

EYRING-KRAMERS FORMULA FOR THE MEAN EXIT TIME OF NON-GIBBSIAN ELLIPTIC PROCESSES: THE NON CHARACTERISTIC BOUNDARY CASE

DORIAN LE PEUTREC, LAURENT MICHEL, AND BORIS NECTOUX

ABSTRACT. In this work, we derive a new sharp asymptotic equivalent in the small temperature regime $h \rightarrow 0$ for the mean exit time from a bounded domain for the non-reversible process $dX_t = b(X_t)dt + \sqrt{h}dB_t$ under a generic orthogonal decomposition of b and when the boundary of Ω is assumed to be *non characteristic*. The main contribution of this work lies in the fact that we do not assume that the process $(X_t, t \geq 0)$ is *Gibbsian*. In this case, a new correction term characterizing the *non-Gibbsianness* of the process appears in the equivalent of the mean exit time. The proof is mainly based on tools from spectral and semi-classical analysis.

Keywords. Eyring-Kramers type formulas, spectral analysis, mean exit time, principal eigenvalue, non-reversible elliptic processes.

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1. INTRODUCTION AND MAIN RESULT

1.1. Setting and assumptions. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space, where the filtration satisfies the usual conditions, and $(B_t, t \geq 0)$ be a $\mathbf{R}^d$ -standard Brownian motion. Let $(X_t, t \geq 0)$ be the process solution to the following elliptic Itô stochastic differential equation on $\mathbf{R}^d$,

$$dX_t = b(X_t)dt + \sqrt{h}dB_t, \quad (1.1)$$

where $h > 0$ and $b : \mathbf{R}^d \rightarrow \mathbf{R}^d$ is a smooth vector field. The parameter $h > 0$ is the temperature of the system. In this work, we consider the small temperature regime, i.e. $h \ll 1$, see Section 1.2, and we make the following assumptions on the vector field b . There exist $\ell : \mathbf{R}^d \rightarrow \mathbf{R}^d$ and $f : \mathbf{R}^d \rightarrow \mathbf{R}$ which are both $\mathcal{C}^\infty$ such that:

[A₁] The vector field b satisfies the following orthogonal decomposition:

$$\text{for all } x \in \mathbf{R}^d, \quad b(x) = -(\nabla f(x) + \ell(x)) \quad \text{and} \quad \ell(x) \cdot \nabla f(x) = 0.$$

[A_{x₀}] The function f has a unique critical point x_0 in $\mathbf{R}^d$. Moreover, the Hessian matrix $\text{Hess } f(x_0)$ at x_0 is positive definite.

$[\mathbf{A}_\infty]$ All the derivatives of f (resp. of ℓ) of order larger or equal than 2 (resp. than 1) are bounded over $\mathbf{R}^d$. In addition, f is bounded from below and there exist $C, R > 0$ such that

$$\forall |x| \geq R, |\nabla f(x)| \geq C.$$

Assumption $[\mathbf{A}_\perp]$ is rather generic as explained in Section 5.3 below. Besides, the non local assumptions on the vector field b made in $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$ are unnecessary for our main result to hold, see the note just after Theorem 1. In all this work, we always assume that $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$ are satisfied. In addition, throughout this work, Ω is a $\mathcal{C}^\infty$ bounded subdomain¹ of $\mathbf{R}^d$ containing $x_0 \in \Omega$. The quantity of interest in this work is the first exit time from Ω for the process (1.1) which will be denoted by τ_Ω , i.e.

$$\tau_\Omega := \inf\{t \geq 0, X_t \notin \Omega\}.$$

Let us mention that the vector field b does not vanish over $\partial\Omega$. In this case, $\partial\Omega$ is said to be non characteristic and the so-called *generalized saddle points* z of f on $\partial\Omega$ (see the set $\mathcal{P}_{\text{sp}}$ defined (1.4)) will play a crucial role in the asymptotic formula of the mean exit time $\mathbf{E}[\tau_\Omega]$ from Ω . Such critical points are associated with edge shaped barriers (or reflected barriers) and were considered by Kramers in its celebrated work [26], see also [40, p. 836-837].

1.2. Purpose of this work. In this work, we derive a new sharp asymptotic formula in the limit $h \rightarrow 0$ (and actually prove a complete asymptotic expansion in power of h) of the mean exit time $\mathbf{E}[\tau_\Omega]$ from Ω for the process (1.1) when $X_0 = x \in \Omega$ (uniformly in the relevant compact sets), see our main result below, Theorem 1.

Contrary to previous contributions in this field (see Section 1.6), we do not assume the process (1.1) to be Gibbsian, which is the main novelty of this work. More precisely, we do not assume that the Gibbs measure

$$\mu_{\text{Gibbs}}(dx) := e^{-\frac{2}{h}f} dx$$

is invariant for the process (1.1) (where dx is the Lebesgue measure over $\mathbf{R}^d$), or equivalently that $\text{div}(\ell) = 0$ over $\mathbf{R}^d$.

This has a strong impact on the Eyring-Kramers formula we derive in Theorem 1 for the mean exit time from Ω for the process (1.1). Indeed, compared to the Gibbsian case, the new (non local) terms $\exp[\int_0^{+\infty} \text{div}(\ell)(\psi_t(z))dt]$, which are

¹We recall that by definition, a domain is a nonempty connected open set.

attached to each generalized saddle point z of f over $\partial\Omega$, appear in the pre-factor as a measure of the non-Gibbsianness of the process, see (1.6). Let us emphasize that these terms can be greater or bigger than 1, see the comments following Theorem 1. Hence, in comparison with the Gibbsian case, both an acceleration or a deceleration of the exit time from Ω can occur. A very similar formula was already derived in [8] with formal computations when the saddle points z are critical points of f . In this case, the difference appears in the fact that the integral of $\operatorname{div}(\ell)(\psi_t(z))$ runs from $-\infty$ to $+\infty$. The non-Gibbsianness of the process (1.1) introduces several difficulties in the analysis of the precise h -limit of the mean exit time from Ω . This follows in particular from the fact that a sufficiently good approximation (at least in $\mathcal{C}^1$ -norm) of the non explicit invariant probability density p_h is needed.

In addition, the spectral analysis performed here is more involved than in our previous work in the Gibbsian case [32], with the construction of a different and much more precise quasi-mode for the principal eigenfunction of the infinitesimal generator of (1.1) than the one built in [32, Section 5.2]. As a by-product, to derive a sharp asymptotic formula in the limit $h \rightarrow 0$ for $\mathbf{E}[\tau_\Omega]$, we do not assume that the vector field ℓ vanishes at each generalized saddle point z of f on $\partial\Omega$ as it was the case in [32, Section 5.1]. In particular, when the process (1.1) is Gibbsian, Theorem 1 improves the statement of [32, Theorem 2] in the case where the vector field b does not vanish over $\partial\Omega$ ².

Let us finally mention that we do not neither assume $b \cdot n_\Omega < 0$ over $\partial\Omega$, where $n_\Omega(z)$ denotes the unit outward normal vector to $\partial\Omega$ at $z \in \partial\Omega$, as it was the case for technical reasons in the classical pioneering works dealing with the exit event from a bounded domain when $h \rightarrow 0$, see e.g. [15, 12, 13] or [20, Chapter 4].

Remark 1. *This work opens up several possibilities, such as considering the case where b has a critical saddle point on the boundary of Ω . The behavior as $h \rightarrow 0$ of the invariant measure near such a point is tricky to derive. Such a situation is left to a future work.*

1.3. Direct consequences of Assumptions $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$. In this section, we give some consequences of $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$ which will be used in this work. On the one hand, note that $[\mathbf{A}_\infty]$ implies that f is coercive, i.e.

$$f(x) \rightarrow +\infty \text{ as } |x| \rightarrow +\infty. \quad (1.2)$$

Furthermore, $[\mathbf{A}_\perp]$ and $[\mathbf{A}_\infty]$ imply that for all $x \in \mathbf{R}^d$, the curve $t \mapsto \varphi_t(x)$ solution on $\mathbf{R}^d$ to

$$\frac{d}{dt}\varphi_t(x) = b(\varphi_t(x)), \quad \varphi_0(x) = x, \quad (1.3)$$

²Let us mention that this technical issue did not appear at a critical point.

is well defined over $\mathbf{R}_+$. Moreover, when in addition $[\mathbf{A}_{x_0}]$ is satisfied, x_0 is the unique local minimum of the function f and it holds $\{z \in \mathbf{R}^d, b(z) = 0\} = \{x_0\}$ (in particular $\ell(x_0) = 0$). For all $x \in \mathbf{R}^d$, the ω -limit set $\omega(x)$ of x is reduced to $\{x_0\}$ (see [32, Section 1.3]), i.e.

$$\omega(x) = \{x_0\}.$$

In addition, thanks to [6, Lemma 1.4] (see also [30] for a similar result), the matrix $\text{Jac } b(x_0) = -{}^t(\text{Hess } f(x_0) + {}^t \text{Jac } \ell(x_0))$ admits precisely d eigenvalues which all have a negative real part. Finally, the relation $\ell \cdot \nabla f = 0$ yields that the matrix $\text{Hess}(f)(x_0) \text{Jac } \ell(x_0)$ is antisymmetric and hence $\text{div } \ell(x_0) = \text{Tr}(\text{Jac } \ell(x_0)) = 0$.

1.4. Preliminary analysis and generalized saddle points.

1.4.1. *The domain Ω is a well of potential.* Recall that Ω is assumed to be a $\mathcal{C}^\infty$ bounded domain of $\mathbf{R}^d$ containing x_0 . Let us define

$$\mathcal{C}_{\min} := \Omega \cap \{f < \min_{\partial\Omega} f\},$$

where for $\mu \in \mathbf{R}$, we use the notation $\{f < \mu\} := \{x \in \mathbf{R}^d, f(x) < \mu\}$. The sets $\{f \leq \mu\}$ and $\{f = \mu\}$ are defined similarly.

When $[\mathbf{A}_{x_0}]$ and $[\mathbf{A}_\infty]$ are satisfied, we are thus in the following geometrical situation which shows that the domain Ω looks like a single well of the potential function $f : \mathbf{R}^d \rightarrow \mathbf{R}$.

Lemma 2. *Assume that $[\mathbf{A}_{x_0}]$ and $[\mathbf{A}_\infty]$ are satisfied. Then:*

- [a] *The function f admits for sole global minimum point x_0 in $\mathbf{R}^d$ and thus in $\overline{\Omega}$.*
- [b] *The set $\mathcal{C}_{\min}$ is equal to $\{f < \min_{\partial\Omega} f\}$; it contains x_0 and is connected. In addition, $\partial\mathcal{C}_{\min} \cap \partial\Omega \neq \emptyset$.*

Proof. The proof is elementary. Recall first that thanks to (1.2) following from $[\mathbf{A}_\infty]$, f admits a global minimum on $\mathbf{R}^d$, and using $[\mathbf{A}_{x_0}]$, it has for only global minimum point x_0 . This implies Item [a].

Notice moreover that $\mathcal{C}_{\min}$ contains x_0 and satisfies $\mathcal{C}_{\min} = \overline{\Omega} \cap \{f < \min_{\partial\Omega} f\}$. It is thus nonempty and both open and closed in $\{f < \min_{\partial\Omega} f\}$. Note also that for any $\lambda > f(x_0)$, the nonempty (and relatively compact) open set $\{f < \lambda\}$ is connected. In particular, the set $\{f < \min_{\partial\Omega} f\}$ is connected, which implies that

$$\mathcal{C}_{\min} = \{f < \min_{\partial\Omega} f\} \quad \text{is connected.}$$

Finally, since $f : \mathbf{R}^d \rightarrow \mathbf{R}$ does not have any local minimum on $\{f = \min_{\partial\Omega} f\}$, it follows that

$$\partial\mathcal{C}_{\min} = \{f = \min_{\partial\Omega} f\},$$

which implies that $\partial\mathcal{C}_{\min} \cap \partial\Omega = \{f = \min_{\partial\Omega} f\} \cap \partial\Omega \neq \emptyset$. $\square$

1.4.2. *Set of generalized saddle points of f .* When $[\mathbf{A}_{x_0}]$ and $[\mathbf{A}_\infty]$ are satisfied, we define the (nonempty) set

$$\mathcal{P}_{\text{sp}} := \partial\mathcal{C}_{\min} \cap \partial\Omega. \quad (1.4)$$

Note that every $z \in \mathcal{P}_{\text{sp}}$ is a global minimum of $f|_{\partial\Omega}$. A point $z \in \mathcal{P}_{\text{sp}}$ is a so-called generalized saddle point. This is due to the fact that $\partial_{n_\Omega} f(z) > 0$ and, for that reason, when the potential function f is extended by $-\infty$ outside $\overline{\Omega}$ (notice that this extension is the one which is compatible with the absorbing boundary condition on $\partial\Omega$), the point z is geometrically a first-order saddle point of f . These points have a crucial role in the asymptotic equivalents of the mean exit time from Ω , see indeed Theorem 1.

