabla\phi_{1,h}$, (5.26), (5.33), and (5.4) that on $v(\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z))$:

$$\begin{aligned}(2\boldsymbol{\ell}\cdot\nabla_{f,h}\mathsf{f}_{1,h})\circ v^{-1} &= -\frac{he^{-\frac{1}{h}(\hat f+\theta)}}{N_zZ_{1,h}}\ \chi(v_d)\Big(\big[2\hat{\boldsymbol{\ell}}(0)+2\operatorname{Jac}\hat{\boldsymbol{\ell}}(0)v\big]\cdot{}^tJ^{-1}\,\mathsf{e}_d+O(|v|^2)\Big)\\ &= -\frac{he^{-\frac{1}{h}(\hat f+\theta)}}{N_zZ_{1,h}}\ \chi(v_d)\big(2\hat{\boldsymbol{\ell}}(0)\cdot{}^tJ^{-1}\,\mathsf{e}_d+2\operatorname{Jac}\hat{\boldsymbol{\ell}}(0)v\cdot J(0)\,\mathsf{e}_d+O(|v|^2)\big).\end{aligned} \tag{5.35}$$

To go further in the computation of $P_h\mathsf{f}_{1,h}$ on $v(\mathsf{V}_{\overline{\Omega}}^{\delta_1,\delta_2}(z))$, let us consider the two cases $\nabla f(z)\neq 0$ and $\nabla f(z)=0$ separately.

Case 1, when $\nabla f(z)\neq 0$: Since $G = (\delta_{ij})+O(|v|)$ (see (5.4)), $\partial_{v_j}\hat f = O(|v|)$ when $1\le j\le d-1$, and $\partial_{v_d}\hat f = \mu(z)$ (see (5.5)), we have

$$\sum_{j=1}^d G^{jd}\partial_{v_j}\hat f = G^{dd}\mu(z)+O(|v|^2).$$

Since moreover $\theta'(v_d) = -2\mu(z)$, we deduce from (5.34) that

$$(\Delta_{f,h}\mathsf{f}_{1,h})\circ v^{-1} = \frac{h\chi(v_d)\,e^{-\frac{1}{h}(\hat f+\theta)}}{N_zZ_{1,h}}\big[O(h)+O(|v|^2)\big]+\frac{h^2e^{-\frac{1}{h}(\hat f+\theta)}}{N_zZ_{1,h}}G^{dd}\chi'(v_d).$$

Recall that (**Normal**) implies $\hat{\boldsymbol{\ell}}(0) = 0$. Hence, a Taylor expansion around $v = 0$ of the relation $\hat{\boldsymbol{\ell}} \cdot {}^tJ^{-1}\nabla \hat f = (\boldsymbol{\ell} \cdot \nabla f) \circ v^{-1} = 0$ (see (5.26)) shows that, for all $v \in \mathbb{R}^d$, $\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)\, v \cdot {}^tJ^{-1}(0)\nabla \hat f(0) = 0$, and then, using (5.4) and (5.5), $\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)\, v \cdot J(0)\mathsf{e}_d = 0$. Thus, using (5.35),

$$(2\boldsymbol{\ell} \cdot \nabla_{f,h}\mathsf{f}_{1,h}) \circ v^{-1} = -\frac{he^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}\ \chi(v_d) \times O(|v|^2).$$

Consequently,

$$(P_h\mathsf{f}_{1,h}) \circ v^{-1} = \frac{h\chi(v_d)\, e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}\big[O(h) + O(|v|^2)\big] + \frac{h^2e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}} G^{dd}\chi'(v_d).$$

