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Semi-classical estimate of the residues of the scattering amplitude for long-range potentials

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Abstract
In this paper, we study the residue of the scattering amplitude for the Schrödinger operator with long-range perturbation of the Laplacian, in the case where there are resonances exponentially close to the real axis. If the resonances are simple and under a separation condition, one proves that the residue of the scattering amplitude associated with a resonance ξ is bounded by $C(h)|\text{Im}\,\xi|$. Here $C(h)$ denotes an explicit constant depending polynomially on h^{-1} and the number of resonances in a fixed box. This generalizes a recent result of Stefanov concerning compactly supported perturbations and isolated resonances.

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1. Introduction

The aim of this paper is to study the residues of the scattering amplitude for the semi-classical Schrödinger operator, in the case where there are resonances exponentially close to the real axis. This problem was treated by Lahmar-Benbernou and Martinez [9, 10] in the particular case of a 'well in a island' with non-degenerate local minimum. Under the assumptions specified in [10], they proved that the residue $f_\xi^{\text{res}}(\theta, \omega, h)$ of the scattering amplitude $f(\theta, \omega, \lambda, h)$ which is associated with a pole ξ satisfies

$$f_\xi^{\text{res}}(\theta, \omega, h) = \mathcal{O}(h^N)|\text{Im}\,\xi|$$

for some fixed N. More recently, Stefanov [18] examined the general situation of black-box compactly supported perturbations of the Laplacian. In this paper, Stefanov deals with the case where $z_0(h)$ is a simple isolated resonance of $P(h)$. Then, for $(\omega, \theta) \in S^{n-1} \times S^{n-1}$, one can write the scattering amplitude $f(\theta, \omega, \lambda, h)$ near $z_0(h)$ as

$$f(\theta, \omega, \lambda, h) = \frac{f^{\text{res}}(\theta, \omega, h)}{z - z_0(h)} + f^{\text{hol}}(\theta, \omega, z, h) \tag{1.1}$$

where $f^{\mathrm{hol}}(\theta, \omega, z, h)$ is holomorphic near $z_0(h)$. Under some additional hypotheses, Stefanov proved that

$$|f^{\mathrm{res}}(\theta, \omega, h)| \leqslant Ch^{-\frac{n-1}{2}}|\mathrm{Im}\, z_0(h)| \quad \text{and} \quad |f^{\mathrm{hol}}(\theta, \omega, z, h)| \leqslant Ch^{-\frac{n-1}{2}} \text{ for } z \text{ close to } z_0(h).$$

In this paper, we will show that these estimates still hold in a more general setting. In particular, we extend the result of Stefanov to the case of long-range perturbations and domains containing many resonances.

Let us now state the problem more precisely. Consider the Schrödinger operator $P(h) = -\frac{1}{2}h^2\Delta + V$, in $\mathbb{R}^n$, $n \geqslant 2$, $0 < h \leqslant 1$. The potential $V(x)$ is assumed to satisfy the following condition for some $\rho > 0$.

Assumption (V)$_\rho$. V is a real C^∞-smooth function such that

$$\forall \alpha \in \mathbb{N}^n \qquad \forall x \in \mathbb{R}^n \qquad \left|\partial_x^\alpha V(x)\right| \leqslant C_\alpha \langle x\rangle^{-\rho-|\alpha|} \qquad \text{where} \quad \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}.$$

The operator $P(h)$ with domain $D(P(h)) = H^2(\mathbb{R}^n)$ is self-adjoint in $L^2(\mathbb{R}^n)$. We can define the scattering matrix $S(\lambda, h)$ related to $P_0(h) = -\frac{1}{2}h^2\Delta$ and $P(h)$, as a unitary operator:

$$S(\lambda, h) : L^2(S^{n-1}) \longrightarrow L^2(S^{n-1}).$$

Next, introduce the operator $T(\lambda, h)$ by $S(\lambda, h) = Id - 2\mathrm{i}\pi T(\lambda\ h)$. It is well known (see [7]) that $T(\lambda, h)$ has a kernel $T(\theta, \omega, \lambda, h)$, smooth in $(\theta, \omega) \in S^{n-1} \times S^{n-1}\backslash\{\theta = \omega\}$ and the scattering amplitude is given by

$$f(\theta, \omega, \lambda, h) = c(\lambda, h)T(\theta, \omega, \lambda, h)$$

with

$$c(\lambda, h) = -2\pi(2\lambda)^{-\frac{n-1}{4}}(2\pi h)^{\frac{n-1}{2}}\,\mathrm{e}^{-\mathrm{i}\frac{(n-3)\pi}{4}}. \tag{1.2}$$

Moreover, in [7], Isozaki and Kitada gave a representation formula that we will recall in the next section. In [4], Gérard and Martinez used this representation formula to prove that the scattering amplitude has a meromorphic continuation, from the lower half-plane to a conic neighbourhood of the real axis. This continuation, which we will explain in the next section, was established for $\theta \neq \omega$ and under the following hypothesis.

Assumption (Hol$_\infty$). We assume that there exist $\theta_0 \in [0, \pi[$ and $R > 0$ such that the potential V extends holomorphically to the domain

$$D_{R,\theta_0} = \{z \in \mathbb{C}^n;\ |z| > R, |\mathrm{Im}\, z| \leqslant \tan\theta_0 |\mathrm{Re}\, z|\}$$

and

$$\exists \beta > 0 \quad \exists C > 0 \quad \forall x \in D_{R,\theta_o} \qquad |V(x)| \leqslant C|x|^{-\beta}.$$

Let us note that this hypothesis allows also the resonances to be defined by complex scaling (see [14, 15]). Near the real axis, the resonances coinciding with the poles of the scattering amplitude and the multiplicity are the same. We will denote by $\mathrm{Res}(P(h))$ the set of resonances of $P(h)$ lying in $\{\mathrm{Im}\, z < 0\}$.

Now, we will formulate our statement on the resonances. Let $E_1(h), E_2(h)$ be such that, $\forall h \in]0, 1]$, $0 < L^{-1} < E_1(h) \leqslant E_2(h) \leqslant L < +\infty$ where $L \gg 1$ is constant independent of h. Assume that $\omega(h), S(h) > 0$ satisfy

$$\lim_{h\to 0} \omega(h) = 0 \qquad \text{and} \qquad S(h) \leqslant h^{\frac{3n+5}{2}}\omega(h). \tag{1.3}$$

Let us set

$$\Omega_0(h) = \{z \in \mathbb{C};\ E_1(h) - \omega(h) \leqslant \mathrm{Re}\, z \leqslant E_2(h) + \omega(h), 0 \leqslant -\mathrm{Im}\, z \leqslant S(h)\}. \tag{1.4}$$

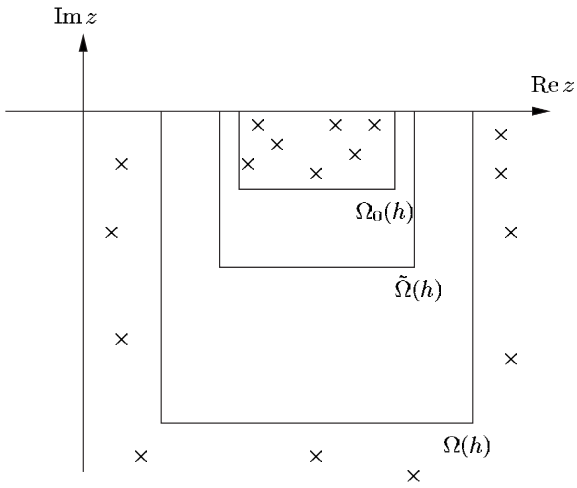

Figure 1. Isolated resonances.

We will say that a resonance is simple, if it is a simple pole of the scattering amplitude. Until the end of this paper, we will assume that each $\xi \in \Omega_0(h) \cap \mathrm{Res}(P(h))$ is a simple resonance and we denote

$$\Lambda(h) = \Omega_0(h) \cap \mathrm{Res}(P(h)) \text{ and } K(h) = \sharp\Lambda(h).$$

We will also assume that the set of resonances $\Lambda(h)$ is isolated in the sense that

$$\mathrm{Res}(P(h)) \cap (\Omega(h)\backslash\Omega_0(h)) = \emptyset \tag{1.5}$$

where

$$\Omega(h) = \{z \in \mathbb{C};\ E_1(h) - 7\omega(h) \leqslant \mathrm{Re}\, z \leqslant E_2(h) + 7\omega(h), 0 \leqslant -\mathrm{Im}\, z \leqslant 4h^{-n-2}S(h)\}. \tag{1.6}$$

Let us note that if $\omega(h)$ satisfies $0 < \omega(h) < h^{n+\alpha}$ with $\alpha > 0$, then $E_1(h)$ and $E_2(h)$ can be chosen so that

$$\mathrm{Res}(P(h)) \cap ([E_1 - 7\omega, E_2 + 7\omega] + \mathrm{i}[0, -S(h)]) = \mathrm{Res}(P(h)) \cap \Omega_0(h). \tag{1.7}$$

This is a direct consequence of the fact that

$$\sharp(\mathrm{Res}(P(h)) \cap ([L^{-1}, L] + \mathrm{i}[-h^{-n-2}S(h), 0])) = \mathcal{O}(h^{-n})$$

which comes from the trace formula proved in [14, 15]. Then, to ensure that (1.5) holds, it suffices to prove that

$$\mathrm{Res}(P(h)) \cap ([E_1 - 7\omega, E_2 + 7\omega] + \mathrm{i}[-S(h), -4S(h)h^{-n-2}]) = \emptyset.$$

We will explain further how this can be done in some special situations.

Under the above assumptions, the scattering amplitude takes the form

$$f(\theta, \omega, z, h) = \sum_{\xi \in \Lambda(h)} \frac{f_\xi^{\mathrm{res}}(\theta, \omega, h)}{z - \xi} + f^{\mathrm{hol}}(\theta, \omega, z, h) \tag{1.8}$$

where $f^{\mathrm{hol}}(\theta, \omega, z, h)$ is holomorphic in $\Omega(h)$ (see figure 1). Our aim is to estimate the residues $f_\xi^{\mathrm{res}}(\theta, \omega, h)$ and the holomorphic part $f^{\mathrm{hol}}(\theta, \omega, z, h)$. For this purpose, we need a separation assumption on the resonances of $P(h)$. We will suppose that there exists $\epsilon > 0$ such that the following condition is satisfied.

Assumption ($\mathbf{Sep}_\epsilon$). For all $\xi, \xi' \in \Omega_0(h) \cap \mathrm{Res}(P(h))$ with $\xi \neq \xi'$, we have

$$|\xi - \xi'| \geqslant \epsilon S(h).$$

Now, we are in a position to announce the main result of this paper.

