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# Scattering amplitude and scattering phase for the Schrödinger equation with strong magnetic field

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In this paper we consider the Schrödinger equation with constant magnetic field of strength $b>0$ in all dimension. We study the behavior of the scattering amplitude and the scattering phase when the parameter $b$ goes to infinity and the energy is far from the Landau levels. © *2005 American Institute of Physics.*
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## I. INTRODUCTION

In this paper, we are interested in the Schrödinger equation with magnetic field

$$i\partial_t \psi = H(b)\psi,$$

where $H(b)=H_0(b)+V(x)$ with $V\in L^\infty(\mathbb{R}^n)$ and

$$H_0(b) = |i\nabla_x - bA(x)|^2. \tag{1}$$

Here, $A:\mathbb{R}^n\to\mathbb{R}^n$ is the magnetic potential and $b$ is a strictly positive real parameter. Our aim is to study the scattering matrix associate to the pair $(H(b),H_0(b))$. In order to lighten the notations we drop the parameter $b$ and write $H$ instead of $H(b)$. There is a wide literature dealing with the Schrödinger equation with magnetic field (see for instance Refs. 2, 7, and 14 for general properties dealing with our problem). In this paper, we consider the case where the magnetic field is constant. More precisely, denoting $A(x)=(A_1(x),\dots,A_n(x))$ with $A_j:\mathbb{R}^n\to\mathbb{R}$, $j=1,\dots,n$, the magnetic field $B$ can be identified with the antisymmetric matrix $B(x)=(\partial_{x_i}A_j(x)-\partial_{x_j}A_i(x))_{i,j}$. Here, we consider the case where the magnetic field $B(x)$ does not depend on $x$. Regarding $B$ as an antisymmetric linear map on $\mathbb{R}^n$, we set $2d=\dim \operatorname{Ran} B$ and $k=n-2d=\dim \operatorname{Ker} B$. As we are interested in the case where $B\neq 0$, we suppose that $d\neq 0$. On the other hand, as we study scattering problems we do not consider the case $n=2d$ where the spectrum of $H_0$ is pure point. Hence, we suppose that $k=n-2d\neq 0$. Under this assumption there exists Cartesian coordinates in which the reference Hamiltonian takes the form

$$H_0 = \sum_{j=1}^{d} [(i\partial_{x_{2j-1}} - b\mu_j x_{2j})^2 + (i\partial_{x2j} + b\mu_j x_{2j-1})^2] - \Delta_{x\parallel},$$

where $x=(x_1,x_2,\dots,x_{2d-1},x_{2d},x_\parallel)=(x_\perp,x_\parallel)\in\mathbb{R}^{2d}\times\mathbb{R}^{n-2d}$ and $\mu_1,\dots,\mu_d$ are strictly positive real numbers (see Ref. 14 for details). Under suitable assumptions on $V$, it is well known (see Ref. 2) that the scattering operator $\mathbf{S}=\mathbf{S}(b)$ associated to the pair $(H_0,H)$ is well defined. Our aim is to describe this operator when the parameter $b$ goes to infinity.

Before going further, let us recall some works concerning such problems. First, we would like to mention some results concerning the scattering matrix in the case where $A(x)$ is a long range potential [i.e., $A(x)$ decreases faster than $|x|^{-\rho}$ for some $\rho>0$ when $|x|$ goes to infinity]. In that

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case, there is a scattering theory for the pair $(H,-\Delta)$ (cf. Refs. 10, 13, 16, and 18) and it is possible to describe the behavior of the scattering amplitude in the high energy limit (cf. the work of Nicoleau[12]).

On the other hand, there are some recent works of Bruneau–Pushnitski–Raikov[5] and Bruneau–Dimassi,[4] concerning the Schrödinger equation with constant magnetic field. In Ref. 5, the authors study the spectral shift function associate to this equation in dimension 3 and they describe its behavior in several asymptotic regimes. In particular, they investigate deeply the case where $b$ goes to infinity and the distance from the energy to the set of Landau levels behaves as $b$. There are also some works of Kostrykin–Kvitsinsky–Mekuyriev,[8] where the authors study partial scattering matrix in dimension 3 near the Landau levels ($b$ being fixed). Moreover, in Ref. 8 the authors make some symmetry assumption on the potential $V$. Here, we would like to treat the case of general potential in dimension $n$ in the asymptotic regime considered in Ref. 5.

First, we need to give an exact definition of the scattering amplitude in the present situation. Let us consider the Schrödinger operator with constant magnetic field in dimension $2d$,

$$\hat{H}_0 = \sum_{j=1}^{d} [(i\partial_{x_{2j-1}} - b\mu_j x_{2j})^2 + (i\partial_{x_{2j}} + b\mu_j x_{2j-1})^2] \tag{2}$$

acting on $L^2(\mathbb{R}^{2d})$. As $\mu_1\cdots\mu_d \neq 0$, it is well known that the spectrum of $\hat{H}_0$ is pure point.[2] For $q=(q_1,\ldots,q_d)\in\mathbb{N}^d$ we denote $\Lambda_q=(2_{q1}+1)\mu_1+\cdots+(2q_d+1)\mu_d$, so that the spectrum of $\hat{H}_0$ is given by the sequence of Landau levels

$$\mathbb{L} = \sigma(\hat{H}_0) = \sigma_{pp}(\hat{H}_0) = \{b\Lambda_q, q \in \mathbb{N}^d\}.$$

In particular, the bottom of the spectrum is given by $b\Lambda_0=b(\mu_1+\cdots+\mu_d)$. Let us denote by $\Upsilon_q\subset L^2(\mathbb{R}^{2d})$ the eigenspace associated to the eigenvalue $b\Lambda_q$ and $\Pi_q: L^2(\mathbb{R}^{2d})\to\Upsilon_q$ the corresponding projector. Denoting $L^2_\alpha(\mathbb{R}^{n-2d})=\{f\in L^2(\mathbb{R}^{n-2d});\langle x_\parallel\rangle^\alpha f\in L^2(\mathbb{R}^{n-2d})\}$; we define $\tilde{\mathcal{F}}_0(\lambda): L^2_\alpha(\mathbb{R}^{n-2d})\to L^2(S^{n-2d-1})$ by

$$\tilde{\mathcal{F}}_0(\lambda)\varphi(\xi) = \frac{\lambda^{(n-2d-2)/4}}{\sqrt{2}(2\pi)^{(n-2d)/2}}\int_{\mathbb{R}^{n-2d}} e^{-i\sqrt{\lambda}\langle x,\xi\rangle}\varphi(a)\mathrm{d}x,$$

and we set

$$\mathcal{F}_0(\lambda): L^2(\mathbb{R}^{2d}, L^2_\alpha(\mathbb{R}^{n-2d})) \to L^2(\mathbb{R}^{2d}\times S^{n-2d-1}),$$

$$\varphi \mapsto \sum_{b\Lambda_q\leqslant\lambda} \Pi_q \otimes \tilde{\mathcal{F}}_0(\lambda - b\Lambda_q)\varphi.$$

Let us introduce the space

$$L^\infty_\alpha(\mathbb{R}^n) = \{u:\mathbb{R}^n \to \mathbb{R}, \langle x_\parallel\rangle^\alpha u \in L^\infty(\mathbb{R}^n)\}$$

and for $u\in L^\infty_\alpha(\mathbb{R}^n)$ let us set $\|u\|_{\infty,\alpha}=\|\langle x_\parallel\rangle^\alpha u\|_{L^\infty}$. The assumption that we make on the potential $V$ is the following.

*Assumption 1:* We assume that $V(x_\parallel,x_\perp)=V^\infty(x_\parallel)+W(x_\parallel,x_\perp)$ with$V^\infty$ and $W$ in $L^\infty_\rho(\mathbb{R}^n)$ for some $\rho>1$, $V^\infty\geqslant 0$ and $\sup\{|W(n)|, 1\times 1\geqslant R\}\to 0$ when $R\to+\infty$.

It follows from the general results of Ref. 2 that under this assumption the wave operators associated to the pair $(H_0,H)$ exist and are complete. Therefore, the scattering operator $\mathbf{S}(b)$ is well defined and by the mean of $\mathcal{F}_0$, we can define the scattering matrix. More precisely, recall that the absolute continuous spectrum of $H_0$ is $\sigma_{ac}(H_0)=]b\Lambda_0,+\infty[$. Then for all $\lambda>b\Lambda_0$ there exists

$$S(\lambda,b): L^2(\mathbb{R}^{2d}\times S^{n-2d-1}) \to L^2(\mathbb{R}^{2d}\times S^{n-2d-1})$$

such that

$$S(\lambda,b)\mathcal{F}_0(\lambda)=\mathcal{F}_0(\lambda)\mathbf{S}(b).$$

Let us denote $\mathcal{H}_\alpha = L^2(\mathbb{R}^n, \langle x_\parallel\rangle^\alpha\, dx)$ and $\|\cdot\|_{\mathcal{H}_\alpha}$ the corresponding norm. Our first result gives a representation formula for the scattering matrix very similar to that obtained for the Schrödinger equation.[1]

**Theorem 1:** *Suppose that Assumption 1 is satisfied and denote by $\sigma_{pp}(H)$ the point spectrum of $H$. Then, for all $\lambda \in ]b\Lambda_0, +\infty[\setminus(L\cup\sigma_{pp}(H))$, one has*

$$S(\lambda,b)-Id=-2i\pi\mathcal{F}_0(\lambda)V(x)\mathcal{F}_0(\lambda)^* + 2i\pi\mathcal{F}_0(\lambda)V(x)R(\lambda+i0)V(x)\mathcal{F}_0(\lambda)^*, \tag{3}$$

