SHARP SPECTRAL ASYMPTOTICS FOR NON-REVERSIBLE METASTABLE DIFFUSION PROCESSES

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ABSTRACT. Let $U_h : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field and consider the associated overdamped Langevin equation

$$dX_t = -U_h(X_t)\,dt + \sqrt{2h}\,dB_t$$

in the low temperature regime $h \to 0$. In this work, we study the spectrum of the associated diffusion $L = -h\Delta + U_h \cdot \nabla$ under the assumptions that $U_h = U_0 + h\nu$, where the vector fields $U_0 : \mathbb{R}^d \to \mathbb{R}^d$ and $\nu : \mathbb{R}^d \to \mathbb{R}^d$ are independent of $h \in (0, 1]$, and that the dynamics admits $e^{-\frac{V}{h}}$ as an invariant measure for some smooth function $V : \mathbb{R}^d \to \mathbb{R}$. Assuming additionally that V is a Morse function admitting n_0 local minima, we prove that there exists $\epsilon > 0$ such that in the limit $h \to 0$, L admits exactly n_0 eigenvalues in the strip $\{\mathrm{Re}(z) < \epsilon\}$ which have moreover exponentially small moduli. Under a generic assumption on the potential barriers of the Morse function V, we also prove that the asymptotic behaviors of these small eigenvalues are given by Eyring-Kramers type formulas.

Contents

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1. Introduction

Let $d \geq 2$, $U_h : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field depending on a small parameter $h \in (0,1]$, and consider the associated overdamped Langevin equation

$$dX_t \;=\; -U_h(X_t)\,dt + \sqrt{2h}\,dB_t\,, \tag{1.1}$$

where $X_t \in \mathbb{R}^d$ and $(B_t)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^d$. The associated Kolmogorov (backward) and Fokker-Planck equations are then the evolution equations

$$\partial_t\, u + L(u) \;=\; 0 \quad \text{and} \quad \partial_t\, \rho + L^\dagger(\rho) \;=\; 0\,, \tag{1.2}$$

where the elliptic differential operator

$$L \;=\; -h\Delta + U_h \cdot \nabla$$

is the infinitesimal generator of the process (1.1),

$$L^\dagger \;=\; -\operatorname{div} \circ (h\nabla + U_h)$$

denotes the formal adjoint of L, and for $x \in \mathbb{R}^d$ and $t \geq 0$: $u(t,x) = \mathbb{E}^x[f(X_t)]$ is the expected value of the observable $f(X_t)$ when $X_0 = x$ and $\rho(t,\cdot)$ is the probability density (with respect to the Lebesgue measure on $\mathbb{R}^d$) of presence of $(X_t)_{t\geq 0}$. In this setting, the Fokker–Planck equation, that is the second equation of (1.2), is also known as the Kramers-Smoluchowski equation.

Throughout this paper, we assume that the vector field U_h decomposes as

$$U_h \;=\; U_0 + h\nu$$

for some real smooth vector fields U_0 and ν independent of h. Moreover, we consider the case where the above overdamped Langevin dynamics admits a specific stationary distribution satisfying the following assumption:

Assumption 1. *There exists a smooth function $V : \mathbb{R}^d \to \mathbb{R}$ such that $L^\dagger(e^{-\frac{V}{h}}) = 0$ for every $h \in (0,1]$.*

A straightforward computation shows that Assumption 1 is satisfied if and only if the vector field $U_h = U_0 + h\nu$ satisfies the following relations, where we denote $b := U_0 - \nabla V$,

$$b \cdot \nabla V \;=\; 0\,, \quad \operatorname{div}(\nu) \;=\; 0\,, \quad \text{and} \quad \operatorname{div}(b) \;=\; \nu \cdot \nabla V\,. \tag{1.3}$$

Using this decomposition, the generator L writes

$$L_{V,b,\nu} \;:=\; L \;=\; -h\Delta + \nabla V \cdot \nabla + b_h \cdot \nabla\,, \tag{1.4}$$

where

$$b_h \;:=\; b + h\nu \;=\; U_0 - \nabla V + h\nu \;=\; U_h - \nabla V\,. \tag{1.5}$$

Note moreover that the two following particular cases enter in the framework of Assumption 1:

1. The case where

$$b\cdot\nabla V \;=\; 0\,,\quad \operatorname{div} b \;=\; 0 \;\text{ and }\; \nu \;=\; 0\,, \tag{1.6}$$

which is in particular satisfied when $\nu = 0$ and b has the form $b = J(\nabla V)$, where J is a smooth map from $\mathbb{R}^d$ into the set of real antisymmetric matrices of size d such that $\operatorname{div}\big(J(\nabla V)\big) = 0$. For instance, this later condition holds if $J(x) = \tilde J\circ V(x)$ for some antisymmetric matrices $\tilde J(y)$ depending smoothly on $y\in\mathbb{R}$.

2. The case where

$$b \;=\; J(\nabla V) \quad\text{and}\quad \nu \;=\; \Big(\sum_{i=1}^{d}\partial_i J_{ij}\Big)_{1\leq j\leq d}, \tag{1.7}$$

where J is a smooth map from $\mathbb{R}^d$ into the set of real antisymmetric matrices of size d.

In the case of (1.7), $L_{V,b,\nu}$ has in particular the following supersymmetric-type structure,

$$L_{V,b,\nu} \;=\; -h\,e^{\frac{V}{h}}\,\operatorname{div}\circ\Big(e^{-\frac{V}{h}}\big(I_d - J\big)\nabla\Big)\,, \tag{1.8}$$

and both cases coincide when b_h has the form $b_h = b = J(\nabla V)$ for some constant antisymmetric matrix J. In the case of (1.6), the structure (1.8) fails to be true in general and we refer to [19] for more details on these questions. Let us also point out that under Assumption 1, the vector field b_h defined in (1.5) is very close to the transverse vector field introduced in [1] and next used in [14].