Since $\chi' = 0$ in a neighborhood of 0 in $\mathbb{R}^-$ (see (5.13)), we obtain from (5.7), (5.24), (5.28), and the Laplace method that when $h \to 0$:

$$\begin{aligned}\int_{\mathsf{V}^{\delta_1,\delta_2}_{\Omega}(z)} |P_h\mathsf{f}_{1,h}|^2 &= \frac{1}{N_z^2 Z_{1,h}^2}\int_{v\left(\mathsf{V}^{\delta_1,\delta_2}_{\Omega}(z)\right)} O(h^4 + h^2|v|^4)e^{-\frac{2}{h}(\hat f - 2\mu(z)v_d)}dv + O(e^{-\frac{c}{h}})\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}\\ &= O(h^{\frac{5}{2}})\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} = O(h^2)\int_{\mathsf{V}^{\delta_1,\delta_2}_{\Omega}(z)} |\nabla \mathsf{f}_{1,h}|^2\, e^{-\frac{2}{h}f},\end{aligned}\tag{5.36}$$

where we used (5.29) to get the last equality.

Case 2, when $\nabla f(z) = 0$**:** From (5.35) and $\hat{\boldsymbol{\ell}}(0) = 0$ (see (1.9)), we have

$$(2\boldsymbol{\ell} \cdot \nabla_{f,h}\mathsf{f}_{1,h}) \circ v^{-1} = -\frac{he^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}\ \chi(v_d)\big(2v \cdot {}^t\,\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)J(0)\mathsf{e}_d + O(|v|^2)\,\big).$$

Therefore, using (5.34), $G = (\delta_{ij}) + O(|v|)$ (see (5.4)) and $\partial_{v_j}\hat f = O(|v|)$ for all $j = \{1, \dots, d\}$:

$$\begin{aligned}(P_h\mathsf{f}_{1,h}) \circ v^{-1} &= \frac{he^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}\chi(v_d)\Big[O(h) - 2G^{dd}|\mu(z)|v_d - 2\sum_j G^{jd}\partial_{v_j}\hat f\\ &\qquad - 2v \cdot {}^t\,\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)J(0)\mathsf{e}_d + O(|v|^2)\Big] + \frac{h^2e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}G^{dd}\chi'(v_d)\\ &= \frac{2he^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}\chi(v_d)\Big[O(h) - |\mu(z)|v_d - \partial_{v_d}\hat f\\ &\qquad - v \cdot {}^t\,\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)J(0)\mathsf{e}_d + O(|v|^2)\Big] + \frac{h^2e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}G^{dd}\chi'(v_d).\end{aligned}$$

We have moreover $\partial_{v_d}\hat f = v \cdot \mathrm{Hess}\,\hat f(0)\mathsf{e}_d + O(|v|^2)$ and (**Normal**) implies $[\mathrm{Hess}\, f(z) + {}^t\,\mathrm{Jac}\,\boldsymbol{\ell}(z)]n_\Omega(z) = \mu(z)n_\Omega(z)$, which becomes in the v-coordinates, using (5.4) (see also (5.6)):

$$\begin{aligned}\big(\,\mathrm{Hess}\,\hat f(0) + {}^t\,\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)\, J(0)\big)\mathsf{e}_d &= {}^tJ(0)\big(\,\mathrm{Hess}\, f(z) + {}^t\,\mathrm{Jac}\,\boldsymbol{\ell}(z)\big)n_\Omega(z)\\ &= \mu(z){}^tJ(0)J(0)\mathsf{e}_d = \mu(z)\mathsf{e}_d.\end{aligned}$$

It follows that $|\mu(z)|v_d + \partial_{v_d}\hat f + v \cdot {}^t\,\mathrm{Jac}\,\hat{\boldsymbol{\ell}}(0)J(0)\mathsf{e}_d = O(|v|^2)$ and consequently,

$$(P_h\mathsf{f}_{1,h}) \circ v^{-1} = \frac{2h\chi(v_d)\, e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}[O(h) + O(|v|^2)] + \frac{h^2e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}}G^{dd}\chi'(v_d).$$