Theorem 1. *Assume that the potential V satisfies hypotheses $(\mathbf{V})_\rho$ with $\rho > 0$, $(\mathbf{Hol}_\infty)$ and $(\mathbf{Sep}_\epsilon)$ with $\epsilon > 0$. Assume that all the resonances in $\Omega_0(h)$ are simple and that $\mathrm{Res}(P(h)) \cap (\Omega(h)\backslash\Omega_0(h)) = \emptyset$. Let $(\theta, \omega) \in S^{n-1} \times S^{n-1}$ with $\theta \neq \omega$. Then, there exist $C_\epsilon > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$, we have*

$$\left|f_\xi^{\mathrm{res}}(\theta, \omega, h)\right| \leqslant C_\epsilon h^{-\frac{n-1}{2}} K(h)^{\frac{24}{\epsilon^2}} |\mathrm{Im}\,\xi| \qquad \forall \xi \in \Lambda(h)$$

$$|f^{\mathrm{hol}}(\theta, \omega, z, h)| \leqslant C_\epsilon h^{-\frac{n-1}{2}} K(h)^{\frac{24}{\epsilon^2}} \log(1 + K(h)) \qquad \forall z \in \tilde{\Omega}(h)$$

where

$$\tilde{\Omega}(h) = \{z \in \mathbb{C};\ E_1(h) - \omega(h) \leqslant \mathrm{Re}\, z \leqslant E_2(h) + \omega(h), 0 \leqslant -\mathrm{Im}\, z \leqslant 2S(h)\}.$$

Let us make a comparison between our result and theorem 1 in [18]. First, our theorem holds for long-range potentials whereas Stefanov's result is proved for compactly supported perturbations of the Laplacian. This creates some difficulties due to the fact that, in the long-range case, we do not have some simple representation formula for f.

The second important difference concerns the density of resonances that we deal with. In [18], it is assumed that $z_0(h)$ is the only resonance in $\Omega(h)$. Here we consider the case where the number $K(h)$ of resonances is larger than one. As $K(h)$ may behave like h^{-n} when h goes to 0, our aim is to prove that the bound on the residues depends polynomially on $K(h)$, while it is easier to obtain a bound depending exponentially on $K(h)$.

Let us note that our result cannot be obtained as a direct consequence of Stefanov's. Indeed, one could try to cover $\Omega(h)$ with some boxes containing only one resonance and to apply Stefanov's theorem on each box. If one follows this approach, one has to make a separation assumption necessary to apply Stefanov's estimate. Roughly speaking, one has to suppose $(\mathbf{Sep}_\epsilon)$ with $\epsilon = h^{-\frac{3n+4}{2}}$ so that the hypotheses become more restrictive than in theorem 1.

Now, let us make some comments on the term $K(h)$. It is easy to deduce from the trace formula proved in [14, 15] that there exists $\tilde{n} \in \mathbb{N}$ such that $K(h) = \mathcal{O}(h^{-\tilde{n}})$. Therefore, theorem 1 yields

$$\left|f_\xi^{\mathrm{res}}(\theta, \omega, h)\right| \leqslant C_\epsilon h^{-n_\epsilon} |\mathrm{Im}\,\xi| \qquad \forall \xi \in \Lambda(h)$$

$$|f^{\mathrm{hol}}(\theta, \omega, z, h)| \leqslant C_\epsilon h^{-1-n_\epsilon} \qquad \forall z \in \tilde{\Omega}(h)$$

with $n_\epsilon \in \mathbb{N}$. In particular $|f^{\mathrm{hol}}|$ and $\left|f_\xi^{\mathrm{res}}\right| \big/ |\mathrm{Im}\,\xi|$ are polynomially bounded with respect to h^{-1}. If we assume additionally that the number $K(h)$ is bounded with respect to h, theorem 1 shows that $|f^{\mathrm{hol}}|$ and $\left|f_\xi^{\mathrm{res}}\right| \big/ |\mathrm{Im}\,\xi|$ are bounded by $Ch^{-\frac{n-1}{2}}$. Therefore, the bound found by Stefanov in the case $K(h) = 1$ is available in the case where $K(h)$ is bounded.

In conclusion, let us discuss briefly the existence of the Breit–Wigner formula for the scattering amplitude. Starting from formula (1.8) and differentiating with respect to z, one obtains

$$\partial_z f(\theta, \omega, z, h) = -\sum_{\xi \in \Lambda(h)} \frac{f_\xi^{\mathrm{res}}(\theta, \omega, h)}{(z - \xi)^2} + \partial_z f^{\mathrm{hol}}(\theta, \omega, z, h).$$

Introducing the term Im ξ in this formula we get

$$\partial_z f(\theta, \omega, z, h) = \sum_{\xi\in\Lambda(h)} c(\xi, h)\frac{-\mathrm{Im}\,\xi}{|z-\xi|^2} + \partial_z f^{\mathrm{hol}}(\theta, \omega, z, h)$$

where $|c(\xi, h)| = \frac{|f_\xi^{\mathrm{res}}(\theta,\omega,h)|}{|\mathrm{Im}\,\xi|} \leqslant Ch^{-\frac{n-1}{2}}$. Moreover, the term $\partial_z f^{\mathrm{hol}}$ can be estimated by using theorem 1 and Cauchy's formula. In particular, if $S(h) \geqslant Ch^M$ for some $C, M > 0$, we obtain

$$\partial_z f(\theta, \omega, z, h) = \sum_{\xi\in\Lambda(h)} c(\xi, h)\frac{-\mathrm{Im}\,\xi}{|z-\xi|^2} + \mathcal{O}(h^{-N})$$

where N is a positive constant. In the case where $\Lambda(h) = \{\xi_0(h)\}$ one obtains

$$\partial_z f(\theta, \omega, z, h) = c(\xi_0, h)\frac{-\mathrm{Im}\,\xi_0}{|z-\xi_0|^2} + \mathcal{O}(h^{-N})$$

with $c(\xi_0, h) = \mathcal{O}\big(h^{-\frac{n-1}{2}}\big)$. Therefore, we will obtain a Breit–Wigner formula, if we can bound the coefficient $c(\xi_0, h)$ from below. In the general case, it is not sufficient to prove a lower bound for the coefficients $c(\xi, h)$. Indeed, we do not control the argument of these complex numbers and there could be some cancellation between different terms of the sum. This is a difficult open problem.

We finish this introduction by giving some examples of potentials satisfying the assumptions of theorem 1.

Example 1. We consider the case of a 'well in a island'. For some fixed energy λ, the potential $V(x)$ is assumed to satisfy

$$\{x \in \mathbb{R}^n;\, V(x) > \lambda\} = U\backslash\{x_0\}$$

where U is bounded and connected and x_0 is a point of U. It is also required that $V''(x_0)$ is positive definite. More precisely, we assume that after a symplectic change of coordinate, the symbol $\sigma_P(x, \xi)$ of $P(h)$ can be written as

$$\sigma_P(x, \xi) = \sum_{j=1}^{n} \frac{\lambda_j}{2}\left(\xi_j^2 + x_j^2\right) + \mathcal{O}((x, \xi)^3)$$

where the λ_j are strictly positive and linearly independent of $\mathbb{Z}$. In that case, for all $\alpha > 0$ and $\delta > 0$, the form of the resonance of $P(h)$ in $\mathcal{O}_{\alpha,\delta}(h) = [\lambda, \lambda + \alpha h] - \mathrm{i}[0, \delta]$ is well known (see [5, 8, 13]). In that situation, we are in a position to verify all the hypotheses required in theorem 1. First, we know from [8] that the resonance $\xi(h) \in \mathrm{Res}(P(h)) \cap \mathcal{O}_{\alpha,\delta}(h)$ have the following expansion:

$$\xi(h) = \lambda + h\sum_{j=1}^{n}\left(k_j + \frac{1}{2}\right)\lambda_j + \mathcal{O}(h^2) \tag{1.9}$$

with $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $|k| \leqslant C$. Moreover, we know from theorem 10.11 in [5] that there exists $S_0 > 0$ such that

$$\forall \xi \in \mathrm{Res}(P(h)) \cap \mathcal{O}_{\alpha,\delta}(h) \qquad |\mathrm{Im}\,\xi| = \mathcal{O}(\mathrm{e}^{-S_0/h}). \tag{1.10}$$

Denoting $m = \inf\big\{\big|\sum_{j=1}^n \lambda_j k_j\big|;\, k \in \mathbb{Z}^n, |k| \leqslant C\big\} > 0$, we deduce from (1.9) that if $\xi \neq \xi'$ are two resonances in $\mathcal{O}_{\alpha,\delta}(h)$ we have

$$|\xi - \xi'| \geqslant h\left|\sum_{j=1}^{n}(k_j - k_j')\lambda_j\right| - \mathcal{O}(h^2) \geqslant mh - \mathcal{O}(h^2) \geqslant Ch. \tag{1.11}$$

Now, let us set $\omega(h) = h^{n+1}$ and $S(h) = h^{\frac{3n+5}{2}}\omega(h)$. As was noted before assumption (**Sep**$_\epsilon$), we can choose $\lambda + 7\omega(h) < E_1(h) < E_2(h) < \lambda + \alpha h - 7\omega(h)$ such that

$$\mathrm{Res}(P(h)) \cap ([E_1 - 7\omega, E_1] - \mathrm{i}[0, \delta]) = \emptyset$$

and

$$\mathrm{Res}(P(h)) \cap ([E_2, E_2 + 7\omega] - \mathrm{i}[0, \delta]) = \emptyset.$$

Combining these properties and (1.10), it follows that $\Omega(h)$ and $\Omega_0(h)$ defined by (1.6) and (1.4) satisfy

$$\Omega(h) \subset \mathcal{O}_{\alpha,\delta}(h) \qquad \text{and} \qquad \mathrm{Res}(P(h)) \cap \Omega(h) \subset \Omega_0(h).$$

Moreover, it follows from (1.11) that for all $\epsilon > 0$ (**Sep**$_\epsilon$) is verified with $S(h)$ as above, so that we have verified all the hypotheses required in theorem 1. Finally, we note that in the present case, the number $K(h)$ is bounded with respect to h. This is not true in general and in the following example, we describe such a situation.