In this paper, we are interested in the spectral analysis of the operator $L_{V,b,\nu}$ and in its connections with the long-time behaviour of the dynamics (1.1) when $h\to 0$. In this regime, the process $(X_t)_{t\geq 0}$ solution to (1.1) is typically metastable, which is characterized by a very slow return to equilibrium. We refer in particular in this connection to the related works [1,14] dealing with the mean transition times between the different wells of the potential V for the process $(X_t)_{t\geq 0}$. In view of Assumption 1, we thus look at $L_{V,b,\nu}$ acting in the natural weighted Hilbert space $L^2(\mathbb{R}^d, m_h)$, where

$$m_h(dx) \;:=\; Z_h^{-1}e^{-\frac{V(x)}{h}}\,dx \quad\text{and}\quad Z_h \;:=\; \int_{\mathbb{R}^d} e^{-\frac{V(x)}{h}}\,dx\,. \tag{1.9}$$

Note that we assume here that $e^{-\frac{V}{h}}\in L^1(\mathbb{R}^d)$ for every $h\in(0,1]$, which will be a simple consequence of our further hypotheses.

In this setting, a first important consequence of (1.3) is the following identity, easily deduced from the relation $\operatorname{div}(b_h e^{-\frac{V}{h}}) = 0$,

$$\forall\, u,v\,\in\,\mathcal{C}_c^\infty(\mathbb{R}^d)\,,\quad \langle L_{V,b,\nu}u, v\rangle_{L^2(m_h)} \;=\; \langle u, L_{V,-b,-\nu}v\rangle_{L^2(m_h)}\,.$$

In particular, using (1.4), it holds

$$\begin{aligned}\operatorname{Re}\langle L_{V,b,\nu}u,u\rangle_{L^2(m_h)} \;&=\; \langle(-h\Delta + \nabla V\cdot\nabla)u, u\rangle_{L^2(m_h)}\\ &=\; h\,\|\nabla u\|^2_{L^2(m_h)} \;\geq\; 0\end{aligned} \tag{1.10}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and the operator $L_{V,b,\nu}$ acting on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, m_h)$ is hence accretive.

Let us now introduce the following confining assumptions at infinity on the functions V, b, and ν that we will consider in the rest of this work.

Assumption 2. *There exist $C > 0$ and a compact set $K \subset \mathbb{R}^d$ such that it holds*

$$V \;\geq\; -C \quad \text{on} \;\; \mathbb{R}^d$$

and, for all $x \in \mathbb{R}^d \setminus K$,

$$|\nabla V(x)| \;\geq\; \frac{1}{C} \quad \text{and} \quad |\mathrm{Hess}\, V(x)| \;\leq\; C|\nabla V(x)|^2 \,. \tag{1.11}$$

Moreover, there exists $C > 0$ such that the vector fields $b = U_0 - \nabla V$ and ν satisfy the following estimate for all $x \in \mathbb{R}^d$:

$$|b(x)| + |\nu(x)| \;\leq\; C\,(1 + |\nabla V(x)|). \tag{1.12}$$

One can show that when V is bounded from below and the first estimate of (1.11) is satisfied, it also holds, for some $C > 0$, $|V(x)| \geq C|x|$ outside a compact set (see for example [18, Lemma 3.14]). In particular, when Assumption 2 is satisfied, then $e^{-\frac{V}{h}} \in L^1(\mathbb{R}^d)$ for all $h \in (0,1]$ (which justifies the definition of Z_h in (1.9)).

In order to study the operator $L_{V,b,\nu}$ in $L^2(\mathbb{R}^d, m_h)$, it is often useful to work with its counterpart in the flat space $L^2(\mathbb{R}^d, dx)$ by using the unitary transformation

$$\Omega \,:\, L^2(\mathbb{R}^d, dx) \longrightarrow L^2(\mathbb{R}^d, m_h)\,, \quad \Omega u \;=\; m_h^{-\frac{1}{2}} u = \; Z_h^{\frac{1}{2}} e^{\frac{V}{2h}}\, u\,.$$

Defining $\phi := \frac{V}{2}$, we then have the unitary equivalence

$$\begin{aligned} \Omega^*\, h\, L_{V,b,\nu}\, \Omega \;&=\; -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi + \, b_h \cdot d_{\phi,h} \\ &=\; \Delta_\phi + b_h \cdot d_\phi\,, \end{aligned} \tag{1.13}$$

where

$$d_\phi \;:=\; d_{\phi,h} \;:=\; h\nabla + \nabla\phi \;=\; he^{-\frac{\phi}{h}} \nabla e^{\frac{\phi}{h}}$$

and

$$\Delta_\phi \;:=\; \Delta_{\phi,h} \;:=\; -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi \;=\; -h^2 e^{\frac{\phi}{h}}\, \mathrm{div}\, e^{-\frac{\phi}{h}} d_\phi$$

denotes the usual semiclassical Witten Laplacian acting on functions (throughout we will sometimes consider d_ϕ as on operator sending functions to 1-forms, but most of the time $d_\phi u$ will simply denote a vector of $\mathbb{R}^d$). It is thus equivalent to study $L_{V,b,\nu}$ acting in the weighted space $L^2(\mathbb{R}^d, m_h)$ or

$$P_\phi \;:=\; P_{\phi,b,\nu} \;:=\; \Delta_\phi + b_h \cdot d_\phi \tag{1.14}$$

acting in the flat space $L^2(\mathbb{R}^d, dx)$.

