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Semi-Classical Behavior of the Scattering Amplitude for Trapping Perturbations at Fixed Energy

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Abstract. We study the semi-classical behavior as $h \to 0$ of the scattering amplitude $f(\theta, \omega, \lambda, h)$ associated to a Schrödinger operator $P(h) = -\frac{1}{2}h^2\Delta + V(x)$ with short-range trapping perturbations. First we realize a spatial localization in the general case and we deduce a bound of the scattering amplitude on the real line. Under an additional assumption on the resonances, we show that if we modify the potential $V(x)$ in a domain lying behind the barrier $\{x : V(x) > \lambda\}$, the scattering amplitude $f(\theta, \omega, \lambda, h)$ changes by a term of order $\mathcal{O}(h^\infty)$. Under an escape assumption on the classical trajectories incoming with fixed direction ω, we obtain an asymptotic development of $f(\theta, \omega, \lambda, h)$ similar to the one established in the non-trapping case.

1 Introduction

The purpose of this paper is to study the asymptotic behavior when $h \to 0$, of the scattering amplitude $f(\theta, \omega, \lambda, h)$ associated to the semi-classical Schrödinger operator $P(h) = -\frac{1}{2}h^2\Delta + V(x)$ with a short range potential satisfying $|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}$, $\rho > 1$. We are interested in two problems. First, we examine how the asymptotic behavior of $f(\theta, \omega, \lambda, h)$ changes when we modify the potential V in a suitable region. Secondly, we wish to obtain an asymptotic development of $f(\theta, \omega, \lambda, h)$ when h tends to 0.

The second problem has been treated by Vainberg [25, 26] for $V \in C_0^\infty(\mathbb{R}^n)$, $\lambda > \sup_{x\in\mathbb{R}^n} V(x)$ and λ non-trapping. Robert and Tamura, in [20], generalized this result to the case where the potential does not have compact support, and the energy level $\lambda > 0$ is non-trapping (but not necessarily $\lambda > \sup(V)$) and θ, ω are fixed so that $\theta \neq \omega$. Moreover, the coefficients of this expansion depend only on the values of $V(x)$ in $\{x : V(x) \leq \lambda\}$. It follows that if we modify the potential V in a region lying in $\{x : V(x) > \lambda\}$, the scattering amplitude remains unchanged modulo $\mathcal{O}(h^\infty)$.

The trapping case is more complicated and there are only a few works treating this case. Let us mention two results dealing with the first problem. Nakamura [16] studied the case of two short range potentials V and $\tilde{V}$, with $\rho > \frac{n+1}{2}$, such that $V = \tilde{V}$ on the unbounded connected component of $\{x : V(x) < \lambda + \epsilon\}$. Assuming additionally that λ is weakly trapping for both potentials, *i.e.*, $\|(P(h) - (\lambda + i0))^{-1}\|_{L^2_\alpha, L^2_{-\alpha}} = \mathcal{O}(h^{-M})$ for some $M \in \mathbb{R}$ and $\alpha > \frac{1}{2}$, he proved that $f(\theta, \omega, \lambda, h) - \tilde{f}(\theta, \omega, \lambda, h) = \mathcal{O}(e^{-\frac{d}{h}})$. Here L^2_α denotes the weighted L^2 space $L^2(\mathbb{R}^n, (1+|x|^2)^{\alpha/2}dx)$

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and $(P(h) - (\lambda + i0))^{-1} = \lim_{\epsilon \to 0, \epsilon > 0}(P(h) - (\lambda + i\epsilon))^{-1}$, where the limit exists in the space of bounded operators $\mathcal{L}(L^2_\alpha, L^2_{-\alpha})$.

On the other hand, Lahmar-Benbernou and Martinez examined in [10], the case of "a well in an island", where the existence of resonances converging exponentially to real axis forbids a polynomial estimate of the resolvent. In this case, which will be detailed below, their modified potential is non-trapping for λ and it is equal to V on the unbounded connected component of $\{x : V(x) < \lambda + \epsilon\}$. Under these conditions, they proved that $\tilde{f}(\theta, \omega, \lambda_j, h) - f(\theta, \omega, \lambda_j, h) = \alpha h^{\frac{n+1}{2}} + \mathcal{O}(h^{\frac{n+3}{2}})$ for some $\alpha \neq 0$ and λ_j converging to λ.

Concerning the second problem, we have two results according to Yajima [27] and to the author [13]. In these papers, asymptotics in average forms have been established, under an escape assumption in a fixed direction without any hypothesis on the growth of the resolvent. The average avoids the problem due to resonances converging exponentially to the real axis. For instance, in the general case, an integral estimate of the resolvent has been proved in [13] and we have

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \|R(\lambda \pm i0\|_{\alpha,-\alpha} d\lambda = \mathcal{O}(h^{-M}),\ \alpha > \frac{1}{2}. \tag{1.1}$$

In the above estimate, $R(z) = (P(h) - z)^{-1},\ z \in \mathbb{C} \setminus \mathbb{R}$ is the resolvent of $P(h)$ and $R(\lambda \pm i0) = \lim_{\epsilon \to 0, \epsilon > 0} R(\lambda \pm i\epsilon)$. Here we take the limit in the spaces of bounded operators $\mathcal{L}(L^2_\alpha, L^2_{-\alpha}), \alpha > \frac{1}{2}$ and for $\alpha, \beta \in \mathbb{R}$, $\|\cdot\|_{\alpha,\beta}$ is the natural norm on $\mathcal{L}(L^2_\alpha, L^2_\beta)$. The estimate (1.1) is one of the crucial points in the proof of [13], and the fact that $\|R(\lambda \pm i0)\|_{\alpha,-\alpha} = \mathcal{O}(e^{Ch^{-n}})$ when the energy λ is fixed at a trapping level is one of the main difficulties.

One of the differences between the short-range case and the case where V has compact support is the form of $f(\theta, \omega, \lambda, h)$. If V is in C_0^∞, we have a representation formula which involves only the truncated resolvent. More precisely,

$$\begin{aligned}(1.2)\quad & f(\theta, \omega, \lambda, h) = \\ & c_n h^{-n} \lambda^{\frac{n-2}{2}} \left\langle [h^2\Delta, \chi_1] R(\lambda + i0) [h^2\Delta, \chi_2] \chi_3 e^{ih^{-1}\langle \cdot, \omega\rangle}, \chi_3 e^{ih^{-1}\langle \cdot, \theta\rangle} \right\rangle_{L^2}, f\end{aligned}$$

where $\chi_j,\ j = 1, 2, 3$ belong to $C_0^\infty(\mathbb{R}^n)$ (see [17]). In the short-range case, this formula is not available and we are going to use the representation of Isozaki-Kitada (see section 2 for more details) where the resolvent is applied to functions belonging to L^2_α. In the non-trapping case, the approach of Robert and Tamura is based on a localization with principal term involving only the truncated resolvent [20];

$$f(\theta, \omega, \lambda, h) = c_n h^{-n} \lambda^{\frac{n-2}{2}} \left\langle R(\lambda + i0) g_{-b} e^{ih^{-1}\Phi_-(.,\omega)}, g_{+a} e^{ih^{-1}\Phi_+(.,\theta)} \right\rangle + \mathcal{O}(h^\infty) \tag{1.3}$$

This spatial localization was done by exploiting the resolvent estimate

$$\|R(\lambda \pm i0)\|_{\alpha,-\alpha} = \mathcal{O}(h^{-1}),\ \alpha > \frac{1}{2}. \tag{1.4}$$

In the general case (without the non-trapping assumption), this estimate fails to be true. In a recent work, N. Burq [2] gave a polynomial estimate of the truncated resolvent,

$$\|\chi_1(x)R(\lambda+i0)\chi_2(x)\|_{L^2} = \mathcal{O}(h^{-1}), \tag{1.5}$$

where supp $\chi_i \subset \{x: \ R_1 < |x| < R_2\}, \ i = 1,2$ and $0 < R_1 < R_2$ are sufficiently large. Applying this estimate, we prove, in the general short-range case, that the scattering amplitude can be written in the form (1.3). Moreover, we deduce from this localization and from Burq's estimate (1.5), that in the general case the scattering amplitude is bounded by $\mathcal{O}(h^{-\frac{n-1}{2}})$. This spatial localization is the main step in our analysis of both problems that we deal with. In the case where we assume $\rho > 1$, such localization permits to obtain a result similar to that of Nakamura. On the other hand, using some ideas developed in [13], we extend the result of [20] to the case of weakly trapping potentials.

Let us now state the problem more precisely. Consider the Schrödinger operator $P(h) = -\frac{1}{2}h^2\Delta + V$, in $\mathbb{R}^n$, $n \geq 2$, $0 < h \leq 1$. The potential $V(x)$ is assumed to satisfy the following condition with $\rho > 1$:

Assumption $(\mathbf{V}_\rho)$ V is a real C^∞-smooth function such that

$$\forall \alpha \in \mathbb{N}^n, \ \forall x \in \mathbb{R}^n, \ |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \text{ where } \langle x \rangle = (1+|x|^2)^{\frac{1}{2}}.$$

The operator $P(h)$ with domain $D(P(h)) = H^2(\mathbb{R}^n)$ is self-adjoint in $L^2(\mathbb{R}^n)$. Moreover, we can define the scattering matrix $S(\lambda, h)$ related to $P_0(h) = -\frac{1}{2}h^2\Delta$ and $P(h)$, as a unitary operator:

$$S(\lambda, h) \ : \ L^2(S^{n-1}) \longrightarrow L^2(S^{n-1}).$$

Next, introduce the operator $T(\lambda, h)$ by $S(\lambda, h) = Id - 2i\pi T(\lambda, h)$. It is well-known (see [8]) that $T(\lambda, h)$ has a kernel $T(\theta, \omega, \lambda, h)$, smooth in $(\theta, \omega) \in S^{n-1} \times S^{n-1} \setminus \{\theta = \omega\}$ and the scattering amplitude is given by

$$f(\theta, \omega, \lambda, h) = c(\lambda, h) T(\theta, \omega, \lambda, h),$$

with

$$c(\lambda, h) = -2\pi(2\lambda)^{-\frac{n-1}{4}}(2\pi h)^{\frac{n-1}{2}} e^{-i\frac{(n-3)\pi}{4}}.$$

Robert and Tamura [20] have studied the asymptotic behavior of $f(\theta, \omega, \lambda, h)$ as $h \to 0$ for fixed $\theta, \omega \in S^{n-1}, \ \theta \neq \omega$, in the case where the energy λ is fixed in a non-trapping interval. More precisely, denote by $\big(q(t,x,\xi), p(t,x,\xi)\big)$ the solution to the system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -\nabla_x V(q), \end{cases} \tag{1.6}$$

with initial data (x, ξ) at $t = 0$, and recall the following non-trapping condition.

Assumption **(NT)** We say that the energy λ is non-trapping for the symbol $\frac{1}{2}|\xi|^2 + V(x)$ if for any $R > 0$ large enough, there exists $T = T(R)$ such that $|q(t, x, \xi)| > R$ for $|t| > T$ when $|x| < R$ and $\lambda = \frac{1}{2}|\xi|^2 + V(x)$.

For ω and θ fixed in S^{n-1} and for λ satisfying the non-trapping condition, Robert and Tamura obtained for f an asymptotics

$$f(\theta, \omega, \lambda, h) = \sum_{j=1}^{l} \hat{\sigma}(z^j)^{-\frac{1}{2}} e^{ih^{-1}S_j - i\mu_j \frac{\pi}{2}} + \mathcal{O}(h), \tag{1.7}$$

where $\hat{\sigma}$, z^j, S_j and μ_j will be defined below.

In order to understand better what happens in the trapping case, let us introduce the resonances by complex scaling as it is done in [2], [21] and [22]. For this, we need an hypothesis of analyticity of V at infinity.

Assumption **($\mathbf{Hol}_\infty$)** We assume that there exist $\theta_0 \in [0, \pi[$ and $R > 0$ such that the potential V extends holomorphically to the domain

$$D_{R,\theta_0} = \{z \in \mathbb{C}^n : \ |z| > R, \ |Im\, z| \leq \tan\theta_0 |Re\, z|\}$$

and

$$\exists \beta > 0, \exists M > 0, \ \forall x \in D_{R,\theta_o}, \ |V(x)| \leq C|x|^{-\beta}.$$

Following [21], we define the resonances in the upper half-plane by complex scaling. Recall that the resonances coincide with the poles of the meromorphic continuation of the resolvent $(P(h) - z)^{-1} \colon L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$ from the lower half-plane to a conic neighborhood of the positive half axis in the upper half-plane. We denote by $\mathrm{Res}(P(h))$ the set of resonances of $P(h)$.