*where*

$$R(\lambda+i0)=\lim_{\mu\to 0^+}(H-\lambda-i\mu)^{-1}$$

*exists in the space $\mathcal{L}(\mathcal{H}_\alpha,\mathcal{H}_{-\alpha})$ for $\alpha>1/2$.*

*Remark 1.1: In the case where the potential $V$ is compactly supported with respect to the variable $x_\parallel$, the scattering matrix takes a form that could be interesting for other applications. More precisely, suppose that $V\in L^\infty(\mathbb{R}^n)$ and there exists a compact $K\subset\mathbb{R}^{n-2d}$ such that $\forall x_\parallel \notin K$, $V(.,x_\parallel)=0$. Let $\chi_1$, $\chi_2\in C_0^\infty(\mathbb{R}^{n-2d})$ such that $\chi_1=1$ in a neighborhood of $K$ and $\chi_2=1$ on* supp $\chi_1$*. Then, using some ingrations by parts, it is not hard to prove that*

$$S(\lambda,b)-Id=-2i\pi\mathcal{F}_0(\lambda)[\Delta_{x_\parallel},\chi_1]R(\lambda+i0)[\Delta_{x_\parallel},\chi_2]\mathcal{F}_0(\lambda)^*.$$

Using Theorem 1, we can describe the behavior of $S(\lambda,b)$. Let us set $T(\lambda,b)=S(\lambda,b)-Id$, then $T(\lambda,b)$ has a kernel

$$(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}\mapsto T(\omega,\omega',\lambda,b)\in\mathcal{L}(L^2(R^{2d})).$$

Denote by $\hat{V}^\parallel$ the partial Fourier transform of $V$ with respect to the variable $x_\parallel$. We need to introduce two additional assumptions.

*Assumption 2:* We suppose that $V\in L^\infty_\rho(\mathbb{R}^n)$ for some $\rho>n-2d$, and that $\hat{V}^\parallel\in L^\infty_r(\mathbb{R}^n)$ for some $r>0$.

*Assumption 3:* We suppose that $\hat{V}^\parallel\in C^1(\mathbb{R}^n)$ and that $\sup_{\mathbb{R}^n}|\partial_{x_\perp}\hat{V}^\parallel|<\infty$.

Now we are in position to state our main result on the scattering amplitude. In the following we denote

$$\tilde{\mathbb{L}}=\{\Lambda_q, q\in\mathbb{N}^d\}=b^{-1}\mathbb{L}.$$

and

$$\tilde{Q}(\mathcal{E})=\{q\in\mathbb{N}^d;\Lambda_q\leqslant\mathcal{E}\},$$

which is a finite set, thanks to the fact that $\mu_1,\ldots,\mu_d>0$. In this paper we denote by $\|\cdot\|$ the $L^2$ norm and the norm on the space of linear bounded operators on $L^2$. We have the following theorem.

**Theorem 2:** *Suppose that Assumptions 1 and 2 are satisfied and let $\lambda>b\Lambda$.*

(i) *Denote $\delta := dist(\lambda,\mathbb{L})$ and suppose that $\delta>\|V\|_{\infty,\rho}$, then*

$$\sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}\left\| T(\omega,\omega',\lambda,b) + \frac{i}{2(2\pi)^{n-2d+1}}\sum_{b\Lambda_q\leq\lambda}(\lambda - b\Lambda_q)^{(n-2d-2)/2}\Pi_q\hat{V}^{\parallel}(x_\perp,\sqrt{\lambda - b\Lambda_q}\right.$$
$$\left.\times(\omega-\omega'))\Pi_q\right\| \leq C\lambda b^{-1}\delta^{[n-2d-2-\min(1,r)]/2}, \quad (4)$$

where $C$ depends only on $\|\hat{V}^{\parallel}\|_{\infty,r}$ and $\|V\|_{\infty,\rho}$.

(ii) *We suppose additionally that Assumption 3 is satisfied. Let* $\mathcal{E}\in]\Lambda_0,+\infty[\setminus\tilde{\mathbb{L}}$ *and* $\Delta\subset\mathbb{R}$ *be a bounded interval. When* $b$ *tends to infinity, one has*

$$\sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}\sup_{\lambda\in\Delta}\left\| T(\omega,\omega',\mathcal{E}b+\lambda,b) + \frac{ib^{(n-2d-2)/2}}{2(2\pi)^{n-2d+1}}\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-2}\hat{V}^{\parallel}(x_\perp,b^{1/2}\beta_q(\omega\right.$$
$$\left.-\omega'))\Pi_q\right\|$$
$$\leq Cb^{[n-2d-2-\min(1,r)]/2}, \quad (5)$$

*where*

$$\beta_q = (\mathcal{E}-\Lambda_q)^{1/2} = (\mathcal{E}-\mu_1 q_1 - \cdots - \mu_d q_d)^{1/2}. \quad (6)$$

From this theorem we can also deduce the following inverse scattering result.

*Corollary 1.2: Suppose that* $V_1$, $V_2$ *satisfy Assumptions* 1, 2, *and* 3. *Assume that the associate scattering operators* $S_1$ *and* $S_2$ *are equal. Then* $V_1=V_2$.

We can also use the representation formula of Theorem 1 to study the scattering phase $s(\lambda,b)$ associate to the pair $(H,H_0)$. Let us recall briefly how to define this function. Assume that the operator $T(\lambda,b)$ is trace class. Then the determinant $\det(I+T(\lambda,b))$ is well defined. Moreover, $S(\lambda,b)$ being unitary, this determinant is of modulus 1 so that the function $s(.,b)$ can be defined modulo $2\pi$ by

$$\det S(\lambda,b) = e^{-2i\pi s(\lambda,b)}. \quad (7)$$

Assume additionally that for $b$ large enough, $\|T(\lambda,b)\|<1$ uniformly with respect to $\lambda$. Then $s(\lambda,b)=(-1/2i\pi)\ln\det S(\lambda,b)$ can be determined uniquely by the following process. Consider the function

$$f:\sigma\in[0,1]\mapsto\det(I+\sigma T)\in\mathbb{C}$$

which is holomorphic with respect to $\sigma$. From the assumption $\|T\|<1$ we deduce that the spectrum of $T$ is contained in $]-1,1[$ and the function $f$ is nonvanishing. Therefore, the function $\ln(f)$ such that $\ln(f)(0)=0$ is uniquely defined and it follows that $s=\ln(f)(1)=(1/2i\pi)\ln\det(I+T)$ is well defined. Moreover, by construction, we have

$$2i\pi s(\lambda,b) = \int_0^1\frac{\mathrm{d}}{\mathrm{d}\sigma}\ln(\det(Id+\sigma T(\lambda,b)))\mathrm{d}\sigma = \int_0^1\operatorname{tr}(T(\lambda,b)(Id+\sigma T(\lambda,b))^{-1})\mathrm{d}\sigma. \quad (8)$$

Before we state our results, let us recall the link between the scattering phase $s(\lambda,b)$ and the spectral shift function $\xi(\lambda,b)$ (in short SSF). Assume that the difference $(H+\lambda_0)^{-\gamma}-(H_0+\lambda_0)^{-\gamma}$ is trace class for some $\lambda_0,\gamma>0$ large enough [for instance, if $\langle x\rangle^{\delta/2}V\in L^2(\mathbb{R}^n)$ with $\delta>n$, this assumption is satisfied in view of Theorem XI.21 of Ref. 15 and the diamagnetic inequality]. Therefore, the spectral shift function can be defined (see Refs. 17 and 9 ) in the sense of distribution by:

$$\langle \xi'(.,b)\,.\,f\rangle = \mathrm{tr}(f(H)-f(H_0)), \quad \forall f \in C_0^\infty(\mathbb{R})$$

and $\xi(\lambda,b)=0$ for $\lambda$ below the infimum spectrum of $H$. Moreover, we know from the Birman–Krein theory (see Refs. 3 and 17) that

$$\det S(\lambda,b) = e^{-2i\pi\xi(\lambda,b)}. \tag{9}$$

Comparing Eqs. (9) and (7), it follows that $\xi(\lambda,b)=s(\lambda,b)+c(\lambda,b)$ with $c(\lambda,b)\in\mathbb{Z}$. In Ref. 5, Bruneau, Pushnitski, and Raikov studied the asymptotics of $\xi(\lambda,b)$ far from the Landau levels. In the two next theorems we lead such a study for the scattering phase.

**Theorem 3:** *Suppose that Assumption 1 is satisfied and that $V\in L^1(\mathbb{R}^n)$. Let $\mathcal{E}\in]\Lambda_0,+\infty[\setminus\tilde{\mathbb{L}}$ and $\Delta\subset\mathbb{R}$ be a bounded interval. When $b\to+\infty$, one has $\sup_{\lambda\in\Delta}\|T(\mathcal{E}b+\lambda)\|\leqslant Cb^{-1/2}$ and the scattering phase defined by (8) satisfies*

$$\sup_{\lambda\in\Delta}\left| s(\mathcal{E}b+\lambda,b) + b^{(n-2)/2}\frac{mes(S^{n-2d-1})}{2(2\pi)^{n-2d+1}}\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-1}\int_{\mathbb{R}^n}V(x)\mathrm{d}x\right| = \mathcal{O}(b^{(n-3)/2}), \tag{10}$$

*where $\beta_q$ is given by (6) and $mes(S^{n-2d-1})$ denotes the Lebesgue measure of $S^{n-2d-1}$.*

Let us remark that in the asymptotic regime that we consider the scattering phase and the spectral shift function differ from a constant independent on $\lambda$ and $b$. Indeed, it is clear that these functions are continuous far from the Landau levels. Hence, for $\mathcal{E}\in]\Lambda_0,+\infty[\setminus\tilde{\mathbb{L}}$, the function $c(\mathcal{E}b+\lambda,b)$ is continuous with respect to $(\lambda,b)\in\Delta\times]b_0,+\infty[$ for $b_0$ large enough. As it takes its values in $\mathbb{Z}$ it follows that $c$ is constant. Therefore, it follows from (10) that under the preceding assumptions we have

$$\sup_{\lambda\in\Delta}\left| \xi(\mathcal{E}b+\lambda,b) + b^{(n-2)/2}\frac{mes(S^{n-2d-1})}{2(2\pi)^{n-2d+1}}\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-1}\int_{\mathbb{R}^n}V(x)\mathrm{d}x\right| = \mathcal{O}(b^{(n-3)/2}).$$

Remark that this result generalizes Theorem 2.1 of Ref. 5 in several directions. First it holds in all dimension whereas Bruneau, Pushnitski, and Raikov work in dimension 3. Moreover, it needs less regularity on the potential. Let us also remark that the method we use to prove it is completely different from that of Ref. 5 as it stands on the study of the scattering phase. However, we can notice that for $n=3$, we obtain the same asymptotics than in Ref. 5.