Example 2. For $a > 0$, let $\phi_a \in C_0^\infty(\mathbb{R}^n)$ be such that $\Phi_a(x) = 1$ for $|x| \leqslant 2a$. Let $b > 0$, $y_0 \in \mathbb{R}^n$ and set

$$V(x) = \Phi - a(x - y_0)(|x - y_0|^2 + b).$$

In that situation, it is shown in [1] (cf the example following theorem 6) that

$$\begin{aligned} &\forall\lambda \in]b, b + a^2[\qquad \exists C_\lambda, \delta_\lambda > 0 \\ &\sharp\mathrm{Res}(P(h)) \cap ([\lambda - \delta_\lambda h, \lambda + \delta_\lambda h] - \mathrm{i}[0, \delta_\lambda h]) \geqslant C_\lambda h^{1-n}. \end{aligned} \tag{1.12}$$

Now, we fix two energy levels $b < E_0 < E_3 < b + a^2$. Denoting $\sigma_P(x, \xi) = \frac{1}{2}|\xi|^2 + V(x)$ the symbol of the operator $P(h)$, we assume that E_0 and E_3 are no-critical values of σ_P. Denoting W_{ext} as the unbounded connected component of $\sigma_P^{-1}([E_0, E_3])$, we assume that all points in W_{ext} are non-trapping in the sense of [12]. Under the above assumptions, Stefanov proved in [16] that for all $M > 0$, there exists a function $0 < \alpha(h) = \mathcal{O}(h^\infty)$ such that for h small enough

$$\mathrm{Res}(P(h)) \cap ([E_0, E_3] + \mathrm{i}[-Mh, -\alpha(h)]) = \emptyset. \tag{1.13}$$

Moreover, we have seen that if we set $\omega(h) = h^{n+\alpha}, \alpha > 0$ and $0 < S(h) < h^{\frac{3n+5}{2}}\omega(h)$, we can choose $E_0 < E_1(h) < E_2(h) < E_3$ such that $|E_1(h) - E_2(h)| \geqslant \frac{E_3 - E_0}{2}$ and (1.7) holds. Combining (1.13) and (1.7), assumption (1.5) is immediately satisfied (see figure 2).

On the other hand, if we assume that (**Sep**$_\epsilon$) is satisfied and that the resonances are simple then we can apply theorem 1 to get

$$\left|f_\xi^{\mathrm{res}}(\theta, \omega, h)\right| \leqslant C_\epsilon h^{-\frac{n-1}{2}} K(h)^{\frac{24}{\epsilon^2}} |\mathrm{Im}\,\xi| \qquad \forall\xi \in \Lambda(h).$$

To conclude, let us note that combining (1.13) and (1.12), it comes easily that $K(h) \geqslant Ch^{1-n}$. Therefore, the estimate $K(h) \leqslant Ch^{-n}$ is almost sharp and it follows that

$$\left|f_\xi^{\mathrm{res}}(\theta, \omega, h)\right| \leqslant C_\epsilon h^{\frac{1}{2} - n(\frac{1}{2} + \frac{24}{\epsilon^2})} |\mathrm{Im}\,\xi| \qquad \forall\xi \in \Lambda(h).$$

In our analysis we deal with a representation formula for the scattering amplitude. In the next section, we recall the representation given by Isozaki and Kitada [7], for λ real and its extension to a conic neighbourhood of the real axis due to Gérard and Martinez [4].

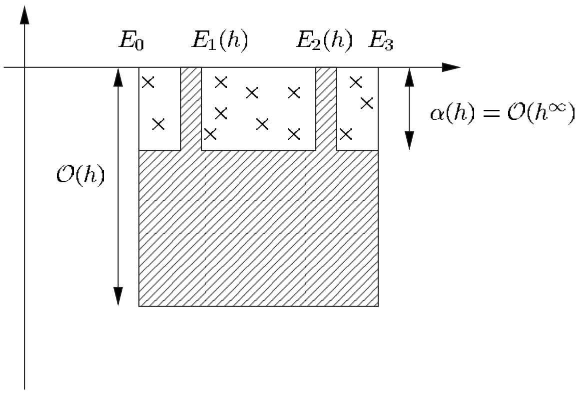

Figure 2. Resonances associated with a non-trapping potential outside a bounded region.

2. Review on the representation formula and the meromorphic continuation of $T(\theta, \omega, \lambda, h)$

2.1. The formula of Isozaki–Kitada

The first step towards the proof of theorem 1 is to establish a representation formula for $T(\theta, \omega, \lambda, h)$ in the long-range case. Such a formula has been obtained in [7] and it was used in [12] to prove an asymptotic expansion of the scattering amplitude in the non-trapping case with $\rho > 1$. We begin with some notation.

Definition 1. *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. For $m, u \in \mathbb{R}$ and $k \in \mathbb{Z}$, we denote by $A_k^{m,u}(\Omega)$ the class of symbols $a(x, \xi, h)$ such that $(x, \xi) \mapsto a(x, \xi, h)$ belongs to $C^\infty(\Omega)$ and*

$$\forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n \qquad \exists C > 0 \qquad \forall (x, \xi) \in \Omega \qquad \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \leqslant C h^k \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{u-|\beta|}$$

and set $A_k^{m,\infty}(\Omega) = \bigcap_{u\in\mathbb{R}} A_k^{m,u}(\Omega)$. In the case where $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, we will write $A_k^{m,u}$ instead of $A_k^{m,u}(\Omega)$.

We also use the incoming and outgoing subsets of the phase space having the form

$$\Gamma_\pm(R, d, \sigma) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : |x| > R, d^{-1} < |\xi| < d, \pm\cos(x, \xi) > \pm\sigma\}$$

for $R > 1$, $d > 1$ and $\sigma \in]-1, 1[$, where $\cos(x, \xi) = \frac{\langle x,\xi\rangle}{|x||\xi|}$. For $\alpha > \frac{1}{2}$, introduce $F_0(\lambda, h) : L^2_\alpha(\mathbb{R}^n) \longrightarrow L^2(S^{n-1})$, by

$$(F_0(\lambda, h) f)(\omega) = c_0(\lambda, h) \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}h^{-1}\sqrt{2\lambda}\langle x,\omega\rangle} f(x)\, \mathrm{d}x \qquad \lambda > 0.$$

The idea of Isozaki and Kitada was to approximate the wave operators by Fourier integral operators $I_h(a_\pm, \Phi_\pm)$ with phases $\Phi_\pm$ and symbols $a_\pm$. Formally, with

$$I_h(a_\pm, \Phi_\pm)(f)(x) = (2\pi h)^{-n} \iint \exp(\mathrm{i}h^{-1}(\Phi_\pm(x, \xi) - \langle y, \xi\rangle)) a_\pm(x, \xi) f(y)\, \mathrm{d}y\, \mathrm{d}\xi$$

the phases $\Phi_\pm$ have to solve the eikonal equation

$$\tfrac{1}{2}|\nabla_x \Phi_\pm(x, \xi)|^2 + V(x) = \tfrac{1}{2}|\xi|^2$$

and the symbols $a_\pm$ are the solution to

$$\left(-\tfrac{1}{2}h^2\Delta + V(x) - \tfrac{1}{2}|\xi|^2\right)\left(a_\pm\, \mathrm{e}^{\mathrm{i}h^{-1}\Phi_\pm}\right) \sim 0. \qquad (2.1)$$

Let $R_0 \gg 1$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$ and $0 < \sigma_2^- < \sigma_1^- < \sigma_0^- < \sigma_0^+ < \sigma_1^+ < \sigma_2^+ < 1$. Denote $\tau_j^\pm = -\sigma_j^\mp$ for $j = 0, 1, 2$, so that we have also $-1 < \tau_2^- < \tau_1^- < \tau_0^- < \tau_0^+ < \tau_1^+ < \tau_2^+ < 0$. According to proposition 2.4 of [6], we can find real C^∞ smooth functions $\Phi_{\pm a}$ satisfying the following properties:

(φ1) $\Phi_{\pm a}(x,\xi)$ solves the eikonal equation $\frac{1}{2}|\nabla_x\Phi_{\pm a}(x,\xi)|^2 + V(x) = \frac{1}{2}|\xi|^2$ in $\Gamma_\pm\big(R_0, d_0, \tau_0^\pm\big)$.

(φ2) $\Phi_{\pm a}(x,\xi) - \langle x,\xi\rangle$ belongs to $A_0^{\epsilon,0}$ for all $\epsilon > 0$.

(φ3) For all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$, $\left|\frac{\partial^2\Phi_{\pm a}}{\partial_{x_j}\partial_{\xi_k}}(x,\xi) - \delta_{jk}\right| < \epsilon(R_0)$, δ_{jk} being the Kronecker symbol, where $\epsilon(R_0)$ can be made as small as we wish by taking R_0 large enough.

Next, we determine $a_\pm$ in the form

$$a_\pm(x,\xi,h) = \sum_{j\geqslant 0} a_{\pm j}(x,\xi)h^j.$$

Replacing $a_\pm$ by this expansion in (2.1) and identifying the power of h, we obtain the following transport equations:

$$\begin{cases}\langle\nabla_x\Phi_{\pm a}, \nabla_x a_{\pm 0}\rangle + \frac{1}{2}\Delta_x\Phi_{\pm a}a_{\pm 0} = 0 \\ \langle\nabla_x\Phi_{\pm a}, \nabla_x a_{\pm j}\rangle + \frac{1}{2}\Delta_x\Phi_{\pm a}a_{\pm j} = \frac{\mathrm{i}}{2}\Delta_x a_{\pm j-1} \qquad j\geqslant 1\end{cases} \tag{2.2}$$

with the conditions at infinity

$$a_{\pm 0}\to 1 \qquad \text{and} \qquad a_{\pm j}\to 0 \qquad j\geqslant 1 \quad \text{as} \quad |x|\to 0. \tag{2.3}$$

These equations are solved by the standard characteristic curve method (see [6, 7, 12]) and finally, we find some symbols $a_{\pm j}$ such that: (s0) $a_{\pm j}$ belongs to $A_0^{-j,-\infty}$. (s1) $\mathrm{supp}(a_{\pm j}) \subset \Gamma_\pm\big(3R_0, d_1, \tau_1^\pm\big)$. (s2) $a_{\pm j}$ solves equation (2.2) with (2.3) in $\Gamma_\pm\big(4R_0, d_2, \tau_2^\pm\big)$. (s3) $a_{\pm j}$ solves equation (2.2) in $\Gamma_\pm\big(4R_0, d_1, \tau_2^\pm\big)$. Now, fix an integer N large enough (to be chosen in the following) and set $a_\pm(x,\xi,h) = \sum_{j=0}^N a_{\pm j}(x,\xi)h^j \in A_0^{0,-\infty}$. Then the operator $J_{\pm a}(h) = I_h(a_\pm, \Phi_{\pm a})$ is well defined and the operator $K_{\pm a}$ given by $K_{\pm a} = P(h)J_{\pm a} - J_{\pm a}P_0(h)$ is also a F.I.O. In fact, $K_{\pm a} = I_h(k_{\pm a}, \Phi_{\pm a})$ with

$$k_{\pm a} = \mathrm{e}^{-\mathrm{i}h^{-1}\Phi_\pm}\left(-\tfrac{1}{2}h^2\Delta + V(x) - \tfrac{1}{2}|\xi|^2\right)\left(\mathrm{e}^{\mathrm{i}h^{-1}\Phi_\pm}a_\pm\right).$$

It follows that the symbol $k_{\pm a}$ has the following properties: (k0) $k_{\pm a}$ belongs to $A_1^{-1,-\infty}$. (k1) $\mathrm{supp}(k_{\pm a}) \subset \Gamma_\pm\big(3R_0, d_1, \tau_1^\pm\big)$. (k2) $k_{\pm a}$ belongs to $A_{N+2}^{-(N+2),-\infty}\big(\Gamma_\pm\big(4R_0, d_1, \tau_2^\pm\big)\big)$.

