Hypoelliptic random walk

L. Michel
(joint work with G. Lebeau)

Laboratoire J.-A. Dieudonné
Université de Nice

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Semiclassical random walk

Let (M, g) be a smooth connected compact manifold and $\mathcal{X} = \{X_1, \ldots, X_p\}$ be a family of smooth vector fields on M.
In all the following $h > 0$ will denote a small parameter.
A natural random walk associated to $\mathcal{X}$ is the following. Assume the walk stands at $m_n \in M$ at time n. Then we construct m_{n+1} as follows:

- choose a vector field at random in $\mathcal{X}$ (i.e. choose a number k at random between 1 and p.)
- pick up the point m_{n+1} at random on the curve $e^{tX_k}(m_n)$, $t \in [-h, h]$.

The Markov operator

The Markov operator T_h associated to this walk can be written as $T_h = \frac{1}{p}\sum_{j=1}^{p} T_{j,h}$ where

$$T_{j,h}f(x) = \frac{1}{2h}\int_{-h}^{h} f(e^{tX_j}(x))dt$$

for any continous function f.

- This operator acts continuously on $C^0(M)$ and hence, its transpose T_h^t acts on Borel measure by duality.
- In the following, we will denote by $t_h(x, dy)$ the distribution kernel of T_h.

Assumptions on $\mathcal{X}$

Assume that M is endowed with a probability measure μ. For any $x \in M$, let $\mathcal{G}_x$ be the Lie algebra generated by $\mathcal{X}$ at point x.

Hypotheses on $\mathcal{X}$

- the vector fields X_j are divergence free with respect to μ:

$$\int_M X_j(f) d\mu = 0, \ \forall f \in C^1(M).$$

- The family $\mathcal{X}$ enjoys the Hörmander condition

$$\mathcal{G}_x = T_x M, \ \forall x \in M.$$

General properties

Under the above assumptions it is easy to prove the following properties.

- T_h is markovian: $T_h(1) = 1$.
- T_h is reversible for μ, i.e.

$$\int_M T_h(f)\, g d\mu = \int_M f\, T_h(g) d\mu,\ \forall f, g \in C^0(M).$$

In particular μ is stationnary for T_h: $T_h^t(\mu) = \mu$.

- For any $p \in [1, \infty]$, T_h acts continuously on $L^p(M)$ and

$$\|T_h\|_{L^p \to L^p} = 1$$

- T_h is self-adjoint on $L^2(M, d\mu)$ and it spectrum is contained in $[-1, 1]$.

Goals

Let us denote $t_h^n(x, dy)$ the kernel of the iterated operator T_h^n, $n \in \mathbb{N}$. Our aim is

- Describe the spectral theory of T_h.
- Study the convergence of $t_h^n(x, dy)$ towards the stationnary distribution $d\mu$, when $\rightarrow \infty$.

The reference operator

Let $\mathcal{H}^1(\mathcal{X})$ be the Hilbert space

$$\mathcal{H}^1(\mathcal{X}) = \{u \in L^2(M),\ \forall j = 1, \dots, p,\ X_j u \in L^2(M)\}$$

and

$$\mathcal{E}(u) = \frac{1}{6} \int_M \sum_{k=1}^{p} |X_k u|^2 d\mu$$

be the associated Dirichlet form. Let $L = -\frac{1}{6p} \sum_k X_k^2$ be the positive Laplacian associated to the Dirichlet form $\mathcal{E}(u)$.

Theorem [Hörmander-Chow]

The following holds true

- there exists $s > 0$ such that $\mathcal{H}^1(\mathcal{X}) \subset H^s(M)$.
- the operator L has compact resolvant
- Denoting (ν_k) the increasing sequence of eigenvalues and m_k the associated multiplicities, we have $\nu_0 = 0$ and $m_0 = 1$.

Our first result is the following

Theorem [Lebeau-Michel]

There exists $h_0 > 0$, $\delta_1, \delta_2 > 0$, $A > 0$, and constants $C_i > 0$ such that for any $h \in]0, h_0]$, the following holds true.