Our last assumption is the following

$$[\mathbf{A}_{\mathcal{P}_{\text{sp}}}] \text{ For all } z \in \mathcal{P}_{\text{sp}}, \det \text{Hess}(f|_{\partial\Omega})(z) \neq 0.$$

Observe that when $[\mathbf{A}_{\mathcal{P}_{\text{sp}}}]$ holds, the set $\mathcal{P}_{\text{sp}}$ has a finite number of elements. We say that Assumption $[\mathbf{A}]$ holds when all the four assumptions above are satisfied.

1.5. **Main result.** In this section, we state our main result which is Theorem 1. We first recall some notation. For every $x \in \Omega$, on sets $t_x := \inf\{t \geq 0, \varphi_t(x) \notin \Omega\} > 0$ the first time the curve $\varphi_t(x)$ exits Ω and we define the domain of attraction of a subset F of Ω by

$$\mathcal{A}_\Omega(F) := \{x \in \Omega, t_x = +\infty \text{ and } \omega(x) \subset F\}.$$

Note that when $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$ hold, $\mathcal{C}_{\min} \subset \mathcal{A}_\Omega(\{x_0\})$.

Throughout the paper, we shall say that a family of scalar $(a^h)_{h \in]0,1]}$ admits a h -classical expansion, if there exists a sequence $(a_n)_{n \in \mathbf{N}}$ such that for all $N \in \mathbf{N}$, $a^h = \sum_{n=0}^N a_n h^n + O(h^{N+1})$. Such an expansion will be denoted by $a^h \sim \sum_{n \geq 0} a_n h^n$. More generally, we shall say that a family of smooth functions $(u^h)_{h \in]0,1]}$ on an open set Ω of $\mathbf{R}^d$ or of $\mathbf{R}_-^d := \mathbf{R}^{d-1} \times \mathbf{R}_-$ admits a h -classical expansion on Ω , if there exists a sequence $(u_n)_{n \in \mathbf{N}}$ of smooth functions on Ω such that for all compact $K \subset \Omega$, for all $k \in \mathbf{N}$, and for all $N \in \mathbf{N}$, $u^h = \sum_{n=0}^N u_n h^n + O(h^{N+1})$ in the $C^k(K)$ topology. This expansion is also denoted $u^h \sim \sum_{n \geq 0} u_n h^n$ and when $u_n \equiv 0$ for all $n \in \mathbf{N}$, we write $u^h = O(h^\infty)$.

Theorem 1. *Assume that [A] holds. Then for every compact subset $K \in \mathcal{A}_\Omega(x_0)$, it holds uniformly in $x \in K$,*

$$\mathbf{E}_x[\tau_\Omega] = \kappa(h) \sqrt{h} e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}, \quad (1.5)$$

where $\kappa(h)$ admits a h -classical expansion $\kappa(h) \sim \sum_{j \geq 0} \kappa_j h^j$ and

$$\frac{1}{\kappa_0} = \frac{\sqrt{\det \text{Hess } f(x_0)}}{\sqrt{\pi}} \sum_{z \in \mathcal{P}_{\text{sp}}} \frac{\partial_{n_\Omega} f(z)}{\sqrt{\det \text{Hess } f|_{\partial\Omega}(z)}} \exp \left[\int_0^{+\infty} \text{div}(\ell)(\psi_t(z)) dt \right], \quad (1.6)$$

where $t \geq 0 \mapsto \psi_t(x)$ is the (global) solution on $\mathbf{R}^d$ to

$$\frac{d}{dt} \psi_t(x) = -(\nabla f - \ell)(\psi_t(x)), \quad \psi_0(x) = x. \quad (1.7)$$

Let us make two comments on this theorem.

1. Given a domain Ω , the mean exit time $\mathbf{E}[\tau_\Omega]$ only depends on the killed process $(X_t, 0 \leq t < \tau_\Omega)$ and thus only on the values of b in $\overline{\Omega}$ (roughly speaking, it does not depend on the non local assumptions in [A]). For that reason, the Eyring-Kramers formula we derive in Theorem 1 above does not depend on the values of b outside $\overline{\Omega}$.
2. It appears that for each $z \in \mathcal{P}_{\text{sp}}$, the term $\int_0^{+\infty} \text{div}(\ell)(\psi_t(z))$, which in some sense measures the non-Gibbsianness of the process when it is not 0, can be positive or negative. Hence, compared with the reversible case $\ell = 0$ or with the Gibbsian case $\text{div } \ell = 0$ [32], both can occur, acceleration or deceleration of the (mean) exit time from Ω .

To observe that both situations can occur, consider, in $\mathbf{R}^2$, the open disc $\Omega = D((0, -1), 2)$ of center $(0, -1)$ and radius 2 and the fields $b_\pm = -(\nabla f + \ell_\pm)$, where $f(x) = \frac{1}{2}|x|^2$ on $\mathbf{R}^2$ and $\ell_\pm(x) = \pm(x_1 x_2, -x_1^2)$ on $\overline{\Omega}$. In this setting, the assumption [A] is satisfied with $x_0 = 0$ (once $\ell_\pm$ have been conveniently defined outside $\overline{\Omega}$), $\mathcal{C}_{\min} = D(0, 1)$, and $\mathcal{P}_{\text{sp}} = \partial \mathcal{C}_{\min} \cap \partial\Omega = \{z\}$, where $z = (0, 1)$. In addition, we have $\psi_t(z) = (0, e^{-t})$ for all $t \geq 0$ and $\text{div } \ell_\pm(x) = \pm x_2$ for all $x \in \overline{\Omega}$. It follows that $\int_0^{+\infty} \text{div}(\ell_\pm)(\psi_t(z)) = \pm 1$ can be negative or positive.

1.6. Related results. In the boundary case, the first works on the mean exit time were probably those of Freidlin–Wentzell, where in the one well setting, and when $h \rightarrow 0$, the limit of $h \ln \mathbf{E}_x[\tau_\Omega]$ has been derived in [20, Chapter 4] when $b \cdot n_\Omega < 0$ over $\partial\Omega$. In this setting, it has also been proved in [12] that $\mathbb{E}_x[\tau_\Omega] \lambda_h = 1 + o(1)$

when $h \rightarrow 0$, where λ_h is the principal eigenvalue of (minus) the generator L_h of (1.1), see Section 3.1 below. Closely related results have also been derived in [15, 43] still in this setting. We also refer to [39, 47] where asymptotic formulas for $\mathbf{E}[\tau_\Omega]$ when $h \rightarrow 0$ have been obtained through formal computations in different geometrical settings. A comprehensive review of the literature until the 2010s on this topic can be found in [2]. Moreover, in the case when $b \cdot n_\Omega \neq 0$ on $\partial\Omega$, the asymptotic behavior of the solution to the parabolic equation $\partial_t u = L_h u$ has been studied in [24, 25] (where the quasilinear case is also treated). In the reversible case (i.e. when $\ell = 0$), Eyring-Kramers type formulas have been derived in [44] when f has critical saddle points on $\partial\Omega$, see also [38, 49, 22, 36, 17, 33, 37] for related results in the reversible case.

Let us also mention that asymptotic estimates on eigenvalues and on mean transition times in the boundary less case have been widely studied. When the process is reversible, we refer to [10, 9, 4, 3, 21, 41, 1]. In the non-reversible case and when the process (1.1) is Gibbsian (i.e. when $\operatorname{div}(\ell) = 0$), sharp equivalent of mean transition times have been derived in [29, 34, 27] (see also [28, 35]), and sharp asymptotic formulas for the smallest eigenvalues of L_h have been obtained in [31] (see also [23, 6, 7]). Finally, as already mentioned, the generalization of the Eyring-Kramers formula for mean transition times to non Gibbsian diffusion processes have been derived in [8] with formal calculations.

2. ASYMPTOTIC BEHAVIOR OF THE STATIONARY DISTRIBUTION

2.1. Asymptotic expansion of the stationary distribution. In this section, we recall and improve a result of Sheu [48] and Mikami [42] (see also [13]) on the properties of the stationary measure of (1.1), see Theorem 2.

For a measure ν over a subset $\mathcal{M}$ of $\mathbf{R}^d$, the set $H^k(\mathcal{M}, \nu(dx))$ stands for the usual (weighted) Sobolev space of regularity $k \geq 0$ over $\mathcal{M}$ for the measure $\nu(dx)$. The infinitesimal generator of the diffusion (1.1) is $\frac{h}{2}\Delta + b \cdot \nabla$ and will rather work with minus this operator, namely with

$$L_h = -\frac{h}{2}\Delta + (\nabla f + \ell) \cdot \nabla. \quad (2.1)$$

The formal adjoint of L_h in $L^2(\Omega, dx)$ is denoted by L_h^* . It is the operator acting on smooth function $u : \mathbf{R}^d \rightarrow \mathbf{R}$ as

$$L_h^* u = -\frac{h}{2}\Delta u + \operatorname{div}(b u). \quad (2.2)$$

For all $h > 0$ fixed, the existence of an invariant probability measure μ_h for the process (1.1) follows from [48, Lemma 1.2] (note that according to the discussion in

Section 1.3, $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$ are simply a shorten formulation Assumption (A) in [48]). It is well-known that since the vector field b is smooth, μ_h is the unique invariant probability measure. This follows from the fact that the process (1.1) is topologically irreducible and strongly Feller [46, Theorem 1.1]. Moreover, μ_h has a smooth density $p_h : \mathbf{R}^d \rightarrow \mathbf{R}_+$ w.r.t. the Lebesgue measure dx , see e.g. [45, 5], which is positive over $\mathbf{R}^d$.

Theorem 2. *Assume $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$. Then, the positive function R^h defined by*

$$R^h : x \in \mathbf{R}^d \mapsto p_h(x) h^{d/2} e^{\frac{2}{h}(f(x)-f(x_0))} \quad (2.3)$$

admits a real-valued h -classical expansion on $\mathbf{R}^d$. More precisely, we have $R^h \sim \sum_{k \geq 0} h^k R_k$ for a sequence of real-valued functions $(R_k)_{k \in \mathbf{N}} \subset C^\infty(\mathbf{R}^d)$ which satisfy

$$-(\nabla f - \ell) \cdot \nabla R_0 + R_0 \operatorname{div}(\ell) = 0, \quad R_0(x_0) = c_0, \quad (2.4)$$

and for, any $k \geq 1$,

$$-(\nabla f - \ell) \cdot \nabla R_k + R_k \operatorname{div}(\ell) = -\frac{1}{2} \Delta R_{k-1}, \quad (2.5)$$

where $c_0 > 0$ is defined by $c_0 \int_{\mathbf{R}^d} e^{-x \cdot \operatorname{Hess} f(x_0) x} dx = 1$, that is by

$$c_0 = |\det \operatorname{Hess} f(x_0)|^{\frac{1}{2}} \pi^{-\frac{d}{2}}. \quad (2.6)$$

We recall that $R^h \sim \sum_{k \geq 0} h^k R_k$ means that for every $n \in \mathbf{N}$, for every compact set K of $\mathbf{R}^d$, and for every $\alpha \in \mathbf{N}^d$, the smooth function $W_n^h := R^h - \sum_{k=0}^n h^k R_k$ satisfies

$$\sup_{x \in K} |\partial^\alpha W_n^h(x)| \leq C_{K,\alpha} h^{n+1} \quad (2.7)$$

when $h \rightarrow 0$, for some constant $C_{K,\alpha}$ independent of h . As we shall see in the proof, the case when $|\alpha| = 0$ is already known [48, 42]. To perform our analysis we need a stronger control on the derivatives of the W_n^h 's. This is the reason we extend it to the case $|\alpha| > 1$.

Proof. From [48, Lemma 1.1], one deduces that $f = \frac{1}{2}V$ where V is the so-called Freidlin–Wentzell quasi-potential defined by

$$V(x) = \frac{1}{2} \inf_{\phi(0)=x, \phi(+\infty)=x_0} \int_0^\infty |\dot{\phi}(s) + b(\phi(s))|^2 ds.$$

In particular, the function V is C^∞ on $\mathbf{R}^d$. In addition, Assumptions (A.2), (A.3), and (A.4.r) in [42] are satisfied. Hence, by [42, Theorem 1.3], we get exactly all

the assertions of the theorem except the estimates (2.7) when $|\alpha| \geq 1$, which we prove now (observe indeed that (2.7) with $|\alpha| = 0$ is proved in [42]).