One of the consequences of the non-trapping hypothesis is the estimate (1.4) which is related to the fact that for non-trapping perturbations, there are no resonances in

$$\{z \in \mathbb{C} : \ 0 < a < Re(z) < b, \ 0 \leq Im(z) \leq N\, h \log(\frac{1}{h})\}, \ \forall N, 0 < h < h_N.$$

We refer to [11] for more details. In the trapping case, the resolvent is not necessarily analytic in the above domain. Moreover, in many cases, there are resonances (that is poles of the scattering amplitude) in any strip of width $e^{-d/h}$. In the case where $z_0(h)$ is a simple isolated pole of the resolvent, one can decompose the scattering amplitude into a singular and holomorphic part around $z_0(h)$:

$$f(\theta, \omega, z, h) = \frac{f^{res}(\theta, \omega, h)}{z - z_0(h)} + f^{hol}(\theta, \omega, z, h), \tag{1.8}$$

for z near $Re\, z_0(h)$. In particular if $z = \lambda$ is fixed and $z_0(h)$ tends to λ exponentially fast, the scattering amplitude could blow up exponentially (that is $|f(\theta, \omega, \lambda, h| \sim e^{C/h}$ when $h \to 0$). After a spatial localization in the general case, we prove that the scattering amplitude can note behave as $e^{C/h}$. More precisely, we have the following Theorem.

Theorem 1.1 *Fix an energy* $\lambda > 0$ *and assume that the potential* V *satisfies* $(\mathbf{V}_\rho)$ *with* $\rho > 1$ *and* $(\mathbf{Hol}_\infty)$. *Then we have*

$$\forall(\omega, \theta) \in S^{n-1} \times S^{n-1} \setminus \{\theta = \omega\}, \; f(\theta, \omega, \lambda, h) = \mathcal{O}(h^{-\frac{n-1}{2}}). \tag{1.9}$$

Remarks 1.1 In a recent paper, Stefanov [23] studied the behavior of f^{res} and f^{hol} for a compactly supported potential V. Assuming that $z_0(h)$ is a simple isolated resonance, he proved that

$$|f^{res}(\theta, \omega, h)| \leq C\, h^{-\frac{n-1}{2}} |Im\, z_0(h)| \text{ and } |f^{hol}(\theta, \omega, z, h)| \leq C\, h^{-\frac{n-1}{2}}$$

in a neighbourhood of $Re\, z_0(h)$ containing $z_0(h)$. In [15], we use Theorem 1.1 to generalize the result of Stefanov to the case of long range potentials.

For some special trapping potentials, it is possible to exhibit some new phemomena. For example, Lahmar-Benbernou and Martinez [10], have studied the case where there exist resonances converging exponentially fast with respect to h to real axis. In a very particular situation, they showed that the presence of such resonances leads to a different behavior of the scattering amplitude. More precisely they consider the case where the potential $V(x)$ is a "well in an island", *i.e.*, there exist $\lambda_0 > 0$, a connected bounded open set $\ddot{O} \subset \mathbb{R}^n$ and $x_0 \in \ddot{O}$ such that

(i) $V(x_0) = \lambda_0$ and $V''(x_0) > 0$,
(ii) $V > \lambda_0$ on $\ddot{O} \setminus \{x_0\}$ and $V < \lambda_0$ on $\mathbb{R}^n \setminus \overline{\ddot{O}}$,
(iii) λ_0 is non-trapping for V outside $\ddot{O}$.

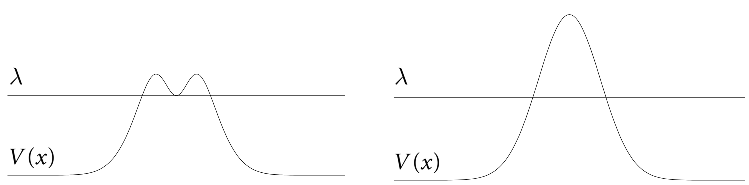

Figure 1: A "well in an island" transformed into a non-trapping potential.

Starting with V, they construct a non-trapping potential $\tilde{V}$ equal to V in $\mathbb{R}^n \setminus \overline{\ddot{O}}$ (see Figure 1) and they prove that for suitable $\theta, \omega \in S^{n-1}$ there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that:

$$f(\theta, \omega, Re(\rho_j), h) - \tilde{f}(\theta, \omega, Re(\rho_j), h) = \alpha h^{\frac{n+1}{2}} + \mathcal{O}(h^{\frac{n+2}{2}}),$$

where $\rho_j = \rho_j(h)$ is a resonance converging exponentially fast to λ_0 and $\tilde{f}$ is the scattering amplitude associated to $\tilde{V}$.

The second goal of this article is to show that if we modify the potential V in a suitable region, then the scattering amplitude remains unchanged modulo $\mathcal{O}(h^\infty)$, even for trapping potentials V, provided there is a resonance-free zone of width h^M near the real axis.

More precisely, introduce $W_\lambda = \{x \in \mathbb{R}^n; V(x) < \lambda\}$. As $\lim_{|x|\to+\infty} V(x) = 0$, the domain W_λ has a unique unbounded connected component denoted by W_{ext}. Let us set $W_{int} = W_\lambda \setminus W_{ext}$ and let F be a compact set such that $W_{int} \subset F \subset \mathbb{R}^n \setminus \overline{W_{ext}}$. We assume that $\tilde{V} \in C^\infty(\mathbb{R}^n)$ is a potential such that $V = \tilde{V}$ on $\mathbb{R}^n \setminus F$. Let us denote $\tilde{P}(h) = -\frac{1}{2}h^2\Delta + \tilde{V}(x)$ and let $\tilde{f}(\theta, \omega, \lambda, h)$ be the scattering amplitude associated to $\tilde{P}(h)$. The following theorem compares $\tilde{f}(\theta, \omega, \lambda, h)$ with $f(\theta, \omega, \lambda, h)$ in the case where we assume only $\rho > 1$.

Theorem 1.2 *Assume the following conditions:*

(i) $(\mathbf{V}_\rho)$ *with* $\rho > 1$.
(ii) $(\mathbf{Hol}_\infty)$.
(iii) *There exist* $\epsilon > 0, C > 0$ *and* $M > 0$ *such that*

$$\big(Res(\tilde{P}(h)) \cup Res(P(h))\big) \cap \big([\lambda - \epsilon, \lambda + \epsilon] + i[0, Ch^M]\big) = \varnothing.$$

Then for any $(\theta, \omega) \in S^{n-1} \times S^{n-1} \setminus \{\theta = \omega\}$ *we have the following estimate*

$$\tilde{f}(\theta, \omega, \lambda, h) = f(\theta, \omega, \lambda, h) + \mathcal{O}(h^\infty).$$

Remarks 1.2 (1) The assumption (ii) of the Theorem 1.2 allows us to use Burq's result [2], and to establish a spatial localization.

(2) Following the work of Lahmar-Benbernou and Martinez [10], it is obvious that the assumption (iii) is necessary. If we make an hypothesis only on the resonances of $P(h)$ (and not on both $Res(P(h))$ and $Res(\tilde{P}(h))$), the result of Theorem 1.2 is not true.

The previous theorem does not give any asymptotic development of f. In the last part of this paper we will prove an asymptotics similar to (1.7) for trapping potentials. As in [13] and [20], we begin by some results of classical mechanics, and for more details we refer to the books [3], [18]. Let $(p(t), q(t))$ be a solution to (1.6) and assume that $|q(t)| \to +\infty$ as $|t| \to +\infty$. Then there exists $(r_\pm, v_\pm) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0$ such that

$$\lim_{t\to\pm\infty} |q(t) - v_\pm t - r_\pm| + |p(t) - v_\pm| = 0.$$

From now up to the end of this paper, we will consider a fixed $\omega \in S^{n-1}$ and we denote by Λ_ω the plane orthogonal to ω and passing through 0. As ω is fixed, we can assume that $\omega = (0, \ldots, 0, 1)$ and we can write the coordinates in Λ_ω as $z = (z_1, \ldots, z_{n-1})$. We will also use the notation $\hat{z} = (z, 0)$ for $z \in \Lambda_\omega$. Then there exists a unique solution $(q_\infty(t, z, \omega), p_\infty(t, z, \omega))$ of (1.6) such that

$$\begin{cases} \lim_{t\to-\infty} |p_\infty(t, z, \omega) - \sqrt{2\lambda}\omega| = 0, \\ \lim_{t\to-\infty} |q_\infty(t, z, \omega) - \sqrt{2\lambda}\omega t - \hat{z}| = 0, \end{cases} \tag{1.10}$$

which depends smoothly on the parameter z. Given $\lambda > 0$, we will replace the condition (**NT**) by the following weaker one.

Assumption $(\mathbf{H}_\omega)$ For all z in Λ_ω, $\lim_{t\to+\infty}|q_\infty(t,z,\omega)| = +\infty$.

Let (q_∞, p_∞) be as above and take $\lambda > 0$ satisfying $(\mathbf{H}_\omega)$. Then, there exist $\xi_\infty(z) \in S^{n-1}$ and $r_\infty(z) \in \mathbb{R}^n$ such that

$$(1.11)\qquad \begin{cases} \lim_{t\to+\infty}|p_\infty(t,z,\omega) - \sqrt{2\lambda}\xi_\infty(z)| = 0, \\ \lim_{t\to+\infty}|q_\infty(t,z,\omega) - \sqrt{2\lambda}\xi_\infty(z)t - r_\infty(z)| = 0. \end{cases}$$

Moreover, one can show that $\Lambda_{\omega'} \ni z \to \xi_\infty(z) \in S^{n-1}$ is C^∞ (see [18]), and we may define

$$\hat{\sigma}(z) = |\det(\xi_\infty, \partial_{z_1}\xi_\infty, \ldots, \partial_{z_{n-1}}\xi_\infty)|.$$

Definition 1.1 We will say that $\theta \in S^{n-1}$ is regular for ω, if $\theta \neq \omega$ and $\forall z \in \Lambda_\omega$, $\xi_\infty(z) = \theta \Longrightarrow \hat{\sigma}(z) \neq 0$.

If θ is regular for ω, we deduce from the implicit functions theorem that there exists a finite set $\{z^1, \ldots, z^l\}$ included in Λ_ω such that $\xi_\infty(z) = \theta \Longleftrightarrow z \in \{z^1, \ldots, z^l\}$. Now we can state our second result.

Theorem 1.3 *Suppose that the potential V satisfies (V_ρ) with $\rho > 1$ and $(\mathbf{Hol}_\infty)$. Let $\omega \in S^{n-1}$, and $\lambda > 0$. Assume the following conditions:*

(1) $(\mathbf{H}_\omega)$.
(2) *θ is regular for ω.*
(3) *There exist $\epsilon > 0$, $C > 0$ and $M > 0$ such that*

$$\mathrm{Res}(P(h)) \cap \big([\lambda-\epsilon, \lambda+\epsilon] + i[0, Ch^M]\big) = \varnothing.$$

Then we have the following asymptotics

$$f(\theta,\omega,\lambda,h) = \sum_{j=1}^{l} \hat{\sigma}(z^j)^{-\frac{1}{2}} e^{ih^{-1}S_j - i\mu_j\frac{\pi}{2}} + \mathcal{O}(h),$$

where

$$(1.12)\quad S_j = \int_{-\infty}^{+\infty}\Big(\frac{1}{2}|p_\infty(t,z^j,\omega)|^2 - V(q_\infty(t,z^j,\omega)) - \lambda\Big)\,dt - \langle r_\infty(z^j), \sqrt{2\lambda}\theta\rangle$$

and $\mu_j \in \mathbb{Z}$ is the Maslov index of the trajectory $\big(q_\infty(t,z^j,\omega), p_\infty(t,z^j,\omega)\big)$ on the Lagrangian manifold

$$\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; x = q_\infty(t,z,\omega),\ \xi = p_\infty(t,z,\omega),\ z \in \Lambda_\omega,\ t \in \mathbb{R}\}.$$

Example There exist potentials satisfying our assumptions. For example for $n = 2$, set $V_0(x, y) = 1 + x^2 - y^2$ and take $\rho \in C_0^\infty(\mathbb{R}^+)$ decreasing such that $\rho(t) = 1$ for $t \in [0, 1]$ and $\rho(t) = 0$ for $t \geq 2$. Consider $V(x, y) = V_0(x, y)\rho(x^2 + y^2)$ (see Figure 2). For $\lambda > 1$ close to 1, there exist trapped trajectories for the potential V. Indeed, for $0 < y < 1$, the trajectory $q(t, \lambda, y)$ with $q(0, \lambda, y) = (0, y)$ and $\dot{q}(0, \lambda, y) = (\sqrt{2(\lambda - 1)}, -\sqrt{2}y)$ has the properties:

$$\forall t > 0,\ q(t, \lambda, y) = (\sqrt{(\lambda - 1)} \sin \sqrt{2}t, ye^{-\sqrt{2}t}), \lim_{t \to -\infty} |q(t, \lambda, y)| = +\infty.$$

Applying the work of Gérard and Sjöstrand [5], one can also show that there are no resonances in a box containing λ and having size h^M, that is hypothesis (iii) of Theorem 1.2 is satisfied.

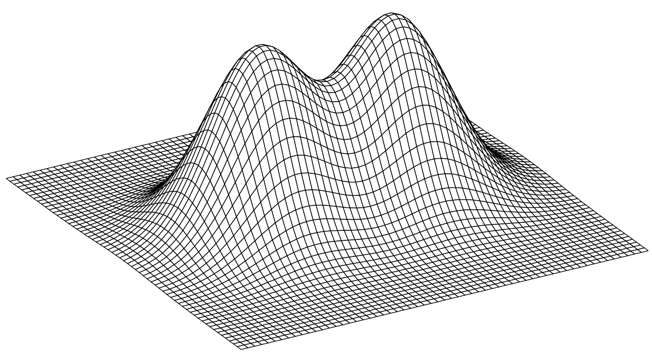

Figure 2: A trapping potential satisfying Gérard-Sjöstrand's assumptions .