Similarly, we define $J_{\pm b} = I_h(b_\pm, \Phi_{\pm b})$ for the region $\Gamma_\pm\big(5R_0, d_3, \sigma_1^\pm\big)$. First, we define the phase functions $\Phi_{\pm b}\in\mathcal{C}^\infty(\mathbb{R}^{2n})$ verifying (φ1) in $\Gamma_\pm\big(R_0, d_0, \sigma_0^\pm\big)$, ($\varphi$2) and ($\varphi$3). Next, we define a symbol

$$b_\pm(x,\xi,h) = \sum_{j=0}^N b_{\pm j}(x,\xi)h^j$$

satisfying (s0), (s1) for the region $\Gamma_\pm\big(5R_0, d_3, \sigma_1^\pm\big)$, (s2) for $\Gamma_\pm\big(6R_0, d_4, \sigma_2^\pm\big)$ and (s3) for $\Gamma_\pm\big(6R_0, d_3, \sigma_2^\pm\big)$. Using the same arguments as above, we define $K_{\pm b}(h) = P(h)J_{\pm b}(h) - J_{\pm b}(h)P_0(h) = I_h(k_{\pm b}, \Phi_{\pm b})$, with

$$k_{\pm b} = \mathrm{e}^{-\mathrm{i}h^{-1}\Phi_{\pm b}}\left(-\tfrac{1}{2}h^2\Delta + V(x) - \tfrac{1}{2}|\xi|^2\right)\left(\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm b}}b_\pm\right). \tag{2.4}$$

Then $k_{\pm b}$ satisfies (k0), (k1) for $\Gamma_\pm\big(5R_0, d_3, \sigma_1^\pm\big)$ and (k2) for $\Gamma_\pm\big(6R_0, d_3, \sigma_2^\pm\big)$. Now, the Isozaki–Kitada formula is stated in the following proposition.

Proposition 1 (Isozaki–Kitada [7]). *For* $\lambda \in \left]\frac{d_4^{-2}}{2}, \frac{d_4^2}{2}\right[$, *we have*

$$T(\lambda, h) = T_1(\lambda, h) - T_2(\lambda, h) \tag{2.5}$$

with

$$T_1(\lambda, h) = F_0(\lambda, h)(J_{+a}^*(h) + J_{-a}^*(h))(K_{+b}(h) + K_{-b}(h))F_0^*(\lambda, h) \tag{2.6}$$

and

$$T_2(\lambda, h) = F_0(\lambda, h)(K_{+a}^*(h) + K_{-a}^*(h))R(\lambda + \mathrm{i}0)(K_{+b}(h) + K_{-b}(h))F_0^*(\lambda, h). \tag{2.7}$$

In formula (2.7), $R(\lambda + \mathrm{i}0)$ is the limit of the resolvent on the real line. More precisely, let us denote $R(z) = (P(h) - z)^{-1}, z \in \mathbb{C}\backslash\mathbb{R}$ the resolvent of $P(h)$, then $R(\lambda \pm \mathrm{i}0) = \lim_{\epsilon\to 0, \epsilon>0} R(\lambda \pm \mathrm{i}\epsilon)$. Here we take the limit in the spaces of bounded operators $\mathcal{L}\big(L_\alpha^2, L_{-\alpha}^2\big), \alpha > \frac{1}{2}$ with $L_\alpha^2 = \{f : \langle x\rangle^\alpha f \in L^2(\mathbb{R}^n)\}$ and for $\alpha, \beta \in \mathbb{R}$, $\|\cdot\|_{\alpha,\beta}$ is the natural norm on $\mathcal{L}\big(L_\alpha^2, L_\beta^2\big)$.

Using this formula and a resolvent estimate proved by Burq [2] and improved by Vodev [22] and Cardoso–Vodev [3], it was proved in [11] that the scattering amplitude is polynomially bounded with respect to h. More precisely, one has the following theorem.

Theorem 2. *Fix an energy* $\lambda > 0$ *and assume that the potential* V *satisfies* $(\mathbf{V})_\rho$ *with* $\rho > 0$ *and* $(\mathbf{Hol}_\infty)$. *Then we have*

$$\forall(\omega, \theta) \in S^{n-1} \times S^{n-1}\backslash\{\theta = \omega\} \qquad f(\theta, \omega, \lambda, h) = \mathcal{O}\big(h^{-\frac{n-1}{2}}\big). \tag{2.8}$$

Let us remark that this result is not exactly the same as in [11], where it is assumed that $\rho > 1$. Nevertheless, it is not hard to verify that the proof given in [11], still works in the case $\rho > 0$.

2.2. Meromorphic continuation of the scattering amplitude and estimates for complex energies

Here, we recall briefly how Gérard and Martinez [4] extend the formula of Isozaki and Kitada to a conic neighbourhood of the real axis in the complex plane. Starting from this formula, we establish some estimates of the scattering amplitude in a conic neighbourhood of $\mathbb{R}_+^*$. Let us begin with some notation. For $R > 0$ large enough, $d > 0$, $\epsilon > 0$ and $\sigma \in]0, 1[$, we denote

$$\begin{aligned}\Gamma^\pm(R, d, \epsilon, \sigma) = \{&(x, \xi) \in \mathbb{C}^{2n}; |\mathrm{Re}\, x| > R, d^{-1} < |\mathrm{Re}\,\xi| < d,\\ &\pm \cos(\mathrm{Re}\, x, \mathrm{Re}\,\xi) \geqslant \pm\sigma, |\mathrm{Im}\, x| \leqslant \epsilon\langle \mathrm{Re}\, x\rangle, |\mathrm{Im}\,\xi| \leqslant \epsilon\langle \mathrm{Re}\,\xi\rangle\}.\end{aligned}$$

From propositions 2.1 and 3.1 in [4], we deduce that the phases $\Phi_{\pm a}$, $\Phi_{\pm b}$ and the symbols $a_\pm$ and $b_\pm$ can be constructed so that the following propositions hold.

Proposition 2. *For each* $\epsilon > 0$, *there exists* $R_0 > 0$ *such that the phase function* $\Phi_{\pm a}$ *(resp.* $\Phi_{\pm b}$*) has a holomorphic continuation in* $\Gamma^\pm\big(R_0, d_0, \epsilon, \tau_0^\pm\big)$ *(resp.* $\Gamma^\pm(R_0, d_0, \epsilon, \sigma_0^\pm)$*) and satisfies*

$$(\nabla_x \Phi_\pm(x, \xi))^2 + V(x) = \xi^2 \qquad \Phi_\pm(x, \xi) - \langle x, \xi\rangle = \mathcal{O}(\langle x\rangle + \langle\xi\rangle)^{1-\rho}\langle\xi\rangle^{-1}$$

uniformly in $\Gamma^\pm\big(R_0, d_0, \epsilon, \tau_0^\pm\big)$ *(resp.* $\Gamma^\pm(R_0, d_0, \epsilon, \sigma_0^\pm)$*).*

Proposition 3. *For* $R_0 > 0$ *large enough and* $\epsilon > 0$ *small enough, there exists* $\alpha > 0$ *such that* $a_\pm$ *has an extension to* $\Gamma^\pm\big(3R_0, d_1, \epsilon, \tau_1^\pm\big)$ *which is holomorphic in* $\Gamma^\pm\big(4R_0, d_2, \epsilon, \tau_2^\pm\big)$.

Moreover, $a_\pm(x,\xi,h)$ *is bounded uniformly with respect to* $(x,\xi) \in \Gamma^\pm\big(3R_0, d_1, \epsilon, \tau_1^\pm\big)$, $h \in]0,1]$ *and we have the following estimates:*

$$\begin{aligned} &a_\pm(x,\xi,h) = 1 + \mathcal{O}(\langle x\rangle^{-\rho}) \\ &k_{\pm a}(x,\xi,h) = \mathrm{e}^{-\mathrm{i}h^{-1}\Phi_\pm(x,\xi)}\left(P(h) - \tfrac{1}{2}\xi^2\right)\left(\mathrm{e}^{\mathrm{i}h^{-1}\Phi_\pm(x,\xi)}a_\pm(x,\xi,h)\right) = \mathcal{O}(\mathrm{e}^{-\alpha\langle x\rangle\langle\xi\rangle/h}) \end{aligned} \tag{2.9}$$

uniformly with respect to $h \in]0,1]$ *and* $(x,\xi) \in \Gamma^\pm\big(4R_0, d_2, \epsilon, \tau_2^\pm\big)$*. Similarly, the preceding statement is true for the symbol* $b_\pm$ *and the domains* $\Gamma^\pm\big(5R_0, d_3, \epsilon, \sigma_1^\pm\big)$, $\Gamma^\pm\big(6R_0, d_4, \epsilon, \sigma_2^\pm\big)$ *respectively.*

Now, using proposition 1, we can write the scattering matrix as

$$S(\lambda,h) = c(\lambda,h)(T_1(\lambda,h) - T_2(\lambda,h))$$

where T_1 and T_2 are given by (2.6), (2.7) and are associated with our new symbols. Denote by $T_1(\theta,\omega,\lambda,h)$ the kernel of $T_1(\lambda,h)$ and by $T_2(\theta,\omega,\lambda,h)$ the kernel of $T_2(\lambda,h)$. Let us set

$$\psi_{\pm b}^{\pm a}(x,\theta,\omega) = \Phi_{\pm b}(x,\sqrt{2\lambda}\omega) - \Phi_{\pm a}(x,\sqrt{2\lambda}\theta).$$

It is easy to see that for $\lambda > 0$ we have

$$T_1(\theta,\omega,\lambda,h) = \big(T_{1,+b}^{+a} + T_{1,-b}^{+a} + T_{1,+b}^{-a} + T_{1,-b}^{-a}\big)(\theta,\omega,\lambda,h) \tag{2.10}$$

with

$$T_{1,\pm b}^{\pm a}(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2 \int \mathrm{e}^{\mathrm{i}h^{-1}\psi_{\pm b}^{\pm a}(x,\theta,\omega)} k_{\pm b}(x,\sqrt{2\lambda}\omega)\bar{a}_\pm(x,\sqrt{2\lambda}\theta)\,\mathrm{d}x \tag{2.11}$$

and

$$T_2(\theta,\omega,\lambda,h) = \big(T_{2,+b}^{+a} + T_{2,-b}^{+a} + T_{2,+b}^{-a} + T_{2,-b}^{-a}\big)(\theta,\omega,\lambda,h) \tag{2.12}$$

with

$$\begin{aligned} T_{2,\pm b}^{\pm a}(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2\big\langle &R(\lambda+\mathrm{i}0)k_{\pm b}(\cdot,\sqrt{2\lambda}\omega)\,\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm b}(\cdot,\sqrt{2\lambda}\omega)}, \\ &k_{\pm a}(\cdot,\sqrt{2\lambda}\theta)\,\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm a}(\cdot,\sqrt{2\lambda}\theta)}\big\rangle. \end{aligned} \tag{2.13}$$

At the end of this section we will explain how we can extend the previous expression for complex energies. As can be easily seen, in the above expressions of T_1 and T_2, it is natural to use the analytic continuation of the symbols involved in these formulae. Moreover, to extend the term T_2, it is essential to holomorphically continue the resolvent to complex energies. This is done by complex scaling, using hypothesis ($\mathbf{Hol}_\infty$). We do not recall here the construction of the complex scaled operator (see [14, 15]), we just give the main properties of this operator. For $\mu_0 > 0$ small enough $\epsilon_0 > 0$ and $0 < \mu < \mu_0$, there exists $f_\mu : \mathbb{R}^+ \to \mathbb{C}$ which is injective for every μ and satisfies the following properties:

(i) $f_\mu(t) = t$ for $0 \leqslant t \leqslant 7R_0$,
(ii) $0 \leqslant \arg f_\mu(t) \leqslant \mu$ and $\partial_t f_\mu(t) \neq 0\ \forall t$
(iii) $\arg f_\mu(t) \leqslant \arg \partial_t f_\mu(t) \leqslant \arg f_\mu(t) + \epsilon_0$
(iv) $\arg f_\mu(t) = \mathrm{e}^{\mathrm{i}\mu}t$, for $t \geqslant 8R_0$.