In the following $C > 0$ denotes a constant independent of $h > 0$ that may change from one occurrence to another. The Sobolev space $H^k(\mathbf{R}^d, dx)$ is simply denoted by H^k and its norm by $\|\cdot\|_{H^k}$. The scalar product in $L^2(\mathbf{R}^d, dx)$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$. For ease of notation, we set $b^* = 2(\nabla f - \ell)$. Then, Equations (2.4) and (2.5) rewrite

$$b^* \cdot \nabla R_k - 2 \operatorname{div}(\ell) R_k - \Delta R_{k-1} = 0$$

for all $k \geq 0$, with the convention $R_{-1} = 0$. On the other hand, since p_h is the invariant density of the process (1.1), it follows that $L_h^* p_h = 0$ (see (2.2)). Hence, we deduce that

$$h \Delta R^h - b^* \cdot \nabla R^h + 2 \operatorname{div}(\ell) R^h = 0.$$

Combining these two identities, we get for every $n \in \mathbf{N}$,

$$(h \Delta - b^* \cdot \nabla + 2 \operatorname{div}(\ell)) W_n^h = -h^{n+1} \Delta R_n. \quad (2.8)$$

By Sobolev embedding, in order to prove (2.7), it is sufficient to prove that for any $\chi \in C_c^\infty(\mathbf{R}^d)$ and any $\alpha \in \mathbf{N}^d$, there exists $C > 0$, such that

$$\|\chi \partial^\alpha W_n^h\|_{L^2} \leq C h^{n+1}. \quad (2.9)$$

To prove (2.9), we first show the following *a priori* estimate:

$$\forall \alpha \in \mathbf{N}^d, \forall \chi \in C_c^\infty(\mathbf{R}^d), \exists C > 0, \|\chi \partial^\alpha W_n^h\|_{L^2} \leq C h^{n+1-\frac{|\alpha|}{2}}. \quad (2.10)$$

We prove this estimate by induction on $s = |\alpha|$. To lighten the notation, the function $\partial^\alpha W_n^h$ will be simply denoted by w^α . As explained above, the case $s = 0$ holds true since it is proved in [42]. Let us now assume that the estimate (2.10) is true for all $s' \leq s$, for some $s \geq 0$. Let $\chi \in C_c^\infty(\mathbf{R}^d)$ be a real-valued function and let $\alpha \in \mathbf{N}^d$ be such that $|\alpha| = s$. Differentiating α times Equation (2.8), w^α satisfies

$$(h \Delta - b^* \cdot \nabla) w^\alpha = -h^{n+1} \partial^\alpha \Delta R_n + \sum_{\beta, |\beta| \leq |\alpha|} a_{\alpha, \beta} w^\beta, \quad (2.11)$$

where the functions $a_{\alpha, \beta}$, which are linear combinations of derivatives of $\operatorname{div}(\ell)$ and b^* , are smooth functions over $\mathbf{R}^d$ and are independent of h . Since the $a_{\alpha, \beta}$ are bounded on $\operatorname{supp}(\chi)$, using the Cauchy–Schwarz inequality, one deduces that

$$|\langle (h \Delta - b^* \cdot \nabla) w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| \leq C \left(\sum_{\beta, |\beta| \leq |\alpha|} \|\chi w^\beta\|_{L^2} + h^{n+1} \|\chi \partial^\alpha \Delta R_n\|_{L^2} \right) \|\chi w^\alpha\|_{L^2}.$$

Since all the derivatives of ΔR_n are bounded on any compact set and using the induction hypothesis, this implies

$$|\langle (h\Delta - b^* \cdot \nabla)w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| \leq Ch^{2n+2-s},$$

and hence

$$|\langle h\Delta w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| \leq |\langle b^* \cdot \nabla w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| + Ch^{2n+2-s}. \quad (2.12)$$

On the other hand, integrating by parts, we get

$$\langle b^* \cdot \nabla w^\alpha, \chi^2 w^\alpha \rangle_{L^2} = -\langle b^* \cdot \nabla w^\alpha, \chi^2 w^\alpha \rangle_{L^2} - \langle \operatorname{div}(b^* \chi^2) w^\alpha, w^\alpha \rangle_{L^2},$$

so that $2\langle b^* \cdot \nabla w^\alpha, \chi^2 w^\alpha \rangle_{L^2} = -\langle \operatorname{div}(b^* \chi^2) w^\alpha, w^\alpha \rangle_{L^2}$. Since $\operatorname{div}(b^* \chi^2)$ has a compact support, using again the induction hypothesis, it follows that

$$|\langle b^* \cdot \nabla w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| \leq Ch^{2n+2-s}.$$

Combined with (2.12), this finally yields

$$|\langle \Delta w^\alpha, \chi^2 w^\alpha \rangle_{L^2}| \leq Ch^{2n+2-s-1}. \quad (2.13)$$

On the other hand, by successive integration by parts, one gets

$$\begin{aligned} -\langle \Delta w^\alpha, \chi^2 w^\alpha \rangle_{L^2} &= \langle \nabla w^\alpha, \chi \nabla \chi w^\alpha \rangle_{L^2} + \langle \chi \nabla w^\alpha, \nabla(\chi w^\alpha) \rangle_{L^2} \\ &= \langle \nabla w^\alpha, \chi \nabla \chi w^\alpha \rangle_{L^2} + \|\nabla(\chi w^\alpha)\|_{L^2}^2 - \langle \nabla \chi w^\alpha, \nabla(\chi w^\alpha) \rangle_{L^2} \\ &= \|\nabla(\chi w^\alpha)\|_{L^2}^2 - \|\nabla(\chi) w^\alpha\|_{L^2}^2. \end{aligned}$$

Combining this previous identity with (2.13) and using once again the induction hypothesis, we deduce that $\|\nabla(\chi w^\alpha)\|_{L^2}^2 = O(h^{2n+2-s-1})$. Therefore, it holds:

$$\|\chi \nabla w^\alpha\|_{L^2} = O(h^{n+1-\frac{s+1}{2}}). \quad (2.14)$$

This proves the estimate (2.10) at rank $s+1$. Let us now improve this bound, and more precisely let us show (2.9). We observe that for any $n \leq n' \in \mathbf{N}$, one has

$$\chi \partial^\alpha W_n^h = \chi \partial^\alpha W_{n'}^h + \sum_{k=n+1}^{n'} h^k \chi \partial^\alpha R_k.$$

Consequently, since the R_k are C^∞ and independent of h , for any $s \in \mathbf{N}$, one deduces from (2.10) that

$$\|\chi \partial^\alpha W_n^h\|_{L^2} \leq \|\chi \partial^\alpha W_{n'}^h\|_{L^2} + \sum_{k=n+1}^{n'} h^k \|\chi \partial^\alpha R_k\|_{L^2} \leq C_s h^{n'+1-\frac{|\alpha|}{2}} + Ch^{n+1}$$

We finally obtain (2.9) by taking $n' = n + \lceil \frac{|\alpha|}{2} \rceil$, which completes the proof. $\square$

We now discuss the term R_0 which will play a crucial role in the Eyring-Kramers formula for the mean exit time. As already noticed in [48, Theorem 3.1] (see also [13]), it is standard to show that R_0 actually admits the expression

$$R_0(x) = c_0 \exp \left[\int_0^{+\infty} \operatorname{div}(\ell)(\psi_t(x)) dt \right], \quad \forall x \in \mathbf{R}^d, \quad (2.15)$$

where we recall that $t \mapsto \psi_t(x)$ is the global solution to (1.7). Indeed, using (2.4), it holds

$$\frac{d}{dt} R_0(\psi_t(x)) + \operatorname{div}(\ell)(\psi_t(x)) R_0(\psi_t(x)) = 0, \quad \forall t \geq 0.$$

Hence, for $t \geq 0$, $R_0(\psi_t(x)) = R_0(x) \exp[-\int_0^t \operatorname{div}(\ell)(\psi_s(x)) ds]$. The identity (2.15) then follows taking the limit $t \rightarrow +\infty$. Note that $\operatorname{div}(\ell)(\psi_s(x)) \rightarrow 0$ exponentially fast as $s \rightarrow +\infty$ since $\psi_s(x) \rightarrow x_0$ exponentially fast as $s \rightarrow +\infty$ (x_0 being non-degenerate), $\operatorname{div}(\ell)$ is globally Lipschitz, and $\operatorname{div}(\ell)(x_0) = 0$. Note also that the formula (2.15) implies that $R_0(x)$ is positive over $\mathbf{R}^d$.

3. SPECTRAL POINT OF VIEW

In the two next sections we recall some results from [32] that will be used to prove our main result. We first note that since p_h is smooth and positive over $\mathbf{R}^d$ and because Ω is bounded, it holds for every $h > 0$ and $k \geq 0$,

$$H^k(\Omega, e^{-\frac{2}{h}f} dx) = H^k(\Omega, dx) = H^k(\Omega, p_h dx), \quad (3.1)$$

and all the involved norms are equivalent (with constants depending on $h > 0$). In the following, the norm (resp. the scalar product) of the space $L^2(\Omega, \nu(dx))$ is denoted by $\|\cdot\|_{L^2(\nu(dx))}$ (resp. $\langle \cdot, \cdot \rangle_{L^2(\nu(dx))}$) and we write $H^k(\Omega, \nu(dx))$ to indicate that the ambient space under consideration is the space $L^2(\Omega, \nu(dx))$.

3.1. Spectral results at fixed $h > 0$. In [32, Proposition 3], we proved that the operator L_h with domain $\mathcal{D}_1 = H^2(\Omega, e^{-\frac{2}{h}f} dx) \cap H_0^1(\Omega, e^{-\frac{2}{h}f} dx)$ had the following spectral properties at $h > 0$ fixed, which are quite standard for an elliptic operator over a bounded domain:

1. The operator $L_h|_{\mathcal{D}_1}$ is maximal quasi-accretive and has a compact resolvent. Its spectrum is thus discrete.
2. The operator $L_h|_{\mathcal{D}_1}$ has a principal eigenvalue $\lambda_h \in \mathbf{R}_+^*$, i.e. λ_h has algebraic multiplicity one, and $\operatorname{Re} \mu > \lambda_h$ for every $\mu \in \sigma(L_h) \setminus \{\lambda_h\}$.

In addition, any associated eigenfunction u_h has a sign in Ω . The normalized positive one is called the principal eigenfunction.

On the other hand, we also notice that for all real function $u \in \mathcal{D}_0^\infty := \{g \in \mathcal{C}^\infty(\overline{\Omega}), g = 0 \text{ on } \partial\Omega\}$,

$$\int_{\Omega} L_h u^2(x) p_h(x) dx = 0,$$

stemming from the identity $L_h^* p_h = 0$ together with the fact that $\nabla u^2 \cdot n_\Omega = 0$ on $\partial\Omega$. Consequently, we have for any such real function u the identity

$$\langle u, L_h u \rangle_{L^2(p_h dx)} = \int_{\Omega} u(x) L_h u(x) p_h(x) dx = \frac{h}{2} \int_{\Omega} |\nabla u(x)|^2 p_h(x) dx. \quad (3.2)$$

As simple as it may seem, this formula is a key ingredient in our analysis. Moreover, the gradient structure (3.2) strongly suggests to rather work with the operator L_h with domain $\mathcal{D}_2 = H^2(\Omega, p_h dx) \cap H_0^1(\Omega, p_h dx)$. In view of (3.1), one can also consider L_h with domain $\mathcal{D}_3 = H^2(\Omega, dx) \cap H_0^1(\Omega, dx)$. It is actually easy to see that the spectrum of L_h is the same on each domain $\mathcal{D}_i$, $i = 1, 2, 3$ (associated with their respective scalar product), as well as both the algebraic and geometric multiplicities of an eigenvalue. In addition, Items 1 and 2 above are satisfied for each $L_h|_{\mathcal{D}_i}$, $i = 1, 2, 3$, with the same principal eigenvalue λ_h and associated eigenspace $\text{Span}(u_h)$. Note also that the identity (3.2) extends by density to every real $u \in \mathcal{D}_i$, $i = 1, 2, 3$.

In the following, for all $h > 0$, we choose u_h such that

$$u_h > 0 \text{ in } \Omega \text{ and } \int_{\Omega} |u_h|^2 p_h = 1.$$

3.2. Spectral results when $h \rightarrow 0$. In this section, we recall the following result which will be the starting point of the proof of Theorem 1.

Theorem 3 ([32]). *Assume $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$. Then, there exists $c_1 > 0$ such that, for all $c_2 \in (0, c_1)$, there exist $h_0 > 0$ and $C > 0$ such that, for all $z \in \{z \in \mathbb{C}, \text{Re } z \leq c_1, |z| \geq c_3\}$ and $h \in (0, h_0]$,*

$$L_h - z \text{ is invertible and } \|(L_h - z)^{-1}\|_{L^2(p_h dx)} \leq C. \quad (3.3)$$

In addition, there exist $c > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$,

$$\sigma(L_h) \cap \{z \in \mathbb{C}, \text{Re } z \leq c\} = \{\lambda_h\} \quad \text{and} \quad \lim_{h \rightarrow 0} h \ln \lambda_h = -2 \left(\min_{\partial\Omega} f - f(x_0) \right). \quad (3.4)$$

Moreover, for every compact subset K of $\mathcal{A}(\{x_0\})$, there exist $c > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$ and $x \in K$,

$$\lambda_h \mathbf{E}_x[\tau_\Omega] = (1 + O(e^{-\frac{c}{h}})) \text{ uniformly in } x \in K. \quad (3.5)$$

Note. It has also been proved in [32] that the law of $\lambda_h \tau_\Omega(x)$ converges exponentially fast to the exponential law of mean 1, uniformly in x in the compact subsets of $\mathcal{A}(\{x_0\})$.