The paper is organized as follows. In Section 2 we recall the representation formula of Isozaki-Kitada [8] for the scattering amplitude. In Section 3 we establish a spatial localization for the scattering amplitude, without any assumption on the resonances and we prove Theorem 1.1. In Section 4 we use the hypothesis on the resonances to get some resolvent estimate which is necessary for the analysis in the next sections. Section 5 is devoted to the proof of Theorem 1.2 and in Section 6 we complete the proof of Theorem 1.3. Finally, in the appendix we collect some results concerning the semi-classical wave front set and the microlocal resolvent estimates.

2 Review of the Representation of the Scattering Matrix

In this section, we recall some results concerning the representation of the scattering matrix. In particular, we will try to emphasize the difference between the case where the potential is short range with $\rho > \frac{n+1}{2}$ and the case where it is only short range with $\rho > 1$.

Denote by $\psi_0(x, \lambda, \omega, h)$ the generalized eigenfunction to $P_0(h) = -h^2\Delta$:

$$\psi_0(x, \lambda, \omega, h) = exp(ih^{-1}\sqrt{2\lambda} < x, \omega >).$$

If the potential $V(x)$ satisfies $(\mathbf{V}_\rho)$ with $\rho > \frac{n+1}{2}$, then the function $V(x)\psi_0(x, \lambda, \omega, h)$ belongs to $L^2_\alpha(\mathbb{R}^n)$, where $\alpha = \rho - \frac{n}{2} > \frac{1}{2}$. Therefore, the outgoing eigenfunction of $\psi_+(x, \lambda, \omega, h)$ of $P(h)$ is given by

$$\psi_+ = \psi_0 - R(\lambda + i0)V\psi_0 \tag{2.1}$$

and the kernel $T(\theta, \omega, \lambda, h)$ is simply written as

$$T(\theta, \omega, \lambda, h) = c_0(\lambda, h)^2 \langle V\psi_+(\,\cdot\,, \lambda, \omega, h), \psi_0(\,\cdot\,, \lambda, \theta, h)\rangle_{L^2(\mathbb{R}^n)}, \tag{2.2}$$

with

$$c_0(\lambda, h) = (2\pi h)^{-\frac{n}{2}}(2\lambda)^{\frac{n-2}{4}}.$$

In the general case $\rho > 1$, we cannot define $T(\theta, \omega, \lambda, h)$ as above. For example, notice that $R(\lambda + i0)V\psi_0$ is well defined if and only if V belongs to L^2_α for some $\alpha > \frac{1}{2}$, *i.e.*, $V(x) \leq C\langle x\rangle^{-\rho}$ with $\rho > \frac{n+1}{2}$. Thus, the first step towards the proof of Theorem 1.1 and 1.2 is to establish a representation formula for $T(\theta, \omega, \lambda, h)$ in the case $1 < \rho \leq \frac{n+1}{2}$. Such a formula has been obtained in [8], and it was used in [20] to prove an asymptotic expansion of the scattering amplitude in the non-trapping case with $\rho > 1$. We present below this representation formula as done in [20]. We begin with some notations.

Definition 2.1 Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. For $m, u \in \mathbb{R}$ and $k \in \mathbb{Z}$, we denote by $A^{m,u}_k(\Omega)$ the class of symbols $a(x, \xi, h)$ such that $(x, \xi) \mapsto a(x, \xi, h)$ belongs to $C^\infty(\Omega)$ and

$$\forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n,\ \exists C > 0,\ \forall(x, \xi) \in \Omega, |\partial^\alpha_x\partial^\beta_\xi a(x, \xi)| \leq C\, h^k\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{u-|\beta|}$$

and set $A^{m,\infty}_k(\Omega) = \bigcap_{u\in\mathbb{R}} A^{m,u}_k(\Omega)$, i.e., $a(x, \xi) \in A^{m,\infty}_k(\Omega)$ if and only if

$$\forall L > 1, \forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n,\ \exists C > 0,\ \forall(x, \xi) \in \Omega,$$
$$|\partial^\alpha_x\partial^\beta_\xi a(x, \xi)| \leq Ch^k\langle x\rangle^{m-|\alpha|}\langle\xi\rangle^{-L}.$$

In the case where $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, we will write $A^{m,u}_k$ instead of $A^{m,u}_k(\Omega)$.

We use also the incoming and outgoing subsets of the phase space having the form:

$$\Gamma_\pm(R,d,\sigma) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : |x| > R, d^{-1} < |\xi| < d, \pm\cos(x,\xi) > \pm\sigma\}$$

for $R > 1$, $d > 1$ and $\sigma \in]-1,1[$, where $\cos(x,\xi) = \frac{\langle x,\xi\rangle}{|x||\xi|}$. For $\alpha > \frac{1}{2}$, introduce $F_0(\lambda,h) : L^2_\alpha(\mathbb{R}^n) \longrightarrow L^2(S^{n-1})$, by

$$\big(F_0(\lambda,h)f\big)(\omega) = c_0(\lambda,h)\int_{\mathbb{R}^n} e^{-ih^{-1}\sqrt{2\lambda}\langle x,\omega\rangle} f(x)\,dx, \lambda > 0.$$

The idea of Isozaki and Kitada was to approximate the Wave Operators by Fourier Integral Operators $I_h(a_\pm,\Phi_\pm)$ with phase $\Phi_\pm$ and symbol $a_\pm$. Formally, with

$$I_h(a_\pm,\Phi_\pm)(f)(x) = (2\pi h)^{-n}\iint exp(ih^{-1}(\Phi_\pm(x,\xi) - \langle y,\xi\rangle))a_\pm(x,\xi)f(y)\,dyd\xi,$$

the phase $\Phi_\pm$ have to solve the eikonal equation

$$\frac{1}{2}|\nabla_x\Phi_\pm(x,\xi)|^2 + V(x) = \frac{1}{2}|\xi|^2$$

and the symbols $a_\pm$ are solution to

$$\big(-\frac{1}{2}h^2\Delta + V(x) - \frac{1}{2}|\xi|^2\big)\,(a_\pm e^{ih^{-1}\Phi_\pm}) \sim 0. \tag{2.3}$$

Let $R_0 \gg 1$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 < 1$. According to Proposition 2.4 of [7], we can find a real C^∞-smooth function $\Phi_\pm$ satisfying the following properties:

(φ1) $\Phi_\pm(x,\xi)$ is a solution of the eikonal equation $\frac{1}{2}|\nabla_x\Phi_\pm(x,\xi)|^2 + V(x) = \frac{1}{2}|\xi|^2$, in $\Gamma_\pm(R_0,d_0,\pm\sigma_0)$.

(φ2) $\Phi_\pm(x,\xi) - \langle x,\xi\rangle$ belongs to $A_0^{\epsilon,0}$, for all $\epsilon > 0$.

(φ3) For all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $|\frac{\partial^2\Phi_\pm}{\partial_{x_j}\partial_{\xi_k}}(x,\xi) - \delta_{jk}| < \epsilon(R_0)$, where δ_{jk} are the Kronecker symbols, and $\epsilon(R_0)$ can be made as small as we wish by taking R_0 large enough.

Next, we determine $a_\pm$ in the form

$$a_\pm(x,\xi,h) = \sum_{j\geq 0} a_{\pm j}(x,\xi)h^j.$$

Replacing $a_\pm$ by this expansion in (2.3) and identifying the power of h, we obtain the following transport equations

$$\begin{cases} \langle\nabla_x\Phi_\pm,\nabla_x a_{\pm 0}\rangle + \frac{1}{2}\Delta_x\Phi_\pm a_{\pm 0} = 0 \\ \langle\nabla_x\Phi_\pm,\nabla_x a_{\pm j}\rangle + \frac{1}{2}\Delta_x\Phi_\pm a_{\pm j} = \frac{i}{2}\Delta_x a_{\pm j-1},\ j \geq 1 \end{cases} \tag{2.4}$$

with the conditions at infinity

$$a_{\pm 0} \to 1 \text{ and } a_{\pm j} \to 0,\ j \geq 1 \text{ as } |x| \to 0. \tag{2.5}$$

These equations are solved by the standard characteristic curve method (see [20]) and finally, we find some symbols $a_{\pm j}$ such that:

(s0) $a_{\pm j}$ belongs to $A_0^{-j,\infty}$.
(s1) $\mathrm{supp}(a_{\pm j}) \subset \Gamma_\pm(3R_0, d_1, \mp\sigma_1)$.
(s2) $a_{\pm j}$ solves equation (2.4) with (2.5) in $\Gamma_\pm(4R_0, d_2, \mp\sigma_2)$.
(s3) $a_{\pm j}$ solves equation (2.4) in $\Gamma_\pm(4R_0, d_1, \mp\sigma_2)$.

Now, fix an integer N large enough (to be chosen in the following) and set

$$a_\pm(x, \xi, h) = \sum_{j=0}^{N} a_{\pm j}(x, \xi) h^j \in A_0^{0,\infty}.$$

Then the operator $J_{\pm a}(h) = I_h(a_\pm, \Phi_\pm)$ is well-defined and the operator $K_{\pm a}$ given by $K_{\pm a} = P(h) J_{\pm a} - J_{\pm a} P_0(h)$ is also a F.I.O. In fact, $K_{\pm a} = I_h(k_{\pm a}, \Phi_\pm)$ with

$$k_{\pm a} = e^{-ih^{-1}\Phi_\pm} \Big(-\frac{1}{2} h^2 \Delta + V(x) - \frac{1}{2}|\xi|^2 \Big) (e^{ih^{-1}\Phi_\pm} a_\pm).$$

It follows that the symbol $k_{\pm a}$ has the following properties:

(k0) $k_{\pm a}$ belongs to $A_1^{-1,\infty}$.
(k1) $\mathrm{supp}(k_{\pm a})) \subset \Gamma_\pm(3R_0, d_1, \mp\sigma_1)$.
(k2) $k_{\pm a}$ belongs to $A_{N+2}^{-(N+2),\infty}(\Gamma_\pm(4R_0, d_1, \mp\sigma_2))$.

Similarly, we define $J_{\pm b} = I_h(b_\pm, \Phi_\pm)$ for the region

$$\Gamma_\pm(5R_0, d_3, \pm\sigma_4) \subset \Gamma_\pm(3R_0, d_1, \mp\sigma_1),$$

where the symbol $b_\pm(x, \xi, h) = \sum_{j=0}^{N} b_{\pm j}(x, \xi) h^j$ satisfies (s0) and (s1) for $\Gamma_\pm(5R_0, d_3, \pm\sigma_4)$ and (s2) for $\Gamma_\pm(6R_0, d_4, \pm\sigma_3)$. Following the same argument as above, we define $K_{\pm b}(h) = P(h) J_{\pm b}(h) - J_{\pm b}(h) P_0(h) = I_h(k_{\pm b}, \Phi_\pm)$, with

$$k_{\pm b} = e^{-ih^{-1}\Phi_\pm} \Big(-\frac{1}{2} h^2 \Delta + V(x) - \frac{1}{2}|\xi|^2 \Big) (e^{ih^{-1}\Phi_\pm} b_\pm).$$

Then $k_{\pm b}$ satisfies (k0), (k1) for $\Gamma_\pm(5R_0, d_3, \pm\sigma_4)$ and (k2) for $\Gamma_\pm(6R_0, d_3, \pm\sigma_3)$. Now, the Isozaki-Kitada formula is stated in the following proposition.

Proposition 2.1 (Isozaki-Kitada [8].) *For* $\lambda \in \left] \frac{d_4^{-2}}{2}, \frac{d_4^2}{2} \right[$, *we have*

$$T(\lambda, h) = T_{+1}(\lambda, h) + T_{-1}(\lambda, h) - T_2(\lambda, h),$$

with

$$T_{\pm 1}(\lambda, h) = F_0(\lambda, h) J_{+a}^*(h) K_{\pm b}(h) F_0^*(\lambda, h)$$

and

$$T_2(\lambda, h) = F_0(\lambda, h) K_{+a}^*(h) R(\lambda + i0, h) \big(K_{+b}(h) + K_{-b}(h) \big) F_0^*(\lambda, h).$$

Remarks 2.1 The above formula is available provided the symbols $a_\pm$, $b_\pm$ are constructed by the process that we have described above. In particular, the integer N used to define $a_\pm$ and $b_\pm$ can be chosen *as large as we want.*

Denote by $T_{\pm 1}(\theta, \omega, \lambda, h)$ the kernel of $T_{\pm 1}(\lambda, h)$ and by $T_2(\theta, \omega, \lambda, h)$ the kernel of $T_2(\lambda, h)$. It is easy to see that

$$T_{\pm 1}(\theta, \omega, \lambda, h) = c_0(\lambda, h)^2 \int e^{ih^{-1}\psi_\pm(x,\theta,\omega)} k_{\pm b}(x, \sqrt{2\lambda}\omega)\overline{a}_+(x, \sqrt{2\lambda}\theta)\, dx,$$

where $\psi_\pm(x, \theta, \omega) = \Phi_\pm(x, \sqrt{2\lambda}\omega) - \Phi_+(x, \sqrt{2\lambda}\theta)$. For $\theta \neq \omega$, one can integrate by parts, to get $T_{\pm 1}(\theta, \omega, \lambda, h) = \mathcal{O}(h^\infty)$. In a such way, assuming $\theta \neq \omega$, we need only to study $T_2(\theta, \omega, \lambda, h) = (T_{2+} + T_{2-})(\theta, \omega, \lambda, h)$ where $T_{2\pm}$ is given by

$$T_{2,\pm}(\theta, \omega, \lambda, h) = c_0(\lambda, h)^2 \times \left\langle R(\lambda + i0)k_{\pm b}(\,\cdot\,, \sqrt{2\lambda}\omega)e^{ih^{-1}\Phi_\pm(\,\cdot\,,\sqrt{2\lambda}\omega)}\ ,\ k_{+a}(\,\cdot\,, \sqrt{2\lambda}\theta)e^{ih^{-1}\Phi_-(\,\cdot\,,\sqrt{2\lambda}\theta)}\right\rangle.$$

3 Spatial Localization

The results in this section are established without any assumption on the distribution of the resonances of the operator $P(h)$. We use the following result of Burq.