Using this representation, we can also give a complete asymptotics expansion of scattering phase. For a sake of simplicity, we formulate the theorem only in the case $n=3$ (and hence we can suppose that $\mu_1=1$), but the proof is the same in the case where $n=2d+1$. We also prove the Theorem for $V$ in the Schwartz class whereas it certainly holds for more general $C^\infty$ potentials going to zero at infinity as well as their derivatives.

**Theorem 4:** *Suppose that $V\in\mathcal{S}(\mathbb{R}^3)$. Let $\mathcal{E}\in\mathbb{R}_+^*\setminus\{2q+1,q\in\mathbb{N}\}$ and $\Delta\subset\mathbb{R}$ be a bounded interval. There exists a sequence of coefficients $(a_j(\lambda,\mathcal{E},V))_{j\in\mathbb{N}}$ such that one has the following expansion when $b\to+\infty$:*

$$\sup_{\lambda\in\Delta}\left| s(\mathcal{E}b+\lambda,b) - b^{\frac{1}{2}}\sum_{j=0}^{\infty}a_j(\lambda,\mathcal{E},V)b^{-j}\right| = \mathcal{O}(b^{-\infty}). \tag{11}$$

*Moreover, the coefficients $a_j$ can be computed explicitly. Setting $\gamma_j(\mathcal{E})=\sum_{q=1}^{[(\varepsilon-1)]/2}(\mathcal{E}-2q-1)^{-\frac{1}{2}-j}$, one has*

$$a_0(\lambda,\mathcal{E},V) = -\frac{\gamma_0(\mathcal{E})}{4\pi^2}\int_{\mathbb{R}^3}V(x)\mathrm{d}x,$$

$$a_1(\lambda,\mathcal{E},V) = \frac{\gamma_1(\mathcal{E})}{16\pi^2}\left(2\lambda\int_{\mathbb{R}^3} V(x)\mathrm{d}x - \int_{\mathbb{R}^3} V(x)^2\ \mathrm{d}x\right).$$

The plan of the paper is the following. In the next section we use the spectral resolution of $H_0$ to obtain a representation formula for the scattering matrix. In Sec. III, we study the scattering amplitude whereas the results concerning the scattering phase are proved in Sec. IV.

## II. REPRESENTATION OF THE SCATTERING MATRIX

In this section, we recall some basic facts on the spectral resolution of $H_0$ and the limiting absorption principle and we prove Theorem 1. Let us denote $\partial\widetilde{E}_0/\partial\lambda : L^2(\mathbb{R}^{n-2d}) \to L^2(\mathbb{R}^{n-2d})$ the spectral resolution of $-\Delta_{x_\parallel}$ on $\mathbb{R}^{n-2d}$. Then, it is well known that the spectral resolution of $H_0$ is given by

$$\frac{\partial E_0}{\partial\lambda} = \sum_{b\Lambda_q \leqslant \lambda} \Pi_q \otimes \frac{\partial\widetilde{E}_0}{\partial\lambda}(\lambda - \Lambda_q). \tag{12}$$

Moreover, one knows that $\partial\widetilde{E}_0/\partial\lambda = \widetilde{\mathcal{F}}_0(\lambda)^*\widetilde{\mathcal{F}}_0(\lambda)$ so that (12) yields

$$\frac{\partial E_0}{\partial\lambda} = \mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda). \tag{13}$$

For $z \in \mathbb{C}$ with $\mathrm{Im}\, z \neq 0$, we set $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ which are holomorphic with respect to $z \in \mathbb{C}\backslash\mathbb{R}$. We denote by $\sigma_{pp}(H)$, the point spectrum of $H$. The following proposition gives the limiting absorption principle for the operators $H_0$ and $H$.

*Proposition 2.1:* (i) *Assume that* $\lambda \in ]b\Lambda_0, +\infty[\backslash\mathbb{L}$, *then the following limit exists in the space of bounded operators* $\mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_{-\alpha})$ *for any* $\alpha > 1/2$:

$$R_0(\lambda \pm i0) = \lim_{\mu\to 0^+} R_0(\lambda \pm i\mu).$$

(ii) *Suppose that Assumption* 1 *is satisfied and that* $\lambda \in \mathbb{R}_+^*\backslash(\sigma_{pp}(H)\cup\mathbb{L})$, *then there exists*

$$R(\lambda \pm i0) = \lim_{\mu\to 0^+} R(\lambda \pm i\mu)$$

*in* $\mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_{-\alpha})$ *for any* $\alpha > 1/2$.

*Proof:* Using (12), it is clear that for all $z \in \mathbb{C}\backslash\mathbb{R}$, one has

$$R_0(z) = \sum_{q\in\mathbb{N}^d} \Pi_q \otimes (-\Delta_{x_\parallel} - (z - b\Lambda_q))^{-1}, \tag{14}$$

where the series converges in $\mathcal{L}(\mathcal{H}_\alpha, \mathcal{H}_{-\alpha})$ for any $\alpha > 1/2$. Assume that $\lambda \in ]b\Lambda_0, +\infty[\backslash\mathbb{L}$, then

$$R_0(\lambda \pm i\mu) = \sum_{b\Lambda_q\leqslant\lambda} \Pi_q \otimes (-\Delta_{x_\parallel} - (\lambda \pm i\mu - b\Lambda_q))^{-1} + W(\lambda \pm i\mu),$$

with

$$\|W(\lambda \pm i\mu) - W(\lambda \pm i\mu')\|^2 \leqslant C|\mu - \mu'|^2 \sum_{b\Lambda_q\geqslant\lambda} \|\Pi_q \otimes Id\|^2 \leqslant C|\mu - \mu'|^2. \tag{15}$$

Moreover, using the limiting absorption principle for the free Laplacian on $\mathbb{R}^{n-2d}$ it is clear that for any $\lambda \in ]b\Lambda_0, +\infty[\backslash\mathbb{L}$, there exists

$$\lim_{\mu\to 0^+}\sum_{b\Lambda_q\leqslant\lambda}\Pi_q\otimes(-\Delta_{x_\parallel}-(\lambda\pm i\mu-b\Lambda_q))^{-1}$$

and the proof of (i) is complete.

The proof of (ii) is very close to the proof of Agmon[1] for Schrödinger operator. For $\mathrm{Im}\, z > 0$, let us denote $R_\infty(z)=(H_0+V^\infty-z)^{-1}$. The potential $V^\infty$ being independent on $x_\perp$, it commutes with the projectors $\Pi_q$ so that

$$\forall\, \mathrm{Im}\, z>0,\ R_\infty(z)=\sum_{q\in\mathbb{N}^d}\Pi_q\otimes(-\Delta_{x_\parallel}+V^\infty(x_\parallel)-(z-b\Lambda_q))^{-1}.$$

As $V^\infty$ is non negative the spectrum of $-\Delta_{x_\parallel}+V^\infty$ is contained in $\mathbb{R}^+$ and we deduce from the limiting absorption principle for the Schrödinger operator that for any $\lambda\in\,]b\Lambda_0,+\infty[\setminus\mathbb{L}$,

$$R_\infty(\lambda\pm i0)=\lim_{\mu\to 0+}R_\infty(\lambda\pm i\mu)=\sum_{b\Lambda_q\leqslant\lambda}\Pi_q\otimes(-\Delta_{x_\parallel}+V^\infty-(\lambda\pm i0-b\Lambda_q))^{-1}$$

exists in $\mathcal{L}(\mathcal{H}_\alpha,\mathcal{H}_{-\alpha})$. Now for $\mathrm{Im}\ z>0$ we can write

$$R(z)=R_\infty(z)(Id+WR_\infty(z))^{-1}$$

As in Ref. 1, the only thing we have to check is that for all $z\in\mathbb{C}$ with $\mathrm{Im}\,(z)\geqslant 0$, $K(z)=W(x)R_\infty(z)$ is compact from $\mathcal{H}_\alpha$ into $\mathcal{H}_\alpha$ for some $\alpha>1/2$. On the other hand,

$$K(z)=W(x)R_0(z)(Id-V^\infty R_\infty(z))$$

and it follows from the limiting absorption principle for $R^\infty(z)$ that $(Id-V^\infty R_\infty(z))$ can be continued to $\mathrm{Im}\, z\geqslant 0$ into a bound operator on $\mathcal{H}_\alpha$ for $1/2<\alpha<\rho/2$. Hence the proof is reduced to show that $W\,R_0(z)$ is compact from $\mathcal{H}_\alpha$ into $\mathcal{H}_\alpha$. Using the diamagnetic inequality ( see Ref. 14, Lemma 2.1), the compactness of $K(z)$ is a straightforward consequence of the same property for the Schrödinger operator.