Proof. All the statements of Theorem 3 have been proved in [32], see indeed Theorems 1 and 4 there³. Let us mention that concerning the second estimate in (3.3), we actually proved in [32] that for such complex numbers z and for all $h > 0$ small enough,

$$\|(L_h - z)^{-1}\|_{L^2(e^{-\frac{2}{h}f} dx)} \leq C,$$

namely that for all $\phi \in L^2(\Omega, e^{-\frac{2}{h}f} dx)$, $\int_\Omega |(L_h - z)^{-1} \phi|^2 e^{-\frac{2}{h}f} \leq C^2 \int_\Omega |\phi|^2 e^{-\frac{2}{h}f}$, or equivalently (see (2.3) and Theorem 2)

$$\int_\Omega |(L_h - z)^{-1} \phi|^2 \frac{p_h}{R_0 + o(1)} \leq C^2 \int_\Omega |\phi|^2 \frac{p_h}{R_0 + o(1)},$$

where $o(1)$ is uniform over $\overline{\Omega}$. The fact that $\|(L_h - z)^{-1}\|_{L^2(p_h dx)} = O(1)$ then follows noticing that over Ω , there exists $r_1, r_2 > 0$ such that $r_1 \leq R_0 \leq r_2$. $\square$

In the next section we state the spectral counterpart of Theorem 1.

3.3. Sharp asymptotics of the principal eigenvalue. The following theorem provides the sharp equivalent of λ_h in the limit $h \rightarrow 0$.

Theorem 4. *Assume that [A] holds. Then, there exists $h_0 > 0$ and a sequence $(\zeta_j)_{j \in \mathbb{N}}$ of real numbers such that for all $h \in]0, h_0]$, one has*

$$\lambda_h = h^{-\frac{1}{2}} \zeta(h) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}, \quad (3.6)$$

where $\zeta(h)$ admits a h -classical expansion $\zeta(h) \sim \sum_{j \geq 0} \zeta_j h^j$ with

$$\zeta_0 = \frac{\sqrt{\det \text{Hess } f(x_0)}}{\sqrt{\pi}} \sum_{z \in \mathcal{P}_{\text{sp}}} \frac{\partial_{n_\Omega} f(z)}{\sqrt{\det \text{Hess } f|_{\partial\Omega}(z)}} \exp \left[\int_0^{+\infty} \text{div}(\ell)(\psi_t(z)) dt \right], \quad (3.7)$$

and where we recall that $t \geq 0 \mapsto \psi_t(x)$ is the solution to (1.7).

Note that combining Theorem 4 with (3.5) yields the assertion of Theorem 1. The rest of this work is thus dedicated to the proof of Theorem 4.

³In the setting of [32], Ω is subdomain of the d -dimensional torus. It is actually straightforward to see that the assertions [32, Theorems 1 and 4] are indeed still valid under $[\mathbf{A}_\perp]$, $[\mathbf{A}_{x_0}]$, and $[\mathbf{A}_\infty]$.

4. CONSTRUCTION OF AN ACCURATE QUASI-MODE FOR u_h

The goal of this section is to construct a very precise quasi-mode u_h^{app} for the principal eigenvalue λ_h of L_h in $L^2(\Omega, p_h dx)$, namely a function approximating the principal eigenfunction u_h sufficiently well so that we can compute asymptotically λ_h as $h \rightarrow 0$. The conditions on u_h^{app} are listed in Proposition 7 below. Roughly speaking, we want to choose the function u_h^{app} equal to 1 on a very large part of $\mathcal{C}_{\min}$ but satisfying the boundary condition $u_h^{\text{app}} = 0$ on $\partial\Omega$. This requires the construction of a quasi-mode u_h^{app} realizing the appropriate transition from 1 to 0 around $\partial\Omega$. The delicate part of this construction occurs around $\mathcal{P}_{\text{sp}} = \partial\mathcal{C}_{\min} \cap \partial\Omega$ (see (1.4)), where we use suitable local coordinates near each $z \in \mathcal{P}_{\text{sp}}$.

From now on, we assume $[\mathbf{A}]$.

4.1. Local coordinates near $\mathcal{P}_{\text{sp}}$. It turns out that the system of coordinates introduced in [22] near such points is well appropriate for defining u_h^{app} and the upcoming computations. Recall that $\mathcal{P}_{\text{sp}} = \partial\mathcal{C}_{\min} \cap \partial\Omega \neq \emptyset$ (see (1.4)) and that thanks to $[\mathbf{A}_{\mathcal{P}_{\text{sp}}}]$, $\mathcal{P}_{\text{sp}}$ has a finite cardinality. In the following, we consider $z \in \mathcal{P}_{\text{sp}}$.

Then, there exists a neighborhood $\mathcal{U}_z$ of z in $\bar{\Omega}$ and a coordinate system

$$x \in \mathcal{U}_z \mapsto v = (v', v_d) = (v_1, \dots, v_{d-1}, v_d) \in \mathbf{R}_-^d = \mathbf{R}^{d-1} \times \mathbf{R}_- \quad (4.1)$$

such that

$$v(z) = 0, \quad \{x \in \mathcal{U}_z, v_d(x) < 0\} = \Omega \cap \mathcal{U}_z, \quad \{x \in \mathcal{U}_z, v_d(x) = 0\} = \partial\Omega \cap \mathcal{U}_z, \quad (4.2)$$

and

$$\forall i, j \in \{1, \dots, d\}, \quad g_z\left(\frac{\partial}{\partial v_i}(z), \frac{\partial}{\partial v_j}(z)\right) = \delta_{ij} \quad \text{and} \quad \frac{\partial}{\partial v_d}(z) = n_\Omega(z),$$

where g_z is the metric tensor in the new coordinates. We denote by $G = (G_{ij})_{1 \leq i, j \leq d}$ its matrix, by $G^{-1} = (G^{ij})_{1 \leq i, j \leq d}$ the inverse of G , and by $|G| = \det G$ its determinant. We also denote the canonical basis of $\mathbf{R}^d$ by $(e_1, \dots, e_d)$. Then, defining $J := \text{Jac } v^{-1}$, we have:

$$G = {}^t J J, \quad G(0) = (\delta_{ij}) \quad \text{i.e.} \quad {}^t J(0) = J^{-1}(0), \quad \text{and} \quad n_\Omega(z) = J(0)e_d. \quad (4.3)$$

Let us now determine the operator L_h in the above coordinates, see (2.1). Throughout this work, for any function u defined on $\mathcal{U}_z$, we denote $\hat{u} = u \circ v^{-1}$ where v is the above change of coordinates. For any $u \in C^\infty(\mathcal{U}_z)$, we have $\widehat{L_h u} = \hat{L}_h \hat{u}$ with

$$\hat{L}_h = -\frac{h}{2\sqrt{|G|}} \text{div} \circ \sqrt{|G|} G^{-1} \circ \nabla + (G^{-1} \nabla \hat{f} + J^{-1} \hat{\ell}) \cdot \nabla. \quad (4.4)$$

We can write this operator as follows:

$$\hat{L}_h = -\frac{h}{2} \operatorname{div} \circ G^{-1} \circ \nabla - (b^\circ + h\rho^\circ) \cdot \nabla, \quad (4.5)$$

where b° and ρ° are the following smooth vector fields over $\mathcal{U}_z$:

$$b^\circ = -G^{-1} \nabla \hat{f} - J^{-1} \hat{\ell} \quad \text{and} \quad \rho^\circ = \frac{1}{2\sqrt{|G|}} \nabla(\sqrt{|G|}) G^{-1}. \quad (4.6)$$

Let us now introduce the notation $\ell^\circ = J^{-1} \hat{\ell}$, so that b° rewrites

$$b^\circ = -G^{-1} \nabla \hat{f} - \ell^\circ. \quad (4.7)$$

In addition, according for example to [22, Section 3.4] (see also [43]), the v -coordinates can be chosen such that:

$$\hat{f}(v', v_d) = f(z) + \mu_z v_d + \frac{1}{2} v' \cdot H_z v', \quad (4.8)$$

where

$$\mu_z := \partial_{n_\Omega} f(z) > 0 \quad \text{and} \quad H_z := \operatorname{Hess} f|_{\partial\Omega}(z). \quad (4.9)$$

Moreover, thanks to $[\mathbf{A}_{\mathcal{P}_{\text{sp}}}]$, 0 is a non degenerate (global) minimum of $\hat{f}|_{\{v_d=0\}}$. For $\delta_1 > 0$ and $\delta_2 > 0$ small enough, one finally defines the following neighborhood of z in $\bar{\Omega}$ (see (4.1)-(4.2)),

$$\mathcal{U}_z^\delta = \{x \in \mathcal{U}_z, |v'(x)| \leq \delta_2 \text{ and } v_d(x) \in [-2\delta_1, 0]\}, \quad \delta = (\delta_1, \delta_2). \quad (4.10)$$

The set defined in (4.10) is a cylinder centered at z in the v -coordinates. The parameters $\delta_1, \delta_2 > 0$ will be reduced a finite number of times to ensure several properties needed in Section 7 to perform computations. Recall also that $f(z) = \min_{\partial\Omega} f > f(x_0)$. Then, up to choosing $\delta_1 > 0$ and $\delta_2 > 0$ smaller, we can assume that

[C₁^δ] The sets $\mathcal{U}_z^\delta$, $z \in \mathcal{P}_{\text{sp}}$, are pairwise disjoint, so in particular

$$\operatorname{argmin}_{\mathcal{U}_z^\delta \cap \partial\Omega} f = \{z\}.$$

[C₂^δ] $\min_{\mathcal{U}_z^\delta} f > f(x_0)$, so in particular $x_0 \notin \mathcal{U}_z^\delta$.

Finally, according to [C₁^δ] and using a continuity argument, once $\delta_2 > 0$ is fixed, one can choose $\delta_1 > 0$ small enough such that

[C₃^δ] There exists $r > 0$ such that

$$\{x \in \mathcal{U}_z, |v'(x)| = \delta_2 \text{ and } v_d(x) \in [-2\delta_1, 0]\} \subset \{f \geq f(z) + r\}.$$

We refer to Figure 4.1 for a schematic representations of the sets $\mathcal{U}_z^\delta$.

4.2. General form of the quasi-mode near $\mathcal{P}_{\text{sp}}$. Let $\chi \in C^\infty(\mathbf{R}_-, [0, 1])$ be a cut-off function such that (see (4.9)):

$$\text{supp } \chi \subset \left[-\frac{\delta_1}{2}\mu_z, 0 \right] \text{ and } \chi = 1 \text{ on } \left[-\frac{\delta_1}{4}\mu_z, 0 \right]. \quad (4.11)$$

Define now, for each $z \in \mathcal{P}_{\text{sp}}$, the cylinder $\mathcal{V}_z^\delta$ by

$$\mathcal{V}_z^\delta := v(\mathcal{U}_z^\delta) = \{v = (v', v_d) \in \mathbf{R}^d, |v'| \leq \delta_2, -2\delta_1 \leq v_d \leq 0\}. \quad (4.12)$$

The set $\mathcal{V}_z^\delta$ is a neighborhood of 0 in $\mathbf{R}_-^d$. For every $z \in \mathcal{P}_{\text{sp}}$, we look for a quasi-mode u_h^{app} defined on the cylinder $\mathcal{U}_z^\delta$ by

$$\forall x \in \mathcal{U}_z^\delta, u_h^{\text{app}}(x) := \varphi_z(v(x)) \quad (4.13)$$

with a function φ_z defined on the set $\mathcal{V}_z^\delta$ by

$$\forall v \in \mathcal{V}_z^\delta, \varphi_z(v) := \frac{1}{N_{z,h}} \int_{\xi^z(v,h)}^0 \chi(t) e^{\frac{t}{h}} dt, \quad (4.14)$$

where for every $h \in]0, 1]$, $v \in \mathcal{V}_z^\delta \mapsto \xi^z(v, h)$ is a real nonpositive smooth function which will be constructed later, and $N_{z,h}$ is the normalizing constant

$$N_{z,h} := \int_{-\infty}^0 \chi(t) e^{\frac{t}{h}} dt = h + O(e^{-\frac{c}{h}}). \quad (4.15)$$

We now turn to the construction of an appropriate function ξ^z vanishing on $\{v_d = 0\}$ and such that u_h^{app} satisfies the Dirichlet boundary condition on $\partial\Omega$.

4.3. Construction of the function ξ^z . We begin this section by deriving equations that shall satisfy ξ^z in order to make u_h^{app} sufficiently close to the principal eigenfunction u_h near each z . To this end, let us fix $z \in \mathcal{P}_{\text{sp}}$.