Denoting by κ_μ the map given by

$$\kappa_\mu : \mathbb{R}^n \ni x = t\omega \longmapsto f_\mu(t)\omega \qquad t = |x|$$

one defines $\Gamma_\mu = \kappa_\mu(\mathbb{R}^n)$ and $U_\mu : L^2(\mathbb{R}^n) \to L^2(\Gamma_\mu)$ by $U_\mu\varphi(x) = J_\mu(x)\varphi(\kappa_\mu(x))$ where $J_\mu(x)$ is the Jacobian associated with the transformation κ_μ. Next, we define the modified operator by $P_\mu(h) = U_\mu P(h) U_\mu^{-1}$. This is an unbounded non self-adjoint operator on $L^2(\Gamma_\mu)$ and the resonances of $P(h)$ are exactly the eigenvalues of any $P_\mu(h)$. Moreover, the resolvent

$(P_\mu - \lambda)^{-1}$ has a meromorphic continuation to $\{\lambda;\ |\mathrm{Im}\,\lambda| \leqslant \mu\langle \mathrm{Re}\,\lambda\rangle\}$. Using estimates (2.9) for $k_{\pm a}$ and $k_{\pm b}$ and the properties of the phases $\Phi_{\pm a}$, $\Phi_{\pm b}$, it is easy to show that there exists $\epsilon_1 > 0$ such that for $\mathrm{Im}\,\lambda > 0$, we have

$$U_\mu\big(\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm b}(x,\sqrt{2\lambda}\omega)}k_{\pm b}(x,\sqrt{2\lambda}\omega)\big) = \mathcal{O}(\mathrm{e}^{-\epsilon_1\langle x\rangle/h}) \tag{2.14}$$

uniformly with respect to $|x| \geqslant 6R_0$, $\omega \in S^{n-1}$, $h \in]0,1]$ and $|\mathrm{Im}\,\lambda| \leqslant \mu\langle \mathrm{Re}\,\lambda\rangle$. Similarly, if we denote by $U_{-\mu}$ the operator associated with the conjugate deformation $\bar{f}_\mu$, then for all $|x| \geqslant 4R_0$, $\omega \in S^{n-1}$, $h \in]0,1]$ and $|\mathrm{Im}\,\lambda| \leqslant \mu\langle \mathrm{Re}\,\lambda\rangle$, we have

$$U_{-\mu}\big(\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm a}(x,\sqrt{2\lambda}\theta)}k_{\pm a}(x,\sqrt{2\lambda}\theta)\big) = \mathcal{O}(\mathrm{e}^{-\epsilon_2\langle x\rangle/h}) \tag{2.15}$$

where ϵ_2 is a strictly positive constant. Therefore, using the analyticity of these quantities with respect to μ, it is not hard to prove that

$$\begin{aligned} T^{\pm a}_{2,\pm b}(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2\big\langle R_\mu(\lambda,h)U_\mu\big(k_{\pm b}(\cdot,\sqrt{2\lambda}\omega)\,\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm b}(\cdot,\sqrt{2\lambda}\omega)}\big),\\ U_{-\mu}\big(k_{\pm a}(\cdot,\sqrt{2\lambda}\theta)\,\mathrm{e}^{\mathrm{i}h^{-1}\Phi_{\pm a}(\cdot,\sqrt{2\lambda}\theta)}\big)\big\rangle \end{aligned} \tag{2.16}$$

for $\lambda > 0$, where $R_\mu(\lambda,h) = (P_\mu(h) - \lambda)^{-1}$ is the resolvent of the modified operator. For $\mu > 0$ fixed, Sjöstrand [15] showed that $R_\mu(\lambda,h)$ is analytic in the region $\{\mathrm{Im}\,\lambda > 0\}$ and is meromorphic in the sector $\mathrm{e}^{-\mathrm{i}[0,\mu]}]0,+\infty[$. By definition, the resonances of $P(h)$ are the poles of $R_\mu(\lambda,h)$. It follows from (2.16) that the poles of $T_2(\theta,\omega,\lambda,h)$ coincide with the resonances of $P(h)$.

The next step is to extend $T^{\pm a}_{1,\pm b}$ to complex energies. We need to extend $T^{\pm a}_{1,\pm b}$ as a function, so that we do not have to recall the general construction of [4]. More precisely, we work in the case where $\omega,\theta \in S^{n-1}$ are fixed and $\omega \neq \theta$. As mentioned in [4], we can choose the parameters $\sigma_2^\pm$ sufficiently close to 1 and $\delta > 0$ small enough, such that

$$\forall y \in \mathbb{R}^n \qquad \cos(y,\omega) \geqslant \sigma_2^- - \delta \quad\Longrightarrow\quad \frac{\langle y,\omega-\theta\rangle}{|y|} \geqslant 2\alpha > 0. \tag{2.17}$$

We will use this property at the end of the demonstration, but for the moment we simply recall that for $\lambda \in \mathbb{R}_+^*$, $T^{\pm a}_{1,\pm b}(\theta,\omega,\lambda,h)$ is given by

$$T^{\pm a}_{1,\pm b}(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2\int \mathrm{e}^{\mathrm{i}h^{-1}(\sqrt{2\lambda}\langle\omega-\theta,x\rangle + r(x,\lambda))}k_{\pm b}(x,\sqrt{2\lambda}\omega)\bar{a}_\pm(x,\sqrt{2\lambda}\theta)\,\mathrm{d}x$$

where $r(x,\lambda) = r^{\pm a}_{\pm b}(x,\lambda) = \mathcal{O}(\langle x\rangle^{1-\rho}\langle\sqrt{\lambda}\rangle^{1-\rho})$. Working as in [4], we can split $T^{2,\pm a}_{1,\pm b}(\theta,\omega,\lambda,h)$ into the sum of two terms

$$T^{\pm a}_{1,\pm b}(\theta,\omega,\lambda,h) = f_1(\theta,\omega,\lambda,h) + f_2(\theta,\omega,\lambda,h)$$

where f_1 is given by

$$f_1(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2\int_{|x|\leqslant 6R_0} \mathrm{e}^{\mathrm{i}h^{-1}(\sqrt{2\lambda}\langle\omega-\theta,x\rangle + r(x,\lambda))}k_{\pm b}(x,\sqrt{2\lambda}\omega)\bar{a}_\pm(x,\sqrt{2\lambda}\theta)\,\mathrm{d}x. \tag{2.18}$$

Using propositions 2 and 3, it is obvious that the functions $(r,\rho) \mapsto k_{\pm b}(rx,\rho\omega)\bar{a}_\pm(rx,\rho\theta)$ are holomorphic with respect to $r \in \{|r| \geqslant 5R_0\} \cap \{|\mathrm{Im}\,r| \leqslant \epsilon\langle\mathrm{Re}\,r\rangle\}$ and $\rho \in \big\{d_2^{-1} \leqslant |\rho| \leqslant d_2\big\} \cap \{|\mathrm{Im}\,\rho| \leqslant \epsilon\langle\mathrm{Re}\,\rho\rangle\}$. Hence, we obtain that f_1 has a holomorphic continuation to

$$\Lambda_{d_2,\epsilon} = \left\{\lambda \in \mathbb{C};\ |\mathrm{Im}\,\lambda| \leqslant \epsilon\langle\mathrm{Re}\,\lambda\rangle,\ \frac{d_2^{-2}}{2} \leqslant |\mathrm{Re}\,\lambda| \leqslant \frac{d_2^2}{2}\right\}.$$

Moreover, for $\lambda \in \Lambda_{d_2,\epsilon}$ we have $\frac{d_2}{\sqrt{2\lambda}} \geqslant 1$ and we can write $f_2 = f_3 + f_4$ with

$$f_3(\theta,\omega,\lambda,h) = c_0(\lambda,h)^2\int_{6R_0\leqslant|x|\leqslant\frac{7R_0d_2}{\sqrt{2\lambda}}} \mathrm{e}^{\mathrm{i}h^{-1}(\sqrt{2\lambda}\langle\omega-\theta,x\rangle + r(x,\lambda))}k_{\pm b}(x,\sqrt{2\lambda}\omega)\bar{a}_\pm(x,\sqrt{2\lambda}\theta)\,\mathrm{d}x$$

which gives after a change of variables

$$f_3(\theta,\omega,\lambda,h)=\frac{c_0(\lambda,h)^2}{\lambda^{n/2}}\int_{6R_0\sqrt{2\lambda}\leqslant|y|\leqslant 7R_0d_2}\mathrm{e}^{\mathrm{i}h^{-1}(\langle\omega-\theta,y\rangle+r_\pm(y/\sqrt{2\lambda},\lambda))}\times k_{\pm b}\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda}\omega\right)\bar{a}_\pm\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda}\theta\right)\mathrm{d}y. \tag{2.19}$$

As in the case of f_1, this expression has a holomorphic continuation to the domain $\Lambda_{d_2,\epsilon}$ and it remains to examine

$$f_4(\theta,\omega,\lambda,h)=\frac{c_0(\lambda,h)^2}{\lambda^{n/2}}\int_{|y|\geqslant 7R_0d_2}\mathrm{e}^{\mathrm{i}h-1(\langle\omega-\theta,y\rangle+r_\pm(y/\sqrt{2\lambda,\lambda}))}\times k_{\pm b}\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda\omega}\right)\bar{a}_\pm\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda\theta}\right)\mathrm{d}y. \tag{2.20}$$

For this purpose, let us fix $\sigma_3^\pm$ such that $0<\sigma_2^- -\delta<\sigma_3^-<\sigma_2^-<\sigma_2^+<\sigma_3^+<1$, where δ is given by (2.17). We introduce a cut-off function χ_ω such that

$$\operatorname{supp}\chi_\omega\subset\left\{|y|\geqslant 7R_0d_2,\cos(y,\omega)\in\left[\sigma_3^-,\sigma_3^+\right]\right\}$$

and

$$\chi_\omega=1\text{ on}\left\{|y|\geqslant 8R_0d_2,\cos(y,\omega)\in\left[\sigma_2^-,\sigma_2^+\right]\right\}.$$