In the next proposition we recall some estimates of the resolvent proved in Ref. 5.

*Proposition 2.2:* (i) *Assume that* $\lambda\in\,]b\Lambda_0,+\infty[\setminus\mathbb{L}$, *then*

$$\|\langle x_\parallel\rangle^{-\alpha}R_0(\lambda\pm i0)\langle x_\parallel\rangle^{-\alpha}\|\leqslant\frac{C}{dist(\lambda,\mathbb{L})^{1/2}},\ \forall\ \alpha>1/2.$$

(ii) *Suppose that Assumption 1 is verified and that* $\lambda\in\,]b\Lambda_0,+\infty[$ *satisfies* $dist(\lambda,\mathbb{L})>\|V\|_{\infty,\rho}$. *Then* $\lambda\notin\sigma_{pp}(H)$ *and*

$$\|\langle x_\parallel\rangle^{-\alpha}R(\lambda\pm i0)\langle x_\parallel\rangle^{-\alpha}\|\leqslant\frac{C}{dist(\lambda,\mathbb{L})^{1/2}},\ \forall\ 1/2<\alpha<\rho/2.$$

*Proof:* The point (i) is a direct consequence of the well-known high-energy estimates of the resolvent of the Schrödinger equation. The claim (ii) follows easily from Birman–Schwinger principle and from the following formula:

$$R(\lambda\pm i0)=R_0(\lambda\pm i0)(Id+V\,R_0(\lambda\pm i0))^{-1}.$$

■

Now, we are in position to give the proof of Theorem 1 which is an adaptation of the demonstration given in the case of the Schrödinger operator (cf. Ref. 6 ). We start with a simple lemma.

*Lemma 2.3: Suppose that* $\lambda\in\,]b\Lambda_0,+\infty[\setminus\mathbb{L}$, *then*

$$R_0(\lambda+i0)-R_0(\lambda-i0)=2i\pi\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda).$$

*Proof:* The proof is based on the fact that this result holds for the Schrödinger operator,

$$\forall \lambda > 0, (-\Delta_{x_\parallel} - \lambda - i0)^{-1} - (-\Delta_{x_\parallel} - \lambda + i0)^{-1} = 2i\pi \tilde{\mathcal{F}}_0(\lambda)^* \tilde{\mathcal{F}}_0(\lambda). \tag{16}$$

On the other hand, for $\lambda \in ]b\Lambda_0, +\infty[ \setminus \mathbb{L}$, it follows from (15) that

$$R_0(\lambda + i0) - R_0(\lambda - i0) = \sum_{b\Lambda_q \leqslant \lambda} (-\Delta_{x_\parallel} - \lambda + b\Lambda_q - i0)^{-1} \otimes \Pi_q - (-\Delta_{x_\parallel} - \lambda + b\Lambda_q + i0)^{-1} \otimes \Pi_q.$$

Using (16), we obtain

$$R_0(\lambda + i0) - R_0(\lambda - i0) = 2i\pi \sum_{b\Lambda_q \leqslant \lambda} \tilde{\mathcal{F}}_0(\lambda)^* \tilde{\mathcal{F}}_0(\lambda) \otimes \Pi_q = 2i\pi \mathcal{F}_0(\lambda)^* \mathcal{F}_0(\lambda),$$

and the proof is complete. ■

Using this lemma, we can prove Theorem 1. Let us denote $W_\pm$ the wave operators for the pair $(H, H_0)$ and take $f$, $g$ in the absolute continuous subspace of $H_0$. Then

$$\begin{aligned}\langle (\mathbf{S} - Id) f, g \rangle &= \langle (W_- - W_+) f, W_{+g} \rangle \\ &= -i \int_{-\infty}^{+\infty} \langle e^{itH} V(x) e^{-itH_0} f, W_{+g} \rangle \mathrm{d}t = -i \int_{-\infty}^{+\infty} \langle V(x) e^{-itH_0} f, W_+ e^{-itH_0} g \rangle \mathrm{d}t.\end{aligned}$$

Moreover, one knows that

$$W_+ - Id = i \int_0^{+\infty} e^{i\sigma H} V(x) e^{-i\sigma H_0} \mathrm{d}\sigma.$$

Therefore,

$$\begin{aligned}\langle (\mathbf{S} - Id) f, g \rangle &= i \int_0^{+\infty} i \int_{-\infty}^{+\infty} \langle V(x) e^{-i\tau H_0} f, e^{i\sigma H} V(x) e^{-i(\sigma+\tau)H_0} g \rangle \mathrm{d}\tau \, \mathrm{d}\sigma - i \int_{-\infty}^{+\infty} \langle V(x) e^{-itH_0} f, e^{-itH_0} g \rangle \mathrm{d}t \\ &= \lim_{\mu, \mu' \to 0^+} i \int_0^{+\infty} e^{-\mu\sigma} i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \langle e^{i(\sigma+\tau)H_0} V(x) e^{-i\sigma H} V(x) e^{-i\tau H_0} f, g \rangle \mathrm{d}\tau \\ &\quad \times \mathrm{d}\sigma - i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \langle V(x) e^{-itH_0} f, e^{-itH_0} g \rangle \mathrm{d}t \\ &= \lim_{\mu, \mu' \to 0} i \int_0^{+\infty} e^{-\mu\sigma} i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) e^{-i\sigma(H-\lambda)} V(x) e^{-i\tau(H_0-\lambda)} f, \mathcal{F}_0(\lambda) g \rangle \, \mathrm{d}\lambda \\ &\quad \times \mathrm{d}\tau \, \mathrm{d}\sigma - i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) e^{-it(H_0-\lambda)} f, \mathcal{F}_0(\lambda) g \rangle \mathrm{d}\lambda \; \mathrm{d}t \\ &= \lim_{\mu' \to 0^+} i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) R(\lambda + i0) V(x) e^{-i\tau(H_0-\lambda)} f, \mathcal{F}_0(\lambda) g \rangle \mathrm{d}\lambda \; \mathrm{d}\tau \\ &\quad - i \int_{-\infty}^{+\infty} e^{-\mu'|\tau|} \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) e^{-it(H_0-\lambda)} f, \mathcal{F}_0(\lambda) g \rangle \mathrm{d}\lambda \; \mathrm{d}t \\ &= \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) R(\lambda + i0) V(x) (R_0(\lambda + i0) - R_0(\lambda - i0)) f, \mathcal{F}_0(\lambda) g \rangle \mathrm{d}\lambda \; \mathrm{d}\tau \\ &\quad - \int_{b\Lambda_0}^{+\infty} \langle \mathcal{F}_0(\lambda) V(x) (R_0(\lambda + i0) - R_0(\lambda - i0)) f, \mathcal{F}_0(\lambda) g \rangle \mathrm{d}\lambda \; \mathrm{d}t.\end{aligned}$$

Using Lemma 2.3, we obtain

$$\langle (\mathbf{S}-Id)f,g\rangle = 2i\pi\int_{b\Lambda_0}^{+\infty}\langle \mathcal{F}_0(\lambda)V(x)R(\lambda+i0)V(x)\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)f,\mathcal{F}_0(\lambda)g\rangle \mathrm{d}\lambda\ \mathrm{d}\tau - 2i\pi\int_{b\Lambda_0}^{+\infty}\langle \mathcal{F}_0(\lambda)V(x)\mathcal{F}_0(\lambda)^*\mathcal{F}_0(\lambda)f,\mathcal{F}_0(\lambda)g\rangle \mathrm{d}\lambda\ \mathrm{d}t$$

and the proof of Theorem 1 is complete. ■

## III. SCATTERING AMPLITUDE IN STRONG MAGNETIC FIELD

In this section, we prove Theorem 2. The first step is to write the scattering amplitude under a convenient form. Let us denote by $\langle .,.\rangle_{L^2(\mathbb{R}^{n-2d})}$ the scalar product on $L^2(\mathbb{R}^{n-2d})$. From Theorem 1 and Assumption 2, it is clear that for $\lambda\in]b\Lambda_0,+\infty[\setminus(\mathbb{L}\cup\sigma_{pp}(H))$, $T(\omega,\omega',\lambda,b)$ can be decomposed into $T=T_1+T_2$ with

$$T_1(\omega,\omega',\lambda,b) = -\frac{i\pi}{(2\pi)^{n-2d}}\sum_{b\Lambda_p\leq\lambda}\sum_{b\Lambda_p\leq\lambda}(\lambda-b\Lambda_q)^{(n-2d-2)/4}(\lambda-b\Lambda_p)^{(n-2d-2)/4} \times \Pi_p\langle V(x_{\perp,.})e^{i\sqrt{\lambda-b\Lambda_q}\langle .,\omega'\rangle},e^{i\sqrt{\lambda-b\Lambda_p}\langle .,\omega\rangle}\rangle_{L^2(\mathbb{R}^{n-2d})}\Pi_q$$

and

$$T_2(\omega,\omega',\lambda,b) = \frac{i\pi}{(2\pi)^{n-2d}}\sum_{b\Lambda_p\leq\lambda}\sum_{b\Lambda_q\leq\lambda}(\lambda-b\Lambda_q)^{(n-2d-2)/4}(\lambda-b\Lambda_p)^{(n-2d-2)/4} \times\int_{\mathbb{R}^{n-2d}}\Pi_pV(x_\perp,x_\parallel)e^{-i\sqrt{\lambda-b\Lambda_p}\langle x_\parallel,\omega\rangle}R(\lambda+i0)V(x_\perp,x_\parallel)e^{i\sqrt{\lambda-b\Lambda_q}\langle x_\parallel,\omega'\rangle}\Pi_q\mathrm{d}x_\parallel,$$

where the last integral converges in the space of bounded operator on $L^2(\mathbb{R}^{2d})$. From Proposition 2.2, it follows that for $\delta>\|V\|_{\infty,\rho}$,