Since λ_h is exponentially small, we look for a function ξ^z such that u_h^{app} is an approximate solution of $L_h u_h^{\text{app}} = 0$. More precisely, we look for a smooth function ξ^z admitting a h -classical expansion $\xi^z \sim \sum_{j \geq 0} h^j \xi_j^z$ in $\mathcal{V}_z^\delta$ such that $\xi_0^z \neq 0$, ξ^z vanishes on $\{v_d = 0\}$, and $L_h u_h^{\text{app}}(x) = O(h^\infty)$ in $\mathcal{U}_z^\delta$. The latter relation reads in the v -coordinates

$$\hat{L}_h \varphi_z = O(h^\infty).$$

By (4.14), one has $\nabla \varphi_z = -\chi(\xi^z) e^{\xi^z/h} \nabla \xi^z / N_{z,h}$. Consequently, using (4.5), one has

$$\begin{aligned} \hat{L}_h \varphi_z &= -\frac{h}{2} \text{div}(G^{-1} \nabla \varphi_z) - (b^\circ + h\rho^\circ) \cdot \nabla \varphi_z \\ &= \frac{\chi(\xi^z) e^{\xi^z/h}}{N_{z,h}} \left[\frac{1}{2} (G^{-1} \nabla \xi^z) \cdot \nabla \xi^z + [b^\circ \cdot \nabla \xi^z + h(\rho^\circ \cdot \nabla \xi^z + \frac{1}{2} \text{div}(G^{-1} \nabla \xi^z))] \right] \\ &\quad + \frac{h}{2N_{z,h}} (G^{-1} \nabla \xi^z) \cdot \nabla \xi^z \chi'(\xi^z) e^{\xi^z/h}. \end{aligned} \quad (4.16)$$

Note moreover that (4.11) and (4.15) imply that the last term of (4.16) is of the order $O(e^{-\frac{c}{h}})$ for some $c > 0$. Hence, in order to ensure $\hat{L}_h \varphi_z = O(h^\infty)$, it is sufficient to choose $\xi^z \sim \sum_{j \geq 0} h^j \xi_j^z$ such that

$$\frac{1}{2}(G^{-1} \nabla \xi^z) \cdot \nabla \xi^z + b^\circ \cdot \nabla \xi^z + h(\rho^\circ \cdot \nabla \xi^z + \frac{1}{2} \operatorname{div}(G^{-1} \nabla \xi^z)) = O(h^\infty). \quad (4.17)$$

Identifying the powers of h in (4.17), this amounts to the following equations:

$$G^{-1} \nabla \xi_0^z \cdot \nabla \xi_0^z + 2b^\circ \cdot \nabla \xi_0^z = 0 \quad (\text{E})$$

and, for all $j \geq 1$,

$$(G^{-1} \nabla \xi_0^z + b^\circ) \cdot \nabla \xi_j^z = \mathcal{Q}_j, \quad (\text{T-j})$$

where $\mathcal{Q}_j$ is a function which depends smoothly on the functions ξ_k^z and their derivatives for $k \in \{0, \dots, j-1\}$. Equation (E) is an eikonal equation while Equations (T-j), $j \geq 1$, are transport equations.

The existence of functions ξ_j^z , $j \geq 0$, satisfying these equations follows from standard results on non-linear first order PDE with non-characteristic boundary (see for example [18, pages 7 to 9] or [19, Section 3.2 in Part I]). We are more specific below.

4.3.1. Resolution of the eikonal equation. In this section, we look for a solution $\xi_0^z \not\equiv 0$ of (E) which vanishes in a neighborhood of 0 in the hyperplane $\{v_d = 0\}$.

The fact that $\{v_d = 0\}$ is non-characteristic near 0 means that the vector field b° involved in (E) is transverse to $\{v_d = 0\}$ near 0. Indeed, we have $b^\circ = -G^{-1} \nabla \hat{f} - \ell^\circ$, $G(0) = \operatorname{Id}$, and $\nabla \hat{f}(0) = \mu_z e_d$ according respectively to (4.7), (4.3), and (4.8), where we recall that $(e_1, \dots, e_d)$ denotes the canonical basis of $\mathbf{R}^d$. Since moreover $\nabla \hat{f}(0) \cdot \ell^\circ = 0$ according to [A $\perp$] and $\mu_z > 0$ (see (4.9)), it follows that

$$b^\circ(0) = -\mu_z e_d - \ell^\circ(0) \quad \text{and} \quad e_d \cdot \ell^\circ(0) = 0. \quad (4.18)$$

Hence, the vector b° is transverse to $\{v_d = 0\}$ at 0, and thus near 0 by continuity.

We can thus apply [18, Theorem 1.5] to

$$p(v, \eta) := G^{-1}(v) \eta \cdot \eta + 2b^\circ(v) \cdot \eta \quad \text{around } (0, \eta^*) \in \mathbf{R}^d \times \mathbf{R}^d$$

for any $\eta^* = (\eta'^*, \eta_d^*) \in \mathbf{R}^d$ satisfying $p(0, \eta^*) = 0$: for any smooth real function ψ defined near 0 in $\mathbf{R}^{d-1}$ such that $\nabla_{x'} \psi(0) = \eta'^*$, there exists a unique smooth real function ξ_0^z defined around $0 \in \mathbf{R}^d$ such that on this neighborhood $p(x, \nabla \xi_0^z(x)) = 0$, $\xi_0^z(x', 0) = \psi(x')$, and $\nabla \xi_0^z(0) = \eta^*$. Since we look for $\xi_0^z \not\equiv 0$ vanishing on $\{v_d = 0\}$,

this amounts to choose $\psi \equiv 0$ and thus $\eta'^* = 0$, and η_d^* as the nonzero solution of $p(0, (0, \eta_d^*)) = |\eta^*|^2 + 2b^\circ(v) \cdot \eta^* = 0$, that is to take $\psi \equiv 0$, $\eta'^* = 0$, and

$$\eta_d^* := -2b^\circ(0) \cdot e_d = 2\mu_z.$$

Note also that Taylor's theorem with integral form of the remainder then implies that ξ_0^z factorizes as $\xi_0^z = v_d a$, where a is a smooth function defined around $0 \in \mathbf{R}^d$. Since $\nabla \xi_0^z(0) = \eta^*$, we have in addition $a(0) = \eta_d^* = 2\mu_z$, and we have thus proved the

Proposition 3. *There exists a function $a \in C^\infty(\mathbf{R}^d)$ satisfying $a(0) = 2\mu_z$ such that the function ξ_0^z defined by $\xi_0^z(v) = v_d a(v)$ satisfies the eikonal equation (E) in a neighborhood of 0.*

4.3.2. *Resolution of the transport equations.* The following proposition permits to solve the equations (T-j).

Proposition 4. *There exists a neighborhood $\mathcal{V}$ of 0 in $\mathbf{R}_-^d$ and a sequence of functions $(\xi_j^z)_{j \geq 1}$ such that for all $j \geq 1$, $\xi_j^z \in C^\infty(\mathcal{V})$ satisfies (T-j) on $\mathcal{V}$ and vanishes on $\mathcal{V} \cap \{v_d = 0\}$.*

Proof. We proceed by induction on $j \geq 1$. All the transport equations have the same structure, only the right hand side of (T-j) depending on the preceding step. Hence, it is sufficient to prove that there exists a neighborhood $\mathcal{V}$ of 0 such that, for any smooth function $\mathcal{Q}$ defined on $\mathcal{V}$, we can find a smooth function u on $\mathcal{V}$ which vanishes on $\{v_d = 0\}$ and solves

$$(G^{-1} \nabla \xi_0^z + b^\circ) \cdot \nabla u = \mathcal{Q}. \quad (4.19)$$

Let us recall that $\nabla \xi_0^z(0) = 2\mu_z e_d$ (see Proposition 3 and the lines above) and $b^\circ(0) = -\mu_z e_d - \ell_0^\circ(0)$ with $e_d \cdot \ell_0^\circ(0) = 0$ (see (4.18)). Thus, the vector field $F^\circ := G^{-1} \nabla \xi_0^z + b^\circ$ satisfies $F^\circ(0) = \mu_z e_d - \ell_0^\circ$ and is transverse to $\{v_d = 0\}$ around 0.

Hence, the characteristics curves

$$\frac{d}{dt} y_t = F^\circ(y_t), \quad y_0(v') = v'$$

define a smooth coordinate system $(t, v') \in \mathbf{R}_- \times \mathbf{R}^{d-1} \mapsto y_t(v')$ near 0 in $\mathbf{R}_-^d$. We then define u in these coordinates by $u(y_t(v')) = \int_0^t \mathcal{Q}(y_s(v')) ds$. It is straightforward to deduce that the function u solves (4.19). Moreover taking $t = 0$ in the above equation, one gets $u(v', 0) = 0$. The proof of the proposition is complete. $\square$

A Borel construction then leads to the existence of a solution ξ^z to (4.17).

Proposition 5. *Let $(\xi_j^z)_{j \geq 0}$ be the sequence of functions given by Propositions 3 and 4. Then, there exist a neighborhood $\mathcal{V}$ of 0 in $\mathbf{R}_-^d$ and a family of smooth functions $\xi^z = \xi^z(\cdot, h)$, $h \in]0, 1]$, admitting the h -classical expansion $\xi^z \sim \sum_{j \geq 0} h^j \xi_j^z$ on $\mathcal{V}$, which vanishes on $\mathcal{V} \cap \{v_d = 0\}$ and satisfies on $\mathcal{V}$,*

$$\frac{1}{2}(G^{-1} \nabla \xi^z) \cdot \nabla \xi^z + b^\circ \cdot \nabla \xi^z + h(\rho^\circ \cdot \nabla \xi^z + \frac{1}{2} \operatorname{div}(G^{-1} \nabla \xi^z)) = O(h^\infty).$$

In particular, we have indeed built up a function φ_z in the v -coordinates such that $\hat{L}_h \varphi_z = O(h^\infty)$ near 0 in $\mathbf{R}_-^d$.

4.4. Extra conditions on the size parameters δ_1, δ_2 . Recall that for $\delta_1, \delta_2 > 0$, $\delta = (\delta_1, \delta_2)$ measures the sizes of the cylinders $\mathcal{V}_z^\delta = v(\mathcal{U}_z^\delta)$ (see (4.12)). In this section, we adjust the size parameters δ_1, δ_2 to get the extra conditions $[\mathbf{C}_4^\delta]$ to $[\mathbf{C}_8^\delta]$ below which will be needed in the quasi-modal estimates of Section 5.1 and in the definition of the quasi-mode u_h^{app} (see (4.31) below). This adjustment is made while preserving the properties $[\mathbf{C}_1^\delta]$ to $[\mathbf{C}_3^\delta]$ of these neighborhoods which were imposed in Section 4.1.

First of all, introduce the following notation (see (4.9)) for $v \in \mathcal{V}_z^\delta$:

$$Q(v) := \mu_z v_d + \frac{1}{2} H_z v' \cdot v' \text{ and } Q^+(v) = -\mu_z v_d + \frac{1}{2} H_z v' \cdot v'. \quad (4.20)$$

Note that $Q^+ \geq 0$. Recall also that $\mu_z > 0$, H_z is a positive definite matrix, and that according to Proposition 3,

$$\xi_0^z = 2\mu_z v_d + v_d \mathcal{A}(v),$$

where $\mathcal{A}$ is a smooth function such that $\mathcal{A}(0) = 0$. Therefore, using (4.20), we get for $v \in \mathcal{V}_z^\delta$:

$$Q(v) - \xi_0^z(v) = Q^+(v) - v_d \mathcal{A}(v) = -v_d(\mu_z + \mathcal{A}(v)) + \frac{1}{2} H_z v' \cdot v'. \quad (4.21)$$

Note also that $|v_d \mathcal{A}(v)| \leq c|v_d||v| \leq c|v_d|^2 + c|v_d||v'| \leq c(1 + \epsilon^{-1})|v_d|^2 + c\epsilon|v'|^2$ for some $c > 0$ and any $\epsilon > 0$. Thus, by (4.21), one has:

$$Q(v) - \xi_0^z(v) \geq -\mu_z v_d - c(1 + \epsilon^{-1})|v_d|^2 + \frac{1}{2} H_z v' \cdot v' - c\epsilon|v'|^2.$$

On the other hand, recall that the support of $1 - |\chi|^2$ is included in the set $\{t \in \mathbf{R}, t \leq -\mu_z \delta_1/4\}$ (see (4.11)). Consequently, choosing $\epsilon > 0$ small enough above and up to decreasing $\delta_1, \delta_2 > 0$, the two following conditions hold:

$$[\mathbf{C}_4^\delta] \operatorname{argmin}_{\mathcal{V}_z^\delta} (Q - \xi_0^z) = \{0\}.$$

$$[\mathbf{C}_5^\delta] (1 - |\chi|^2)(\xi_0^z(v)) \neq 0 \text{ and } v \in \mathcal{V}_z^\delta \text{ imply that}$$

$$v_d \leq -\delta_1/9 \text{ and } Q(v) - \xi_0^z(v) \geq \mu_z \delta_1/10.$$

In addition, recall (see Proposition 5) that the family of functions $\xi^z(\cdot, h) \sim \sum_{j \geq 0} h^j \xi_j^z$ is defined in a h -independent neighborhood of 0 in $\mathbf{R}_-^d$. Therefore, up to decreasing again $\delta_1, \delta_2 > 0$, there exists $h_0 > 0$ such that for all $h \in]0, h_0]$:

$$[\mathbf{C}_6^\delta] \text{ The function } v = (v', v_d) \mapsto \xi^z(v, h) \text{ is well defined and satisfies Proposition 5 on } \mathcal{V}_z^\delta, \text{ and for every } v \in \mathcal{V}_z^\delta, \xi^z((v', v_d), h) < 0 \text{ when } v_d < 0.$$

$$[\mathbf{C}_7^\delta] \text{ For any } v = (v', v_d) \in \mathcal{V}_z^\delta, \xi^z((v', v_d), h) \leq -\delta_1 \mu_z \text{ when } v_d \leq -\delta_1.$$

$$[\mathbf{C}_8^\delta] \text{ For any } v \in \mathcal{V}_z^\delta, \text{ one has}$$

$$\xi^z(v, h) \in \operatorname{supp} \chi' \Rightarrow v_d \leq -\frac{\delta_1}{9} \text{ and } Q(v) - \xi^z(v, h) \geq \mu_z \delta_1/10.$$

Note that in order to deduce that $\xi^z((v', v_d), h) < 0$ when $v_d < 0$ for $v \in \mathcal{V}_z^\delta$, we used the fact that $\xi^z((v', v_d), h) = v_d(2\mu_z + O(v) + O(h))$.