- $Spec(T_h) \subset [-1+\delta_1, 1]$, 1 is a simple eigenvalue of T_h, and $Spec(T_h) \cap [1-\delta_2, 1]$ is discrete. For any $0 \leq \lambda \leq \delta_2 h^{-2}$,

$$\sharp\sigma(T_h) \cap [1-h^2\lambda, 1] \leq C_1(1+\lambda)^A.$$

- for any $R > 0$ and $\varepsilon > 0$ small enough, there exists $h_1 > 0$ such that for all $h \in]0, h_1]$

$$Spec(\frac{1-T_h}{h^2}) \cap]0, R] \subset \cup_{j\geq 1}[\nu_j - \varepsilon, \nu_j + \varepsilon]$$

and the number of eigenvalues of $\frac{1-T_h}{h^2}$ with multiplicities, in the interval $[\nu_j - \varepsilon, \nu_j + \varepsilon]$, is equal to m_j.

Given μ, ν two probabilities measure on M, the total variation distance between ν and μ is defined by

$$\|\nu - \mu\|_{TV} = \sup_A |\nu(A) - \mu(A)|$$

where the sup is over all Borel sets A.

Theorem [Lebeau-Michel]

The following estimate holds true for all integer n

$$sup_{x\in M}\|t_h^n(x,dy) - d\mu(y)\|_{TV} \leq C_4 e^{-ng(h)}$$

where $g(h) = dist(1, Spect(T_h) \setminus \{1\})$.

Lemma 1

Let $f \in C^4(M)$, then

$$(1 - T_h)f(x) = h^2 Lf(x) + \mathcal{O}(h^4 \|f\|_{C^4(M)})$$

with $L = -\frac{1}{6}\sum_j X_j^2$.

Proof. Let $f \in C^4(M)$, then we get by Taylor expansion

$$T_{j,h}f(x) = \frac{1}{2h}\int_{-h}^{h} f(e^{tX_j}x)dt$$

$$= \frac{1}{2h}\int_{-h}^{h}\Big(f(x) + tX_jf(x) + \frac{t^2}{2}X_j^2 f(x) + \frac{t^3}{6}X_j^3 f(x) + t^4 r(x,t)\Big)dt$$

with $\|r(x,t)\|_{L^\infty} \leq C\|f\|_{C^4(M)}$. By parity argument, we get

$$T_{j,h}f(x) = f(x) + \frac{h^2}{6}X_j^2 f(x) + \mathcal{O}(h^4\|f\|_{C^4(M)}).$$

We conclude by summing over $j = 1, \ldots, p$.

Quasimodes for T_h

Let $\lambda \in Spec(L)$ and let f be such that $Lf = \lambda f$. By Hörmander Theorem, f is C^∞ and it follows from the preceding lemma that

$$T_h f = (1 - h^2\lambda)f + \mathcal{O}(h^4).$$

Using mini-max principle this shows that for any $k \in \mathbb{N}$, there exists $C, h_0 > 0$ s.t., ,for all $h \in]0, h_0]$

$$\sharp Spec(\frac{1 - T_h}{h^2}) \cap [\nu_j - Ch^2, \nu_j + Ch^2] \geq m_j.$$

Reverse inequality

Let $R > 0$ be fixed and consider a family $(\lambda_h, u_h) \in [0, R] \times L^2(\Omega)$ such that $\|u_h\|_{L^2} = 1$ and

$$T_h u_h = (1 - h^2 \lambda_h) u_h$$

We want to show that λ_h converges to an eigenvalue of L when $h \to 0$. For this purpose, we need some compactness on $(u_h)_{h \in]0, h_0]}$.

The fondamental Lemma

Let us introduce the Dirichlet form associated to T_h

$$\mathcal{E}_h(u) = h^{-2}\langle (1 - T_h)u, u\rangle_{L^2(M,d\mu)}$$

The most difficult part of our analysis is contained in the following lemma (proof postponed to the end of the talk).