The Witten Laplacian $\Delta_\phi \;=\; P_{\phi,0,0}$, which is the counterpart of the weighted Laplacian

$$L_{V,0,0} \;=\; -h\Delta + \nabla V \cdot \nabla = h\nabla^*\nabla$$

(the adjoint is considered here with respect to m_h) acting in the flat space $L^2(\mathbb{R}^d, dx)$ is moreover essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ (see [7, Theorem 9.15]). We still denote by Δ_ϕ its unique self-adjoint extension and by $D(\Delta_\phi)$ the domain of this extension. In addition, it is clear that for every $h \in (0,1]$, it holds $\Delta_\phi e^{-\frac{\phi}{h}} = 0$ in the distribution sense. Hence, under Assumption 2, since $\phi = \frac{V}{2}$ satisfies the relation (1.11), it holds $e^{-\frac{\phi}{h}} \in L^2(\mathbb{R}^d)$ and the essential self-adjointness of Δ_ϕ then implies that $e^{-\frac{\phi}{h}} \in D(\Delta_\phi)$ so that $0 \in \operatorname{Ker} \Delta_\phi$. It follows moreover from (1.11) and from [8, Proposition 2.2] that there exists $h_0 > 0$ and $c_0 > 0$ such that for all $h \in (0, h_0]$, it holds

$$\sigma_{ess}(\Delta_\phi) \subset [c_0, \infty[.$$

Coming back to the more general operator $P_\phi = P_{\phi,b,\nu}$ defined in (1.14), or equivalently to the operator $L_{V,b,\nu}$ according to the relation (1.13), the following proposition gathers some of its basic properties which specify in particular the preceding properties of Δ_ϕ (and their equivalents concerning the weighted Laplacian $L_{V,0,0}$). It will be proven in the following section.

Proposition 1.1. *Under Assumption 1, the operator P_ϕ with domain $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is accretive. Moreover, assuming in addition Assumption 2, there exists $h_0 \in (0,1]$ such that the following hold true for every $h \in (0, h_0]$:*

i) The closure of $(P_\phi, \mathcal{C}_c^\infty(\mathbb{R}^d))$, that we still denote by P_ϕ, is maximal accretive, and hence its unique maximal accretive extension.

ii) The operator P_ϕ^* is maximal accretive and $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is a core for P_ϕ^*. We have moreover the inclusions

$$D(\Delta_\phi) \subset D(P_\phi) \cap D(P_\phi^*) \subset D(P_\phi) \cup D(P_\phi^*) \subset \{u \in L^2(\mathbb{R}^d),\ d_\phi u \in L^2(\mathbb{R}^d)\},$$

where, for any unbounded operator A, $D(A)$ denotes the domain of A. In addition, for $\mathbf{P}_\phi \in \{P_\phi, P_\phi^*\}$, we have the equality

$$\forall\, u \in D(\mathbf{P}_\phi), \quad \operatorname{Re}\langle \mathbf{P}_\phi u, u\rangle = \|d_\phi u\|^2.$$

iii) There exists $\Lambda_0 > 0$ such that, defining

$$\Gamma_{\Lambda_0} := \Big\{ \operatorname{Re}(z) \geq 0 \text{ and } |\operatorname{Im} z| \leq \Lambda_0 \max\Big(\operatorname{Re}(z), \sqrt{\operatorname{Re}(z)}\Big)\Big\} \subset \mathbb{C},$$

the spectrum $\sigma(P_\phi)$ of P_ϕ is included in Γ_{Λ_0} and

$$\forall\, z \in \Gamma_{\Lambda_0}^c \cap \{\operatorname{Re}(z) \geq 0\}, \ \|(P_\phi - z)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\operatorname{Re}(z)}.$$

iv) There exists $c_1 > 0$ such that the map $z \mapsto (P_\phi - z)^{-1}$ is meromorphic in $\{\operatorname{Re}(z) < c_1\}$ with finite rank residues. In particular, the spectrum of P_ϕ in $\{\operatorname{Re}(z) < c_1\}$ is made of isolated eigenvalues with finite algebraic multiplicities.

v) It holds $\operatorname{Ker} P_\phi = \operatorname{Ker} P_\phi^* = \operatorname{Span}\{e^{-\frac{\phi}{h}}\}$ and 0 is an isolated eigenvalue of P_ϕ (and then of P_ϕ^*) with algebraic multiplicity one.

From (1.13) and the last item of Proposition 1.1, note that $\operatorname{Ker} L_{V,b,\nu} = \operatorname{Span}\{1\}$ and that 0 is an isolated eigenvalue of $L_{V,b,\nu}$ with algebraic multiplicity one. Moreover, according to Proposition 1.1 and to the Hille-Yosida theorem, the operators $L_{V,b,\nu}$ and its adjoint $L^*_{V,b,\nu}$ (in $L^2(\mathbb{R}^d, m_h)$) generate, for every $h > 0$ small enough, contraction semigroups $(e^{-tL_{V,b,\nu}})_{t\geq 0}$ and $(e^{-tL^*_{V,b,\nu}})_{t\geq 0}$ on $L^2(\mathbb{R}^d, m_h)$ which permit to solve (1.2).

In order to describe precisely, in particular by stating Eyring-Kramers type formulas, the spectrum around 0 of $L_{V,b,\nu}$ (or equivalently of P_ϕ) in the regime $h \to 0$, we will assume from now on that V is a Morse function:

Assumption 3. *The function V is a Morse function.*

Under Assumption 3 and thanks to Assumption (1.11), the set $\mathcal{U}$ made of the critical points of V is finite. In the following, the critical points of V with index 0 and with index 1, that is its local minima and its saddle points, will play a fundamental role, and we will respectively denote by $\mathcal{U}^{(0)}$ and $\mathcal{U}^{(1)}$ the sets made of these points. Throughout the paper, we will moreover denote

$$n_0 \;:=\; \mathrm{card}(\mathcal{U}^{(0)})\,.$$

From the pioneer work by Witten [23], it is well-known that for every $h \in (0,1]$ small enough, there is a correspondance between the small eigenvalues of Δ_ϕ and the local minima of $\phi = \frac{V}{2}$. More precisely, we have the following result (see in particular [6, 8, 11] or more recently [21]).