Hence, since $\chi' = 0$ around 0, it follows from (5.8), (5.24), (5.30), (5.31), and from the Laplace method that when $h \to 0$,

$$\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} |P_h \mathsf{f}_{1,h}|^2 = \frac{h^{\frac{d}{2}} O(h^4)}{h\, h^{\frac{d}{2}}}\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} + O(e^{-\frac{c}{h}})\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}$$

$$(5.37)\qquad = O(h^3) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} = O(h^2) \int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} |\nabla \mathsf{f}_{1,h}|^2\, e^{-\frac{2}{h} f}.$$

Plugging (5.36) and (5.37) into (5.32), and using (5.25) and (E1), then leads to:

$$\int_{\Omega} |P_h\, \mathsf{f}_{1,h}|^2 = O(h^2) \langle P_h \mathsf{f}_{1,h}, \mathsf{f}_{1,h} \rangle.$$

Therefore $\mathsf{f}_{1,h}$ satisfies (E2).

Step 3. The function $\mathsf{f}_{1,h}$ satisfies (E3).

Recall that $P_h^* = \Delta_{f,h} - 2\boldsymbol{\ell} \cdot \nabla_{f,h}$ according to Proposition 3 and to (**Div-free**). Therefore, the computations of the previous step show that, on any $v(\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z))$, $z \in \partial \mathbf{C}_{\min} \cap \partial\Omega$:

$$(P_h^* \mathsf{f}_{1,h}) \circ v^{-1} = \begin{cases} \frac{h\chi(v_d)\, e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}} [O(h) + O(|v|^2)] + \frac{h^2 e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}} G^{dd} \chi'(v_d) & \text{when } \nabla f(z) \neq 0, \\ \frac{h\chi(v_d)\, e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}} [O(h) + O(|v|)] + \frac{h^2 e^{-\frac{1}{h}(\hat f + \theta)}}{N_z Z_{1,h}} G^{dd} \chi'(v_d) & \text{when } \nabla f(z) = 0. \end{cases}$$

It follows that, when $h \to 0$,

$$\int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} |P_h^*\, \mathsf{f}_{1,h}|^2 = \begin{cases} O(h^2) \int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} |\nabla \mathsf{f}_{1,h}|^2\, e^{-\frac{2}{h} f} & \text{when } \nabla f(z) \neq 0, \\ O(h) \int_{\mathsf{V}_{\Omega}^{\delta_1,\delta_2}(z)} |\nabla \mathsf{f}_{1,h}|^2\, e^{-\frac{2}{h} f} & \text{when } \nabla f(z) = 0, \end{cases}$$

and hence, according to (5.32) (with P_h replaced by P_h^*), (5.29), and (5.31):

$$\int_{\Omega} |P_h^*\, \mathsf{f}_{1,h}|^2 = \begin{cases} O(h)\, h\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} & \text{when } \kappa_1^P = 0, \\ O(h^2)\, h^{\frac{1}{2}}\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} & \text{when } \kappa_2^P = 0, \\ O(h^{\frac{3}{2}})\, h^{\frac{1}{2}}\, e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} & \text{when } \kappa_1^P \neq 0 \text{ and } \kappa_2^P \neq 0. \end{cases}$$

This proves that $\mathsf{f}_{1,h}$ satisfies (E3) and completes the proof of Proposition 22. □

Appendix

In this appendix, we prove Proposition 3.

Proof of Proposition 3. Let $h > 0$ be fixed. Let us first prove the first item in Proposition 3 and take $u \in D(P_h) = H^2(\Omega) \cap H_0^1(\Omega)$. Since $\boldsymbol{\ell} \cdot \nabla f = 0$ and then $\boldsymbol{\ell} \cdot \nabla_{f,h} = h\boldsymbol{\ell} \cdot \nabla$ according to (**Ortho**) and to the relation $\nabla_{f,h} := h\, e^{-\frac{f}{h}} \nabla e^{\frac{f}{h}} = h\nabla + \nabla f$, it holds

$$\int_{\Omega} (\boldsymbol{\ell} \cdot \nabla_{f,h} u)\, \overline{u} = -\int_{\Omega} u\, (\boldsymbol{\ell} \cdot \nabla_{f,h} \overline{u}) - h \int_{\Omega} (\operatorname{div} \boldsymbol{\ell}) |u|^2.$$