Lemma 2

There exists $C, h_0 > 0$ such that the following holds true for all $h \in]0, h_0]$: for all $u \in L^2(M, d\mu)$ such that

$$\|u\|_{L^2}^2 + \mathcal{E}_h(u) \leq 1$$

there exists $v_h \in \mathcal{H}^1(\mathcal{X})$ and $w_h \in L^2$ such that

$$u = v_h + w_h, \quad \forall j,\ \|X_j v_h\|_{L^2} \leq C, \quad \|w_h\|_{L^2} \leq Ch$$

Proof of reverse inequality

Let $R > 0$ be fixed and consider a family $(\lambda_h, u_h) \in [0, R] \times L^2(M)$ such that $\|u_h\|_{L^2} = 1$ and

$$|\Delta|_h u_h = \lambda_h u_h \text{ with } |\Delta|_h := \frac{1 - T_h}{h^2}$$

- Fondamental Lemma $\Longrightarrow u_h = v_h + w_h$ with $\|w_h\|_{L^2} = O(h)$ and v_h bounded in $\mathcal{H}^1(\mathcal{X})$.
- We can assume $v_h \rightharpoonup v$ in $\mathcal{H}^1(\mathcal{X})$ and $\lambda_h \to \lambda$. Hence $u_h \to v$ in L^2.
- Lemma 1 implies that for any $f \in C^\infty(M)$,

$$\begin{aligned} \lambda\langle f, v\rangle &= \lim_{h\to 0}\langle f, \lambda_h u_h\rangle = \lim_{h\to 0}\langle |\triangle_h|(f), u_h\rangle \\ &= \lim_{h\to 0}\langle Lf + \mathcal{O}(h^2), u_h\rangle = \langle Lf, v\rangle = \langle f, Lv\rangle \end{aligned}$$

Hence, $(L-\lambda)v = 0$ and since $v \in \mathcal{H}^1(\mathcal{X})$, then $\lambda \in Spec(L)$.

Proof of total variation estimates

Let Π_0 be the orthogonal projector in $L^2(\Omega)$ on the space of constant functions

$$\Pi_0(u)(x) = \int_M u(y) d\mu(y). \tag{1}$$

Then, by definition

$$2 \sup_{x_0 \in M} \| t_h^n(x_0, dy) - d\mu(y) \|_{TV} = \| T_h^n - \Pi_0 \|_{L^\infty \to L^\infty}. \tag{2}$$

Thus, we have to prove that for $h > 0$ small and any n, one has

$$\| T_h^n - \Pi_0 \|_{L^\infty \to L^\infty} \leq C_0 e^{-ng(h)}. \tag{3}$$

Since $g(h) = O(h^2)$, we can suppose that $nh^2 >> 1$.

Denote $\lambda_{j,h}$ the eigenvalues of T_h and Π_j the associated spectral projector. Fix $\alpha > 0$ small and use the spectral decomposition $T_h - \Pi_0 = T_{h,1} + T_{h,2}$ with

$$T_{h,1} = \sum_{1-\alpha<\lambda_{j,h}<1} \lambda_{j,h}\Pi j$$

and $T_{h,2}$ spectrally localized in $[-1+\delta_0, 1-\alpha]$. It is easy to see that

$$\|T_h^n - \Pi_0\|_{L^2\to L^2} \leq e^{-ng(h)}.$$

Since, we deal with $L^\infty \to L^\infty$ norm, we need:

- to control $\|\Pi_j\|_{L^2\to L^\infty}$
- a bound on the number of eigenvalues in any interval $[\alpha_h, 1]$ with $1-\delta_0 < \alpha_h < 1 - Ch^2$.

Control of small eigenvalues

For this purpose, we show that there exists $C, \delta_0, A, D > 0$ s.t.