Proposition 1.2. *Assume that* (1.11) *and Assumption 3 hold true. Then, there exist $\epsilon_0 > 0$ and $h_0 > 0$ such that for every $h \in (0, h_0]$, Δ_ϕ has precisely n_0 eigenvalues (counted with multiplicity) in the interval $[0, \epsilon_0 h]$. Moreover, these eigenvalues are actually exponentially small, that is live in an interval $[0, Che^{-2\frac{S}{h}}]$ for some $C, S > 0$ independent of $h \in (0, h_0]$.*

Since the operator $P_\phi = \Delta_\phi + b_h \cdot d_\phi$ is not self-adjoint (when $b_h \neq 0$), the analysis of its spectrum is more complicated that the one of the spectrum of Δ_ϕ. The following result states a counterpart of Proposition 1.2 in this setting.

Theorem 1.3. *Assume that Assumptions 1 to 3 hold true, and let $\epsilon_0 > 0$ be given by Proposition 1.2. Then, for every $\epsilon_1 \in (0, \epsilon_0)$, there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, the set $\sigma(P_\phi) \cap \{\mathrm{Re}\, z < \epsilon_1 h\}$ is finite and consists in*

$$n_0 \;=\; \mathrm{card}\big(\,\sigma(\Delta_\phi) \,\cap\, \{\mathrm{Re}\, z < \epsilon_0 h\}\,\big)$$

eigenvalues counted with algebraic multiplicity. Moreover, there exists $C > 0$ such that for all $h \in (0, h_0]$,

$$\sigma(P_\phi) \cap \{\mathrm{Re}\, z < \epsilon_1 h\} \;\subset\; D(0, Ch^{\frac{1}{2}} e^{-\frac{S}{h}})\,,$$

where S is given by Proposition 1.2. Eventually, for every $\epsilon \in (0, \epsilon_1)$, one has, uniformly with respect to z,

$$\forall\, z \,\in\, \{\mathrm{Re}\, z < \epsilon_1 h\} \cap \{|z| > \epsilon h\}\,, \quad \|(P_\phi - z)^{-1}\|_{L^2 \to L^2} \;=\; \mathcal{O}(h^{-1})\,.$$

Lastly, all the above conclusions also hold for P^*_ϕ.

This theorem will be proved in the next section using Proposition 1.2 and a finite dimensional reduction. In order to give sharp asymptotics of the small eigenvalues of P_ϕ, that is the ones in $D(0, Ch^{\frac{1}{2}} e^{-\frac{S}{h}})$, we will introduce some additional, but generic, topological assumptions on the Morse function V (see Assumption 4 below). To this end, we first recall the general labelling of [12] (see in particular Definition 4.1 there) generalizing the labelling of [8] (and of [2,3]). The main ingredient is the notion of separating saddle point, defined after the following observation. Here and in the sequel, we define, for $a \in \mathbb{R}$,

$$\{V < a\} := V^{-1}\big((-\infty, a)\big) \quad \text{and} \quad \{V \le a\} := V^{-1}\big((-\infty, a]\big),$$

and $\{V > a\}$, $\{V \ge a\}$ in a similar way. The following lemma recalls the local structure of the sublevel sets of a Morse function. A proof can be found in [8].

Lemma 1.4. *Let $z \in \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ be a Morse function. Then, for every $r > 0$ small enough, $B(z, r) \cap \{V < V(z)\}$ has at least two connected components if and only if z is a saddle point of V, i.e. if and only if $z \in \mathcal{U}^{(1)}$. In this case, $B(z, r) \cap \{V < V(z)\}$ has precisely two connected components.*

Definition 1.5. *i) We say that the saddle point $\mathbf{s} \in \mathcal{U}^{(1)}$ is a separating saddle point of V if for every $r > 0$ small enough, the two connected components of $B(\mathbf{s}, r) \cap \{V < V(\mathbf{s})\}$ (see Lemma 1.4) are contained in different connected components of $\{V < V(\mathbf{s})\}$. We will denote by $\mathcal{V}^{(1)}$ the set made of these points.*

ii) We say that $\sigma \in \mathbb{R}$ is a separating saddle value of V if it has the form $\sigma = V(\mathbf{s})$ for some $\mathbf{s} \in \mathcal{V}^{(1)}$.

iii) Moreover, we say that $E \subset \mathbb{R}^d$ is a critical component of V if there exists $\sigma \in V(\mathcal{V}^{(1)})$ such that E is a connected component of $\{V < \sigma\}$ satisfying $\partial E \cap \mathcal{V}^{(1)} \neq \emptyset$.

Let us now describe the general labelling procedure of [12]. We will omit details when associating local minima and separating saddle points below, but the following proposition (cf. [5, Proposition 18]) can be helpful to well understand the construction.

Proposition 1.6. *Assume that V is a Morse function with a finite number of critical points and such that $V(x) \to +\infty$ when $|x| \to +\infty$. Let $\lambda \in \mathbb{R}$ and $\mathcal{C}$ be a connected component of $\{V < \lambda\}$. Then, it holds*

$$\mathcal{C} \cap \mathcal{V}^{(1)} \neq \emptyset \quad \textit{iff} \quad \operatorname{card}(\mathcal{C} \cap \mathcal{U}^{(0)}) \ge 2\,.$$

Let us also define

$$\sigma := \max_{\mathcal{C} \cap \mathcal{V}^{(1)}} V$$

with the convention $\sigma := \min_{\mathcal{C}} V$ when $\mathcal{C} \cap \mathcal{V}^{(1)} = \emptyset$. It then holds:

i) For every $\mu \in (\sigma, \lambda]$, the set $\mathcal{C} \cap \{V < \mu\}$ is a connected component of $\{V < \mu\}$.

ii) If $\mathcal{C} \cap \mathcal{V}^{(1)} \neq \emptyset$, then $\mathcal{C} \cap \mathcal{U}^{(0)} \subset \{V < \sigma\}$ and all the connected components of $\mathcal{C} \cap \{V < \sigma\}$ are critical.