Theorem 3.1 (Burq [2]) *Assume that the potential V satisfies $(\mathbf{V}_\rho)$ with $\rho > 1$ and $(\mathbf{Hol}_\infty)$. Let $K \subset \mathbb{R}_+^*$ be a compact set. Then, there exists $R_1 > 0$ such that for any $R_2 > R_1$, there exist $C > 0$ and $h_0 > 0$ such that for any $0 < h \leq h_0$ and any $\lambda \in K$, we have*

$$\|1_{R_1 \leq |x| \leq R_2} R(\lambda \pm i0) 1_{R_1 \leq |x| \leq R_2}\| \leq Ch^{-1}. \tag{3.1}$$

Take $\chi(\,\cdot\,, R, R') \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ for $|x| \leq R$ and $\chi = 0$ for $|x| \geq R'$ and denote $\chi(x, R) = \chi(x, R, R + 1)$. Then, Bruneau and Petkov [1] have proved the following.

Lemma 3.1 *Under hypothesis $(\mathbf{V}_\rho)$, $\rho > 1$, there exists $\rho_0 > 0$ such that for $R \geq \rho_0$, λ is a non-trapping energy level for the operator $\hat{P}(h) = -\frac{1}{2}h^2\Delta + (1 - \chi(x, R))V(x)$.*

Set $\chi_a(x) = \chi(x, 20R_0)$ and $\chi_b(x) = \chi(x, 10R_0)$. Using Lemma 3.1 and the estimate (3.1), we can prove the following.

Proposition 3.1 *Assume the hypotheses $(\mathbf{V}_\rho)$ with $\rho > 1$ and $(\mathbf{Hol}_\infty)$ fulfilled. Then for $\alpha > \frac{n}{2}$ we have the following assertions:*

(i) $\|K_{+a}^*(h)R(\lambda + i0)K_{+b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$,
(ii) $\|K_{+a}^*(h)R(\lambda + i0)(1 - \chi_b)K_{-b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$,
(iii) $\|((1 - \chi_a)K_{+a})^*(h)R(\lambda + i0)\chi_b K_{-b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$.

Here N is given by the construction of the symbols $a_\pm, b_\pm$ and can be chosen arbitrarily large.

Remarks 3.1 The above estimates have been proved in [20] under the assumption that λ is a non-trapping energy level. Here we prove that these estimates hold for all energy levels.

Proof Take $R_0 \gg \rho_0$, where ρ_0 is given by Lemma 3.1. Let $\chi_1(x) = \chi(x, R_0, 2R_0)$, $\chi_2(x) = \chi(x, 2R_0, \frac{5}{2}R_0)$ and $\chi_3(x) = \chi(x, \frac{5}{2}R_0, \frac{11}{4}R_0)$. Using the fact that

$$\hat{P}(h)(1-\chi_1) = P(h)(1-\chi_1),$$

we obtain the following identity,

$$R(z)(1-\chi_2) = (1-\chi_1)\hat{R}(z)(1-\chi_2) + R(z)[P_0(h), \chi_1]\hat{R}(z)(1-\chi_2),$$

where $\hat{R}(z) = (\hat{P}(h) - z)^{-1},\ \forall z \in \mathbb{C} \setminus \mathbb{R}$. Therefore, the limiting absorption principle yields

$$\begin{aligned} &R(\lambda + i0)(1-\chi_2) \\ &\quad = (1-\chi_1)\hat{R}(\lambda + i0)(1-\chi_2) + R(\lambda + i0)[P_0(h), \chi_1]\hat{R}(\lambda + i0)(1-\chi_2) \end{aligned} \tag{3.2}$$

Similarly, we have

$$R(\lambda - i0)(1-\chi_3) = (1-\chi_2)\hat{R}(\lambda - i0)(1-\chi_3) + R(\lambda - i0)[P_0(h), \chi_2]\hat{R}(\lambda - i0)(1-\chi_3)$$

and by taking the adjoint, we get

$$\begin{aligned} &(1-\chi_3)R(\lambda + i0) = \\ &\quad (1-\chi_3)\hat{R}(\lambda + i0)(1-\chi_2) + (1-\chi_3)\hat{R}(\lambda + i0)[P_0(h), \chi_2]^* R(\lambda + i0). \end{aligned} \tag{3.3}$$

Then, multiplying (3.2) by $(1-\chi_3)$ and using (3.3) in the right-hand side of the equation obtained, we have

$$\begin{aligned} (1-\chi_3)&R(\lambda + i0)(1-\chi_2) = \\ &(1-\chi_3)\hat{R}(\lambda + i0)(1-\chi_2)[P_0(h), \chi_1]\hat{R}(\lambda + i0)(1-\chi_2) \\ &\quad + (1-\chi_3)\hat{R}(\lambda + i0)[P_0(h), \chi_2]^* R(\lambda + i0)[P_0(h), \chi_1]\hat{R}(\lambda + i0)(1-\chi_2) \\ &\quad + (1-\chi_3)(1-\chi_1)\hat{R}(\lambda + i0)(1-\chi_2). \end{aligned}$$

Recall that $\chi_2 = 1$ on $\operatorname{supp}(\chi_1)$ and $\chi_3 = 1$ on $\operatorname{supp}(\chi_1)$, so the above equation yields

$$\begin{aligned} &(1-\chi_3)R(\lambda + i0)(1-\chi_2) = (1-\chi_3)\hat{R}(\lambda + i0)(1-\chi_2) \\ &\quad + (1-\chi_3)\hat{R}(\lambda + i0)[P_0(h), \chi_2]^* R(\lambda + i0)[P_0(h), \chi_1]\hat{R}(\lambda + i0)(1-\chi_2) \end{aligned} \tag{3.4}$$

which we will use frequently in the following.

This formula is interesting for the following reasons. A priori, the resolvent $(1-\chi_3)R(\lambda+i0)(1-\chi_2)$ could behave as $e^{C/h}$. The right member of equation (3.4) involves only the modified resolvent $\hat{R}(\lambda+i0)$ and the truncated resolvent $[P_0(h),\chi_2]^*R(\lambda+i0)[P_0(h),\chi_1]$. The energy λ being non-trapping for $\hat{P}$, we will be able to apply the result of [20] to treat the term containing $\hat{R}(\lambda+i0)$. Moreover, the coefficients of the operators $[P_0(h),\chi_i]$, $i=1,2$ are supported in rings as far as we need from the origin. Therefore, we will be able to apply Theorem 3.1 to get a bound of $\|[P_0(h),\chi_2]^*R(\lambda+i0)[P_0(h),\chi_1]\|$ by $\mathcal{O}(h^{-1})$.

Proof of (i) Recall that $\operatorname{supp}(k_{+a})\subset\Gamma_+(3R_0,d_1,-\sigma_1)$ and $\operatorname{supp}(\chi_3)\subset\{|x|\geq\frac{11}{4}R_0\}$, hence $(1-\chi_3)K_{+a}=K_{+a}$, similarly $(1-\chi_2)K_{\pm b}=K_{\pm b}$ and we can multiply (3.4) by K^*_{+a} and K_{+b} to obtain

$$K^*_{+a}R(\lambda+i0)K_{+b}=K^*_{+a}\hat{R}(\lambda+i0)K_{+b}+ \\ K^*_{+a}\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*R(\lambda+i0)[P_0(h),\chi_1]\hat{R}(\lambda+i0)K_{+b}.$$

By Lemma 3.1, λ is non-trapping for $\hat{V}$ and we can apply (i) of Lemma 7.3 to obtain

$$\|K^*_{+a}\hat{R}(\lambda+i0)K_{+b}\|_{-\alpha,\alpha}=\mathcal{O}(h^{\frac{N}{2}}).$$

Hence it suffices to estimate the second term of the right-hand side of the previous equation. Take ψ_1 and ψ_2 in C_0^∞ such that $\psi_1=1$ in $\{x:R_0<|x|<2R_0\}$, $\psi_1=0$ in $\mathbb{R}^n\setminus\{x:\frac{R_0}{2}<|x|<3R_0\}$, $\psi_2=1$ in $\{x:2R_0<|x|<\frac{5}{2}R_0\}$ and $\psi_2=0$ in $\mathbb{R}^n\setminus\{x:R_0<|x|<3R_0\}$. It is clear that $[P_0(h),\chi_j]=\psi_j[P_0(h),\chi_j]$, $\forall j=1,2$ and we must estimate

$$K^*_{+a}\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*\psi_2R(\lambda+i0)\psi_1[P_0(h),\chi_1]\hat{R}(\lambda+i0)K_{+b}.$$

According to Proposition 3.1, we deduce $\|\psi_2R(\lambda+i0)\psi_1\|=\mathcal{O}(h^{-1})$, and by the construction of χ_1 and χ_2 we can apply the results concerning the non-trapping case described in Lemma 7.2 to obtain

$$\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)K_{+b}\|_{-\alpha,\alpha}=\mathcal{O}(h^\infty)\quad\text{and} \\ \|[P_0(h),\chi_2]\hat{R}(\lambda-i0)K_{+a}\|_{\alpha,\alpha}=\mathcal{O}(h^{-1}). \tag{3.5}$$

Using these estimates, we get

$$\begin{aligned}\|K^*_{+a}R(\lambda+i0)K_{+b}\|_{-\alpha,\alpha}&\leq\|K^*_{+a}\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*\|_{\alpha,\alpha}\|\psi_2R(\lambda+i0)\psi_1\|_{\alpha,\alpha}\\&\quad\times\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)K_{+b}\|_{-\alpha,\alpha}+\mathcal{O}(h^{\frac{N}{2}})\\&\leq C\|[P_0(h),\chi_2]\hat{R}(\lambda-i0)K_{+a}\|_{\alpha,\alpha}\|\psi_2R(\lambda+i0)\psi_1\|_{\alpha,\alpha}\\&\quad\times\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)K_{+b}\|_{-\alpha,\alpha}+\mathcal{O}(h^{\frac{N}{2}})\\&\leq Ch^{-1}\times Ch^{-1}\times Ch^\infty+Ch^{\frac{N}{2}}\leq Ch^{\frac{N}{2}}.\end{aligned}$$

Proof of (ii) Our aim is to estimate the operator $K_{+a}^*(h)R(\lambda+i0)(1-\chi_b)K_{-b}(h)$ and, as in the previous case, we use equation (3.4) and the fact that $(1-\chi_3)K_{+a} = K_{+a}$ and $(1-\chi_2)K_{-b} = K_{-b}$ to write

$$K_{+a}^*R(\lambda+i0)(1-\chi_b)K_{-b} = K_{+a}^*\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*R(\lambda+i0)[P_0(h),\chi_1] \times\hat{R}(\lambda+i0)(1-\chi_b)K_{-b} + K_{+a}^*\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}.$$

As above, λ being non-trapping for $\hat{V}$, we deduce from (ii) of Lemma 7.3 that

$$\|K_{+a}^*\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$$

and it remains to estimate

$$K_{+a}^*\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*R(\lambda+i0)[P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}.$$

Now we use the functions ψ_1 and ψ_2 defined previously, and the proof is reduced to the analysis of

$$\begin{aligned} I(h) &= \|K_{+a}^*\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*\psi_2R(\lambda+i0)\psi_1[P_0(h),\chi_1] \\ &\quad\times\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}\|_{-\alpha,\alpha} \\ &\leq \|K_{+a}^*\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*\|_{\alpha,\alpha}\|\psi_2R(\lambda+i0)\psi_1\|_{\alpha,\alpha} \\ &\quad\times\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}\|_{-\alpha,\alpha}. \end{aligned}$$