- **Claim 1:** for any $0 \leq \lambda \leq \delta_0/h^2$,

$$\sharp(Spec(T_h) \cap [1 - h^2\lambda, 1]) \leq C(1+\lambda)^{A/2}.$$

- **Claim 2:** any eigenfuntion $T_h(u) = \lambda u$ with $\lambda \in [1-\delta_0, 1]$ satisfies the bound

$$\|u\|_{L^\infty} \leq Ch^{-D/2}\|u\|_{L^2}.$$

Using these estimates we get easily that there exists $D' > 0$ s.t.

$$\|T_{2,h}^n\|_{L^\infty \to L^\infty} \leq Ch^{-D'}e^{-n(1-\alpha)} << e^{-ng(h)}$$

since $g(h) \sim h^2$.

Nash inequality

Let $E_\alpha = span(e_{j,h}, 1-\alpha < \lambda_{j,h} < 1)$.

Lemma 2 (Nash inequality)

There exists $C, B, \alpha > 0$, s.t. any funtion $u \in E_\alpha$ satisfies:

$$\|u\|_{L^2}^{2+1/B} \leq Ch^{-2}(\|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2 + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/B}.$$

Proof.

- Use the fondamental lemma to show that there exists $p > 2$, $\alpha > 0$ such that any function $u \in E_\alpha$ satisfies

$$\|u\|_{L^p}^2 \leq Ch^{-2}(\mathcal{E}_h(u) + h^2\|u\|_{L^2}^2)$$

- Use the bound $\mathcal{E}_h(u) \leq \langle (1-T_h)u, u\rangle$ on E_α and interpolate between L^p and L^1 to get the L^2 estimate. . □

Control of $T_{h,1}$

We want to control the norm $\|T_{h,1}^n\|_{L^2\to L^\infty} = \|T_{h,1}^n\|_{L^1\to L^2}$.

- Take $g \in L^2$ s.t. $\|g\|_{L^1} = 1$ and denote $c_n = \|T_{h,1}^n g\|_{L^2}^2$. Thanks to the preceding Lemma:

$$c_n^{1+2B} \leq Ch^{-2}(c_n - c_{n+1} + h^2 c_n)$$

  Hence, for $0 \leq n \leq h^{-2}$, $c_n \leq (h^{-2}/(1+n))^{2B}$.
- This permit to show that for some large $n \simeq h^{-2}$,

$$\|T_{h,1}^n\|_{L^2\to L^\infty} = \|T_{h,1}^n\|_{L^1\to L^2} = O(1)$$

Combined with $\|T_h^p\|_{L^2\to L^2} \leq Ce^{-pg(h)}$, this completes the proof.

Let us discuss the proof of two technical results used above where the Hörmander condition enters, namely:

- a priori bound on eigenfunctions
- the fondamental lemma

The proof needs some algebraic material.

The reference Lie algebra

For any family of vector fields $Z_1, \ldots, Z_p$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_p^k$ denote $Z^\alpha = [Z_{\alpha_1}, [Z_{\alpha_2}, \ldots [Z_{\alpha_{k-1}}, Z_{\alpha_k}] \ldots]$

- Let $\mathcal{F}$ denote the free Lie algebra with p generators.
- Let $\mathfrak{r} \in \mathbb{N}$ be the smallest integer such that for any $x \in M$, $\mathcal{G}_x$ is generated by commutators of length at most $\mathfrak{r}$.
- Let $\mathcal{N}$ the free up to step $\mathfrak{r}$ nilpotent Lie algebra generated by p elements $Y_1, \ldots, Y_p$, and let N be the corresponding simply connected Lie group. We have the decomposition

$$\mathcal{N} = \mathcal{N}_1 \oplus \ldots \oplus \mathcal{N}_\mathfrak{r}$$

  where $\mathcal{N}_1$ is generated by $Y_1, \ldots Y_p$ and $\mathcal{N}_j$ is spanned by the commutators Y^α with $|\alpha| = j$ for $2 \leq j \leq \mathfrak{r}$.
- denote $Q = \sum_{j=1}^{\mathfrak{r}} j \dim \mathcal{N}_j$ the homogenous dimension of $\mathcal{N}$.
- $\mathbb{R}_+$ acts on $\mathcal{N}$ by $t \cdot (x_1, \ldots, x_\mathfrak{r}) = (tx_1, t^2 x_2, \ldots, t^\mathfrak{r} x_\mathfrak{r})$.