Under the hypotheses of Proposition 1.6, $V(\mathcal{V}^{(1)})$ is finite. We moreover assume that $n_0 \ge 2$, so that, under the hypotheses of Proposition 1.1 and of Theorem 1.3, 0 is not the only exponentially small eigenvalue of P_ϕ (or equivalently of $L_{V,b,\nu}$) and $\mathcal{V}^{(1)} \neq \emptyset$ by Proposition 1.6. We then denote the elements of $V(\mathcal{V}^{(1)})$ by $\sigma_2 > \sigma_3 > \ldots > \sigma_N$, where $N \geq 2$. For convenience, we also introduce a fictive infinite saddle value $\sigma_1 = +\infty$. Starting from σ_1, we will recursively associate to each σ_i a finite family of local minima $(\mathbf{m}_{i,j})_j$ and a finite family of critical components $(E_{i,j})_j$ (see Definition 1.5).

Let $N_1 := 1$, $\underline{\mathbf{m}} = \mathbf{m}_{1,1}$ be a global minimum of V (arbitrarily chosen if there are more than one), and $E_{1,1} := \mathbb{R}^d$. We now proceed in the following way:

- Let us denote, for some $N_2 \geq 1$, by $E_{2,1}, \ldots, E_{2,N_2}$ the connected components of $\{V < \sigma_2\}$ which do not contain $\mathbf{m}_{1,1}$. They are all critical by the preceding proposition and we associate to each $E_{2,j}$, where $j \in \{1, \ldots, N_2\}$, some global minimum $\mathbf{m}_{2,j}$ of $V|_{E_{2,j}}$ (arbitrarily chosen if there are more than one).
- Let us then consider, for some $N_3 \geq 1$, the connected components $E_{3,1}, \ldots, E_{3,N_3}$ of $\{V < \sigma_3\}$ which do not contain the local minima of V previously labelled. These components are also critical and included in the $E_{2,j} \cap \{V < \sigma_3\}$'s, $j \in \{1, \ldots, N_2\}$, such that $E_{2,j} \cap \{V = \sigma_3\} \cap \mathcal{V}^{(1)} \neq \emptyset$ (and $\sigma_3 = \max_{E_{2,j} \cap \mathcal{V}^{(1)}} V$ for such a j). We then again associate to each $E_{3,j}$, $j \in \{1, \ldots, N_3\}$, some global minimum $\mathbf{m}_{3,j}$ of $V|_{E_{3,j}}$.
- We continue this process until having considered the connected components of $\{V < \sigma_N\}$ after which all the local minima of V have been labelled.

Next, we define two mappings

$$E : \mathcal{U}^{(0)} \to \mathcal{P}(\mathbb{R}^d) \quad \text{and} \quad \mathbf{j} : \mathcal{U}^{(0)} \to \mathcal{P}(\mathcal{V}^{(1)} \cup \{\mathbf{s}_1\}),$$

where, for any set A, $\mathcal{P}(A)$ denotes the power set of A, and $\mathbf{s}_1$ is a fictive saddle point such that $V(\mathbf{s}_1) = \sigma_1 = +\infty$, as follows: for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N_i\}$,

$$E(\mathbf{m}_{i,j}) \ := \ E_{i,j} \tag{1.15}$$

and

$$\mathbf{j}(\underline{\mathbf{m}}) \ := \ \{\mathbf{s}_1\} \quad \text{and, when } i \geq 2, \quad \mathbf{j}(\mathbf{m}_{i,j}) \ := \ \partial E_{i,j} \cap \mathcal{V}^{(1)} \neq \emptyset\,. \tag{1.16}$$

In particular, it holds $E(\underline{\mathbf{m}}) = \mathbb{R}^d$ and

$$\forall\, i \in \{1, \ldots, N\}\,,\ \forall\, j \in \{1, \ldots, N_i\}\,, \quad \emptyset \neq \mathbf{j}(\mathbf{m}_{i,j}) \ \subset \ \{V = \sigma_i\}\,.$$

Lastly, we define the mappings

$$\boldsymbol{\sigma} : \mathcal{U}^{(0)} \to V(\mathcal{V}^{(1)}) \cup \{\sigma_1\} \quad \text{and} \quad S : \mathcal{U}^{(0)} \to (0, +\infty]$$

by

$$\forall\, \mathbf{m} \in \mathcal{U}^{(0)}, \ \ \boldsymbol{\sigma}(\mathbf{m}) := V(\mathbf{j}(\mathbf{m})) \quad \text{and} \quad S(\mathbf{m}) := \boldsymbol{\sigma}(\mathbf{m}) - V(\mathbf{m})\,, \tag{1.17}$$

where, with a slight abuse of notation, we have identified the set $V(\mathbf{j}(\mathbf{m}))$ with its unique element. Note that $S(\mathbf{m}) = +\infty$ if and only if $\mathbf{m} = \underline{\mathbf{m}}$.

Our generic topological assumption is the following one. Assume that V is a Morse function with a finite number $n_0 \geq 2$ of critical points such that $V(x) \to +\infty$ when $|x| \to +\infty$, and let $E : \mathcal{U}^{(0)} \to \mathcal{P}(\mathbb{R}^d)$ and $\mathbf{j} : \mathcal{U}^{(0)} \to \mathcal{P}(\mathcal{V}^{(1)} \cup \{\mathbf{s}_1\})$ be the mappings defined in (1.15) and in (1.16).

Assumption 4. *For every $\mathbf{m} \in \mathcal{U}^{(0)}$, the following hold true:*

i) *the local minimum $\mathbf{m}$ is the unique global minimum of $V|_{E(\mathbf{m})}$,*

ii) *for all $\mathbf{m}' \in \mathcal{U}^{(0)} \setminus \{\mathbf{m}\}$, $\mathbf{j}(\mathbf{m}) \cap \mathbf{j}(\mathbf{m}') = \emptyset$.*

In particular, V uniquely attains its global minimum, at $\underline{\mathbf{m}} \in \mathcal{U}^{(0)}$.

This assumption is slightly more general than the assumption considered in the generic case in [8, 12] (see also [2, 3]) where, for instance, each set $\mathbf{j}(\mathbf{m})$, $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\mathbf{m}\}$, is assumed to only contain one element.