We define also

$$u(y,\lambda,\theta,\omega,h)=\mathrm{e}^{\mathrm{i}h^{-1}r(y/\sqrt{2\lambda},\lambda)}k_{\pm b}\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda}\omega\right)\bar{a}_\pm\left(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda}\theta\right)$$

and we decompose f_4 as $f_4=f_5+f_6$, with

$$f_5(\theta,\omega,\lambda,h)=\frac{c_0(\lambda,h)^2}{\lambda^{n/2}}\int(1-\chi_\omega)(y)\,\mathrm{e}^{\mathrm{i}h^{-1}\langle\omega-\theta,y\rangle}u(y,\lambda,\theta,\omega,h)\,\mathrm{d}y \tag{2.21}$$

and

$$f_6(\theta,\omega,\lambda,h)=\frac{c_0(\lambda,h)^2}{\lambda^{n/2}}\int\chi_\omega(y)\,\mathrm{e}^{\mathrm{i}h^{-1}\langle\omega-\theta,y\rangle}u(y,\lambda,\theta,\omega,h)\,\mathrm{d}y. \tag{2.22}$$

Using the fact that $k_{\pm b}(x,\xi)=\mathcal{O}(\mathrm{e}^{-\epsilon_2\langle x\rangle/h})$ for $\cos(\operatorname{Re}x,\operatorname{Re}\xi)\notin\left[\sigma_2^-,\sigma_2^+\right]$, we show easily that for $\epsilon_3,\epsilon>0$ small enough, $\lambda\in\Lambda_{d_2,\epsilon}$ and $y\in\operatorname{supp}(1-\chi_\omega)$, we have $k_{\pm b}\big(\frac{y}{\sqrt{2\lambda}},\sqrt{2\lambda}\omega\big)=\mathcal{O}(\mathrm{e}^{-\epsilon_3\langle x\rangle/h})$. Moreover, we deduce from proposition 3 that for $\lambda\in\Lambda_{d_2,\epsilon}$ and $y\in\operatorname{supp}(1-\chi_\omega)$ we have

$$|u(y,\lambda,\theta,\omega,h)|\leqslant C\,\mathrm{e}^{-\epsilon_3\langle y\rangle/h}\left|\mathrm{e}^{\mathrm{i}h^{-1}r_\pm(y/\sqrt{2\lambda},\lambda)}\right|\leqslant C\,\mathrm{e}^{-\epsilon_3\langle y\rangle/h+C\langle y\rangle^{1-\rho}/h}.$$

As $\rho>0$, we can take R_0 sufficiently large and ϵ_4 small enough so that

$$\forall\lambda\in\Lambda_{d_2,\epsilon}\qquad\forall y\in\operatorname{supp}(1-\chi_\omega)\qquad|u(y,\lambda,\theta,\omega,h)|\leqslant C\,\mathrm{e}^{-\epsilon_4\langle y\rangle/h}.$$

It follows immediately from this estimate that f_5 has a holomorphic continuation to $\Lambda_{d_2,\epsilon}$ and that

$$\forall\lambda\in\Lambda_{d_2,\epsilon},\ |f_5(\theta,\omega,\lambda,h)|\leqslant Ch^{-n-1}. \tag{2.23}$$

The continuation of f_6 is performed via a change of integration path in formula (2.22). Let χ_0 be a $\mathcal{C}^\infty$-smooth function with $\operatorname{supp}\chi_0\subset\{|y|\geqslant 9R_0d_2\}$ and $\chi_0=1$ on $\{|y|\geqslant 10R_0d_2\}$. For $\epsilon>0$, the new path of integration will be $L_{\epsilon,\chi_0}=\{1+\mathrm{i}\epsilon\chi_0(|y|),\,y\in\mathbb{R}^n\}$. Using (2.17), it is clear that for all $y\in\operatorname{supp}\chi_\omega$, $\langle y,\omega-\theta\rangle\geqslant\alpha|y|$. It follows, for ϵ sufficiently small and $y\in L_{\epsilon,\chi_0}$, that we have $\operatorname{Im}\langle y,\omega-\theta\rangle\geqslant\alpha|y|$ and then

$$\left|\mathrm{e}^{\mathrm{i}h^{-1}\langle y,\omega-\theta\rangle}\right|\leqslant\mathrm{e}^{-\alpha|y|/h}.$$

Therefore, the integral giving f_6 becomes absolutely convergent and we can easily extend f_6 holomorphically, to $\Lambda_{d_2,\epsilon}$, for $\epsilon > 0$ small enough, by

$$f_6(\theta, \omega, \lambda, h) = \frac{c_0(\lambda, h)^2}{\lambda^{n/2}} \int_{L_{\epsilon,\chi_0}} \chi_\omega(y)\, \mathrm{e}^{\mathrm{i}h^{-1}\langle \omega-\theta, y\rangle} u(y, \lambda, \theta, \omega, h)\, \mathrm{d}y. \quad (2.24)$$

Thus, we have extended the kernel $T^{\pm a}_{1,\pm b}(\theta, \omega, \lambda, h)$ to the domains $\Lambda_{d_2,\epsilon}$, for $\epsilon > 0$ small enough. Moreover the continuation can be decomposed into the sum

$$T^{\pm a}_{1,\pm b}(\theta, \omega, \lambda, h) = (f_1 + f_3 + f_5 + f_6)(\theta, \omega, \lambda, h) \quad (2.25)$$

where f_j, $j = 1, 3, 5, 6$ are given by (2.18), (2.19), (2.21) and (2.24) respectively. These formulae permit a bound for $T^{\pm a}_{1,\pm b}$ to be obtained for complex energies.

Proposition 4. *Let ω and θ be fixed in S^{n-1} with $\theta \neq \omega$. Then, there exist $\epsilon_0, h_0 > 0$ and $C > 0$ such that for all $0 < \epsilon < \epsilon_0$ and λ satisfying $|\mathrm{Im}\,\lambda| \leqslant \epsilon\langle \mathrm{Re}\,\lambda\rangle$, $\frac{d_2^{-2}}{2} \leqslant |\mathrm{Re}\,\lambda| \leqslant \frac{d_2^2}{2}$, we have*

$$\forall 0 < h < h_0 \qquad \left|T^{\pm a}_{1,\pm b}(\theta, \omega, \lambda, h)\right| \leqslant C\, \mathrm{e}^{C/h}.$$

Proof. We have just shown that $T^{\pm a}_{1,\pm b} = f_1 + f_3 + f_5 + f_6$, so that we have to control each f_j. We begin by the analysis of f_1. In the following, C will denote a positive constant that may change from line to line. For $\lambda \in \Lambda_{d_2,\epsilon}$, we deduce from equation (2.18) that

$$|f_1(\theta, \omega, \lambda, h)| \leqslant Ch^{-n} \sup_{|y|\leqslant 6R_0} |k_{\pm b}(y, \sqrt{2\lambda}\omega)\bar{a}_\pm(y, \sqrt{2\lambda}\theta)|$$
$$\times \int_{|x|\leqslant 6R_0} \mathrm{e}^{h^{-1}(\mathrm{Im}(\sqrt{2\lambda})|\omega-\theta||x| - |r(x,\lambda)|)}\, \mathrm{d}x.$$

Using the fact that $r(x, \lambda) = \mathcal{O}(\langle x\rangle^{1-\rho}\langle\sqrt{\lambda}\rangle^{1-\rho})$, we obtain for R_0 sufficiently large

$$\forall\lambda \in \Lambda_{d_2,\epsilon} \qquad |f_1(\theta, \omega, \lambda, h)| \leqslant Ch^{-n}\, \mathrm{e}^{C/h} \leqslant C\, \mathrm{e}^{C/h}. \quad (2.26)$$

The case of f_3 is similar and we use the fact that after integration over a compact set we get

$$\forall\lambda \in \Lambda_{d_2,\epsilon} \qquad |f_3(\theta, \omega, \lambda, h)| \leqslant Ch^{-n}\, \mathrm{e}^{C/h} \leqslant C\, \mathrm{e}^{C/h}. \quad (2.27)$$

The estimate of f_5 has already been obtained in (2.23) and treating f_6 remains. By the definition of χ_ω, there exists $\alpha > 0$ such that

$$\left|\chi_\omega(y, \omega)\, \mathrm{e}^{\mathrm{i}h^{-1}\langle y, \omega-\theta\rangle}\right| \leqslant \mathrm{e}^{-\alpha|y|/h}.$$

Moreover, using the definition of $r_\pm$ and proposition 3, we can choose R_0 large enough so that $u(y, \lambda, \theta, \omega, h) \leqslant \mathrm{e}^{\alpha|y|/2h}$. Hence, we deduce from (2.24) that

$$\forall\lambda \in \Lambda_{d_2,\epsilon}, |f_6(\theta, \omega, \lambda, h)| \leqslant Ch^{-n} \int \mathrm{e}^{-\alpha|y|/2h}\, \mathrm{d}y \leqslant Ch^{-n-1}. \quad (2.28)$$

Combining equations (2.26), (2.27), (2.23) and (2.28) we obtain the result. □

3. Residues' estimate

The aim of this section is to prove theorem 1. As in [18], we apply the semi-classical maximum principle to a well-chosen function.

3.1. Preliminary estimates of an auxiliary function

As a preparation, we introduce the following function. For z in $\Omega(h)$, we set

$$F(z,h) = \left(\prod_{\xi\in\Lambda(h)} \frac{z-\xi}{z-\bar{\xi}}\right) f(\theta,\omega,z,h). \tag{3.1}$$

Following [18], we apply the semi-classical maximum principle to this function. The latter was originally proved by Tang and Zworski [20, 21], generalizing lemma 1 in [19]. The following lemma is a refined version of this principle, due to Stefanov [17].

Lemma 1. *For $0 < h < 1$, let $a(h) \leqslant b(h)$. Suppose that $G(z,h)$ is a holomorphic function of z defined in a neighbourhood of*

$$U(h) = [a(h) - 5\omega(h), b(h) + 5\omega(h)] + \mathrm{i}[-S(h)h^{-n-2}, 0]$$

where $0 < S(h) \leqslant \omega(h)h^{\frac{3n+5}{2}}$ and $\omega(h) \to 0$ as $h \to 0$. Assume that $F(z,h)$ satisfies

$$|G(z,h)| \leqslant A\exp(Ah^{-n-1}\log(1/h)) \qquad \text{on} \quad U(h) \tag{3.2}$$

$$|G(z,h)| \leqslant M(h) \qquad \text{on} \quad [a(h) - 6\omega(h), b(h) + 6\omega(h)] \tag{3.3}$$

with $M(h) \to +\infty$ when $h \to 0$. Then, there exists $h_0 > 0$ such that

$$|G(z,h)| \leqslant 2\,\mathrm{e}^3 M(h) \qquad \forall z \in \tilde{U}(h) := [a(h) - \omega(h), b(h) + \omega(h)] + \mathrm{i}[-S(h), 0]$$

for $0 < h < h_0$.