Vector fields on $\mathcal{N}$

- Define the product law $a.b$ on $\mathcal{N}$ by $exp(a.b) = exp(a)exp(b)$.
- For $Y \in T_e\mathcal{N} \simeq \mathcal{N}$, we denote by $\tilde{Y}$ the left invariant vector field on $\mathcal{N}$ such that $\tilde{Y}(o_{\mathcal{N}}) = Y$, i.e

$$\tilde{Y}(f)(x) = \frac{d}{ds}(f(x.sY)|_{s=0}$$

- The right invariant vector field on $\mathcal{N}$ such that $\tilde{Z}(o_{\mathcal{N}}) = Y$ is defined by

$$\tilde{Z}(f)(x) = \frac{d}{ds}(f(sY.x)|_{s=0}$$

Here, sY is the usual product of the vector $Y \in \mathcal{N}$ by the scalar $s \in \mathbb{R}$.

The Rothschild-Stein Theorem

Let $x_0 \in M$ be fixed, let $\Omega_0 \subset \mathcal{N}$ nbhd of $o_{\mathcal{N}}$ and $V_0 \subset M$ nbhd of x_0 and let

$$\Lambda : \Omega_0 \to V_0$$

be a submersion. Then the map $W_\Lambda : \mathcal{C}^\infty(V_0) \to \mathcal{C}^\infty(\Omega_0)$ defined by $W_\Lambda f = f \circ \Lambda$ is injective.

Theorem [Rothschild-Stein]

There exists a local submersion Λ as above and some vector fields $Z_1, \ldots, Z_p$ on Ω_0 such that for any $\alpha \in \mathcal{A}$ we have

- $Z^\alpha W_\Lambda = W_\Lambda X^\alpha$
- $Z^\alpha = \tilde{Y}^\alpha + R_\alpha$ with R_α of order less than $|\alpha| - 1$.

Here we say that a vector field Z is of order less that k if for any function f vanishing at order m in $0_{\mathcal{N}}$, Zf vanishes at order at least $m - k$ (for homogenous norms).

The lifted operator

Using Rothschild-Stein Theorem we are reduced to study the operator $\tilde{T}_h$ defined on $L^2(\mathcal{N})$ by $\tilde{T}_h = \frac{1}{p}\sum_{k=1}^{p} \tilde{T}_{k,h}$ with

$$\tilde{T}_{k,h}g(u) = \frac{1}{2h}\int_{-h}^{h} g(e^{tZ_k}u)dt.$$

In order to simplify we will assume $R_\alpha = 0$ so that

$$\tilde{T}_{k,h}g(u) = \frac{1}{2h}\int_{-h}^{h} g(e^{t\tilde{Y}_k}u)dt.$$

Let us denote $\tilde{t}_h(x, dy)$ the kernel of $\tilde{T}_h$. For any $x \in M$ we define a positive measure $S_h^\epsilon(x, dy)$ on $\mathcal{N}$ by the formula

$$\forall f \in C^0(\mathcal{N}), \quad \int f(y) S_h^\epsilon(x, dy) = h^{-Q} \int_{u \in I_{\epsilon,h}} f(u)\, du$$

where $du = \Pi_\alpha du_\alpha$ is the left and right invariant Haar measure on $\mathcal{N}$ and

$$I_{\epsilon,h} = \{u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha, \quad u_\alpha \in]-\epsilon h^{|\alpha|}, \epsilon h^{|\alpha|}[\ \}.$$

Proposition

There exists $P \in \mathbb{N}$, $\epsilon > 0$, $c > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $x \in M$

$$\tilde{t}_h^P(x, dy) = \rho_h(x, dy) + c S_h^\epsilon(x, dy)$$

where $\rho_h(x, dy)$ is a non-negative Borel measure on $\mathcal{N}$ for all $x \in M$.