Remark 1.7. *One can also show that Assumption 4 implies that for every $\mathbf{m} \in \mathcal{U}^{(0)}$ such that $\mathbf{m} \neq \underline{\mathbf{m}}$, there is precisely one connected component $\widehat{E}(\mathbf{m}) \neq E(\mathbf{m})$ of $\{f < \boldsymbol{\sigma}(\mathbf{m})\}$ such that $\overline{\widehat{E}(\mathbf{m})} \cap \overline{E(\mathbf{m})} \neq \emptyset$. In other words, there exists a connected component $\widehat{E}(\mathbf{m}) \neq E(\mathbf{m})$ of $\{f < \boldsymbol{\sigma}(\mathbf{m})\}$ such that $\mathbf{j}(\mathbf{m}) \subset \partial \widehat{E}(\mathbf{m})$. Moreover, the global minimum $\mathbf{m}'$ of $V|_{\widehat{E}(\mathbf{m})}$ is unique and satisfies $\boldsymbol{\sigma}(\mathbf{m}') > \boldsymbol{\sigma}(\mathbf{m})$ and $V(\mathbf{m}') < V(\mathbf{m})$. We refer [20] or [5] for more details on the geometry of the sublevel sets of a Morse function.*

In order to state our main results, we also need the following lemma which is fondamental in our analysis.

Lemma 1.8. *For $x \in \mathbb{R}^d$, let $B(x) := \mathrm{Jac}_x b$ denote the Jacobian matrix of $b = U_0 - \nabla V$ at x, and consider a saddle point $\mathbf{s} \in \mathcal{U}^{(1)}$.*

i) The matrix $\mathrm{Hess}\, V(\mathbf{s}) + B^*(\mathbf{s}) \in \mathcal{M}_d(\mathbb{R})$ admits precisely one negative eigenvalue $\mu = \mu(\mathbf{s})$, which has moreover geometric multiplicity one.

ii) Denote by $\xi = \xi(\mathbf{s})$ one of the two (real) unitary eigenvectors of $\mathrm{Hess}\, V(\mathbf{s}) + B^*(\mathbf{s})$ associated with μ. The real symmetric matrix

$$M_V \; := \; \mathrm{Hess}\, V(\mathbf{s}) + 2|\mu|\, \xi\, \xi^*$$

is then positive definite and its determinant satisfies:

$$\det M_V \; = \; -\det \mathrm{Hess}\, V(\mathbf{s})\,.$$

iii) Lastly, denoting by $\lambda_1 = \lambda_1(\mathbf{s})$ the negative eigenvalue of $\mathrm{Hess}\, V(\mathbf{s})$, it holds $|\mu| \geq |\lambda_1|$, with equality if and only if $B^*(\mathbf{s})\xi = 0$, and

$$\langle (\mathrm{Hess}\, V(\mathbf{s}))^{-1}\xi, \xi\rangle \; = \; \frac{1}{\mu} \; < \; 0\,.$$

Note that the real matrix $\mathrm{Hess}\, V(\mathbf{s}) + B^*(\mathbf{s})$ of Lemma 1.8 is in general non symmetric. Let us also point out that the statements of Lemma 1.8 already appear in the related work [14] (see in particular the beginning of Section 8 there) and in [15], where proofs are given (see indeed Section 4.1 there). We will nevertheless give a proof in Section 3 for the sake of completeness.

We can now state our main result.

Theorem 1.9. *Suppose that Assumptions 1 to 4 hold true, and let* $\epsilon_0 > 0$ *be given by Proposition 1.2. Then, for all* $\epsilon_1 \in (0, \epsilon_0)$*, there exists* $h_0 > 0$ *such that for all* $h \in (0, h_0]$*, one has, counting the eigenvalues with algebraic multiplicity,*

$$\sigma(L_{V,b,\nu}) \cap \{\operatorname{Re} z < \epsilon_1\} \;=\; \{\lambda(\mathbf{m}, h),\ \mathbf{m} \in \mathcal{U}^{(0)}\},$$

where, denoting by $\underline{\mathbf{m}}$ *the unique absolute minimum of* V*,* $\lambda(\underline{\mathbf{m}}, h) = 0$ *and, for all* $\mathbf{m} \neq \underline{\mathbf{m}}$*,* $\lambda(\mathbf{m}, h)$ *satisfies the following Eyring-Kramers type formula:*

$$\lambda(\mathbf{m}, h) \;=\; \zeta(\mathbf{m})\, e^{-\frac{S(\mathbf{m})}{h}} \left(1 + \mathcal{O}(\sqrt{h})\right), \tag{1.18}$$

where $S : \mathcal{U}^{(0)} \to (0, +\infty]$ *is defined in* (1.17) *and, for every* $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$*,*

$$\zeta(\mathbf{m}) \;:=\; \frac{\det \operatorname{Hess} V(\mathbf{m})^{\frac12}}{2\pi} \Big(\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \frac{|\mu(\mathbf{s})|}{|\det \operatorname{Hess} V(\mathbf{s})|^{\frac12}} \Big), \tag{1.19}$$

where $\mathbf{j} : \mathcal{U}^{(0)} \to \mathcal{P}(\mathcal{V}^{(1)} \cup \{\mathbf{s}_1\})$ *is defined in* (1.16) *and the* $\mu(\mathbf{s})$*'s are defined in Lemma 1.8.*
In addition, it holds

$$\sigma(L_{V,-b,-\nu}) \cap \{\operatorname{Re} z < \epsilon_1\} = \sigma(L^*_{V,b,\nu}) \cap \{\operatorname{Re} z < \epsilon_1\} = \{\overline{\lambda(\mathbf{m}, h)},\ \mathbf{m} \in \mathcal{U}^{(0)}\}.$$