We have already seen that $\|\psi_2R(\lambda+i0)\psi_1\| = \mathcal{O}(h^{-1})$ and

$$\|K_{+a}^*\hat{R}(\lambda+i0)[P_0(h),\chi_2]^*\|_{-\alpha,\alpha} = \mathcal{O}(h^{-1}),$$

consequently

$$I(h) \leq Ch^{-2}\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}\|_{-\alpha,\alpha},$$

and it suffices to estimate $\|[P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\chi_b)K_{-b}\|_{-\alpha,\alpha}$. Let us introduce a function $\omega \in A_0^{0,\infty}(\mathbb{R}^{2n})$ such that $w(x,\xi) = 1$ if $(x,\xi) \in \Gamma_-(10R_0,d_3,\sigma_2)$ and $\mathrm{supp}(\omega) \subset \Gamma_-(10R_0,d_3,\sigma_3)$ and write

$$\begin{aligned} [P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\chi_b)K_{-b} &= [P_0(h),\chi_1]\hat{R}(\lambda+i0)\omega(x,hD_x)\tilde{K}_{-b} \\ &\quad + [P_0(h),\chi_1]\hat{R}(\lambda+i0)(1-\omega(x,hD_x))\tilde{K}_{-b}, \end{aligned}$$

where $\tilde{K}_{-b} = (1-\chi_b)K_{-b}$ is the Fourier Integral Operator with phase Φ_- and symbol $\tilde{k}_{-b}(x,\xi) = (1-\chi_b(x))k_{-b}(x,\xi)$. Using the definition of k_{-b}, it is clear that $\mathrm{supp}(\tilde{k}_{-b}) \subset \Gamma_-(10R_0,d_3,\sigma_4)$ and

$$\tilde{k}_{-b} \in A_{N+2}^{-(N+2),\infty}(\Gamma_-(10R_0,d_3,\sigma_3)). \tag{3.6}$$

Using the fact that $\omega(x, hD_x)$ localizes exactly in $\Gamma_-(10R_0, d_3, \sigma_3)$, we get

$$\|[P_0(h), \chi_1]\hat{R}(\lambda + i0)\omega(x, hD_x)\tilde{K}_{-b}\|_{\alpha,-\alpha} = \mathcal{O}(h^N).$$

On the other hand, $(1 - \omega)(x, hD_x)$ localizes in outgoing domain of the phase space. Then Lemma 7.2 gives

$$\|[P_0(h), \chi_1]\hat{R}(\lambda + i0)(1 - \omega(x, hD_x))\tilde{K}_{-b}\|_{\alpha,-\alpha} = \mathcal{O}(h^\infty)$$

and the proof of (ii) is complete.

Proof of (iii) It is very similar to the proof of (ii) and we just sketch it. We want to estimate $\big((1 - \chi_a)K_{+a}(h)\big)^* R(\lambda + i0)\chi_b K_{-b}(h)$, and taking into account (3.4), one can write

$$\begin{aligned}((1 - \chi_a)K_{+a})^* R(\lambda + i0)\chi_b K_{-b} &= ((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)\chi_b K_{-b} \\ &\quad + ((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)[P_0(h), \chi_2]^* \\ &\quad \times R(\lambda + i0)[P_0(h), \chi_1]\hat{R}(\lambda + i0)\chi_b K_{-b}.\end{aligned}$$

From (iii) of Lemma 7.3 we deduce

$$\|((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)\chi_b K_{-b}\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$$

and it remains to estimate

$$I(h) = \|((1-\chi_a)K_{+a})^*\hat{R}(\lambda+i0)[P_0(h), \chi_2]^* R(\lambda+i0)[P_0(h), \chi_1]\hat{R}(\lambda+i0)\chi_b K_{-b}\|_{-\alpha,\alpha}.$$

Once more using the functions ψ_1 and ψ_2, we get

$$\begin{aligned}I(h) \leq& \|((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)[P_0(h), \chi_2]^*\|_{-\alpha,\alpha}\|\psi_1\hat{R}(\lambda + i0)\psi_2\|_{-\alpha,-\alpha} \\ &\times \|[P_0(h), \chi_1]\hat{R}(\lambda + i0)\chi_b K_{-b}\|_{-\alpha,-\alpha}.\end{aligned} \tag{3.7}$$

We know that $\|\psi_1\hat{R}(\lambda + i0)\psi_2\|_{-\alpha,-\alpha} = \mathcal{O}(h^{-1})$ and, from Lemma 7.2, as in the proof of (ii), one easily obtains that $\|[P_0(h), \chi_1]\hat{R}(\lambda + i0)\chi_b K_{-b}\|_{-\alpha,-\alpha} = \mathcal{O}(h^{-1})$. Therefore, we have the estimate

$$I(h) \leq Ch^{-2}\|((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)[P_0(h), \chi_2]^*\|_{-\alpha,\alpha}. \tag{3.8}$$

On the other hand,

$$\|((1-\chi_a)K_{+a})^*\hat{R}(\lambda+i0)[P_0(h), \chi_2]^*\|_{-\alpha,\alpha} = \|[P_0(h), \chi_2]\hat{R}(\lambda-i0)(1-\chi_a)K_{+a}\|_{-\alpha,\alpha}.$$

Let $\omega(x, \xi) \in A_0^{0,\infty}$ be such that $\omega = 1$ on $\Gamma_+(4R_0, d_1, -\sigma_1)$ and $\mathrm{supp}(\omega) \subset \Gamma_+(4R_0, d_1 - \sigma_2)$ then

$$\begin{aligned}&\|((1 - \chi_a)K_{+a})^*\hat{R}(\lambda + i0)[P_0(h), \chi_2]^*\|_{-\alpha,\alpha} \leq \\ &\qquad \|[P_0(h), \chi_2]\hat{R}(\lambda - i0)(1 - \chi_a)\omega(x, hD_x)K_{+a}\|_{-\alpha,\alpha} \\ &\qquad + \|[P_0(h), \chi_2]\hat{R}(\lambda - i0)(1 - \chi_a)(1 - \omega(x, hD_x))K_{+a}\|_{-\alpha,\alpha}.\end{aligned}$$

As $\operatorname{supp}((1-\chi_a)(1-\omega)) \cap \operatorname{supp}(k_{+a}) \subset \Gamma_-(10R_0, d_1, -\sigma_0)$, Lemma 7.2 yields

$$\|[P_0(h), \chi_2]\hat{R}(\lambda - i0)(1-\chi_a)(1-\omega(x, hD_x))K_{+a}\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

Moreover, by construction, $k_{+a} \in A_{\mathbb{N}+2}^{-(N+2),\infty}$ on $\operatorname{supp}(1-\chi_a)\omega$, then

$$\|[P_0(h), \chi_2]\hat{R}(\lambda - i0)(1-\chi_a)\omega(x, hD_x)K_{+a}\|_{-\alpha,\alpha} = \mathcal{O}(h^N)$$

and we have proved that

$$\|((1-\chi_a)K_{+a})^*\hat{R}(\lambda + i0)[P_0(h), \chi_2]^*\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}}).$$

This estimate combined with (3.8) yields $I(h) = \mathcal{O}(h^\infty)$ and the proof is complete. ■

Let us denote $\Phi_+(x) = \Phi_+(x, \sqrt{2\lambda}\theta)$ and $\Phi_-(x) = \Phi_-(x, \sqrt{2\lambda}\omega)$. The Proposition 3.1 and a simple calculation yield the following theorem

Theorem 3.2 *Assumme* $(\mathbf{V}_\rho)$ *with* $\rho > 1$ *and* $(\mathbf{Hol}_\infty)$*. Then for every* $N \in \mathbb{N}$, R_0 *can be chosen large enough so that*

$$f(\theta, \omega, \lambda, h) = c_1(\lambda, h)G_0(\theta, \omega, \lambda, h) + \mathcal{O}(h^{\frac{N}{3}}), \tag{3.9}$$

with

$$G_0(\theta, \omega, \lambda, h) = \left\langle R(\lambda + i0)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle, \tag{3.10}$$

$$c_1(\lambda, h) = 2\pi(2\lambda)^{(n-3)/4}(2\pi h)^{-(n+1)/2}e^{-i\frac{(n-3)\pi}{4}} \tag{3.11}$$

and

$$g_{-b} = e^{ih^{-1}\Phi_-}[\chi_b, P_0(h)]\left(b_-(\,\cdot\,, \sqrt{2\lambda}\omega)e^{id^{-1}\Phi_-}\right), \tag{3.12}$$

$$g_{+a} = e^{ih^{-1}\Phi_+}[\chi_a, P_0(h)]\left(a_+(\,\cdot\,, \sqrt{2\lambda}\theta)e^{ih^{-1}\Phi_+}\right). \tag{3.13}$$

Notice that g_{-b} and g_{+a} have compact support included in rings which are situated as far as we want from origin. Therefore, modulo some error terms of order $\mathcal{O}(h^\infty)$, we established a representation formula for the scattering amplitude, involving only the truncated resolvent. Now we are in position to prove our first result.

Proof of Theorem 1.1 Because of equations (3.10) and (3.11), the proof is reduced to show that $|G_0(\theta, \omega, \lambda, h)| = \mathcal{O}(h)$. Let us choose $R_1 \gg 1$ so that estimate (3.1) holds for any $R_2 > R_1$ and assume $R_0 > R_1$. Then $g_{-b} = 1_{\{R_0<|x|<30R_0\}}g_{-b}$, $g_{+a} = 1_{\{R_0<|x|<30R_0\}}g_{+a}$, and one can write

$$\begin{aligned}|G_0(\theta, \omega, \lambda, h)| &\le \|g_{+a}\|_{L^2}\,\|1_{\{R_0<|x|<30R_0\}}R(\lambda + i0)1_{\{R_0<|x|<30R_0\}}\|_{0,0}\,\|g_{-b}\|_{L^2}\\ &\le Ch^{-1}\|g_{-b}\|_{L^2}\|g_{+a}\|_{L^2},\end{aligned}$$

where the last inequality comes from estimate (3.1). Moreover, a simple calculation yields

$$g_{-b} = h^2\left(\Delta\chi_b b_- + 2\langle\nabla\chi_b, \nabla b_-\rangle\right) + 2ih\langle\nabla\chi_b, \nabla\Phi_-\rangle b_-,$$
$$g_{+a} = h^2\left(\Delta\chi_a a_+ + 2\langle\nabla\chi_a, \nabla a_+\rangle\right) + 2ih\langle\nabla\chi_a, \nabla\Phi_+\rangle a_+.$$

Using the fact that $\Delta\chi_b$ and $\nabla\chi_b$ are compactly supported, it is clear that $\|g_{-b}\|_{L^2} = \mathcal{O}(h)$ and $\|g_{+a}\|_{L^2} = \mathcal{O}(h)$. Therefore, we obtain $|G_0(\theta, \omega, \lambda, h)| = \mathcal{O}(h)$ and the proof is complete. ■

4 Resolvent Estimate in the Trapping Case

For the proofs of Theorems 1.1 and 1.2, as in [13] and [20], we need an estimate of the resolvent on the real axis. In this section we prove the following.

Proposition 4.1 *Assume* ($\mathbf{Hol}_\infty$) *and hypothesis* (5) *of Theorem 1.3. Then there exists $\tilde{n} \in \mathbb{N}^*$ such that*

$$\|R(\lambda \pm i0)\|_{\alpha,-\alpha} = \mathcal{O}(h^{-\tilde{n}}),\ \alpha > \frac{1}{2}.$$

According to the work of Bruneau and Petkov [1], it suffices to show that a such estimate holds for the truncated resolvent $\chi R(\lambda \pm i0)\chi$, for suitable $\chi \in C_0^\infty(\mathbb{R}^n)$. More precisely we have the following proposition.

Proposition 4.2 *Let $d > 0$. There exists $\rho > 0$ large enough such that for all $\chi \in C_0^\infty(R^n)$, $\chi = 1$ on $B(0, \rho)$ and for h small enough we have*

$$\forall\alpha > \frac{1}{2},\ \|R(z)\|_{\alpha,-\alpha} \leq Ch^{-2}\left(1 + \|\chi R(z)\chi\|\right),$$

uniformly for $z \in \mathcal{B}_\pm = \{z \in \mathbb{C} : (Re\, z, \pm Im\, z) \in]d^{-1}, d[\times]0, 1]\}$.

Remarks 4.1 Notice that the functions χ satisfying the hypotheses of the above proposition do not vanish near 0, so that we can not apply Theorem 1.5.

We need two Lemmas due to Tang and Zworski [24] which are essential to obtain an estimate for the truncated resolvent. For completeness we state these Lemmas below.

Lemma 4.1 ([24]) *Let $F(\cdot, h)$ be a family of functions such that $F(\cdot, h)$ is holomorphic in $\Omega(h) = [\lambda - 5h^{q+\delta}, \lambda + 5h^{q+\delta}] + i[-h^{q+\frac{3n}{2}+3\delta}, h^{q+\frac{n}{2}+2\delta}]$ for some $q, \delta > 0$ and assume that F satisfies the estimates*

$$|F(z,h)| \leq A exp(Ah^{-\frac{n}{2}} log\frac{1}{h})\ in\ \Omega(h), \tag{4.1}$$

$$|F(z,h)| \leq \frac{C}{|Im(z)|}\ in\ \Omega(h) \cap \{Im(z) < 0\}. \tag{4.2}$$

Then there exists $C > 0$ *such that*

$$\forall h \ll 1,\ \forall z \in \tilde{\Omega}(h),\ |F(z,h| \leq Ch^{-q-\frac{3n}{2}-3\delta}, \tag{4.3}$$

where $\tilde{\Omega}(h) = [\lambda - h^{q+\delta}, \lambda + h^{q+\delta}] + i[-h^{q+\frac{3n}{2}+3\delta}, h^{q+\frac{3n}{2}+3\delta}]$.