Proof of the Proposition:

In order to simplify, we assume that $dim(\mathcal{N}) = 3$, $p = 2$ and $(Y_1, Y_2, Y_3 = [Y_1, Y_2])$ basis of $\mathcal{N}$.

- We have to find $c, \epsilon > 0$ independent of h small, such that for any non negative continous function f on M, one has

$$T_h^P f(x) \geq c S_h^\epsilon f(x)$$

- Recall the Campbell-Hausdorff formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\dots}.$$

- Using the above formula we get

$$\tilde{T}_h^6 f(x) \geq (\tilde{T}_{1,h}\tilde{T}_{2,h})(\tilde{T}_{1,h}\tilde{T}_{2,h}\tilde{T}_{1,h}\tilde{T}_{2,h}) f(x) \geq c S_h^\epsilon f(x)$$

Consequence on eigenfunctions

Corollary

There exists $a \in]0,1[$ and $C = C_a > 0$ such that for any $\lambda \in [a,1]$ and any $f \in L^2(M, d\mu)$ we have

$$\tilde{T}_h f = \lambda f \Longrightarrow \|f\|_{L^\infty} \leq C h^{-\frac{Q}{2}} \|f\|_{L^2}$$

Proof.

- Use the Markov property to prove that

$$\|\rho_h(x, dy)\|_{L^\infty \to L^\infty} \leq \gamma < 1$$

- Suppose $\tilde{T}_h f = \lambda f$, then $S_h^\epsilon f = \lambda^P f - \rho_h(f)$ and then

$$\|S_h^\epsilon f\|_{L^\infty} \geq \lambda^P \|f\|_{L^\infty} - \gamma \|f\|_{L^\infty} \geq c_a \|f\|_{L^\infty}$$

Use Cauchy-Schwartz to get (since Λ is a submersion)

$$|S_h^\epsilon f(x)| \leq h^{-Q} \,\mathrm{meas}(I_{\epsilon,h})^{1/2} \Big(\int_{u\in I_{\epsilon,h}} |f(\Lambda(u))|^2 \, du\Big)^{1/2}$$
$$\leq Ch^{-Q/2} \|f\|_{L^2(M)}$$

□

Thanks to the above remark, we assume $M = \mathcal{N}$ and $X_k = \tilde{Y}_k$.
Recall the statement of the Fondamental Lemma

Lemma

There exists $C, h_0 > 0$ such that the following holds true for all $h \in]0, h_0]$: for all $u \in L^2(M, d\mu)$ such that

$$\|u\|_{L^2}^2 + \mathcal{E}_h(u) \leq 1$$

there exists $v_h \in \mathcal{H}^1(\mathcal{X})$ and $w_h \in L^2$ such that

$$u = v_h + w_h, \quad \forall j,\ \|\tilde{Y}_j v_h\|_{L^2} \leq C, \quad \|w_h\|_{L^2} \leq Ch$$

An easy decomposition

We first prove the following:

Lemma 4

For any $j = 1, \ldots, p$, there exists $C, h_0 > 0$ such that the following holds true for all $h \in]0, h_0]$: for all $u \in L^2(\mathcal{N})$ such that

$$\|u\|_{L^2}^2 + \mathcal{E}_h(u) \leq 1$$

there exists $v_{j,h} \in \mathcal{H}^1(\mathcal{X})$ and $w_{j,h} \in L^2$ such that

$$u = v_{j,h} + w_{j,h}, \quad \|\tilde{Y}_j v_{j,h}\|_{L^2} \leq C, \quad \|w_{j,h}\|_{L^2} \leq Ch$$

Remark

Observe that the difference between these two lemmas is that the decomposition in the fondamental lemma is independant on $j = 1, \ldots, p$.