4.5. Definition of the quasi-mode. We are now in position to define u_h^{app} near the generalized saddle point $z \in \mathcal{P}_{\text{sp}}$ (see (1.4)). Let ξ^z be given by Proposition 5 and χ satisfying (4.11). Recall that (see (4.12) and (4.14)):

$$\forall v \in \mathcal{V}_z^\delta, \quad \varphi_z(v) = \frac{1}{N_{z,h}} \int_{\xi^z(v,h)}^0 \chi(t) e^{\frac{t}{h}} dt, \quad \text{where } N_{z,h} := \int_{-\infty}^0 \chi(t) e^{\frac{t}{h}} dt. \quad (4.22)$$

Note that φ_z also depends on $h > 0$ but for ease of notation, we have decided not to indicate this dependency in its notation. By construction of $\xi^z(\cdot, h)$, and using $[\mathbf{C}_6^\delta]$ and $[\mathbf{C}_7^\delta]$, it holds for $h > 0$ small enough:

$$\begin{cases} \varphi_z \in C^\infty(\mathcal{V}_z^\delta, [0, 1]), \\ \varphi_z(v', 0) = 0, \\ \forall (v', v_d) \in \mathcal{V}_z^\delta, \varphi_z(v', v_d) = 1 \text{ when } v_d \in [-2\delta_1, -\delta_1]. \end{cases} \quad (4.23)$$

We now want to glue all these definitions near $z \in \mathcal{P}_{\text{sp}}$ into a globally defined quasi-mode u_h^{app} over $\overline{\Omega}$ vanishing on $\partial\Omega$. Recall the conditions $[\mathbf{C}_1^\delta]$ to $[\mathbf{C}_8^\delta]$. On the other hand, for every $x \in \partial\mathcal{C}_{\min}$, one has $\nabla f(x) \neq 0$, which implies that for every $r > 0$ small enough, $\{f < f(x)\} \cap B(x, r)$ is connected and thus included in $\mathcal{C}_{\min}$.

These considerations imply the existence of the following subsets $\mathcal{C}_-$ and $\mathcal{C}_+$ of Ω .

Proposition 6. *Assume [A]. Then, there exist two $\mathcal{C}^\infty$ connected open sets $\mathcal{C}_-$ and $\mathcal{C}_+$ of Ω satisfying the following properties:*

1. $\overline{\mathcal{C}_{\min}} \subset \mathcal{C}_+ \cup \partial\Omega$.
2. $\overline{\mathcal{C}_+}$ is a neighborhood in $\overline{\Omega}$ of each set $\mathcal{U}_z^\delta$, for $z \in \mathcal{P}_{\text{sp}}$.
3. $\overline{\mathcal{C}_-} \subset \mathcal{C}_+$ and the strip $\overline{\mathcal{C}_+} \setminus \mathcal{C}_-$ satisfies

$$\exists c > 0, \quad f \geq f(x_0) + c \text{ on } \overline{\mathcal{C}_+} \setminus \mathcal{C}_- \quad \text{and} \quad \overline{\mathcal{C}_+} \setminus \mathcal{C}_- = \bigcup_{z \in \mathcal{P}_{\text{sp}}} \mathcal{U}_z^\delta \cup \mathcal{O}, \quad (4.24)$$

where the subset $\mathcal{O}$ of $\overline{\Omega}$ is such that:

$$\exists c > 0, \quad f \geq \min_{\partial\Omega} f + c \text{ on } \mathcal{O}. \quad (4.25)$$

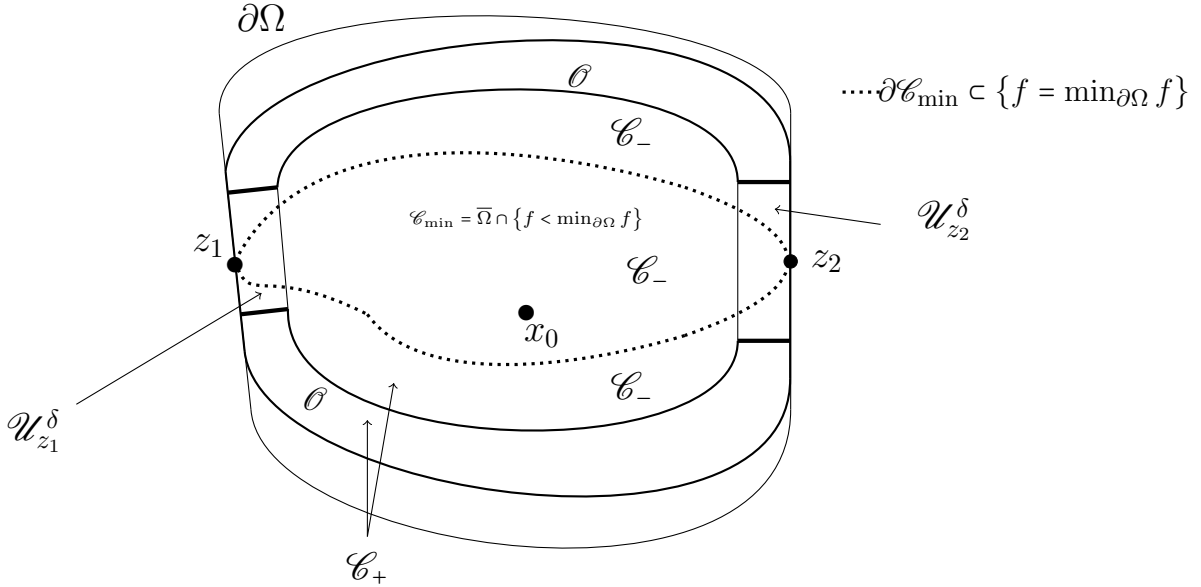


FIGURE 4.1. Schematic representation of $\mathcal{C}_-$, $\mathcal{C}_+$, and $\mathcal{O}$ (see Proposition 6). On the figure, $\mathcal{P}_{\text{sp}} = \{z_1, z_2\}$.

We refer to Figure 4.1 for a schematic representation of $\mathcal{C}_-$, $\mathcal{C}_+$, and $\mathcal{O}$. Note that $x_0 \in \mathcal{C}_-$ and hence,

$$\operatorname{argmin}_{\overline{\mathcal{C}_+}} f = \operatorname{argmin}_{\overline{\mathcal{C}_-}} f = \{x_0\}. \quad (4.26)$$

Using the above sets $\mathcal{C}_+$ and $\mathcal{C}_-$, we define a function

$$w_h^{\text{app}} : \overline{\Omega} \rightarrow [0, 1]$$

as follows:

[−] For every $z \in \mathcal{P}_{\text{sp}}$, w_h^{app} is defined on the cylinder $\mathcal{U}_z^\delta$ (see (4.10)) by

$$\forall x \in \mathcal{U}_z^\delta, \quad w_h^{\text{app}}(x) := \varphi_z(v(x)), \quad \text{see (4.22)}. \quad (4.27)$$

[−] From (4.23), (4.24), and the fact that $\overline{\mathcal{C}}_- \subset \mathcal{C}_+$ (see Proposition 6), the above function w_h^{app} satisfying (4.27) can be extended to $\overline{\Omega}$ so that

$$w_h^{\text{app}} = 0 \text{ on } \overline{\Omega} \setminus \mathcal{C}_+, \quad w_h^{\text{app}} = 1 \text{ on } \mathcal{C}_-, \text{ and } w_h^{\text{app}} \in C^\infty(\overline{\Omega}, [0, 1]). \quad (4.28)$$

Note that $w_h^{\text{app}} = 0$ on $\partial\Omega$. Moreover, in view of (4.22) and (4.24), w_h^{app} can be chosen on $\mathcal{O}$ such that, for some $C > 0$ and for every h small enough,

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \in \{1, 2\}, \quad \|\partial^\alpha w_h^{\text{app}}\|_{L^\infty(\mathcal{O})} \leq Ch^{-2}. \quad (4.29)$$

Notice that (4.28) implies

$$\text{supp } \nabla w_h^{\text{app}} \subset \overline{\mathcal{C}}_+ \setminus \mathcal{C}_-. \quad (4.30)$$

We finally define the normalized quasi-mode u_h^{app} over $\overline{\Omega}$ by

$$u_h^{\text{app}} = w_h^{\text{app}} / Z_h^{\text{app}} \quad \text{where } Z_h^{\text{app}} = \left(\int_{\Omega} |w_h^{\text{app}}|^2 p_h(x) dx \right)^{\frac{1}{2}}. \quad (4.31)$$

5. PROOF OF THEOREM 4

In all this section, we assume [A].

5.1. Action of L_h on the quasi-mode.

Proposition 7. *Assume [A]. There exist $h_0 > 0$ and a family of real numbers $(\zeta(h))_{h \in]0, h_0]}$ admitting a h -classical expansion $\zeta(h) \sim \sum_{j \geq 0} h^j \zeta_j$ with ζ_0 given by (3.7) such that, defining*

$$\lambda_h^{\text{app}} = h^{-\frac{1}{2}} \zeta(h) e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))},$$

the following holds true:

- [1] $\langle L_h u_h^{\text{app}}, u_h^{\text{app}} \rangle_{L^2(p_h dx)} = \lambda_h^{\text{app}} (1 + O(h^\infty))$
- [2] $\|L_h u_h^{\text{app}}\|_{L^2(p_h dx)}^2 = O(h^\infty) \lambda_h^{\text{app}}.$
- [3] $\|L_h^\dagger u_h^{\text{app}}\|_{L^2(p_h dx)}^2 = O(h^{-1} \lambda_h^{\text{app}})$, where $L_h^\dagger$ is the adjoint of the operator L_h in $L^2(\Omega, p_h dx)$.

Proof. In what follows, $C, c > 0$ are constants independent of $h > 0$ and of $x \in \overline{\Omega}$ which can change from one occurrence to another. We start by estimating Z_h^{app} in the limit $h \rightarrow 0$.

Asymptotic equivalent of Z_h^{app} . Recall that $u_h^{\text{app}} = w_h^{\text{app}}/Z_h^{\text{app}}$, see (4.31). From (4.28), we have

$$|Z_h^{\text{app}}|^2 = \int_{\Omega} |w_h^{\text{app}}(x)|^2 p_h(x) dx = \int_{\mathcal{C}_-} |w_h^{\text{app}}(x)|^2 p_h(x) dx + \int_{\mathcal{C}_+ \setminus \mathcal{C}_-} |w_h^{\text{app}}(x)|^2 p_h(x) dx.$$

Recall that from Theorem 2, $p_h(x) = h^{-\frac{d}{2}} R^h(x) e^{-\frac{2}{h}(f(x)-f(x_0))}$, where R^h admits a h -classical expansion $R^h \sim \sum_{j \geq 0} h^j R_j$ on $\mathbf{R}^d$. In particular R^h is uniformly bounded on the compact $\overline{\Omega} \subset \mathbf{R}^d$. Since $f \geq f(x_0) + c$ on $\mathcal{C}_+ \setminus \mathcal{C}_-$ (see (4.24)) and $|w_h^{\text{app}}| \leq 1$, it follows that

$$\int_{\mathcal{C}_+ \setminus \mathcal{C}_-} |w_h^{\text{app}}(x)|^2 p_h(x) dx = O(e^{-\frac{c}{h}}).$$

Moreover, using the fact that $w_h^{\text{app}} = 1$ on $\mathcal{C}_-$, and Lemma 1.2 of [48], we get

$$\int_{\mathcal{C}_-} |w_h^{\text{app}}(x)|^2 p_h(x) dx = \int_{\mathcal{C}_-} p_h(x) dx = \int_{\mathbf{R}^d} p_h(x) dx + O(e^{-\frac{c}{h}}) = 1 + O(e^{-\frac{c}{h}}).$$

We thus obtain that in the limit $h \rightarrow 0$,

$$Z_h^{\text{app}} = 1 + O(e^{-\frac{c}{h}}). \quad (5.1)$$

Proof of [1]. Note that $w_h^{\text{app}} \in \mathcal{D}_i$, for all $i = 1, 2, 3$. Then, from (3.2), and the fact that w_h^{app} is supported in $\overline{\mathcal{C}_+}$ and equal to 1 on $\mathcal{C}_-$, we have

$$\langle L_h w_h^{\text{app}}, w_h^{\text{app}} \rangle_{L^2(p_h dx)} = \frac{h}{2} \int_{\Omega} |\nabla w_h^{\text{app}}(x)|^2 p_h(x) dx = \frac{h}{2} \int_{\mathcal{C}_+ \setminus \mathcal{C}_-} |\nabla w_h^{\text{app}}(x)|^2 p_h(x) dx.$$

Using (4.24), this implies that

$$\langle L_h w_h^{\text{app}}, w_h^{\text{app}} \rangle_{L^2(p_h dx)} = \frac{h}{2} \int_{\mathcal{C}} |\nabla w_h^{\text{app}}(x)|^2 p_h(x) dx + \frac{h}{2} \sum_{z \in \mathcal{P}_{\text{sp}}} \int_{\mathcal{W}_z^\delta} |\nabla w_h^{\text{app}}(x)|^2 p_h(x) dx.$$