Lemma 4.2 ([24]) *Let* U *be a compact neighborhood of* λ *in* $Re(z) > 0$*, and let* $g(h) \ll 1$. *Then there exists* $C > 0$ *such that for* $h \ll 1$ *we have*

$$\|R_\chi(z,h)\| \leq Ce^{Ch^{-n}\log\frac{1}{g(h)}},\ \text{for } z \in U \setminus \bigcup_{\xi \in Res(P(h))} B(\xi, g(h)), \tag{4.4}$$

where for $\chi \in C_0^\infty(\mathbb{R}^n)$ *we set* $R_\chi(z,h) = \chi R(z,h)\chi$.

Proof of Proposition 4.1 We have already seen that it is sufficient to show that $\|R_\chi(\lambda \pm i0)\| = \mathcal{O}(h^{-\tilde{n}})$ and we may apply Lemma 4.1 to $F(z,h) = R_\chi(z,h)$. Take $\delta > 0$ and let $q > 0$ be given by hypothesis (5) of Theorem 1.3. The estimate (4.1) is trivially satisfied. To establish (4.2), we apply Lemma 4.2 with $g(h) = \frac{1}{2}h^q$ and it suffices to show that $\Omega(h) \cap \bigcup_{\lambda_j \in Res(P(h))} B(\lambda_j, g(h)) = \varnothing$. For $\xi \in \Omega(h)$ and $\lambda_j \in Res(P(h))$, by assumption (5) of Theorem 1.3 we have

$$|\xi - \lambda_j| \geq |\lambda - \lambda_j| - |\xi - \lambda| \geq h^q - h^{q+\delta} > \frac{1}{2}h^q = g(h)$$

and the proof is complete. ∎

5 Proof of Theorem 1.2

We start with a representation formula for $\tilde{T}(\lambda,h)$, where obviously $\tilde{T}$ is defined by $\tilde{S}(\lambda,h) = Id - 2i\pi\tilde{T}(\lambda,h)$. We know from Proposition 2.1 that there exist some phases $\tilde{\Phi}_\pm(x,\xi)$ and some symbols $\tilde{a}_\pm$, $\tilde{b}_\pm$ such that

$$\tilde{T}(\lambda,h) = \tilde{T}_{+1}(\lambda,h) + \tilde{T}_{-1}(\lambda,h) - \tilde{T}_2(\lambda,h)$$

with

$$\tilde{T}_{\pm 1}(\lambda,h) = F_0(\lambda,h)J^*_{+\tilde{a}}(h)K_{\pm\tilde{b}}(h)F_0^*(\lambda,h)$$

and

$$\tilde{T}_2(\lambda,h) = F_0(\lambda,h)K^*_{+\tilde{a}}(h)\tilde{R}(\lambda+i0,h)(K_{+\tilde{b}}(h) + K_{-\tilde{b}}(h))F_0^*(\lambda,h).$$

On the other hand, the proof of Isozaki-Kitada [7], shows clearly that the phases $\tilde{\Phi}_\pm(x,\xi)$ and the symbols $\tilde{a}_\pm$, $\tilde{b}_\pm$ depend only of the potential $\tilde{V}$ outside a fixed compact set as large as we want. As $V = \tilde{V}$ in $\mathbb{R}^n \setminus F$, one can take $\tilde{\Phi}_\pm = \Phi_\pm$, $\tilde{a}_\pm = a_\pm$ and $\tilde{b}_\pm = b_\pm$. The same argument as for $T_{\pm 1}$ shows that the kernel of $\tilde{T}_{\pm 1}(\lambda,h)$ satisfies $\tilde{T}_{\pm 1}(\theta,\omega,\lambda,h) = \mathcal{O}(h^\infty)$ for any $\theta \neq \omega$. Moreover, we can compute exactly the same spatial localization for $\tilde{T}_2$ as that we have done for T_2 in section 3. Hence we obtain

$$\tilde{f}(\theta,\omega,\lambda,h) = c_1(\lambda,h)\tilde{G}_0(\theta,\omega,\lambda,h) + \mathcal{O}(h^{\frac{N}{3}}) \tag{5.1}$$

with

$$\tilde{G}_0(\theta, \omega, \lambda, h) = \langle \tilde{R}(\lambda + i0) g_{-b} e^{ih^{-1}\Phi_-}, g_{+a} e^{ih^{-1}\Phi_+} \rangle, \tag{5.2}$$

where g_{-b} and g_{+a} are given by formulae (3.12) and (3.13). Let us take $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ such that $1_F \prec \chi_1 \prec \chi_2 \prec 1_{\mathbb{R}^n \setminus \overline{W_{ext}}}$, *i.e.*, $\chi_1(x) = 1$ for $x \in F$, $\chi_2(x) = 1$ for $x \in \operatorname{supp}(\chi_1)$, $\chi_2(x) = 0$ for $x \in \overline{W_{ext}}$. With this construction we have

$$\tilde{P}(h)(1 - \chi_1) = (-h^2\Delta + \tilde{V})(1 - \chi_1) = (-h^2\Delta + V)(1 - \chi_1) = P(h)(1 - \chi_1),$$

and working as in the previous section, we easily obtain

$$\begin{aligned} \tilde{R}(\lambda + i0)(1 - \chi_2) &= (1 - \chi_1) R(\lambda + i0)(1 - \chi_2) \\ &\quad + \tilde{R}(\lambda + i0)[P_0(h), \chi_1] R(\lambda + i0)(1 - \chi_2). \end{aligned} \tag{5.3}$$

This identity combined with equations (3.9) and (5.1) gives

$$\begin{aligned} &\tilde{f}(\theta, \omega, \lambda, h) - f(\theta, \omega, \lambda, h) = \\ &\qquad c_1(\lambda, h) \Big\langle R(\lambda + i0)[\tilde{P}(h), \chi_1] \tilde{R}(\lambda + i0) g_{-b} e^{ih^{-1}\Phi_-}, g_{+a} e^{ih^{-1}\Phi_+} \Big\rangle . \end{aligned} \tag{5.4}$$

Recall that $\operatorname{supp}(g_{-b}) \subset \{10R_0 < |x| < 10R_0 + 1\}$ where R_0 can be chosen as large as we need. In particular we may assume $\mathbb{R}^n \setminus \overline{W_{ext}} \subset \{|x| < R_0\}$ and take $\psi_1 \in C_0^\infty(\{|x| < 5R_0\})$ such that $\psi_1 = 1$ on $\{|x| < 4R_0\}$. Then, we easily get

$$\begin{aligned} &\tilde{f}(\theta, \omega, \lambda, h) - f(\theta, \omega, \lambda, h) = \\ &\qquad c_1(\lambda, h) \langle R(\lambda + i0) \psi_1 [P_0(h), \chi_1] \tilde{R}(\lambda + i0) g_{-b} e^{ih^{-1}\Phi_-}, g_{+a} e^{ih^{-1}\Phi_+} \rangle. \end{aligned}$$

Let us denote $\tilde{\mathcal{V}}_0 = \tilde{R}(\lambda + i0) g_{-b} e^{ih^{-1}\Phi_-}$ and notice that according to hypothesis (iii), $\tilde{\mathcal{V}}_0$ is in $D'_{sc}(\mathbb{R}^n)$ (see the appendix for the definition). Indeed, we can apply the result of the previous section to $\tilde{V}$ and we obtain $\|\tilde{R}(\lambda + i0)\|_{\alpha, -\alpha} = \mathcal{O}(h^{-M})$. The notion of semi-classical wave front set WF^{sc} will permit us to control $\tilde{\mathcal{V}}_0$ (see section 7 for a precise definition of WF^{sc}). By definition,

$$(\tilde{P}(h) - \lambda)\tilde{\mathcal{V}}_0 = g_{-b} e^{ih^{-1}\Phi_-}.$$

Applying (iii) of Proposition 7.1, we deduce that

$$WF^{sc}(\tilde{\mathcal{V}}_0) \subset WF^{sc}(g_{-b}) \cup \operatorname{Char}^{sc}(\tilde{P}(h) - \lambda)$$

and using (ii) of Proposition 7.1 we get

$$WF^{sc}(\tilde{\mathcal{V}}_0) \subset T^*(\operatorname{supp}(g_{-b})) \cup \operatorname{Char}^{sc}(\tilde{P}(h) - \lambda).$$

By construction, we know that $\operatorname{supp}(g_{-b}) \subset \{4R_0 < |x| < 5R_0\}$ and $\operatorname{Char}^{sc}(\tilde{P}(h)) \subset T^*(\overline{W_{ext}} \cup F)$. Therefore,

$$WF^{sc}(\tilde{\mathcal{V}}_0) \subset T^*(\overline{W_{ext}} \cup F).$$

Moreover, by construction,

$$\operatorname{supp}([P_0(h), \chi_1]) \cap (\overline{W_{ext}} \cup F) = \varnothing$$

and we deduce from (ii) of Proposition 7.1 that

$$WF^{sc}([P_0(h), \chi_1]\tilde{\mathcal{V}}_0) = \varnothing.$$

Using (i) of Proposition 7.1, we obtain

$$\|[P_0(h), \chi_1]\tilde{R}(\lambda + i0)g_{-b}e^{ih^{-1}\Phi_-}\| = \mathcal{O}(h^\infty). \tag{5.5}$$

On the other hand, combining hypothesis (iii) and Proposition 4.1, we get

$$\|g_{+a}R(\lambda + i0)\psi_1\| = \mathcal{O}(h^{-M}). \tag{5.6}$$

Taking together equations (5.4), (5.5) and (5.6) we have

$$f(\theta, \omega, \lambda, h) - \tilde{f}(\theta, \omega, \lambda, h) = \mathcal{O}(h^\infty)$$

and the proof is complete. ■

6 Proof of Theorem 1.3

6.1 Short Time Localization

Starting with (3.10), we wish to replace $R(\lambda + i0)$ by the following representation.

$$R(\lambda + i0) = ih^{-1}\int_0^T e^{ih^{-1}t\lambda}e^{-ih^{-1}tP(h)}dt + e^{ih^{-1}T\lambda}R(\lambda + i0)e^{-ih^{-1}TP(h)}. \tag{6.1}$$

Our goal is to show that for $T > 0$ large enough, we have

$$\left\langle R(\lambda + i0)e^{-ih^{-1}TP(h)}g_{-b}e^{ih^{-1}\Phi_-}\ ,\ g_{+a}e^{ih^{-1}\Phi_+}\right\rangle = \mathcal{O}(h^N). \tag{6.2}$$

As in [13] and [20], the proof is based on an judicious application of Egorov's lemma. For this purpose we need to study the Hamiltonian flow $\Phi_t(x, \xi) = (q(t, x, \xi), p(t, x, \xi))$ associated to $\sigma_P(x, \xi) = \frac{1}{2}|\xi|^2 + V(x)$. This analysis is essentially the same as that given in [13]. For reader's convenience we recall the main steps. Let us denote

$$\Sigma_\lambda = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n :\ \frac{1}{2}|\xi|^2 + V(x) = \lambda\},$$

and for subsets W of S^{n-1} let

$$\Sigma_\lambda(W) = \{(x, \xi) \in \Sigma_\lambda :\ \frac{\xi}{|\xi|} \in \overline{W}\}.$$

As a preliminary step, we check that the assumption $(\mathbf{H}_\omega)$ is an open condition. More precisely, we have the following.

Proposition 6.1 *Assume that* $(\mathbf{H}_\omega)$ *is satisfied. Then there exists a neighborhood* W *of* ω *such that*

$$\forall \omega' \in W,\ \forall z \in \Lambda_{\omega'},\ \lim_{t\to+\infty} |q_\infty(t,z,\omega')| = +\infty.$$

Proof The proof is split into two steps.

First step Assume that $(\mathbf{H}_\omega)$ is satisfied. We begin to prove that for all $R > 0$, we can find a neighborhood W of ω such that

$$\forall \omega' \in W,\ \forall z \in B_{\Lambda_\omega}(0,R),\ \lim_{t\to+\infty} |q_\infty(t,z,\omega')| = +\infty.$$

First, it is not difficult to verify that we can find $R_0 > 0$, $d > 0$ and $\sigma > 0$ such that

$$\forall (x,\xi) \in \Gamma_+(R_0,d,\sigma) \cap \Sigma_\lambda,\ \lim_{t\to+\infty} |q(t,x,\xi)| = +\infty.$$

It follows that it suffices to find $T > 0$ and a neighborhood W of ω such that

$$\forall \omega' \in W,\ \forall z \in B_{\Lambda_\omega}(0,R),\quad \big(q_\infty(T,z,\omega'), p_\infty(T,z,\omega')\big) \in \Gamma_+(R_0,d,\sigma). \tag{6.3}$$

From assumption $(\mathbf{H}_\omega)$, we deduce that for all $z \in B_{\Lambda_\omega}(0,R)$, there exists $T(z) > 0$ such that $(q_\infty,p_\infty)(T(z),z,\omega) \in \Gamma_+(R_0,d,\sigma)$. For all $z \in B_{\Lambda_\omega}(0,R)$, we can use the continuity of the Hamiltonian flow with respect to initial data on every compact time interval to find $r(z) > 0$ and a neighborhood $W(z)$ of ω such that

$$\forall z' \in B_{\Lambda_\omega}(z,r(z)), \forall \omega' \in W(z),\quad (q_\infty,p_\infty)(T(z),z',\omega') \in \Gamma_+(R_0,d,\sigma).$$

Moreover the open sets $\big(B_{\Lambda_\omega}(z,r(z))\big)_{z\in B_{\Lambda_\omega}(0,R)}$ recover $\overline{B}_{\Lambda_\omega}(0,R)$. Using the compactness of $\overline{B}_{\Lambda_\omega}(0,R)$, we easily find $T > 0$ and W satisfying (6.3).