Using this lemma, we can prove the main result of this section which is stated in the following proposition.

Proposition 5. *Under the hypotheses of theorem 1, we can find $h_0 > 0$ small enough and $C > 0$ such that*

$$\forall h \in\,]0, h_0] \qquad \forall z \in \tilde{U}(h) \qquad |F(z,h)| \leqslant Ch^{-\frac{n-1}{2}} \tag{3.4}$$

where

$$\tilde{U}(h) = [E_1(h) - 2\omega(h), E_2(h) + 2\omega(h)] + \mathrm{i}[-2S(h), 0].$$

To prove this proposition we will show that the function $F(z,h)$ satisfies the estimates (3.3) and (3.2). For this purpose, we need to control the norm of the modified resolvent $(P_\mu(h) - z)^{-1}$ near the poles $\xi \in \Lambda(h)$.

Lemma 2. *Under the hypotheses of theorem 1, we can find $\mu_0 > 0$, $h_0 > 0$ small enough and $C > 0$ such that for all $\mu < \mu_0$, $0 < h < h_0$ and $z \in \Omega_{\frac{3}{4}}(h)$ we have*

$$\left\| \left(\prod_{\xi\in\Lambda(h)} \frac{z-\xi}{z-\bar{\xi}}\right) (P_\mu(h) - z)^{-1} \right\|_{L^2(\Gamma_\mu), L^2(\Gamma_\mu)} \leqslant C\,\mathrm{e}^{Ch^{-n-1}} \tag{3.5}$$

where $\Omega_{\frac{3}{4}}(h)$ is the domain

$$\Omega_{\frac{3}{4}}(h) = \left\{ z \in \mathbb{C};\, E_1(h) - \tfrac{21}{4}\omega(h) \leqslant \operatorname{Re} z \leqslant E_2(h) + \tfrac{21}{4}\omega(h), 0 \leqslant -\operatorname{Im} z \leqslant 3h^{-n-2}S(h) \right\}.$$

Proof. The proof is based on the estimate established by Tang and Zworski in the proof of lemma 1 of [20]:

$$\|(P_\mu(h)-z)^{-1}\|_{L^2(\Gamma_\mu),L^2(\Gamma_\mu)} \leqslant C\,\mathrm{e}^{Ch^{-n}\log\frac{1}{g(h)}} \qquad \forall z\in\Omega(h)\Big\backslash\bigcup_{z_j\in\mathrm{Res}(P(h))} D(z_j,g(h)) \tag{3.6}$$

where $0<g(h)\ll 1$. Let us set

$$F_\mu(z,h)=\left(\prod_{\xi\in\Lambda(h)}\frac{z-\xi}{z-\bar\xi}\right)(P_\mu(h)-z)^{-1}.$$

By construction, the resonances of $P(h)$ coincide with the poles of $(P_\mu(h)-z)^{-1}$ with the same multiplicity. As the resonances $\xi\in\Lambda(h)$ are simple, then $F_\mu(\cdot,h)$ is holomorphic in $\Omega(h)$. Hence, applying the maximum principle, it suffices to show that estimate (3.5) holds on the border $\partial\Omega_{\frac34}(h)$. Let us recall that according to Burq's result ([2], theorem 1), there exists $C>0$ such that

$$\mathrm{Res}(P(h))\cap\left(\left[\frac{E_1(h)}{2},\frac{3E_2(h)}{2}\right]+\mathrm{i}[-\mathrm{e}^{-C/h},0]\right)=\emptyset.$$

Let us set $g(h)=\mathrm{e}^{-C/h}\ll 1$. With this choice of $g(h)$ it is easy to prove that all resonances are at least at distance $g(h)$ from $\partial\Omega_{\frac34}(h)$. Indeed, as $\mathrm{Res}(P(h))\cap(\Omega(h)\backslash\Omega_0(h))=\emptyset$, for z in $\partial\Omega_{\frac34}(h)$ we can write

$$\begin{aligned}\mathrm{dist}(z,\mathrm{Res}(P(h))) &\geqslant \min\left(\mathrm{dist}(z,\Lambda(h)),\mathrm{dist}(z,\mathrm{Res}(P(h)\cap\Omega(h)^c))\right)\\ &\geqslant \min\left(S(h),\mathrm{dist}\left(\Omega_{\frac34}(h),\Omega(h)^c\right)\right)\\ &\geqslant \min\left(S(h),\frac{h^{-n-2}}{4}S(h)\right)\geqslant \mathrm{e}^{-C/h}\end{aligned}$$

where the second inequality comes from $S(h)\geqslant -\mathrm{Im}\,\xi\geqslant \mathrm{e}^{-C/h}$, $\forall\xi\in\Lambda(h)$. It follows that we can apply estimate (3.6) for $z\in\partial\frac34\Omega(h)$ to get

$$\forall z\in\partial\Omega_{\frac34}(h)\qquad \|F_\mu(z,h)\|_{L^2(\Gamma_\mu),L^2(\Gamma_\mu)}\leqslant C\left(\prod_{\xi\in\Lambda(h)}\frac{|z-\xi|}{|z-\bar\xi|}\right)\mathrm{e}^{Ch^{-n-1}}\leqslant C\,\mathrm{e}^{Ch^{-n-1}}$$

and the proof is complete. □

Proof of proposition 5. Let us set $a(h)=E_1(h)$, $b(h)=E_2(h)$ and

$$U(h)=[a(h)-6\omega(h),b(h)+6\omega(h)]+\mathrm{i}[-2S(h)h^{-n-2},0].$$

By definition, $0<S(h)\leqslant\omega(h)h^{\frac{3n+5}{2}}$ with $\omega(h)\to 0$ as $h\to 0$. It follows that $U(h)$ is exactly in the form required in lemma 1. As each $\xi\in\Lambda(h)$ is a simple resonance of $P(h)$, $F(z,h)$ is a holomorphic function of z in $\Omega(h)$. We have just checked that the domain $U(h)$ satisfies the hypotheses of this lemma, so that we need only verify estimates (3.2) and (3.3) with $M(h)=h^{-\frac{n-1}{2}}$.

Proof of estimate (3.3). It is based on the estimate of the scattering amplitude for real energies, proved in [11]. First, note that for $\lambda\in\mathbb{R}_+^*$ and $\xi\in\Lambda(h)$, $\left|\frac{\lambda-\xi}{\lambda-\bar\xi}\right|=1$ and

$$|F(\lambda,h)|=|f(\theta,\omega,\lambda,h)|.$$

Now, it suffices to apply theorem 2 to obtain

$$|F(\lambda,h)|=\mathcal{O}\left(h^{-\frac{n-1}{2}}\right)$$

and the proof of estimate (3.3) is complete.

Proof of estimate (3.2). First we choose $h_0 > 0$ such that for all $0 < h < h_0$, $\Omega(h) \subset \Lambda_{d_2,\epsilon}$, where $\Lambda_{d_2,\epsilon}$ is defined in section 2 and we suppose $0 < h < h_0$. For $z \in \Omega(h)$ we have the decomposition

$$F(z,h) = \Pi(z,h)(T_1(\theta,\omega,z,h) - T_2(\theta,\omega,z,h))$$

where T_1 is defined by (2.10) with (2.25), T_2 is defined by (2.10) with (2.16) and

$$\Pi(z,h) = c(z,h)\prod_{\xi\in\Lambda(h)}\frac{z-\xi}{z-\bar{\xi}}.$$

Here $c(z,h)$ is given by formula (1.2) and is chosen to be holomorphic in $\mathbb{C}\backslash]-\infty,0]$. We will estimate successively each term of the right-hand side of this equation. We begin by the estimate of $F_2(z,h) = \Pi(z,h)T_2(\theta,\omega,z,h)$ and we note that

$$\forall z \in \{y\in\mathbb{C};\ \mathrm{Im}\, y < 0\} \qquad |\Pi(z,h)| \leqslant |c(z,h)| \leqslant Ch^{\frac{n-1}{2}}.$$

Using estimates (2.14) and (2.15) in combination with (2.16), it is obvious that

$$|F_2(z,h)| \leqslant C\|\Pi(z,h)(P_\mu(h)-z)^{-1}\|_{L^2(\Gamma_\mu),L^2(\Gamma_\mu)}$$

for $z \in \Omega(h)$. Using the fact that $U(h) \subset \frac{3}{4}\Omega(h)$, we deduce immediately from lemma 2 that $|F_2(z,h)| \leqslant C\,\mathrm{e}^{Ch^{-n-1}}$ for all $z \in U(h)$. Therefore, it remains to estimate $F_1(z,h) = \Pi(z,h)T_1(\theta,\omega,z,h)$. Using proposition 4 and identity (2.10), we get immediately

$$\forall z\in\Omega(h) \qquad |F_1(z,h)| \leqslant \left|ch^{\frac{n-1}{2}}T_1(\theta,\omega,z,h)\right| \leqslant C\,\mathrm{e}^{C/h} \leqslant C\,\mathrm{e}^{Ch^{-n-1}}$$

and the proof of estimate (3.2) is complete. □

3.2. *Proof of theorem 1*

Let us recall that

$$f(\theta,\omega,z,h) = \sum_{\xi\in\Lambda(h)}\frac{f_\xi^{\mathrm{res}}(\theta,\omega,h)}{z-\xi} + f^{\mathrm{hol}}(\theta,\omega,z,h)$$

where $f^{\mathrm{hol}}(\theta,\omega,z,h)$ is holomorphic with respect to $z\in\Omega(h)$. By a simple calculation, we obtain

$$f_\xi^{\mathrm{res}}(\theta,\omega,h) = 2\mathrm{i}\,\mathrm{Im}(\xi)F(\xi,h)\left(\prod_{\zeta\in\Lambda(h)\backslash\{\xi\}}\frac{\xi-\bar{\zeta}}{\xi-\zeta}\right) \qquad \forall\xi\in\Lambda(h) \tag{3.7}$$

and

$$f^{\mathrm{hol}}(\theta,\omega,z,h) = \left(\prod_{\xi\in\Lambda(h)}\frac{z-\bar{\xi}}{z-\xi}\right)F(z,h) - \sum_{\xi\in\Lambda(h)}\frac{f_\xi^{\mathrm{res}}}{z-\xi} \qquad \forall z\in\Omega(h). \tag{3.8}$$

Using proposition 5, it follows that

$$\left|f_\xi^{\mathrm{res}}(\theta,\omega,h)\right| \leqslant Ch^{-\frac{n-1}{2}}|\mathrm{Im}\,\xi|\prod_{\zeta\in\Lambda(h)\backslash\{\xi\}}\frac{|\xi-\bar{\zeta}|}{|\xi-\zeta|} \leqslant Ch^{-\frac{n-1}{2}}|\mathrm{Im}\,\xi|\prod_{\zeta\in\Lambda(h)\backslash\{\xi\}}\left(1+\frac{2|\mathrm{Im}\,\xi|}{|\xi-\zeta|}\right). \tag{3.9}$$