Therefore, one has $2\operatorname{Re} \langle \boldsymbol{\ell} \cdot \nabla_{f,h} u, u \rangle_{L^2(\Omega)} = -h \int_{\Omega} (\operatorname{div} \boldsymbol{\ell}) |u|^2$, and thus, using (1.13) and (1.14):

$$(5.38)\qquad \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \quad \operatorname{Re} \langle P_h u, u \rangle_{L^2(\Omega)} = \int_{\Omega} |\nabla_{f,h} u|^2 - h \int_{\Omega} (\operatorname{div} \boldsymbol{\ell}) |u|^2.$$

This implies that $P_h + h\|\operatorname{div} \boldsymbol{\ell}\|_\infty : H^2(\Omega)\cap H^1_0(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ is accretive. Using moreover the Lax-Milgram Theorem and the elliptic regularity of P_h, the operator $P_h + \lambda$ is invertible for $\lambda > 0$ large enough. Thus, P_h is maximal quasi-accretive and is in particular closed. In addition, from the compact injection $H^1_0(\Omega) \subset L^2(\Omega)$, P_h has a compact resolvent.

Let us now prove that P_h is sectorial. For all $u \in H^2(\Omega) \cap H^1_0(\Omega)$, it holds

$$\operatorname{Im} \langle P_h u, u\rangle_{L^2(\Omega)} = \operatorname{Im} \int_\Omega (2\boldsymbol{\ell} \cdot \nabla_{f,h} u)\, \overline{u}.$$

Consequently, there exists $C > 0$ such that for all $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and all $\varepsilon > 0$, one has

$$|\operatorname{Im} \langle P_h u, u\rangle_{L^2(\Omega)}| \le C\|\nabla_{f,h}u\|_{L^2}\|u\|_{L^2(\Omega)} \le C\Big[\frac{\varepsilon}{2}\|\nabla_{f,h}u\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|u\|^2_{L^2(\Omega)}\Big].$$

Taking $\lambda > 0$ and choosing $\varepsilon > 0$ such that $1 - \varepsilon\frac{\lambda C}{2} \ge \frac12$, one has, using (5.38),

$$\operatorname{Re} \langle P_h u, u\rangle_{L^2(\Omega)} - \lambda|\operatorname{Im} \langle P_h u, u\rangle_{L^2(\Omega)}| \ge \frac12\|\nabla_{f,h}u\|^2_{L^2(\Omega)} - \big(\frac{\lambda C}{2\varepsilon} + h\|\operatorname{div} \boldsymbol{\ell}\|_\infty\big)\|u\|^2_{L^2(\Omega)}.$$

Therefore, for some $a_h \in \mathbb{R}$, $\operatorname{Re} \langle (P_h + a_h)u, u\rangle_{L^2(\Omega)} \ge \lambda|\operatorname{Im} \langle P_h u, u\rangle_{L^2(\Omega)}|$. The numerical range of P_h is then included in the sector $\{\mathsf{z} \in \mathbb{C}, |\operatorname{Im} \mathsf{z}| \le \lambda^{-1}\operatorname{Re} (\mathsf{z} + a_h)\}$, so P_h is sectorial.