Second step We will prove that we can find $R > 0$ and an neighborhood W of ω such that

$$\forall \omega' \in W,\ \forall z \in \Lambda_\omega \cap \{|z| \geq 4R\},\ \lim_{t\to+\infty} |q_\infty(t,z,\omega')| = +\infty. \tag{6.4}$$

First, we easily verify that if the potential $V(x)$ belongs to $\mathcal{C}_0^\infty(\mathbb{R}^n)$, one can find $R > 0$ and a neighborhood W of ω such that

$$\forall \omega' \in W,\ \forall z \in \Lambda_\omega \cap \{|z| \geq 2R\},\quad \forall t \in \mathbb{R},\ q_\infty(t,z,\omega') = \sqrt{2\lambda}\omega' t + z. \tag{6.5}$$

Particularly, if $V \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, then (6.4) is satisfied. In the case where we assume only that V satisfies $(\mathbf{V}_\rho)$ with $\rho > 1$, we proceed by approximation. Given $1 < \rho' < \rho$ and $R > 0$, we can find $\tilde{V} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\tilde{W} \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} V = \tilde{V} + \tilde{W},\ \operatorname{supp} \tilde{V} \subset B(0,R), \\ \forall \alpha \in \mathbb{N}^n,\ \forall x \in \mathbb{R}^n,\ |\partial_\alpha \tilde{W}(x)| \leq C\, R^{\rho'-\rho} \langle x \rangle^{-\rho'-|\alpha|}. \end{cases} \tag{6.6}$$

Let $\omega \in S^{n-1}$, $z \in \Lambda_\omega$ and denote by $(\tilde{p}_\infty, \tilde{q}_\infty)(\,\cdot\,, z, \lambda, \omega)$ the solution to

$$\begin{cases} \dot{\tilde{q}}_\infty = \tilde{p}_\infty, \\ \dot{\tilde{p}}_\infty = -\nabla_x \tilde{V}(\tilde{q}_\infty) \end{cases}$$

such that

$$\begin{cases} \lim_{t\to-\infty} |\tilde{p}_\infty(t, z, \lambda, \omega) - \sqrt{2\lambda}\omega| = 0, \\ \lim_{t\to-\infty} |\tilde{q}_\infty(t, z, \lambda, \omega) - \sqrt{2\lambda}\omega t - z| = 0. \end{cases}$$

We chose W and $R > 0$ such that (6.6) holds. Using (6.5) we can prove that

$$\forall |z| \geq 4R,\ \forall \omega' \in W,\ \forall t \in \mathbb{R},\ |q_\infty(t, z, \omega') - \tilde{q}_\infty(t, z, \omega')| < 1 + \frac{1}{2}\tilde{q}_\infty(t, z, \omega').$$

Using (6.5) again, it follows that (6.4) holds in the general short range case and the proof is complete. ■

Using Proposition 6.1, we can copy the proof of Lemma 4.3 in [13], to obtain the following Lemma.

Lemma 6.1 *Assume* $(\mathbf{H}_\omega)$ *with* $\lambda \in\ ]d^{-1}, d[$. *Then there exist* $R_0 > 0$ *large enough and a neighborhood* W *of* ω *such that for all* $R > 0$ *there exists* $T_0(R) > 0$ *satisfying*

$$\forall (x, \xi) \in \Sigma_\lambda(W),\ 5R_0 < |x| < 6R_0 \Longrightarrow \forall t > T_0(R),\ \Phi_t(x, \xi) \in \Gamma_+(R, d, 0).$$

We recall also a weighted Egorov Lemma, whose proof can be found in [13], [14].

Lemma 6.2 *Let* $w \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, *and* $p \in A_k^{m,u}(\mathbb{R}^n \times \mathbb{R}^n)$, $(m, u, k) \in \mathbb{R} \times \mathbb{R} \times \mathbb{Z}$. *Assume that for* $t \geq 0$, $p(x, \xi)$ *vanishes on* $\Phi_t(\mathrm{supp}(\omega))$. *Then we have*

$$\forall \alpha \in \mathbb{N}^n,\ \|p(x, hD_x)e^{-ith^{-1}P(h)}\omega(x, hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

Applying Lemmas 6.1 and 6.2, we are able to prove the following.

Proposition 6.2 *There exists* $T_0 > 0$ *such that* $\forall T > T_0$, *we have*

$$\left\langle R(\lambda + i0)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+} \right\rangle = \int_0^T e^{ih^{-1}t\lambda} F(t, \theta, \omega, h)\, dt + \mathcal{O}(h^N)$$

with $F(t, \theta, \omega, h) = \left\langle e^{-ith^{-1}P(h)}g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+} \right\rangle$.

Proof Introduce $\omega \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ so that $\omega = 1$ in $U = \Sigma_\lambda(W) \cap (C(R_0) \times \mathbb{R}^n)$ and $\omega = 0$ in the complementary of an open neighborhood V of U. Setting $\omega_b = \chi_b \omega$, we have

$$\begin{aligned} &\left\langle R(\lambda + i0)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+} \right\rangle = \\ &\qquad \left\langle R(\lambda + i0)\omega_b(x, hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+} \right\rangle \\ &\qquad\qquad + \left\langle R(\lambda + i0)(1 - \omega_b)(x, hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+} \right\rangle \\ &\qquad\qquad = I(h) + J(h). \end{aligned}$$

Clearly,

$$(1-\omega_b)(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-} = \sum_{|\alpha|\le M} h^{|\alpha|}(1-\omega_b)^{(\alpha)}(x,\nabla_x\Phi_-)g_\alpha(x) + h^M R_M(x)$$

with $\mathrm{supp}(g_\alpha) \subset \mathrm{supp}(g_{-b})$, and $|R_M(x)| \le C\langle x\rangle^{-M}$ uniformly with respect to h. On the one hand, without a loss of generality, we can assume that $\forall |x| \ge R_0$, $\frac{\nabla_x\Phi_-}{|\nabla_x\Phi_-|} \in W$ and we get $\forall x \in \mathcal{C}(R_0)$, $(x,\nabla_x\Phi_-) \in U$. On the other hand, we have $\mathrm{supp}(g_{-b}) \subset C(R_0)$, and consequently $g_\alpha(x)(1-\omega)^{(\alpha)}(x,\nabla_x\Phi_-) = 0$, $\forall\alpha$. Moreover, for any M large enough, we have $\|R_M\|_\alpha = \mathcal{O}(1)$. This estimate combined with Proposition 4.1 yields

$$\begin{aligned} |J(h)| &= \left|\left\langle R(\lambda+i0)(1-\omega_b)(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle\right| \\ &= h^M\left|\left\langle R(\lambda+i0)R_M, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle\right| \\ &\le h^M\|R_M\|_\alpha\|g_{+a}\|_\alpha\|R(\lambda+i0)\|_{\alpha,-\alpha} \le Ch^{M-\tilde{n}}. \end{aligned}$$

As we may take M as large as we wish, we obtain $J(\lambda,h) = \mathcal{O}(h^\infty)$ and it remains to treat $I(h)$. For this purpose write

$$\begin{aligned} I(h) &= \int_0^T e^{ih^{-1}t\lambda}\left\langle e^{-ih^{-1}tP(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle\, dt \\ &\quad + \langle e^{-ih^{-1}T\lambda}R(\lambda+i0)e^{-ih^{-1}TP(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\rangle \\ &= I_1(T,h) + I_2(T,h). \end{aligned}$$

From the estimate $|(1-\omega_b)(x,hD_x)(g_{-b}e^{ih^{-1}\Phi_-})| \le C_M h^M\langle x\rangle^{-M}$ for every M, we deduce

$$I_1(T,h) = \int_0^T e^{ih^{-1}t\lambda}\left\langle e^{-ih^{-1}tP(h)}g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle\, dt + \mathcal{O}(h^\infty),$$

and it suffices to show that $I_2(T,h) = \mathcal{O}(h^N)$.
Let $\beta \in C^\infty$ be such that

$$\beta(x,\xi) = \begin{cases} 1 \text{ in } \Gamma_+(R,d,0), \\ 0 \text{ in } \Gamma_+(R,d,-\sigma)^c \end{cases}$$

and write

$$\begin{aligned} &I_2(T,h) = \\ &\left\langle e^{-ih^{-1}T\lambda}R(\lambda+i0)\beta(x,hD_x)e^{-ih^{-1}TP(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle \\ &\quad + \left\langle e^{-ih^{-1}T\lambda}R(\lambda+i0)(1-\beta)(x,hD_x)e^{-ih^{-1}TP(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle \\ &\qquad = I_{+2}(T,h) + I_{-2}(T,h). \end{aligned}$$

We begin to estimate $I_{+2}(T,h)$. Choosing $R \geq 20R_0 + 1 > \max(\rho_0, R_1)$, where ρ_0 is given in Lemma 3.1 and R_1 in Theorem 3.1, and working as in Propoosition 3.1, we verify that

$$\|\chi_a R(\lambda + i0)\beta(x, hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

Moreover, using the continuity of $e^{-ih^{-1}TP(h)}$ on the weighted Sobolev spaces (see [13], Lemma B.1 and [14], Proposition 1.3), we know that

$$\|e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)(g_{-b}e^{ih^{-1}\Phi_-})\|_{L^2_\alpha} = \mathcal{O}(1).$$

Hence

$$\begin{aligned}
&I_{+2}(\lambda, T, h) = \\
&\quad \left\langle e^{-ih^{-1}t\lambda}R(\lambda + i0)\beta(x, hD_x)e^{-ih^{-1}tP(h)}\omega_b(x, hD_x)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\right\rangle \\
&\qquad \leq \|\chi_a R(\lambda + i0)\beta(x, hD_x)\|_{-\alpha,\alpha}\|e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)(g_{-b}e^{ih^{-1}\Phi_-})\|_{L^2_\alpha} \\
&\qquad\quad \times \|g_{+a}e^{ih^{-1}\Phi_+}\|_{L^2_\alpha} \leq C_N h^N, \qquad \forall N.
\end{aligned}$$

It remains to estimate $I_{-2}(\lambda, T, h)$. Recall that we have the estimate

$$\|R(\lambda + i0)\|_{\alpha,-\alpha} = \mathcal{O}(h^{-\tilde{n}}).$$

Moreover,

$$\begin{aligned}
|I_{-2}(\lambda)| &\leq \|R(\lambda + i0)\|_{\alpha,-\alpha}\|g_{+a}\|_{L^2_\alpha} \\
&\qquad \times \|(1-\beta)(x, hD_x)e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)\|_{-\alpha,\alpha}\|g_{-b}\|_{L^2_{-\alpha}} \\
&\leq C\|(1-\beta)(x, hD_x)e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)\|_{-\alpha,\alpha}\|R(\lambda + i0)\|_{\alpha,-\alpha} \\
&\leq Ch^{-\tilde{n}}\|(1-\beta)(x, hD_x)e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)\|_{-\alpha,\alpha}
\end{aligned}$$

and it suffices to show that

$$\|(1-\beta)(x, hD_x)e^{-ih^{-1}TP(h)}\omega_b(x, hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty). \tag{6.7}$$

By the construction of ω and from Lemma 6.1 we deduce the existence of $T_0 > 0$ such that

$$\forall T > T_0,\ \forall (x,\xi) \in \operatorname{supp}(\omega_b),\ \Phi_T(x,\xi) \in \Gamma_+(R, d, 0).$$

Then

$$\forall T > T_0,\ \forall (x,\xi) \in \operatorname{supp}(\omega_b),\ \beta(\Phi_T(x,\xi)) = 1,$$

and for $T > T_0$, Lemma 6.2 implies (6.7). ∎

6.2 Second Localization

In this subsection, we follow exactly the same construction as that of [20]. Introduce

$$Z_j = \{z \in \Lambda_\omega : |z - z_j| < \epsilon\}, \quad 1 \le j \le l,$$

for $\epsilon > 0$ small enough, and set

$$Y_j = \{y \in \mathrm{supp}(g_{-b}) : y = q_\infty(s, z_j),\ s < 0\}.$$

For R_0 large enough, one can find $S_1 > S_0 \gg 1$ such that

$$Y_j \subset \Pi_{-j} = \{y : y = q_\infty(s, z),\ -S_1 < s < -S_0,\ z \in Z_j\}.$$

Let π_{-j} be in $C_0^\infty(\Pi_{-j})$, such that $0 \le \pi_{-j} \le 1$, and $\pi_{-j} = 1$ in Y_j. In [20] and [13], using Hamilton-Jacobi theory and Lemma 6.2, it is shown that

$$G_0(\theta, \omega, \lambda, h) = ih^{-1} \sum_{j=1}^{l} \int_0^{T_0} e^{ih^{-1}t\lambda} F_{-j}(t, \theta, \omega, h)\, dt + \mathcal{O}(h^\infty)$$

with

$$F_{-j}(t, \theta, \omega, h) = \left\langle e^{ih^{-1}tP(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-}, g_{+a} e^{ih^{-1}\Phi_+} \right\rangle.$$