Moreover, thanks to Theorem 2 and (4.25), for all $x \in \mathcal{O}$, $p_h(x) \leq C e^{-2(\min_{\partial\Omega} f - f(x_0) + c)/h}$ and thanks to (4.29), one deduces that

$$\langle L_h w_h^{\text{app}}, w_h^{\text{app}} \rangle_{L^2(p_h dx)} = \frac{h}{2} \sum_{z \in \mathcal{P}_{\text{sp}}} \boldsymbol{\vartheta}_{z,h} + O(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}), \quad (5.2)$$

where

$$\boldsymbol{\vartheta}_{z,h} := \int_{\mathcal{U}_z^\delta} |\nabla w_h^{\text{app}}(x)|^2 p_h(x) dx.$$

We now estimate each integral $\boldsymbol{\vartheta}_{z,h}$ when $h \rightarrow 0$. To this end, fix z in $\mathcal{P}_{\text{sp}}$. Recall the coordinates $x \mapsto v(x)$ defined over $\mathcal{U}_z$ in Section 4.1, see (4.1)–(4.3). We also recall that $\hat{f} = f \circ v^{-1}$ (see (4.8)), $\hat{\ell} = \ell \circ v^{-1}$, and that for any smooth function u , we have $(\nabla u)(v^{-1}) = {}^t J^{-1} \nabla \hat{u}$. Using the v -coordinates, together with (4.12) and (4.27), one has

$$\boldsymbol{\vartheta}_{z,h} = \int_{\mathcal{V}_z^\delta} (G^{-1} \nabla \varphi_z) \cdot \nabla \varphi_z \hat{p}_h \sqrt{|G|}(v) dv.$$

Let ξ^z be given by Proposition 5. Recall that by (4.22), one has for all $v \in \mathcal{V}_z^\delta$,

$$\nabla \varphi_z(v) = -\frac{1}{N_{z,h}} \chi(\xi^z(v, h)) e^{\xi^z(v, h)/h} \nabla \xi^z(v, h).$$

Using (2.3), we then obtain that:

$$\begin{aligned} \boldsymbol{\vartheta}_{z,h} &= \frac{h^{-\frac{d}{2}}}{|N_{z,h}|^2} \int_{\mathcal{V}_z^\delta} G^{-1} \nabla \xi^z(v, h) \cdot \nabla \xi^z(v, h) |\chi(\xi^z(v, h))|^2 \\ &\quad \times \hat{R}^h(v) e^{-\frac{2}{h}(\hat{f}(v) - f(x_0) - \xi^z(v, h))} \sqrt{|G|}(v) dv. \end{aligned}$$

Moreover, using (4.8) and the fact that $f(z) = \min_{\partial\Omega} f$, one has $\hat{f}(v) - f(x_0) = \min_{\partial\Omega} f - f(x_0) + Q(v)$, see (4.20). Then, one has

$$\begin{aligned} &\frac{|N_{z,h}|^2}{h^{-\frac{d}{2}}} e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} \boldsymbol{\vartheta}_{z,h} \\ &= \int_{\mathcal{V}_z^\delta} G^{-1} \nabla \xi^z(v, h) \cdot \nabla \xi^z(v, h) |\chi(\xi^z(v, h))|^2 \hat{R}^h(v) e^{-\frac{2}{h}(Q(v) - \xi^z(v, h))} \sqrt{|G|}(v) dv. \end{aligned}$$

Recall that $\xi^z(v, h) \sim \sum_{j \geq 0} h^j \xi_j^z$ and $R^h(v) \sim \sum_{j \geq 0} h^j R_j(v)$, see respectively Proposition 5 and Theorem 2. Recall also that according to Propositions 3 and 4, ξ_0^z solves the eikonal equation (E) and ξ_1^z solves the first transport equation (T-j). We deduce that for any $N \in \mathbf{N}$,

$$\frac{|N_{z,h}|^2}{h^{-\frac{d}{2}}} e^{\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} \boldsymbol{\vartheta}_{z,h} = \sum_{j=0}^N h^j \boldsymbol{\vartheta}_{z,h}^{(j)} + O(h^{N+1}), \quad (5.3)$$

where $(\boldsymbol{\vartheta}_{z,h}^{(j)})_{j \geq 0} \subset \mathbf{R}$ and

$$\boldsymbol{\vartheta}_{z,h}^{(0)} = \int_{\mathcal{V}_z^\delta} G^{-1} \nabla \xi_0^z(v) \cdot \nabla \xi_0^z(v) \hat{R}_0(v) |\chi(\xi_0^z(v))|^2 e^{2\xi_1^z(v)} e^{-\frac{2}{h}(Q(v) - \xi_0^z(v))} \sqrt{|G|}(v) dv.$$

Note that we have used $[\mathbf{C}_4^\delta]$ in see Section 4.4, and more precisely that $Q(v) - \xi_0^z(v) \geq 0$, to obtain the remainder term $O(h^{N+1})$ in (5.3).

Recall (4.21). Using the condition $[\mathbf{C}_5^\delta]$ (see Section 4.4), we deduce that

$$\begin{aligned} \boldsymbol{\vartheta}_{z,h}^{(0)} &= \int_{\mathcal{V}_z^\delta} G^{-1} \nabla \xi_0^z(v) \cdot \nabla \xi_0^z(v) \hat{R}_0(v) e^{2\xi_1^z(v)} e^{-\frac{2}{h}(Q^+(v) - v_d \mathcal{A}(v))} \sqrt{|G|}(v) dv \\ &\quad + O(e^{-\frac{c}{h}}). \end{aligned} \quad (5.4)$$

By Propositions 3 and 4, one has $|\nabla \xi_0^z(0)|^2 = 4|\mu_z|^2$ and $\xi_1^z(0) = 0$. Then, performing a Taylor expansion, we deduce from (2.15) and (4.3) that there exists a sequence $(\boldsymbol{\theta}_\alpha)_{\alpha \in \mathbf{N}^d} \subset \mathbf{R}$ such that for any $K \in \mathbf{N}$:

$$\sqrt{|G|}(v) G^{-1}(v) \nabla \xi_0^z(v) \cdot \nabla \xi_0^z(v) \hat{R}_0(v) e^{2\xi_1^z(v)} = \sum_{|\alpha| \leq K} \boldsymbol{\theta}_\alpha v^\alpha + O(|v|^{K+1}), \quad (5.5)$$

where

$$\boldsymbol{\theta}_0 = 4|\mu_z|^2 c_0 \exp \left[\int_0^{+\infty} \operatorname{div}(\ell)(\psi_t(z)) dt \right] \quad (5.6)$$

and $c_0 > 0$ is given by (2.6). For $\alpha \in \mathbf{N}^d$, we write $\alpha = (\alpha', \alpha_d)$ with $\alpha' \in \mathbf{N}^{d-1}$ and $\alpha_d \in \mathbf{N}$. Set $v_h = (\sqrt{h}v', hv_d)$. Then, combining (5.4) and (5.5), it follows that for any $K \in \mathbf{N}$, using a change of variables,

$$\begin{aligned} \boldsymbol{\vartheta}_{z,h}^{(0)} &= \sum_{|\alpha| \leq 2K} \boldsymbol{\theta}_\alpha \int_{\mathcal{V}_z^\delta} v^\alpha e^{-\frac{2}{h}(Q^+(v) - v_d \mathcal{A}(v))} dv + \int_{\mathcal{V}_z^\delta} O(|v|^{2K+1}) e^{-\frac{2}{h}(Q^+(v) - v_d \mathcal{A}(v))} dv \\ &= h^{\frac{d+1}{2}} \left[\sum_{\substack{|\alpha'| \leq K \\ \frac{|\alpha'|}{2} + \alpha_d \leq K}} \boldsymbol{\theta}_\alpha h^{\frac{|\alpha'|}{2} + \alpha_d} \int_{|v'| \leq \delta_2/\sqrt{h}} \int_{-2\delta_1/h}^0 v^\alpha e^{-2(Q^+(v) - v_d \mathcal{A}(v_h))} dv + O(h^{K+\frac{1}{2}}) \right]. \end{aligned}$$

Recall that $\mathcal{A}(0) = 0$. Hence, performing a second Taylor expansion and denoting, for $k \in \mathbf{N}$, by $(\mathbf{v}_\beta)_{\beta \in \mathbf{N}^d, |\beta|=k}$ the monomial basis of the homogeneous polynomials of order k , there exists a family $(\lambda_\beta)_{\beta \in \mathbf{N}^d} \subset \mathbf{R}$ such that, for every $K \in \mathbf{N}$,

$$2v_d \mathcal{A}(v_h) = \sum_{|\beta| \geq 1, \frac{|\beta'|}{2} + \beta_d \leq K} h^{\frac{|\beta'|}{2} + \beta_d} 2\lambda_\beta v_d \mathbf{v}_\beta(v) + O(h^{K+\frac{1}{2}}).$$

Thus, it holds

$$e^{2v_d \mathcal{A}(v_h)} = 1 + \sum_{|\gamma| \geq 1, |\gamma'|/2 + \gamma_d \leq K} h^{\frac{|\gamma'|}{2} + \gamma_d} \mathcal{A}_\gamma(v) + O(h^{K+\frac{1}{2}}),$$

where each $\mathcal{A}_\gamma(v)$ is a linear combination of monomials of the form $v_d^m \mathbf{v}_{\beta_1}(v) \dots \mathbf{v}_{\beta_m}(v)$ with $m \geq 1$ and where

$$\gamma = (\beta'_1 + \dots + \beta'_m, (\beta_1)_d + \dots + (\beta_m)_d) \in \mathbf{N}^d \quad \text{and} \quad \frac{1}{2}|\gamma'| + \gamma_d = \frac{1}{2} \sum_{i=1}^m |\beta'_i| + \sum_{i=1}^m (\beta_i)_d \geq \frac{1}{2}.$$

Therefore, there exist coefficients $\boldsymbol{\theta}_\alpha^*$ such that $\boldsymbol{\theta}_0^* = \boldsymbol{\theta}_0$ and

$$\begin{aligned} \boldsymbol{\vartheta}_{z,h}^{(0)} &= h^{\frac{d+1}{2}} \left[\sum_{\frac{|\alpha'|}{2} + \alpha_d \leq K} \boldsymbol{\theta}_\alpha^* h^{\frac{|\alpha'|}{2} + \alpha_d} \int_{|v'| \leq \delta_2/\sqrt{h}} \int_{-2\delta_1/h}^0 v^\alpha P_\alpha(v_d) e^{-2Q^+(v)} dv + O(h^{K+\frac{1}{2}}) \right] \\ &= h^{\frac{d+1}{2}} \left[\sum_{\frac{|\alpha'|}{2} + \alpha_d \leq K} \boldsymbol{\theta}_\alpha^* h^{\frac{|\alpha'|}{2} + \alpha_d} \int_{v' \in \mathbf{R}^{d-1}} \int_{v_d < 0} v^\alpha P_\alpha(v_d) e^{-2Q^+(v)} dv + O(h^{K+\frac{1}{2}}) \right], \end{aligned}$$

where for $\alpha \in \mathbf{N}^d$, $P_\alpha \in \mathbf{R}[X]$. For a parity reason, if α' is an odd number, $\int_{v' \in \mathbf{R}^{d-1}} v^\alpha e^{-2Q^+(v)} dv' = 0$. Hence, we deduce that as $h \rightarrow 0$,

$$\boldsymbol{\vartheta}_{z,h}^{(0)} \sim h^{\frac{d+1}{2}} \sum_{k \geq 0} h^k K_{z,k}^{(0)}$$

for some sequence $(K_{z,k}^{(0)})_{k \geq 0} \subset \mathbf{R}$ such that

$$K_{z,0}^{(0)} = \boldsymbol{\theta}_0 \int_{\mathbf{R}^{d-1} \times \mathbf{R}^-} e^{2\mu_z v_d - H_z v' \cdot v'} dv = \frac{\boldsymbol{\theta}_0 \pi^{\frac{d-1}{2}}}{2\mu_z \sqrt{\det H_z}}.$$

Combined with (5.6) and (2.6), this gives

$$K_{z,0}^{(0)} = \frac{2\mu_z}{\sqrt{\pi}} \frac{\sqrt{\det \text{Hess}(f)(x_0)}}{\sqrt{\det H_z}} \exp \left[\int_0^{+\infty} \text{div}(\ell)(\psi_t(z)) dt \right].$$

Similar arguments show that for any $j \geq 1$, there exists a sequence $(K_{z,k}^{(j)})_{k \geq 0} \subset \mathbf{R}$ such that

$$\boldsymbol{\vartheta}_{z,h}^{(j)} \sim h^{\frac{d+1}{2}} \sum_{k \geq 0} h^k K_{z,k}^{(j)}.$$

Using (5.2), (5.3), and (4.15), we finally obtain

$$\langle L_h w_h^{\text{app}}, w_h^{\text{app}} \rangle_{L^2(p_h dx)} \sim \frac{e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}}{\sqrt{h}} \sum_{j \geq 0} h^j \zeta_j(z) \quad \text{with} \quad \zeta_0(z) = \sum_{z \in \mathcal{P}_{\text{sp}}} \frac{1}{2} K_{z,0}^{(0)}.$$

Moreover since $u_h^{\text{app}} = \frac{w_h^{\text{app}}}{Z_h^{\text{app}}}$ and $Z_h^{\text{app}} = 1 + O(e^{-\frac{c}{h}})$ (see (5.1)), we deduce that

$$\langle L_h u_h^{\text{app}}, u_h^{\text{app}} \rangle_{L^2(p_h dx)} \sim \frac{e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}}{\sqrt{h}} \sum_{j \geq 0} h^j \zeta_j(z).$$

This completes the proof of [1].