$$\|T_2(\omega,\omega',\lambda,b)\|\leq C\delta^{-1/2}\sum_{b\Lambda_q\leq\lambda}(\lambda-b\Lambda_q)^{(n-2d-2)/2}\|V\|^2_{\infty,\rho}\leq C\lambda\|V\|^2_{\infty,\rho}b^{-1}\delta^{(n-2d-3)/2},$$

where the constant $C$ does not depend on $\omega$ and $\omega'$. It remains to treat the term $T_1$. Suppose that $p\neq q$. As $\hat{V}^\parallel\in L^\infty_r(\mathbb{R}^n)$, then

$$\sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}|\langle V(x_\perp,.)e^{i\sqrt{\lambda-b\Lambda_q}\langle .,\omega'\rangle},e^{i\sqrt{\lambda-b\Lambda_p}\langle .,\omega\rangle}\rangle_{L^2(\mathbb{R}^{n-2d})}| = \sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}|\hat{V}^\parallel(x_\perp,\sqrt{\lambda-b\Lambda_p}\omega-\sqrt{\lambda-b\Lambda_q}\omega')| \leq C\|\hat{V}^\parallel\|_{\infty,r}\sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}|\sqrt{\lambda-b\Lambda_p}\omega-\sqrt{\lambda-b\Lambda_q}\omega'|^{-r}\leq C\|\hat{V}^\parallel\|_{\infty,r}\delta^{-r/2}. \tag{17}$$

Therefore,

$$\sup_{(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}}\left\|T_1(\omega,\omega',\lambda,b)+\frac{i\pi}{(2\pi)^{n-2d}}\sum_{b\Lambda_q\leq\lambda}(\lambda-b\Lambda_q)^{(n-2d-2)/2}\Pi_q\hat{V}^\parallel(x_\perp,\sqrt{\lambda-b\Lambda_q}(\omega-\omega'))\Pi_q\right\|\leq C\|\hat{V}^\parallel\|_{\infty,r}\lambda b^{-1}\delta^{[n-2d-2-\min(1,r)]/2} \tag{18}$$

and the proof of (i) is complete.

Let us prove (ii). Starting from (4) at the energy $\mathcal{E}b+\lambda$, we get

$$T(\omega,\omega',\mathcal{E}b+\lambda,b)=\frac{-i\pi}{(2\pi)^{n-2d}}b^{(n-2d-2)/2}\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-2}\Pi_q\hat{V}^{\parallel}\Big(x_\perp,b^{\frac{1}{2}}\beta_q(\omega-\omega')\Big)\Pi_q$$
$$+\mathcal{O}(b^{[n-2d-2-\min(1,r)]})/2.$$

On the other hand, we know from Lemma 9.1 in Ref. 5 that

$$\|(1-\Pi_q)\hat{V}^{\parallel}\Big(.,b^{\frac{1}{2}}\beta_q(\omega-\omega')\Big)\Pi_q\|\leqslant Cqb^{-1/2}\sup_{x\in\mathbb{R}^n}|\partial_{x_\perp}\hat{V}^{\parallel}|.$$

Combining these estimates, we obtain the result claimed in (ii).

Finally, let us give the proof of Corollary 1.2. Suppose that $S_1=S_2$, then $T_1=T_2$ and for all $\mathcal{E}\notin\tilde{\mathbb{L}}$, $b>0$ and $(\omega,\omega')\in S^{n-2d-1}\times S^{n-2d-1}$, we have

$$T_1(\omega,\omega',\mathcal{E}b,b)=T_2(\omega,\omega',\mathcal{E}b,b).$$

It follows from Theorem 2 that

$$\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-2}\hat{W}^{\parallel}(x_\perp,b^{1/2}\beta_q(\omega-\omega'))\Pi_q=\mathcal{O}(b^{-[\min(1,r)]/2}),$$

where $W=V_1-V_2$. Now, let $\xi\in\mathbb{R}^{n-2d}$, then for all $b>0$ there exists $\omega,\omega'\in S^{n-2d-1}$ such that $b^{1/2}(\omega-\omega')=\xi$. Therefore,

$$\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-2}\hat{W}^{2}(x_\perp,\beta_q\xi)\Pi_q=\mathcal{O}(b^{-[\min(1,r)]/2}),$$

and taking the limit when $b$ tends to infinity, we obtain

$$\sum_{q\in\tilde{Q}(\mathcal{E})}\beta_q^{n-2d-2}\hat{W}^{2}(x_\perp,\beta_q\xi)\Pi_q=0.$$

Moreover, this equality holds for all $\mathcal{E}\notin\tilde{\mathbb{L}}$, so that for all $q\in\mathbb{N}^d$, the map $x_\perp\mapsto\hat{W}^{\parallel}(x_\perp,-\beta_q\xi)$ belongs to $(\mathrm{Im}\,\Pi_q)^\perp$. As $L^2(\mathbb{R}^{2d})=\oplus_{q\in\mathbb{N}^d}\mathrm{Im}\,\Pi_q$, it follows that $\hat{W}^{\parallel}$ vanishes identically and the proof is complete. ■

## IV. ASYMPTOTICS OF THE SCATTERING PHASE

In this section, we prove Theorems 3 and 4. Starting from formula (8), we must show that the operator $T=T(\mathcal{E}b+\lambda,b)$, $\mathcal{E}\notin\tilde{\mathbb{L}}$ is trace class and to obtain convenient estimates on $\|T\|$. For this purpose, we recall that

$$T(\mathcal{E}b+\lambda,b)=-2i\pi\mathcal{F}_0(\mathcal{E}b+\lambda)V(x)\mathcal{F}_0(\mathcal{E}b+\lambda)^*+2i\pi\mathcal{F}_0(\mathcal{E}b+\lambda)V(x)$$
$$\times R(\mathcal{E}b+\lambda+i0)V(x)\mathcal{F}_0(\mathcal{E}b+\lambda)^*.$$

Moreover, as $\mathcal{E}\notin\tilde{\mathbb{L}}$, Proposition 2.2 shows that $\|\langle x_\parallel\rangle^{-\alpha}R_0(\mathcal{E}b+\lambda+i0)\langle x_\parallel\rangle^{-\alpha}\|$ is bounded by $b^{-1/2}$. Using Assumption 1, the resolvent

$$R(\mathcal{E}b+\lambda+i0)=R_0(\mathcal{E}b+\lambda+i0)(Id+V\,R_0(\mathcal{E}b+\lambda+i0))^{-1}$$

can expand in powers of $V\,R_0$. Combining this argument with the formula giving $\mathcal{F}_0$, it follows that for $L\in\mathbb{N}$,

$$T(\mathcal{E}b+\lambda,b)=\sum_{l=0}^{L}\sum_{q\in\tilde{Q}(\mathcal{E})}T_{q,l}(\mathcal{E}b+\lambda,b)+\mathcal{O}(\|T_{q,L+1}(\mathcal{E}b+\lambda,b)\|_1), \tag{19}$$

where for $l\in\mathbb{N}$ we have defined

$$T_{q,l}(\mathcal{E}b+\lambda,b)=(-1)^{l+1}2i\pi\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)(V(x)R_0(\mathcal{E}b+\lambda+i0))^lV(x)\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)^* \tag{20}$$

and

$$\forall E>b\Lambda_q,\quad \mathcal{F}_{q,0}(E)=\Pi_q\otimes\tilde{\mathcal{F}}_0(E-b\Lambda_q). \tag{21}$$

Let us denote by $S^1$ the space of trace class operators on $L^2(\mathbb{R}^{2d}\times S^{n-2d-1})$ and by $\|.\|_1$ the corresponding norm. For $A\in S^1$, we denote by tr $A$ the trace of $A$. With these notations, we have the following lemma.

*Lemma 4.1: Suppose that V satisfies Assumption* 1. Let $\mathcal{E}\in]\Lambda_0+\infty[\setminus\tilde{\mathbb{L}}$ *and* $\Delta\subset\mathbb{R}$ *be a bounded interval. When b tends to infinity, one has*

(i) $\forall\epsilon>0$,

$$\sup_{\lambda\in\Delta}\|T_{q,l}(\mathcal{E}b+\lambda,b)\|_{L^2(\mathbb{R}^{2d}\times S^{n-2d-1}),L^2(\mathbb{R}^{2d}\times S^{n-2d-1})}\leqslant Cb^{-\frac{3}{4}-\frac{1}{2}+\epsilon},$$

$$\sup_{\lambda\in\Delta}\|T(\mathcal{E}b+\lambda,b)\|_{L^2(\mathbb{R}^{2d}\times S^{n-2d-1}),L^2(\mathbb{R}^{2d}\times S^{n-2d-1})}\leqslant Cb^{-\frac{3}{4}+\epsilon}.$$

(ii) *Suppose additionally that* $V\in L^1(\mathbb{R}^n)$. For *b large enough*, $T_{q,l}(\mathcal{E}b+\lambda,b)$ *and* $T(\mathcal{E}b+\lambda,b)$ *are trace class and*

$$\sup_{\lambda\in\Delta}\|T_{q,l}(\mathcal{E}b+\lambda,b)\|_1\leqslant Cb^{(n-2-l)/2},\quad \sup_{\lambda\in\Delta}\|T(\mathcal{E}b+\lambda,b)\|_1\leqslant Cb^{(n-2)/2}.$$