Hence, we have to estimate the product which appears in the right-hand side of the last equation. If we just write that $|\mathrm{Im}\,\xi| \leqslant S(h)$ and $\forall\zeta\in\Lambda(h)\backslash\{\xi\}$, $|\xi-\zeta| \geqslant \epsilon S(h)$, we obtain

$$\prod_{\zeta\in\Lambda(h)\backslash\{\xi\}}\left(1+\frac{2|\mathrm{Im}\,\xi|}{|\xi-\zeta|}\right) \leqslant (1+\epsilon^{-1})^{K(h)}.$$

As $K(h)$ may grow as h^{-n}, this estimate does not give a polynomial bound on $f_\xi^{\mathrm{res}}\big/|\mathrm{Im}\,\xi|$. To overcome this difficulty, we use the fact that the resonances cannot accumulate in a given area. In the following lemma, $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Lemma 3. *Assume* (**Sep**$_\epsilon$) *with* $0 < \epsilon < 1$ *and let* $\alpha \in [E_1(h) - \omega(h), E_2(h) + \omega(h)]$. *Then we can find* $L_\epsilon(h) \in \left[\frac{\epsilon}{2}K(h), \left(\frac{2}{\epsilon} - 1\right)^{-1} K(h)\right]$ *such that*

$$\Lambda(h) = \bigcup_{j=1}^{L_\epsilon(h)} \bigcup_{i=1}^{[2/\epsilon]} \{z_{ij}\} \tag{3.10}$$

and

$$\forall z \in \Omega(h) \cap \{\mathrm{Re}\, z = \alpha\} \qquad \forall j \geqslant 2 \quad \forall i \in \{1, \ldots, [2/\epsilon]\} \qquad |z - z_{ij}| \geqslant (j-1)\frac{\epsilon S(h)}{6}. \tag{3.11}$$

Let us complete the proof of theorem 1, assuming lemma 3. From here until the end of this paper, C_ϵ will denote a positive constant independent of h, which can change from line to line. Our aim is to give a good estimate of

$$\Pi_1(\xi, h) = \prod_{\zeta \in \Lambda(h)\setminus\{\xi\}} \left(1 + \frac{2|\mathrm{Im}\,\xi|}{|\xi - \zeta|}\right).$$

Let us apply lemma 3 with $\alpha = \mathrm{Re}\,\xi$. Then we can write

$$\Lambda(h) = \bigcup_{j=1}^{L_\epsilon(h)} \bigcup_{i=1}^{[2/\epsilon]} \{z_{ij}\}$$

with $z_{11} = \xi$ and

$$\forall j \geqslant 2 \qquad \forall i \in \{1, \ldots, [2/\epsilon]\} \qquad |\xi - z_{ij}| \geqslant (j-1)\frac{\epsilon S(h)}{6}.$$

Using (**Sep**$_\epsilon$) to separate ξ and z_{1i}, $i = 1, \ldots, [2/\epsilon]$, we obtain

$$\begin{aligned}\Pi_1(\xi, h) &\leqslant \prod_{i=2}^{[2/\epsilon]} \left(1 + \frac{2S(h)}{\epsilon S(h)}\right) \prod_{j=2}^{L_\epsilon(h)} \prod_{i=1}^{[2/\epsilon]} \left(1 + \frac{12S(h)}{(j-1)\epsilon S(h)}\right) \\ &\leqslant \left(1 + \frac{2}{\epsilon}\right)^{[2/\epsilon]-1} \prod_{j=2}^{L_\epsilon(h)} \prod_{i=1}^{[2/\epsilon]} \left(1 + \frac{12}{(j-1)\epsilon}\right) \leqslant C_\epsilon (1 + L_\epsilon(h))^{24/\epsilon^2}.\end{aligned}$$

Here, we have used the elementary estimate $\prod_{j=1}^{N}\left(1 + \frac{\alpha}{j}\right) \leqslant N^\alpha, \forall \alpha > 0$. By construction, we have $L_\epsilon(h) \leqslant \left(\frac{2}{\epsilon} - 1\right)^{-1} K(h) \leqslant K(h)$ and we obtain

$$\Pi_1(\xi, h) \leqslant C_\epsilon (1 + K(h))^{24/\epsilon^2}. \tag{3.12}$$

Finally, we deduce from equations (3.9) and (3.12) that

$$\left|f_\xi^{\mathrm{res}}(\theta, \omega, h)\right| \leqslant C_\epsilon h^{-\frac{n-1}{2}} K(h)^{24/\epsilon^2} |\mathrm{Im}\,\xi|. \tag{3.13}$$

Now we shall estimate the holomorphic part f^{hol} of the scattering amplitude. Let us denote $M_\epsilon(h) = h^{-\frac{n-1}{2}} K(h)^{24/\epsilon^2}$. Starting from formula (3.8) and using estimate (3.13), we obtain

$$|f^{\mathrm{hol}}(\theta, \omega, z, h)| \leqslant \Pi_2(z, h)|F(z, h)| + C_\epsilon M_\epsilon(h) \sum_{\xi \in \Lambda(h)} \frac{|\mathrm{Im}\,\xi|}{|z - \xi|} \qquad \forall z \in \Omega(h) \tag{3.14}$$

where $\Pi_2(z,h) = \prod_{\xi\in\Lambda(h)} \frac{|z-\bar{\xi}|}{|z-\xi|}$. Our aim is to estimate f^{hol} on $\tilde{\Omega}(h)$. This function being analytic on $\Omega(h)$, it suffices to obtain an estimate on $\partial\tilde{\Omega}(h)$. Let $z \in \partial\tilde{\Omega}(h)$ and apply lemma 3 with $\alpha = \mathrm{Re}\, z$ in combination with estimate (3.4)

$$
\begin{aligned}
|f^{\mathrm{hol}}(\theta,\omega,z,h)| &\leqslant Ch^{-\frac{n-1}{2}} \prod_{j=1}^{L_\epsilon(h)} \prod_{i=1}^{[2/\epsilon]} \left(1 + \frac{|2\,\mathrm{Im}\, z_{ij}|}{|z-z_{ij}|}\right) + C_\epsilon M_\epsilon(h) \sum_{j=1}^{L_\epsilon(h)} \sum_{i=1}^{[2/\epsilon]} \frac{|\mathrm{Im}\, z_{ij}|}{|z-z_{ij}|} \\
&\leqslant C_\epsilon M_\epsilon(h) + C_\epsilon M_\epsilon(h) \sum_{i=1}^{[2/\epsilon]} \frac{|\mathrm{Im}\, z_{i1}|}{|z-z_{i1}|} + C_\epsilon M_\epsilon(h) \sum_{j=2}^{L_\epsilon(h)} \sum_{i=1}^{[2/\epsilon]} \frac{6}{(j-1)\epsilon} \\
&\leqslant C_\epsilon M_\epsilon(h) \left(1 + \frac{2}{\epsilon} + \frac{12}{\epsilon^2} \log(L_\epsilon(h)) + \sum_{i=1}^{[2/\epsilon]} \frac{|\mathrm{Im}\, z_{i1}|}{|z-z_{i1}|}\right).
\end{aligned}
$$

Moreover, for $z \in \partial\tilde{\Omega}(h)$ and $z_{i1} \in \Lambda(h)$, we know that $|z - z_{i1}| \geqslant \min(S(h), \omega(h), |\mathrm{Im}\, z_{i1}|)$ and we obtain

$$
|f^{\mathrm{hol}}(\theta,\omega,z,h)| \leqslant C_\epsilon M_\epsilon(h) \left(1 + \frac{4}{\epsilon} + \frac{12}{\epsilon^2} \log(1+\epsilon K(h))\right) \leqslant C_\epsilon M_\epsilon(h) \log(1+K(h)).
$$

This estimate completes the proof of theorem 1.

Proof of lemma 3. First, we number the resonances such that $\Lambda(h) = \bigcup_{j=1}^{K(h)} \{z_j\}$ and $\forall i \leqslant j, \mathrm{Re}\, z_i \leqslant \mathrm{Re}\, z_j$. Let us fix $\alpha \in [E_1(h) - \omega(h), E_2(h) + \omega(h)]$, then we can find $i_0(h) \in \{1, \ldots, K(h)\}$ such that

$$
\forall i \leqslant i_0(h) \quad \mathrm{Re}\, z_i \leqslant \alpha \qquad \text{and} \qquad \forall i \geqslant i_0(h) \quad \mathrm{Re}\, z_i \geqslant \alpha.
$$

By induction, the proof is reduced to show that

$$
\forall i_1 \geqslant i_0 \qquad \forall i \geqslant i_1 + [1/\epsilon] \qquad \mathrm{Re}\, z_i \geqslant \mathrm{Re}\, z_{i_1} + \frac{\epsilon S(h)}{6} \tag{3.15}
$$

and

$$
\forall j_1 \leqslant i_0 \qquad \forall j \leqslant j_1 - [1/\epsilon] \qquad \mathrm{Re}\, z_j \leqslant \mathrm{Re}\, z_{j_1} - \frac{\epsilon S(h)}{6}. \tag{3.16}
$$

We give the proof of (3.15) only, because the demonstration of (3.16) is identical. Suppose that (3.15) does not hold. The sequence $(\mathrm{Re}\, z_i)_i$ being increasing, we can find $i_1 \geqslant i_0$ such that

$$
\forall i \in \{i_1, \ldots, i_1 + [1/\epsilon]\} \qquad \mathrm{Re}\, z_{i_1} \leqslant \mathrm{Re}\, z_i \leqslant \mathrm{Re}\, z_{i_1} + \frac{\epsilon S(h)}{6}.
$$

Let us denote $\alpha_1 = \mathrm{Re}\, z_{i_1}$ and $\Delta_\epsilon = \left[\alpha_1, \alpha_1 + \frac{\epsilon S(h)}{6}\right] + \mathrm{i}[-S(h), 0]$. Then as the surface $S_\epsilon(h)$ of the rectangle Δ_ϵ is given by

$$
S_\epsilon(h) = \frac{\epsilon S(h)^2}{6}. \tag{3.17}
$$

On the other hand, the balls $B\left(z_i, \frac{\epsilon S(h)}{2}\right), i = i_1, \ldots, i_1 + [1/\epsilon]$ do not intercept one another. Denoting $S_{i,\epsilon}(h)$ as the surface of each of these balls, it follows that

$$
S_\epsilon(h) \geqslant \frac{1}{4} \sum_{i=i_1}^{i_1+[1/\epsilon]} S_{i,\epsilon}(h) \geqslant \frac{1}{4\epsilon} \pi \frac{\epsilon^2 S(h)}{4} \geqslant \frac{\pi \epsilon S(h)^2}{16}.
$$

Combining this equation and (3.17), we obtain a contradiction. □

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