Let us now prove the second item in Proposition 3. With the previous arguments, the formal adjoint

$$P^\dagger_h = \Delta_{f,h} - 2h\,\boldsymbol{\ell}\cdot\nabla - 2h\operatorname{div}\boldsymbol{\ell} = \Delta_{f,h} - 2\boldsymbol{\ell}\cdot\nabla_{f,h} - 2h\operatorname{div}\boldsymbol{\ell}$$

of P_h endowed with the domain $D(P_h) = H^2(\Omega)\cap H^1_0(\Omega)$ is also maximal quasi-accretive, with a compact resolvent, and sectorial. To conclude, it thus just remains to show that $(P^\dagger_h, D(P_h)) = (P^*_h, D(P^*_h))$, where $P^*_h : D(P^*_h) \to L^2(\Omega)$ is the adjoint of P_h. But, for any $u, v \in D(P_h) = H^2(\Omega) \cap H^1_0(\Omega)$, we have by integration by parts

$$\langle P_h u, v\rangle_{L^2(\Omega)} = \langle u, P^\dagger_h v\rangle_{L^2(\Omega)}\,,$$

which implies, by definition of $P^*_h : D(P^*_h) \to L^2(\Omega)$, that

$$(P^\dagger_h, D(P_h)) \subset (P^*_h, D(P^*_h)).$$

Since moreover $(P^*_h, D(P^*_h))$ is maximal quasi-accretive (since P_h is) as well as $(P^\dagger_h, D(P_h))$, it necessarily holds $(P^\dagger_h, D(P_h)) = (P^*_h, D(P^*_h))$.

Let us lastly prove the third item in Proposition 3. First, by standard results on elliptic regularity (see e.g. [19, Section 6.3]), any eigenfunction $u \in H^2(\Omega) \cap H^1_0(\Omega)$ of P_h (resp. of P^*_h) belongs to $\mathcal{C}^\infty(\overline{\Omega})$. Moreover, according to [17, Theorems 1.3, 1.4, and 2.7] (see also the slightly weaker result stated in [19, Theorem 3 in Section 6.5.2]), P_h (resp. P^*_h) admits a real eigenvalue $\lambda^P_{1,h}$ (resp. $\lambda^{P^*}_{1,h}$) with algebraic multiplicity one such that:

- there exists an associated eigenfunction $u^P_{1,h}$ (resp. $u^{P^*}_{1,h}$) which is positive within Ω,
- any other eigenvalue λ of P_h (resp. of P^*_h) satisfies $\operatorname{Re}\lambda > \lambda^P_{1,h}$ (resp. $\operatorname{Re}\lambda > \lambda^{P^*}_{1,h}$).

Since in addition $\sigma(P^*_h) = \overline{\sigma(P_h)}$ (see e.g. [30, Section 6.6 in Chapter 3]), we have $\lambda^P_{1,h} = \lambda^{P^*}_{1,h}$ and it thus only remains to show that $\lambda^P_{1,h} > 0$, which is a consequence of the weak maximum principle [19, Theorem 1 in Section 6.4.1]. Indeed, according to (1.14), if it was not the case, the second-order elliptic operator without zeroth-order term $L_h = -\frac{h}{2}\Delta + (\nabla f + \operatorname{div} \boldsymbol{\ell}) \cdot \nabla$ would satisfy

$$L_h\big(e^{\frac{f}{h}} u_{1,h}^P\big) = \frac{\lambda_{1,h}^P}{2h} e^{\frac{f}{h}} u_{1,h}^P \leq 0 \ \text{ in } \ \Omega,$$

which would imply by the weak maximum principle that $\max_{\overline{\Omega}} \left(e^{\frac{f}{h}} u_{1,h}^P\right) = \max_{\partial\Omega} \left(e^{\frac{f}{h}} u_{1,h}^P\right) = 0$, contradicting $u_{1,h}^P > 0$ in Ω. □

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D. Le Peutrec, Institut Denis Poisson, Université d'Orléans

E-mail address: dorian.le-peutrec@univ-orleans.fr

L. Michel, Institut Mathématique de Bordeaux, Université de Bordeaux

E-mail address: laurent.michel@math.u-bordeaux.fr

B. Nectoux, Laboratoire de Mathmatiques Blaise Pascal, UCA

E-mail address: boris.nectoux@uca.fr