*Proof:* Let us start with the point (i). We start by estimating the operator $\mathcal{F}^*_{q,0}(\mathcal{E}b+\lambda)$ which is bounded from $L^2(S^{n-2d-1}\times\mathbb{R}^{2d})$ into $L^2_{-\beta}(\mathbb{R}^{n-2d},L^2(\mathbb{R}^{2d}))$ for all $\beta>\frac{1}{2}$. Moreover, for all $\beta>\frac{1}{2}$ and $\varphi\in L^2(S^{n-2d-1})$, we have

$$\begin{aligned}\|\tilde{\mathcal{F}}_0(\lambda)^*\varphi\|^2_{L^2_{-\beta}}&=\lambda^{(n-2d-2)/2}\int_{\mathbb{R}^{n-2d}}\langle x\rangle^{-\beta}\left|\int_{S^{n-2d-1}}e^{i\sqrt{\lambda}x\omega}\mathrm{d}\omega\right|^2\mathrm{d}x\\&=\lambda^{-1}\int_{\mathbb{R}^{n-2d}}\langle\lambda^{-1/2}x\rangle^{-\beta}\left|\int_{S^{n-2d-1}}e^{ix\omega}d\omega\right|^2\mathrm{d}x\leqslant C\lambda^{(\beta/2)-1}\|\mathcal{F}_0(1)^*\varphi\|^2_{L^2_{-\beta}}\\&\leqslant C\lambda^{(\beta/2)-1}\|\varphi\|^2_{L^2(S^{n-2d-1})}.\end{aligned}$$

From this estimate, one deduces easily that for all $\beta>1/2$,

$$\begin{aligned}\|\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)\|_\beta&:=\|\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)\|_{L^2_\beta(\mathbb{R}^{n-2d},L^2(\mathbb{R}^{2d})),L^2(S^{n-2d-1}\times\mathbb{R}^{2d})}\\&=\|\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)^*\|_{L^2(S^{n-2d-1}\times\mathbb{R}^{2d}),L^2_{-\beta}(\mathbb{R}^{n-2d},L^2(\mathbb{R}^{2d}))}\\&\leqslant C(\mathcal{E}b+\lambda-b\Lambda_q)^{(\beta-2)/4}.\end{aligned} \tag{22}$$

It follows from this estimate, Assumption 1, formula (20) and Proposition 2.2 that for $\epsilon>0$,

$$\|T_{q,l}(b)\|_{L^2(\mathbb{R}^{2d}\times S^{n-2d-1}),L^2(\mathbb{R}^{2d}\times S^{n-2d-1})}$$
$$\leqslant C\|\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)\|^2_{1/2+\epsilon}\times\|V(R_0(\mathcal{E}b+\lambda)V)^l\|_{L^2_{-1/2-\epsilon}(\mathbb{R}^{n-2d},L^2(\mathbb{R}^{2d})),L^2_{1/2+\epsilon}(\mathbb{R}^{n-2d},L^2(\mathbb{R}^2d))}$$
$$\leqslant b^{-\frac{3}{4}-\frac{1}{2}+\epsilon}. \tag{23}$$

This achieves to prove the first estimate of (i). The second one then follows by Eq. (19).

Let us prove (ii). Thanks to the resolvent estimates of Proposition 2.2, it suffices to show that the operator $|V|^{\frac{1}{2}}\mathcal{F}_{q,0}(\lambda)^*:L^2(S^{n-2d-1}\times\mathbb{R}^{2d})\to L^2(\mathbb{R}^{2n})$ belongs to the Hilbert–Schmidt class and that

$$\||V|^{\frac{1}{2}}\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)\|_2\leqslant Cb^{(n-2)/4}, \tag{24}$$

where $\|.\|_2$ denotes the Hilbert–Schmidt norm. For $q=(q_1,\ldots,q_d)\in\mathbb{N}^d$, let us denote by $K_q(y_\perp,x_\perp)$ the kernel of $\Pi_q$ and by $\sigma(.,.)$ the symplectic form on $\mathbb{R}^{2d}$. We have

$$K_q(y_\perp,x_\perp)=\frac{b^d}{2\pi}\exp\left(-\frac{b}{4}[|y_\perp-x_\perp|^2+2i\sigma(y_\perp,x_\perp)]\right)L_q(y_\perp,x_\perp,b), \tag{25}$$

with

$$L_q(y_\perp,x_\perp,b)=\prod_{j=1}^d\tilde{L}_{q_j-1}\left(\frac{b}{2}|y_{\perp,j}-x_{\perp,j}|^2\right),$$

where for $s\in\mathbb{N}$, $\tilde{L}_s$ is the Laguerre polynomial of order $s$ (see Ref. 14 for more details). With these notations, the kernel $N(x_\perp,x_\parallel,x'_\perp,\omega)$ of $|V|^{\frac{1}{2}}\mathcal{F}_{q,0}(\lambda)^*$ satisfies

$$N(x_\perp,x_\parallel,x'_\perp,\omega)=(\mathcal{E}b+\lambda-b\Lambda_q)^{(n-2d-2)/4}|V|^{\frac{1}{2}}K_q(x_\perp,x'_\perp)e^{-i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}\langle x_\parallel,\omega\rangle}$$
$$=\mathcal{O}(b^{(n-2d-2)/4})|V|^{\frac{1}{2}}K_q(x_\perp,x'_\perp)e^{-i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}\langle x_\parallel,\omega\rangle}$$

and

$$\|N\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^{2d}\times S^{n-2d-1})}=\mathcal{O}(b^{(n+2d-2)/2})\int_{\mathbb{R}^{2d}\times\mathbb{R}^{n-2d}\times\mathbb{R}^{2d}\times S^{n-2d-1}}|V(x_\perp,x_\parallel)|e^{-(b/2)|x_\perp-x'_\perp|^2}$$
$$\times\left|L_q\left(\frac{b}{2}|x_\perp-x'_\perp|^2\right)\right|^2\mathrm{d}x_\perp\mathrm{d}x_\parallel\mathrm{d}x'_\perp\mathrm{d}\omega$$
$$\leqslant Cb^{(n+2d-2)/2}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{n-2d}\times\mathbb{R}^{2d}}|V(x_\perp,x_\parallel)|e^{-(b/2)|x_\perp-x'_\perp|^2}$$
$$\times\left|L_q\left(\frac{b}{2}|x_\perp-x'_\perp|^2\right)\right|^2\mathrm{d}x'_\perp\mathrm{d}x_\parallel\mathrm{d}x_\perp.$$

By change of variable, it comes

$$\|N\|^2_{L^2(\mathbb{R}^n\times\mathbb{R}^{2d}\times S^{n-2d-1})}\leqslant Cb^{(n+2d-2)/2}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{n-2d}\times\mathbb{R}^{2d}}|V(x_\perp,x_\parallel)|e^{-b|x_\perp-x'_\perp|^2}\mathrm{d}x'_\perp\mathrm{d}x_\parallel\mathrm{d}x_\perp$$
$$\leqslant Cb^{(n+2d-2)/2}\int_{\mathbb{R}^n}|V(x)|\mathrm{d}x\int_{\mathbb{R}^{2d}}e^{-|x'_\perp|^2}\mathrm{d}x'_\perp\leqslant Cb^{(n-2)/2}$$

which proves (24). Using (24) and Proposition 2.2, it comes

$$\|T_{q,l}(\mathcal{E}b+\lambda)\|_1 \leqslant Cb^{(n-2-l)/2}$$

and the proof of (ii) is complete. ∎

From this lemma, we know that $\|T(\mathcal{E}b+\lambda,b)\|<1$ and by Taylor expansion, we deduce from (8) that for $N\in\mathbb{N}$

$$2i\pi s(\mathcal{E}b+\lambda,b)=\sum_{k=0}^{N}\frac{(-1)^k}{k+1}\mathrm{tr}(T(\mathcal{E}b+\lambda,b)^{k+1})+\mathcal{O}(\|T(\mathcal{E}b+\lambda,b)^{N+2}\|_1). \tag{26}$$

Hence, we must show that for $k\in\mathbb{N}$, $\mathrm{tr}(T(\mathcal{E}b+\lambda,b)^k)$ admits an expansion in powers of $b^{1/2}$. Using the fact that for $p\neq q$, $\Pi_p\Pi_q=0$, we deduce from Eq. (19) that

$$\mathrm{tr}(T(\mathcal{E}b+\lambda,b)^k)=\sum_{q\in\tilde{Q}(\mathcal{E})}\mathrm{tr}\left(\sum_{l=0}^{L}T_{q,l}(\mathcal{E}b+\lambda,b)\right)^k+\mathcal{O}(\|T^k_{q,L+1}(\mathcal{E}b+\lambda,b)\|_1). \tag{27}$$

At this point of the calculus, we can either continue the expansion to get a complete asymptotics or we can stop the expansion at the first order to prove Theorem 3. Indeed, it follows from Lemma 4.1 that the remainder terms in Eqs. (26) and (27) satisfy

$$\|T^N_{q,L+1}(\mathcal{E}b+\lambda,b)\|_1=\mathcal{O}(b^{[n-1-N(L+2)]/2})\ \text{and}\ \|T(\mathcal{E}b+\lambda,b)^{N+2}\|_1=\mathcal{O}(b^{(n-N-3)/2}).$$

Therefore, Eqs. (26) and (19) yield

$$s(\mathcal{E}b+\lambda,b)=\frac{1}{2i\pi}\sum_{q\in\tilde{Q}(\mathcal{E})}\mathrm{tr}(T_{q,0}(\mathcal{E}b+\lambda,b))+\mathcal{O}(b^{(n-3)/2}). \tag{28}$$

On the other hand, a standard calculation shows that the kernel $N_{q,0}$ of $T_{q,0}(\mathcal{E}b+\lambda,b)$ is given by

$$N_{q,0}(\omega',x'_\perp,\omega,x_\perp)=-\frac{i\pi}{(2\pi)^{n-2d}}(\mathcal{E}b+\lambda-b\Lambda_q)^{(n-2d-2)/2}K_q(x_\perp,x'_\perp)\hat{V}^{\parallel}(x_\perp,\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(\omega-\omega')).$$

Using (25), it follows that

$$\begin{aligned}\mathrm{tr}(T_{q,0}(\mathcal{E}b+\lambda,b))&=\int_{S^{n-2d-1}\times\mathbb{R}^{2d}}N_{q,0}(\omega,x_\perp,\omega,x_\perp)\mathrm{d}x_\perp\mathrm{d}\omega\\&=-\frac{i\pi}{(2\pi)^{n-2d+1}}b^d(\mathcal{E}b+\lambda-b\Lambda_q)^{(n-2d-2)/2}\int_{S^{n-2d-1}\times\mathbb{R}^{2d}}\hat{V}^{\parallel}(x_\perp,0)\mathrm{d}x_\perp\mathrm{d}\omega\\&=-\frac{i\pi\, mes(S^{n-2d-1})}{(2\pi)^{n-2d+1}}\int_{\mathbb{R}^n}V(x)\mathrm{d}x(b^{(n-2)/2}(\mathcal{E}-\Lambda_q)^{(n-2d-2)/2}+\mathcal{O}(b^{(n-4)/2})).\end{aligned}$$

Combining this equation with (28), we obtain the result claimed in Theorem 3.