Proof of [2]. Since w_h^{app} is supported in $\overline{\mathcal{C}}_+$, equal to 1 in $\mathcal{C}_-$ and since L_h has no zero order term, then

$$\|L_h w_h^{\text{app}}\|_{L^2(p_h dx)}^2 = \int_{\mathcal{C}_+ \setminus \mathcal{C}_-} |L_h w_h^{\text{app}}(x)|^2 p_h(x) dx.$$

Since $\overline{\mathcal{C}}_+ \setminus \mathcal{C}_- = \bigcup_{z \in \mathcal{P}_{\text{sp}}} \mathcal{U}_z^\delta \cup \mathcal{O}$, thanks to (4.29), (4.25) and Theorem 2, we get

$$\|L_h w_h^{\text{app}}\|_{L^2(p_h dx)}^2 = \sum_{z \in \mathcal{P}_{\text{sp}}} \int_{\mathcal{U}_z^\delta} |L_h w_h^{\text{app}}(x)|^2 p_h(x) dx + O(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}). \quad (5.7)$$

Using the v -coordinates, it follows from (4.27) that

$$J_{z,h} := \int_{\mathcal{U}_z^\delta} |L_h w_h^{\text{app}}(x)|^2 p_h(x) dx = \int_{\mathcal{V}_z^\delta} |\hat{L}_h \varphi_z(v)|^2 \hat{p}_h(v) \sqrt{|G|}(v) dv. \quad (5.8)$$

On the other hand, recall (4.16):

$$\begin{aligned} \hat{L}_h \varphi_z &= \frac{\chi(\xi^z) e^{\xi^z/h}}{N_{z,h}} \left[\frac{1}{2} (G^{-1} \nabla \xi^z) \cdot \nabla \xi^z + [b^\circ \cdot \nabla \xi^z + h(\rho^\circ \cdot \nabla \xi^z + \frac{1}{2} \operatorname{div}(G^{-1} \nabla \xi^z))] \right] \\ &\quad + \frac{h}{2N_{z,h}} (G^{-1} \nabla \xi^z) \cdot \nabla \xi^z \chi'(\xi^z) e^{\xi^z/h}. \end{aligned}$$

Recall also that $\hat{f} = f(z) + Q$ (see (4.8)). Then, using Theorem 2 and Proposition 5, one deduces that (see (4.20))

$$J_{z,h} = \frac{h^{-\frac{d}{2}} e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}}{|N_{z,h}|^2} \int_{\mathcal{V}_z^\delta} |O(h^\infty) + O(h) \chi'(\xi^z)|^2 \hat{R}^h(v) e^{-\frac{2}{h}(Q(v) - \xi^z(v,h))} dv$$

Moreover, thanks to $[\mathbf{C}_8^\delta]$ in Section 4.4, there exists $c > 0$ such that $Q(v) - \xi^z(v,h) \geq c$ for every $v \in \mathcal{V}_z^\delta$ satisfying $\xi^z(v,h) \in \operatorname{supp}(\chi')$. Hence, we have

$$\begin{aligned} J_{z,h} &= \frac{h^{-\frac{d}{2}} e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}}{|N_{z,h}|^2} \int_{\mathcal{V}_z^\delta} O(h^\infty) \hat{R}^h(v) e^{-\frac{2}{h}(Q(v) - \xi^z(v,h))} dv + O(h^\infty e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}) \\ &= O(h^\infty e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}). \end{aligned}$$

In view of (5.7) and (5.8), this yields the desired result, namely [2].

Proof of [3]. We start with the computation of the adjoint $L_h^\dagger$ of L_h in $L^2(p_h dx)$. Set

$$\psi_h = -h \ln(p_h).$$

Recall that the function ψ_h is well-defined and smooth. Note that $\nabla\psi_h = (-h\nabla R^h + 2R^h\nabla f)/R^h$. Hence, according to Theorem 2 and formula (2.15), for every $h > 0$ small enough and $\alpha \in \mathbf{N}^d$, $\partial^\alpha\psi_h$ is bounded uniformly with respect to h on $\overline{\Omega}$. On the other hand, since $L_h^*p_h = 0$ (see (2.2)), a direct computation shows that

$$L_h^\dagger = -\frac{h}{2}\Delta + (b + \nabla\psi_h) \cdot \nabla. \quad (5.9)$$

Now, the same computation as the one leading to (5.7) shows that

$$\|L_h^\dagger w_h^{\text{app}}\|_{L^2(p_h dx)}^2 = \sum_{z \in \mathcal{P}_{\text{sp}}} \int_{\mathcal{U}_z^\delta} |L_h^\dagger w_h^{\text{app}}(x)|^2 p_h(x) dx + O(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}). \quad (5.10)$$

Using the v -coordinates, we have:

$$\|L_h^\dagger w_h^{\text{app}}\|_{L^2(p_h dx)}^2 = \sum_{z \in \mathcal{P}_{\text{sp}}} \int_{\mathcal{V}_z^\delta} |\hat{L}_h^\dagger \varphi_z(v)|^2 \hat{p}_h(v) \sqrt{|G|} dv + O(e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0) + c)}).$$

Note that $\hat{L}_h^\dagger = -\frac{h}{2} \text{div} \circ G^{-1} \circ \nabla + e_h \cdot \nabla$, where e_h is a smooth vector field uniformly bounded with respect to h . In particular, one has for every h small enough

$$\hat{L}_h^\dagger \varphi_z = O(e^{\xi^z/h}/N_{z,h}).$$

Using (4.15) and the fact that

$$\hat{p}_h e^{2\xi^z/h} = h^{-\frac{d}{2}} e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))} \hat{R}^h e^{-\frac{2}{h}(Q - \xi^z)},$$

we get, since $\int_{\mathcal{V}_z^\delta} e^{-\frac{2}{h}(Q - \xi^z)} dv = O(\int_{\mathcal{V}_z^\delta} e^{-\frac{2}{h}(Q - \xi_0^z)} dv) = O(h^{\frac{d+1}{2}})$,

$$\|L_h^\dagger w_h^{\text{app}}\|_{L^2(p_h dx)}^2 = O(h^{-\frac{3}{2}} e^{-\frac{2}{h}(\min_{\partial\Omega} f - f(x_0))}) = O(h^{-1}) \lambda_h^{\text{app}}.$$

This concludes the proof of [3]. □

5.2. Proof of Theorem 4. According to Theorem 3, for all $\beta > 0$ small enough, the projector

$$\pi_h := \frac{1}{2i\pi} \int_{\{|z|=\beta\}} (z - L_h)^{-1} dz$$

is of rank one for all $h > 0$ small enough, and more precisely $\text{Ran } \pi_h = \text{Span}(u_h)$. Moreover, Theorem 3 implies that there exists $C > 0$ such that for all $|z| = \beta$, $\|(z - L_h)^{-1}\|_{L^2(p_h dx)} \leq C$. In particular, one has $\pi_h = O(1)$ and for all $u \in \mathcal{D}_2$, $\|(1 - \pi_h)u\|_{L^2(p_h dx)} \leq C \|L_h u\|_{L^2(p_h dx)}$, which follows from the identity

$$(1 - \pi_h)u = \frac{-1}{2\pi i} \int_{\{|z|=\beta\}} z^{-1} (z - L_h)^{-1} L_h u dz.$$

Thus, thanks to Item [2] in Proposition 7, one obtains that in the limit $h \rightarrow 0$:

$$\pi_h u_h^{\text{app}} = u_h^{\text{app}} + O(h^\infty |\lambda_h^{\text{app}}|^{\frac{1}{2}}) \quad \text{in } L^2(\Omega, p_h dx). \quad (5.11)$$

We thus have for every h small enough,

$$\begin{aligned} \lambda_h &= \frac{\langle L_h \pi_h u_h^{\text{app}}, \pi_h u_h^{\text{app}} \rangle_{L^2(p_h dx)}}{\|\pi_h u_h^{\text{app}}\|_{L^2(p_h dx)}^2} = \frac{\langle L_h u_h^{\text{app}}, u_h^{\text{app}} \rangle_{L^2(p_h dx)} + E_h}{1 + O(e^{-\frac{c}{h}})} \\ &= \lambda_h^{\text{app}}(1 + O(h^\infty)) + E_h(1 + O(e^{-\frac{c}{h}})), \end{aligned}$$

where $E_h := \langle L_h(\pi_h - 1)u_h^{\text{app}}, u_h^{\text{app}} \rangle_{L^2(p_h dx)} + \langle L_h \pi_h u_h^{\text{app}}, (\pi_h - 1)u_h^{\text{app}} \rangle_{L^2(p_h dx)}$ is the so-called projection error. It satisfies

$$|E_h| \leq C \|(\pi_h - 1)u_h^{\text{app}}\|_{L^2(p_h dx)} (\|L_h^\dagger u_h^{\text{app}}\|_{L^2(p_h dx)} + \|L_h u_h^{\text{app}}\|_{L^2(p_h dx)})$$

which, combined with (5.11) and [3] of Proposition 7, yields

$$|E_h| = O(h^\infty) |\lambda_h^{\text{app}}|^{\frac{1}{2}} \times O(h^{-\frac{1}{2}}) |\lambda_h^{\text{app}}|^{\frac{1}{2}} + O(h^\infty) \lambda_h^{\text{app}} = O(h^\infty) \lambda_h^{\text{app}}.$$

This proves the desired result using Item [1] in Proposition 7.

5.3. On the assumption $[\mathbf{A}_\perp]$. In statistical physics, we are typically given a vector field $b : \mathbf{R}^d \rightarrow \mathbf{R}^d$, that we assume to be smooth. Metastability occurs when there exist several stable equilibrium positions for the dynamics (1.3), see e.g. [16] and references therein. Let x_0 be one of them and assume that

$$\text{all the eigenvalues of } \text{Jac } b(x_0) \text{ have negative real parts.} \quad (5.12)$$

Let us denote by B_{x_0} the basin of attraction of x_0 for the dynamics (1.3) and by $V(x) = V(x_0, x)$ the Freidlin-Wentzell quasi-potential from x_0 to $x \in \overline{B_{x_0}}$ of the process (1.1) in $\overline{B_{x_0}}$ (see e.g. [20] or [14, Equation (2.14)]). According to [14, Corollaries 2 and 7], $V(x) > 0$ over $\mathbf{R}^d \setminus \{0\}$ and there exists a connected open and dense subset G_{x_0} of B_{x_0} containing x_0 such that $x \in G_{x_0} \mapsto V(x)$ is $\mathcal{C}^\infty$. By [20, Section 4.3], one deduces that there exists a smooth vector field $\ell : G_{x_0} \rightarrow \mathbf{R}^d$ such that b admits the following transversal decomposition over G_{x_0} :

$$b = -\frac{1}{2} \nabla V + \ell \quad \text{and} \quad \nabla V \cdot \ell = 0.$$

This shows that the assumption $[\mathbf{A}_\perp]$ is generic with $f = V/2$, at least in the (large) set G_{x_0} . Note that though the domain G_{x_0} is very large in B_{x_0} , without any extra assumption, it might not necessarily be the whole space B_{x_0} . We also

mention that by [13, Equation (4.2) and Theorem 3], $\nabla V(x_0) = 0$, the Hessian matrix $\text{Hess } V(x_0)$ at x_0 is positive definite, and all the statements in Theorem 2 holds true uniformly in the compacts of G_{x_0} as well as the formula (2.15).

In conclusion, if $x_0 \in \mathbf{R}^d$ is such that $b(x_0) = 0$ and (5.12) holds, then, for any smooth bounded domain Ω containing x_0 such that $\overline{\Omega} \subset G_{x_0}$ and $[\mathbf{A}_{\mathcal{P}_{\text{sp}}}]$ holds, the Eyring-Kramers formula (1.5) holds uniformly for x in the compact subsets of $\mathcal{A}_{\Omega}(x_0)$.

Remark 8. *The regularity of the quasi-potential is at the heart of the transversal decomposition of b which is possible in the regions of B_{x_0} where V is sufficiently smooth (say at least $\mathcal{C}^1$, see [20, Section 4.3]). More generally, finding the regions where the quasi-potential is smooth is a very challenging task for the success of several asymptotic formulas in statistical physics [14, 11, 13, 8], as it is the case here.*

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D. LE PEUTREC, LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES

Email address: dorian.lepeutrec@univ-nantes.fr

L. MICHEL, INSTITUT MATHÉMATIQUE DE BORDEAUX, UNIVERSITÉ DE BORDEAUX

Email address: laurent.michel@math.u-bordeaux.fr

B. NECTOUX, LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL, UCA

Email address: boris.nectoux@uca.fr