Proof of the easy decomposition

Let us suppose $j = 1$. Since $\tilde{Y}_1$ doesn't vanish we can assume that $\tilde{Y}_1 = \partial_{x_1}$. Denote $\mathcal{F}_1$ the Fourier transform in y_1, then the operator $\tilde{T}_{1,h}$ can be written as $\tilde{T}_{1,h} = G(hD_1)$ where

$$G : \mathbb{R} \to \mathbb{R}, \quad G(s) = \frac{\sin(s)}{s}.$$

Hence, the equation

$$\mathcal{E}_h(u) \leq C\|u\|_{L^2}^2$$

reads

$$\int (1 - \frac{\sin h\xi_1}{h\xi_1})|\mathcal{F}_1 u(\xi_1, y')|^2 \, d\xi_1 dy' \leq C_0' h^2 \|u\|^2$$

Proof of the easy decomposition (continued)

- There exists $c > 0$ such that

$$(1 - \frac{\sin h\xi_1}{h\xi_1}) \geq ch^2\xi_1^2$$

for $h|\xi_1| \leq a$ and

$$(1 - \frac{\sin h\xi_1}{h\xi_1}) \geq c$$

for $h|\xi_1| > a$.

- Then, for any $\chi \in C_0^\infty(\mathbb{R})$ equal to 1 near 0, the decomposition

$$v_{1,h} = \chi(hD_1)g, \quad w_{1,h} = (1-\chi)(hD_1)g$$

works.

From “easy” to “fondamental ” Lemma

In order to prove the Fondamental Lemma, we will construct operators $\Phi, C_j, B_{k,j}, R_l$, depending on h, acting on L^2 functions with support in a small neighborhood of $o_{\mathcal{N}}$ in $\mathcal{N}$, with values in $L^2(\mathcal{N})$, such that $\Phi, C_j, B_{k,j}, C_j h\tilde{Y}_j, B_{k,j} h\tilde{Y}_k$ are uniformly in h bounded on L^2 and

$$1 - \Phi = \sum_{j=1}^{p} C_j h\tilde{Y}_j$$

$$\tilde{Y}_j\Phi = \sum_{k=1}^{p} B_{k,j}\tilde{Y}_k$$

and then we set

$$v_h = \Phi(u), \quad w_h = (1-\Phi)(u)$$

- Let $f * u$ be the convolution on $\mathcal{N}$

$$f * u(x) = \int_{\mathcal{N}} f(x.y^{-1})u(y)dy = \int_{\mathcal{N}} f(z)u(z^{-1}.x)dz$$

where dy is the left (and right) invariant Haar measure on $\mathcal{N}$.

- Let $\tilde{Z}_k$ be the right invariant vector field on $\mathcal{N}$ such that $\tilde{Z}_k(o_{\mathcal{N}}) = Y_k$. Then

$$\tilde{Y}_k f = f * \tilde{Y}_k \delta_e \text{ and } \tilde{Z}_k f = \tilde{Y}_k \delta_e * f.$$

- Introduce the scaling operator $\mathcal{T}_h f(x) = h^{-Q} f(h^{-1} \cdot x)$.
- Let $\varphi \in \mathcal{S}(\mathbb{N})$ be such that $\int_{\mathcal{N}} \varphi = 1$. Let $\varphi_h = \mathcal{T}_h(\varphi)$ and Φ_h defined on $L^2(\mathcal{N})$ by

$$\Phi_h(f) = f * \varphi_h.$$

- We look for $B_{k,j}$ under the form

$$B_{k,j}(f) = f * \mathcal{T}_h(b_{k,j})$$

- Then the equation $\tilde{Y}_j \Phi = \sum_{k=1}^{p} B_{k,j} \tilde{Y}_k$ reads

$$\tilde{Y}_j \varphi = \sum_k \tilde{Z}_k b_{k,j}$$

- Finding $b_{k,j}$ solving this equation is possible since $\int_{\mathcal{N}} \tilde{Y}_j \varphi = 0$.