The end of the paper is devoted to the proof of Theorem 4. We must show that for all $N\in\mathbb{N}^*$ and all $(l_1,\ldots,l_N)\in\mathbb{N}^N,(k_1,\ldots,k_N)\in\mathbb{N}^N$,

$$\mathrm{tr}(T^{k_1}_{q,l_1}\cdots T^{k_N}_{q,l_N})$$

admits an asymptotic expansion in powers of $b^{1/2}$. For this purpose, we work directly on the kernel of these operators that we expand with respect to $b$. For $V\in\mathcal{S}(\mathbb{R})$, let us denote $\hat{V}$ its Fourier transform. The two next lemmas permit us to obtain an expansion of the kernel of $T^{k_1}_{q,l_1}\cdots T^{k_N}_{q,l_N}$ by mean of the expansion of each term of the product.

*Lemma 4.2. Let* $V_1$, $V_2\in\mathcal{S}(\mathbb{R})$ *and for* $\omega$, $\omega'\in\{\pm1\}$, $\lambda>0$ *let*

$$W(\lambda,\omega,\omega') = \sum_{\theta\in\{\pm 1\}} \hat{V}_1(\sqrt{\lambda}(\omega-\theta))\hat{V}_2(\sqrt{\lambda}(\theta-\omega')).$$

*Then, there exists* $V\in\mathcal{S}(\mathbb{R})$ *such that*

$$W(\lambda,\omega,\omega') = \hat{V}(\sqrt{\lambda}(\omega-\omega')) + \mathcal{O}(\lambda^{-\infty}),$$

*when* $\lambda\to+\infty$.

*Proof:* From the properties of the Fourier transform, it is clear that

$$W(\lambda,\omega,\omega') = \begin{cases} (\int V_1(x)\mathrm{d}x)(\int V_2(x)\mathrm{d}x) + \mathcal{O}(\lambda^{-\infty}) & \text{if} \quad \omega=\omega', \\ \mathcal{O}(\lambda^{-\infty}) \quad \text{if} \quad \omega\neq\omega'. \end{cases}$$

Let us set

$$V(x) = \left(\int V_2(y)\mathrm{d}y\right)V_1(x),$$

then it is clear that

$$W(\lambda,\omega,\omega') = \hat{V}(\sqrt{\lambda}(\omega-\omega')) + \mathcal{O}(\lambda^{-\infty}).$$

■

*Lemma 4.3: Let* $V\in\mathcal{S}(\mathbb{R})$ *and for* $l\in\mathbb{N}^*$, $\omega,\omega'\in\{\pm 1\}$, $\lambda>0$ *let*

$$W(\lambda,\omega,\omega') = \int e^{i\sqrt{\lambda}(|x_1-x_2|+\ldots+|x_l-x_{l+1}|+x_{l+1}\omega-x_1\omega')}V(x_1)\cdots V(x_{l+1})\mathrm{d}x_1\cdots\mathrm{d}x_{l+1}.$$

*Then, there exists a sequence* $(V_j)_{j\in\mathbb{N}}$ *of potentials in* $\mathcal{S}(\mathbb{R})$ *such that*

$$W(\lambda,\omega,\omega') = \sum_{j=0}^{+\infty}(i\sqrt{\lambda})^{-j}\hat{V}_j(\sqrt{\lambda}(\omega-\omega')).$$

*Proof:* The integral being absolutely convergent, we have

$$W(\lambda,\omega,\omega') = \int_{\mathbb{R}} e^{i\sqrt{\lambda}(\omega-\omega')y}V(y)\tilde{V}(y)\mathrm{d}y,$$

with

$$\tilde{V}(y) = \int_{\mathbb{R}} e^{i\sqrt{\lambda}(|x|-x\omega')}V(x+y)\mathrm{d}x.$$

Moreover,

$$\begin{aligned}\tilde{V}(y) &= \int_0^{+\infty} V(\omega' x+y)\mathrm{d}x + \int_{-\infty}^0 e^{-2i\sqrt{\lambda}x}V(\omega' x+y)\mathrm{d}x \\ &= V_0(y) - \frac{1}{2i\sqrt{\lambda}}[e^{-2i\sqrt{\lambda}x}V(\omega' x+y)]_{x=-\infty}^{x=0} + \frac{\omega'}{2i\sqrt{\lambda}}\int_{-\infty}^0 e^{-2i\sqrt{\lambda}x}V'(\omega' x+y)\mathrm{d}y \\ &= \tilde{V}_0(y) + \lambda^{-1/2}\tilde{V}_1(y) + \frac{\omega'}{2i\sqrt{\lambda}}\int_{-\infty}^0 e^{-2i\sqrt{\lambda}x}V'(\omega' x+y)\mathrm{d}y,\end{aligned}$$

with $\tilde{V}_0(y)=\int_0^{+\infty}V(\omega' x+y)\mathrm{d}x$ and $\tilde{V}_1(y)=(i/2)V(y)$. In particular, $\tilde{V}_0$ and $\tilde{V}_1$ are $C^\infty$ functions whose derivatives are bounded at all orders. Integrating by parts $N$ times, we obtain

$$\tilde{V}(y) = \tilde{V}_0(y) + \sum_{j=1}^{N} \lambda^{-j/2} \frac{\omega'^{j-1}}{(2i)^j} V^{(j-1)}(y) + \mathcal{O}(\lambda^{-N-1}).$$

Let us set $\tilde{V}_j(y)=[\omega'^{j-1}/(2i)^j]V^{(j-1)}(y)$, then

$$W(\lambda,\omega,\omega') = \sum_{j=0}^{N} \lambda^{-j/2} \int_{\mathbb{R}} e^{i\sqrt{\lambda}(\omega-\omega')y} V(y)\tilde{V}_j(y)\mathrm{d}y + \mathcal{O}(\lambda^{-N-1}) = \sum_{j=0}^{N} \lambda^{-j/2} \hat{V}_j(\sqrt{\lambda}(\omega-\omega')) + \mathcal{O}(\lambda^{-N-1}),$$

with $V_j(y)=V(y)\tilde{V}_j(y)$. As $V\in\mathcal{S}(\mathbb{R})$ and for $j\geqslant 0$, $\tilde{V}_j$ and their derivatives are bounded, it is clear that $V_j\in\mathcal{S}(\mathbb{R})$ and the proof is complete. ■

Now, we give the proof of Theorem 4. Thanks to Eqs. (26) and (27), it suffices to prove that for all $N\in\mathbb{N}^*$ and all $l=(l_1,\ldots,l_N)\in\mathbb{N}^N$, $k=(k_1,\ldots,k_N)\in\mathbb{N}^N$,

$$\mathrm{tr}(T_{q,l_1}^{k_1}\cdots T_{q,l_N}^{k_N})$$

admits an asymptotic expansion in powers of $b^{1/2}$. For this purpose, we will simply show that the kernel of $T_{q,l_1}^{k_1}\cdots T_{q,l_N}^{k_N}$ admits such an expansion. Let us start with $T_{q,l_j}^{k_j}$, $j\in\{1,\ldots,N\}$. Recall that

$$T_{q,l_j}(\mathcal{E}b+\lambda,b) = (-1)^{l_j+1}2i\pi\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)(V(x)R_0(\mathcal{E}b+\lambda+i0))^{l_j}V(x)\mathcal{F}_{q,0}(\mathcal{E}b+\lambda)^*.$$

Moreover, it is well known (see Ref. 11) that for $E>0$, the resolvent $[-(\mathrm{d}^2/\mathrm{d}x^2)-E-i0]^{-1}$ has a kernel $N_0(x,y)$ given by

$$N_0(x,y) = \frac{1}{2i\sqrt{E}}e^{i\sqrt{E}|x-y|}.$$

Therefore, the kernel of $T_{q,l_j}$ takes the form

$$N_{q,l_j}(\omega,\omega',x_\perp,x'_\perp) = \frac{(-1)^{l_j+1}}{(2i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q})^{l_j+1}}K_q(x_\perp,x'_\perp)\int e^{i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(|x_1-x_2|+\cdots+|x_{l_j}-x_{l_j+1}|+x_{l_j+1}\omega-x_1\omega')}$$
$$\times V(x_\perp,x_1)\cdots V(x_\perp,x_{l_j+1})\mathrm{d}x_1\cdots\mathrm{d}x_{l_j+1}. \quad (29)$$