Remark 1.10. *In the case where* V *has precisely two minima* $\underline{\mathbf{m}}$ *and* $\mathbf{m}$ *such that* $V(\underline{\mathbf{m}}) = V(\mathbf{m})$*, the above result can be easily generalized. In this case, using the definitions of* S *and* $\mathbf{j}$ *given in* (1.17) *and in* (1.16) *(note that the choice of* $\underline{\mathbf{m}}$ *among the two minima of* V *is arbitrary in this case), we have, counting the eigenvalues with algebraic multiplicity, for every* $h > 0$ *small enough,*

$$\sigma(L_{V,b,\nu}) \cap \{\operatorname{Re} z < \epsilon_1\} \;=\; \{0, \lambda(\mathbf{m}, h)\},$$

where

$$\lambda(\mathbf{m}, h) \;=\; \zeta(\mathbf{m})\, e^{-\frac{S(\mathbf{m})}{h}} \left(1 + \mathcal{O}(\sqrt{h})\right)$$

with

$$\zeta(\mathbf{m}) \;=\; \frac{\det \operatorname{Hess} V(\mathbf{m})^{\frac12} + \det \operatorname{Hess} V(\underline{\mathbf{m}})^{\frac12}}{2\pi} \Big(\sum_{\mathbf{s} \in \mathbf{j}(\mathbf{m})} \frac{|\mu(\mathbf{s})|}{|\det \operatorname{Hess} V(\mathbf{s})|^{\frac12}} \Big).$$

Moreover, since $\sigma(L_{V,b,\nu}) = \overline{\sigma(L_{V,b,\nu})}$*, the eigenvalue* $\lambda(\mathbf{m}, h)$ *is real.*

Let us make a few comments on the above theorem.

First, observe that if we assume that $U_h = \nabla V$, that is if $b_h = 0$ (see (1.5)), we obtain the precise asymptotics of the small eigenvalues of $L_{V,0,0}$ (or equivalently of Δ_ϕ after multiplication by $\frac{1}{h}$, see (1.13)) and hence recover the results already proved in this reversible setting in [3,8] (see also [18] for an extension to logarithmic Sobolev inequalities). In this case, for every saddle point $\mathbf{s}$ appearing in (1.19), the real number $\mu(\mathbf{s})$ is indeed the negative eigenvalue of $\operatorname{Hess} V(\mathbf{s})$ according to the first item of Lemma 1.8. Let us also point out that under the hypotheses made in [3,8], the set $\mathbf{j}(\mathbf{m})$ actually contains one unique element for every $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$. Moreover, our analysis permits in this case to recover that the error term $\mathcal{O}(\sqrt{h})$ is actually of order $\mathcal{O}(h)$, as proven in [8]. However, it does not permit to prove that this $\mathcal{O}(h)$ actually admits a full asymptotic expansion in h as proven in [8].

To the best of our knowledge, the above theorem is the first result giving sharp asymptotics of the small eigenvalues of the generator $L_{V,b,\nu}$ in the non-reversible case. Similar results were obtained by Hérau-Hitrik-Sjöstrand for the Kramers-Fokker-Planck (KFP) equation in [12]. Compared to our framework, they deal with non-self-adjoint and non-elliptic operators, which makes the analysis more complicated. However, the KFP equation enjoys several symmetries which are crucial in their analysis. First of all, the KFP operator has a supersymmetric structure (for a non-symmetric skew-product $\langle .,. \rangle_{\mathrm{KFP}}$) which permits to write the interaction matrix associated with the small eigenvalues as a square $M = A^*A$, where the adjoint A^* is taken with respect to $\langle .,. \rangle_{\mathrm{KFP}}$. Using this square structure, the authors can then follow the strategy of [8] to construct accurate approximations of the matrices A and A^*. However, since $\langle .,. \rangle_{\mathrm{KFP}}$ is not a scalar product, they cannot identify the squares of the singular values of A with the eigenvalues of M. This difficulty is solved by using an extra symmetry (the PT-symmetry) which permits to modify the skew-product $\langle .,. \rangle_{KFP}$ into a new product $\langle .,. \rangle_{KFPS}$, which is a scalar product when restricted to the "small spectral subspace", and for which the identity $M = A^*A$ remains true with an adjoint taken with respect to $\langle .,. \rangle_{KFPS}$. This permits to conclude as in [8], using in particular the Fan inequalities to estimate the singular values of A.

In the present case, none of these two symmetries are available in general ($L_{V,b,\nu}$, or equivalently P_ϕ, enjoys however a supersymmetric structure when b and ν satisfy the relation (1.7), see indeed (1.8) or Remark 3.2 below in this connection). We then developed an alternative approach based on the construction of very accurate quasimodes and partly inspired by [4] (see also the related constructions made in [2, 14, 17]). This permits the construction of the interaction matrix M as above. However, since we cannot write $M = A^*A$ and use the Fan inequalities as in [8, 12] (and e.g. in [5, 10, 16, 17, 20]), we have to compute directly the eigenvalues of M. To this end, we use crucially the Schur complement method. This leads to Theorem A.4 in appendix, which permits to replace the use of the Fan inequalities to perform the final analysis in our setting. We believe that these two arguments are quite general and may be used in other contexts.

Though it is generic, one may ask if Assumption 4 is necessary to get Eyring-Kramers type formulas as in Theorem 1.9. In the reversible setting, the full general (Morse) case was recently treated by the second author in [20], but applying the methods developed there to our non-reversible setting was not straightforward and we decided to postpone this analysis to future works. Let us point out in this connection that in the general (Morse) case, some tunneling effect between the characteristic wells of V defined by the mapping E (see (1.15)) mixes their corresponding prefactors, see indeed Remark 1.10, or [20] for more intricate situations in the reversible setting.

Note that Theorem 1.9 does not state that the operator $L_{V,b,\nu}$ is diagonalizable when restricted to the spectral subspace associated with its small eigenvalues. Indeed, since $L_{V,b,\nu}$ is not self-adjoint, we cannot exclude the existence of Jordan's blocks. We cannot neither exclude the existence of non-real eigenvalues, but the spectrum of $L_{V,b,\nu}$ is obviously stable by complex conjugation since $L_{V,b,\nu}$ is a partial differential operator with real coefficients. However, in the case where for every $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, the prefactors $\zeta(\mathbf{m}')$ defined in (1.19) are all disctinct for $\mathbf{m}' \in S^{-1}(S(\mathbf{m}))$, the $\lambda(\mathbf{m}, h)$'s, $\mathbf{m} \in \mathcal{U}^{(0)}$, are then real eigenvalues of multiplicity one of $L_{V,b,\nu}$ and its restriction to its small spectral subspace is diagonalizable.