Similarly, we define

$$X_j = \{x \in \mathrm{supp}(g_{+a}) : x = q(t, y, \nabla_x \Phi_-(y)),\ y \in Y_j,\ t > 0\}$$

and there exists $T_0 > T_1 \gg 1$ such that

$$X_j \subset \Pi_{+j} = \{x : x = q(t, y, \nabla_x \Phi_+(y)),\ y \in \Pi_{-j},\ T_1 < t < T_0\}.$$

Thus we may construct $\pi_{+j} \in C_0^\infty(\Pi_{+j})$, such that $0 \le \pi_{+j} \le 1$ and $\pi_{+j} = 1$ in X_j. Repeating the above argument, we obtain

$$G_0(\theta, \omega, \lambda, h) = ih^{-1} \sum_{j=1}^{l} \int_{T_1}^{T_0} e^{ih^{-1}t\lambda} F_j(t, \theta, \omega, h)\, dt + \mathcal{O}(h^\infty) \tag{6.8}$$

with

$$F_j(t, \theta, \omega, h) = bigl\langle e^{ih^{-1}tP(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-}, \pi_{+j} g_{+a} e^{ih^{-1}\Phi_+} \right\rangle.$$

6.3 Approximation of the Unitary Group and Stationary Phase Method

In this section, we repeat without changes the proof given in [20] and we recall only the main steps. First we construct an approximation for

$$\psi_j(t, x, h) = e^{ih^{-1}tP(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-},\ x \in \Pi_{+j},\ T_1 < t < T_0.$$

Lemma 6.3 *The point* $x = q(t, y, \nabla_x \Phi_-(y)) \in \Pi_{+j}$ *with* $y \in \Pi_{-j}$, *is non-focal,* i.e.,

$$D(t, y) = \det(\frac{\partial}{\partial y} q(t, y, \nabla_x \Phi_-(y))) \neq 0,\ T_1 < t < T_0.$$

From Lemma 6.3, we deduce the following representation of $\psi_j(t, x, h),\ T_1 < t < T_0,\ x \in \Pi_{+j}$ (cf. [12], sect. 12):

$$\psi_j(t, x, \lambda, h) = e^{ih^{-1}S_j(t,y) - i\mu_j \frac{\pi}{2}} |D(t, y)|^{-\frac{1}{2}} ih\pi_{-j}(y) g_{0b}(y) + \mathcal{O}(h), \tag{6.9}$$

for $x = q(t, y, \nabla_x \Phi_-(y)),\ y \in \Pi_{-j}$. Here S_j is the action along the trajectory joining the points x and y, *i.e.*,

$$S_j(t, y) = \Phi_-(y) + \int_0^t \Big(\frac{1}{2} |p(\tau, y, \nabla_x \Phi_-)|^2 - V(q(\tau, y, \nabla_x \Phi_-)) \Big) d\tau, \tag{6.10}$$

$\mu_j \in \mathbb{Z}$ is the Maslov index of this trajectory and $g_{0a}, g_{0b} \in C_0^\infty$ depend on $g_{+a}$and g_{-b}. We insert (6.9) into the expression of F_j, and after a change of variables $x = q(t, y, \nabla_x \Phi_-) \to y$, we get

(6.11)

$$G_0(\theta, \omega, \lambda, h) = ih \sum_{j=1}^{l} \int_{T_1}^{T_0} e^{ih^{-1}t\lambda} \int_{\mathbb{R}^n} e^{ih^{-1}\phi_j(t,y) - i\mu_j \frac{\pi}{2}} M_j(t, y) |D(t, y)|^{\frac{1}{2}}\, dy dt + \mathcal{O}(h^2)$$

where

$$\phi_j(t, y) = S_j(t, y) - \Phi_+\big(q(t, y, \nabla_x \Phi_-), \sqrt{2\lambda}\theta\big)$$

and

$$M_j(t, y) = \pi_{-j}(y) g_{0b}(y) \pi_{+j}\big(q(t, y, \nabla_x \Phi_-) g_{0a}(q(t, y, \nabla_x \Phi_-)\big)\,.$$

Thus, the proof of the theorem is reduced to the study on the asymptotic behavior of the integral

$$N_j(\theta, \omega, h) = \int_{T_1}^{T_0} e^{ih^{-1}t\lambda} \int_{\mathbb{R}^n} e^{ih^{-1}\phi_j(t,y) - i\mu_j \frac{\pi}{2}} M_j(t, y) |D(t, y)|^{\frac{1}{2}}\, dy dt, 1 \leq j \leq l.$$

The direction θ is regular for ω, hence we can make a change of variables

$$(z, s) \in Z_j \times]-S_1, -S_0[\longrightarrow y = q_\infty(s, z) \in \Pi_{-j}.$$

We obtain

$$N_j(\theta, \omega, h) = h^2 \int_{T_1}^{T_0} e^{ih^{-1}t\lambda - i\mu_j \frac{\pi}{2}} \int_{-S_1}^{-S_0} I_j(t, s, \theta, \omega, h)\, ds dt,$$

with

$$I_j(t,s,\theta,\omega,h) = \int_{Z_j} e^{ih^{-1}\Phi_j(t,s,z)} f_j(t,s,z) |D_\infty(t+s,z)|^{\frac{1}{2}} |D_\infty(s,z)|^{\frac{1}{2}}\, dz,$$

$$\Phi_j(t,s,z) = S_j(t, q_\infty(s,z)) - \Phi_+(q_\infty(t+s,z), \sqrt{2\lambda}\theta),$$

$$f_j(t,s,z) = \pi_{-j}(q_\infty(s,z)) g_{0b}(q_\infty(s,z)) \pi_{+j}(q_\infty(t+s,z)) g_{0a}(q_\infty(t+s,z)).$$

Now let us apply the stationary phase method to the integral I_j. As in [20], for (t,s) fixed, the only stationary point of the phase $\Phi_j(t,s,z)$ is z^j. We refer to Theorem 7.7.6 in [6] for the stationary phase method for integral depending on parameters. We apply this theorem at each z^j and we use Lemma 4.5 of [20] to get

$$N_j(\theta,\omega,h) = c_2(\lambda,h) e^{ih^{-1}S_j - i\mu_j\frac{\pi}{2}} \hat{\sigma}_j(z^j)^{-\frac{1}{2}} + \mathcal{O}(h),$$

with

$$c_2(\lambda,h) = -(2\lambda)^{-\frac{n-3}{4}} (2\pi h)^{\frac{n-1}{2}} h^2 e^{i(n-1)\frac{\pi}{4}}.$$

This equality combined with (3.9) and (6.2) completes the proof of Theorem 1.3.

7 Appendix

7.1 Semi-Classical Estimates in the Non-Trapping Case

In this section, we recall some results proved by Robert and Tamura and we refer to [19] and [20] for the proofs. Let $\hat{P}(h) = -h^2\Delta + \hat{V}(x)$ be a short-range perturbation of the Laplacian (*i.e.*, $\hat{V}$ satisfies $(\mathbf{V}_\rho)$ with $\rho > 1$) and let $\lambda > 0$ be a *non-trapping* energy level for the potential $\hat{V}$. We have the following lemmas.

Lemma 7.1 *Let $A(x, hD_x)$ be a p-order h-admissible differential operator, $p \leq 2$ and let $\alpha > \frac{1}{2}$. Then we have the following semiclassical estimate*

$$\|A(x,hD_x)\hat{R}(\lambda \pm i0)\|_{\alpha,-\alpha} = \mathcal{O}(h^{-1}).$$

See [1] for the proof.

Lemma 7.2 *Let $\omega_\pm \in A_0^{0,\infty}$ be such that* $\operatorname{supp}(\omega_\pm) \subset \Gamma_\pm(R,d,\sigma_\pm)$. *Then for $\alpha > 1$ we have the following assertions:*

(i) $\|\hat{R}(\lambda \pm i0)\omega_\pm(x,hD_x)\|_{-\alpha+\delta,-\alpha} = \mathcal{O}(h^{-1}),\ \forall \delta > 1.$

(ii) *If $\sigma_+ > \sigma_-$, then*

$$\|\omega_\mp(x,hD_x)\hat{R}(\lambda \pm i0)\omega_\pm(x,hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

(iii) *If $\omega \in A_0^{0,m}$, $m \in \mathbb{R}$ and* $\operatorname{supp}(\omega) \subset \{|x| < \tilde{R}\}$, $\tilde{R} > 0$, *then*

$$\|\omega(x,hD_x)\hat{R}(\lambda \pm i0)\omega_\pm(x,hD_x)\|_{-\alpha,\alpha} = \mathcal{O}(h^\infty).$$

Remarks 7.1 This Lemma is the same as that used by Robert and Tamura in [20], with exception of (iii) where we establish the same estimate for symbols $\omega \in A_0^{0,m}$, $\forall m$, instead of $\omega \in A_0^{0,\infty}$, but the proof works with the same argument.

As a direct consequence, we have the following.

Lemma 7.3 *Under the same assumption as those of the precedent Lemma and for $\alpha > \frac{n}{2}$, we have the following assertions:*

(i) $\|K_{+a}^*(h)\hat{R}(\lambda + i0)K_{+b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$,
(ii) $\|K_{+a}^*(h)\hat{R}(\lambda + i0)(1 - \chi_b)K_{-b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$,
(iii) $\|((1 - \chi_a)K_{+a})^*(h)\hat{R}(\lambda + i0)\chi_b K_{-b}(h)\|_{-\alpha,\alpha} = \mathcal{O}(h^{\frac{N}{2}})$.

7.2 Semi-classical Wave Front Set

The aim of this section is to recall briefly a notion of semi-classical wave front set appropriate to our problem and to give the basic properties that we need in the proof of Theorem 1.2. We refer to [14] for demonstrations and to [4], [9] for other definitions of semiclassical wave front set.

Definition 7.1

(1) We denote by $D'_{sc}(\mathbb{R}^n)$ the set of distributions $u(x, h)$ belonging to $D'(\mathbb{R}^n)$ such that for any $\chi \in C_0^\infty(\mathbb{R}^n)$, there exists $N \in \mathbb{N}$ such that

$$|\mathcal{F}_h(\chi u)(\xi, h)| \leq C_N h^{-N}\langle\xi\rangle^N,$$

where

$$\mathcal{F}_h(u)(\xi, h) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ih^{-1}\langle x,\xi\rangle} u(x, h)\, dx.$$

(2) Given $m, u \in \mathbb{R}$, we set $A_{sc}^{m,u} = \bigcup_{k\in\mathbb{Z}} A_k^{m,u}$ and we denote $L_{sc}^{m,k} = Op_h(A_{sc}^{m,k})$.

Definition 7.2 Let $p \in A_k^{m,u}$ and $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$. We say that the operator $P = p(x, hD_x, h)$ is elliptic in (x_0, ξ_0) if there exist a neighborhood V_0 of x_0 and a conic neighborhood Γ_0 of ξ_0 such that

$$\forall(x, \xi) \in V_0 \times \Gamma_0,\ |p(x, \xi, h)| \geq Ch^k\langle\xi\rangle^u.$$

We denote by $\mathrm{Char}^{sc}(P)$ the set of all points (x, ξ) where P is not elliptic.

Definition 7.3 Let $u \in D'_{sc}(\mathbb{R}^n)$ and let $(x_0, \xi_0) \in T^*(\mathbb{R}^n)$. We say that (x_0, ξ_0) does not belong to $WF^{sc}(u)$ if there exist a neighborhood V_0 of x_0 and a conic neighborhood Γ_0 of ξ_0 such that for all $\chi \in C_0^\infty(V_0)$ satisfying $\chi(x_0) = 1$ and for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that

$$\forall\chi \in \Gamma_0,\ |\mathcal{F}_h(\chi u)(\xi)| \leq C_N h^N\langle\xi\rangle^{-N}.$$

As for the classical wave front set, one has the following proposition.

Proposition 7.1

(i) *Let* $u(x,h) \in D'_{sc}(\mathbb{R}^n)$ *and assume that* $WF^{sc}(u) = \emptyset$. *Then for any* $k, m \in \mathbb{N}$, $u \in h^k H^m_{loc}(\mathbb{R}^n)$, *that is*

$$\forall \chi \in C_0^\infty(\mathbb{R}^n), \exists C > 0, \forall 0 < h < 1, \|\chi u(\,\cdot\,, h)\|_{H^m(\mathbb{R}^n)} \leq Ch^k.$$

(ii) *Let* $u(x,h) \in D'_{sc}(\mathbb{R}^n)$ *and* $\chi \in C_0^\infty(\mathbb{R}^n)$, *then*

$$WF^{sc}(\chi u) \subset T^*(\mathrm{supp}(\chi)) \cap WF^{sc}(u).$$

(iii) *Let* $u(x,h) \in D'_{sc}(\mathbb{R}^n)$ *and* $P \in L^{m,k}_{sc}$, *then*

$$WF^{sc}(Pu) \subset WF^{sc}(u) \subset WF^{sc}(Pu) \cup \mathrm{Char}^{sc}(P).$$

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