By Lemma 4.3 applied in the variable $x_\parallel$, we obtain the following expansion:

$$N_{q,l_j}(\omega,\omega',x_\perp,x'_\perp) = K_q(x_\perp,x'_\perp)(i\sqrt{b})^{-l_j-1}\sum_{m=0}^{+\infty}(i\sqrt{b})^{-m}\hat{V}^{\parallel}_{m,q,l_j}(x_\perp,\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(\omega-\omega')),$$

with $V_{m,q,lj}\in\mathbf{S}(\mathbb{R}^3)$. Using Lemma 4.2, it comes that the kernel $N_{q,l_j,k_j}$ of $T_{q,l_j}^{k_j}$ has the expansion

$$N_{q,l_j,k_j}(\omega,\omega',x_\perp,x'_\perp) = K_q(x_\perp,x'_\perp)(i\sqrt{b})^{-l_j-1}\sum_{m=0}^{+\infty}(i\sqrt{b})^{-m}\hat{V}^{\parallel}_{m,q,l,k}(x_\perp,\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(\omega-\omega')),$$

with $V_{m,q,l_j,k_j}\in\mathbf{S}(\mathbb{R}^3)$. Next, using again Lemma 4.2, it follows by induction that $T_{q,l_1}^{k_1}\cdots T_{q,l_N}^{k_N}$ has a kernel $N_{q,l,k}(\omega,\omega',x_\perp,x'_\perp)$ which admits an expansion in powers of $ib^{-1/2}$,

$$N_{q,l,k}(\omega,\omega',x_\perp,x'_\perp) = (i\sqrt{b})^{-|l|-N}\sum_{m=0}^{+\infty}(i\sqrt{b})^{-m}\hat{V}^{\parallel}_{m,q,l,k}(x_\perp,\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(\omega-\omega'))K_q(x_\perp,x'_\perp),$$

where $|l|:=l_1+\cdots+l_N$. Hence, we get

$$\mathrm{tr}(T_{q,l_1}^{k_1}\dots T_{q,l_N}^{k_N}) = \sum_{\omega=\pm 1}\int_{\mathbb{R}^2} N_{q,l,k}(\omega,\omega,x_\perp,x_\perp)\mathrm{d}x_\perp = 2(i\sqrt{b})^{-|l|-N}\sum_{m=0}^{+\infty}(i\sqrt{b})^{-m}\int_{\mathbb{R}^2}\hat{V}^{\|}_{m,q,l,k}(x_\perp,0)\mathrm{d}x_\perp$$

$$= 2(i\sqrt{b})^{-|l|-N}\sum_{m=0}^{+\infty}(i\sqrt{b})^{-m}\int_{\mathbb{R}^3} V_{m,q,l,k}(x)\mathrm{d}x$$

and the proof of Theorem 4 is almost complete. Indeed, we have shown that there exists a sequence $(\alpha_j)_{j\in\mathbb{N}}$ of real numbers such that

$$s(\mathcal{E}b+\lambda,b) = \frac{1}{2i\pi}b^{(n-2)/2}\sum_{j=0}^{+\infty}\alpha_j(\mathcal{E},\lambda)(i\sqrt{b})^{-j}.$$

Hence, we must prove that for $j\in\mathbb{N}$, $\alpha_{2j}=0$. For this purpose, let us remark that $S(\lambda,b)$ being unitary, $s(\mathcal{E}b+\lambda,b)$ is real valued. Therefore, the coefficients $\alpha_{2j}$, $j\in\mathbb{N}$ vanish and the proof of expansion (11) is complete.

It remains to compute the coefficients $a_0$ and $a_1$. From Eqs. (26) and (27) and Lemma 4.1 we deduce that

$$s(\mathcal{E}b+\lambda,b) = \frac{1}{2i\pi}\sum_{q=0}^{[(\mathcal{E}-1)/2]}\sum_{k=0}^{2}\frac{(-1)^k}{k+1}\mathrm{tr}\left(\sum_{l=0}^{2}T_{q,l}(\mathcal{E}b+\lambda,b)\right)^{k+1} + \mathcal{O}(b^{-1}).$$

Using again Lemma 4.1, we obtain

$$s(\mathcal{E}b+\lambda,b) = \frac{1}{2i\pi}\sum_{q=0}^{[(\mathcal{E}-1)/2]}\left(\mathrm{tr}\ T_{q,0} + \mathrm{tr}\ T_{q,1} + \mathrm{tr}\ T_{q,2} - \frac{1}{2}\mathrm{tr}\ T_{q,0}^2 - \mathrm{tr}\ T_{q,0}T_{q,1} + \frac{1}{3}\mathrm{tr}\ T_{q,0}^3\right) + \mathcal{O}(b^{-1}), \tag{30}$$

and we must compute all the terms of the sum. From the proof of Theorem 3 with $n=3$, $d=1$, we deduce that

$$\mathrm{tr}(T_{q,0}(\mathcal{E}b+\lambda,b)) = -\frac{2i\pi}{4\pi^2}\int_{\mathbb{R}^3}V(x)\mathrm{d}x\left(b^{\frac{1}{2}}(\mathcal{E}-2q-1)^{-\frac{1}{2}} - \frac{\lambda}{2}b^{-\frac{1}{2}}(\mathcal{E}-2q-1)^{-\frac{3}{2}}\right) + \mathcal{O}\left(b^{-\frac{3}{2}}\right).$$

By similar computations, we prove that

$$\mathrm{tr}(T_{q,0}^2(\mathcal{E}b+\lambda,b)) = -\frac{1}{4\pi(\mathcal{E}-2q-1)}\left(\int V(x)\mathrm{d}x\right)^2 + \mathcal{O}(b^{-1})$$

and

$$\mathrm{tr}(T_{q,0}^3(\mathcal{E}b+\lambda,b)) = \frac{i(\mathcal{E}-2q-1)^{-3/2}}{8\pi}b^{-1/2}\left(\int V(x)\mathrm{d}x\right)^3 + \mathcal{O}(b^{-1}).$$

Let us compute $\mathrm{tr}(T_{q,1})$. It follows from Eq. (29) that

$$\mathrm{tr}(T_{q,1}(\mathcal{E}b+\lambda,b)) = \frac{-b}{8\pi(\mathcal{E}b+\lambda-b\Lambda_q)}\sum_{\omega=\pm 1}\int e^{i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(|x_1-x_2|+\omega(x_1-x_2))}V(x_\perp,x_1)V(x_\perp,x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_\perp$$

$$= \frac{-1}{8\pi(\mathcal{E}-2q-1)}\int(1+e^{2i\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}(|x_1-x_2|)})V(x_\perp,x_1)V(x_\perp,x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_\perp$$

$$+\mathcal{O}(b^{-1}) = \frac{-1}{8\pi(\mathcal{E}-2q-1)}\left(\int V(x)\mathrm{d}x\right)^2$$

$$-\frac{2}{8\pi(\mathcal{E}-2q-1)}\int_{x_1\leqslant x_2} e^{2i\sqrt{\mathcal{E}b+\lambda-1\Lambda_q}(x_2-x_1)}V(x_\perp,x_1)V(x_\perp,x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_\perp$$
$$+\mathcal{O}(b^{-1}).$$

Integrating by parts with respect to $x_1$, we obtain

$$\mathrm{tr}(T_{q,1}(\mathcal{E}b+\lambda,b))=\frac{-1}{8\pi(\mathcal{E}-2q-1)}\left(\left(\int V(x)\mathrm{d}x\right)^2+\frac{i}{\sqrt{\mathcal{E}b+\lambda-b\Lambda_q}}\int V(x)^2\mathrm{d}x\right)+\mathcal{O}(b^{-1})$$
$$=\frac{-1}{8\pi(\mathcal{E}-2q-1)}\left(\left(\int V(x)\mathrm{d}x\right)^2-\frac{ib^{-1/2}}{8\pi(\mathcal{E}-2q-1)^{3/2}}\int V(x)^2\mathrm{d}x\right)+\mathcal{O}(b^{-1}).$$

The computations of $\mathrm{tr}(T_{q,2})$ and $\mathrm{tr}(T_{q,0}T_{q,1})$ are similar to the preceding ones. We find

$$\mathrm{tr}(T_{q,0}(\mathcal{E}b+\lambda,b)T_{q,1}(\mathcal{E}b+\lambda,b))=\frac{i(\mathcal{E}-2q-1)^{-3/2}}{16\pi}b^{-1/2}\left(\int V(x)\mathrm{d}x\right)^3+\mathcal{O}(b^{-1})$$

and

$$\mathrm{tr}(T_{q,2}(\mathcal{E}b+\lambda,b))=\frac{i(\mathcal{E}-2q-1)^{-3/2}}{48\pi}b^{-1/2}\left(\int V(x)\mathrm{d}x\right)^3+\mathcal{O}(b^{-1}).$$

Combining these equations with (30), we obtain

$$s(\mathcal{E}b+\lambda,b)=-\frac{\gamma_0(\mathcal{E})}{4\pi^2}b^{1/2}\left(\int V(x)\mathrm{d}x\right)+\frac{\lambda\gamma_1(\mathcal{E})}{8\pi^2}b^{-1/2}\left(\int V(x)\mathrm{d}x\right)$$
$$-\frac{\gamma_1(\mathcal{E})}{16\pi^2}b^{-1/2}\left(\int V(x)^2\mathrm{d}x\right)+\mathcal{O}(b^{-1})$$

with

$$\gamma_j(\mathcal{E})=\sum_{q=0}^{[(\mathcal{E}-1)/2]}(\mathcal{E}-2q-1)^{-1/2-j}.$$

This completes the proof of Theorem 4. ■

To conclude, let us notice that Theorem 4 could be generalized to the case $n-2d>1$ by using stationary phase method in the variable $x_\parallel$. Nevertheless, there are some difficulties due to degenerate phases.


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