Coming back to the contraction semigroups $(e^{-tL_{V,b,\nu}})_{t\geq 0}$ and $(e^{-tL^*_{V,b,\nu}})_{t\geq 0}$ on $L^2(\mathbb{R}^d, m_h)$ introduced just after Proposition 1.1, Theorem 1.9 has the following consequences on the rate of convergence to equilibrium for the process (1.1).

Theorem 1.11. *Assume that the hypotheses of Theorem 1.9 hold and let* $\mathbf{m}^* \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$ *be such that*

$$S(\mathbf{m}^*) \;=\; \max_{\mathbf{m}\in\mathcal{U}^{(0)}} S(\mathbf{m}) \quad \text{and} \quad \zeta(\mathbf{m}^*) \;=\; \min_{\mathbf{m}\in S^{-1}(S(\mathbf{m}^*))} \zeta(\mathbf{m})\,, \tag{1.20}$$

where the prefactors $\zeta(\mathbf{m})$*'s,* $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$*, are defined in* (1.19)*, and* $S : \mathcal{U}^{(0)} \to (0, +\infty]$ *is defined in* (1.17)*. Let us then define, for any* $h > 0$*,*

$$\lambda(h) \;:=\; \zeta(\mathbf{m}^*)\, e^{-\frac{S(\mathbf{m}^*)}{h}}\,.$$

Then, there exist $h_0 > 0$ *and* $C > 0$ *such that for every* $h \in (0, h_0]$*, it holds*

$$\forall\, t \,\geq\, 0\,, \quad \|\, e^{-tL_{V,b,\nu}} - \Pi_0 \,\|_{L^2(m_h)\to L^2(m_h)} \;\leq\; C\, e^{-\lambda(h)(1-C\sqrt{h})t}\,, \tag{1.21}$$

where Π_0 *denotes the orthogonal projector on* $\operatorname{Ker} L_{V,b,\nu} = \operatorname{Span}\{1\}$*:*

$$\forall\, u \,\in\, L^2(m_h)\,, \quad \Pi_0 u \;=\; \langle u, 1\rangle_{L^2(m_h)} \;=\; \int_{\mathbb{R}^d} u\, dm_h\,.$$

Assume moreover that $(X_t)_{t\geq 0}$ *is solution to* (1.1) *and that the probability distribution* ϱ_0 *of* X_0 *admits a density* $\mu_0 \in L^2(\mathbb{R}^d, m_h)$ *with respect to the probability measure* m_h*. Then, for every* $t \geq 0$*, the probability distribution* ϱ_t *of* X_t *admits the density* $\mu_t = e^{-tL^*_{V,b,\nu}}\mu_0 \in L^2(\mathbb{R}^d, m_h)$ *with respect to* m_h*, and for every* $h \in (0, h_0]$*, it holds*

$$\forall\, t \,\geq\, 0\,, \quad \|\, \varrho_t - \nu_h \,\|_{TV} \;\leq\; C\, \|\mu_0\|_{L^2(m_h)}\, e^{-\lambda(h)(1-C\sqrt{h})t}\,, \tag{1.22}$$

where $\|\cdot\|_{TV}$ *denotes the total variation distance.*
Finally, when there exists one unique $\mathbf{m}^*$ *satisfying* (1.20)*, the eigenvalue* $\lambda(\mathbf{m}^*, h)$ *associated with* $\mathbf{m}^*$ *(see* (1.18)*) is real and simple, and the estimates* (1.21) *and* (1.22) *remain valid if one replaces* $\lambda(h)(1 - C\sqrt{h})$ *by* $\lambda(\mathbf{m}^*, h)$ *in the exponential terms.*

Theorems 1.9 and 1.11 describe the metastable behaviour of the dynamics (1.1) from a spectral perspective. A closely related point of view is to study the mean transition times between the different wells of the potential V for the process $(X_t)_{t\geq 0}$ solution to (1.1). In the non-reversible case, this question has been studied recently e.g. in [1, 14], to which we also refer for more details and references on this subject. In [1], an Eyring-Kramers type formula is derived from formal computation relying on the study of the appropriate quasi-potential. This Eyring-Kramers type formula has been proved in [14] by a potential theoretic approach in the case of a double-well potential V when b and ν satisfy the relation (1.7) in such a way that $L_{V,b,\nu}$ has the form (1.8). Moreover, though the mathematical objects considered in [14] and in the present paper are not the same, these two works share some similarities. Nevertheless, we would like to emphasize that our approach permits to go beyond the supersymmetric assumption (1.7) and to treat the case of multiple-well potentials.

To be more precise on the connections between the present paper and [14] (and also [1]), let us conclude this introduction with the corollary below which combines the results given by Theorem 1.9 when V is a double-well potential and [14, Theorem 5.2 and Remarks 5.3 and 5.6]. This result generalizes in particular, in this non-reversible double-well setting, the results obtained in the reversible case in [2, 3] on the relations between the small eigenvalues of $L_{V,b,\nu}$ and the mean transition times of (1.1) when $b = \nu = 0$.

Corollary 1.12. *Assume that the hypotheses of Theorem 1.9 hold with moreover*

$$\lim_{|x|\to+\infty} \frac{x}{|x|}\cdot\nabla V(x) \;=\; +\infty \quad \textit{and} \quad \lim_{|x|\to+\infty} |\nabla V(x)| - 2\Delta V(x) \;=\; +\infty\,,$$

and that V admits precisely two local minima $\underline{\mathbf{m}}$ and $\mathbf{m}$ such that $V(\underline{\mathbf{m}}) < V(\mathbf{m})$ (it then holds $\mathcal{V}^{(1)} = \mathbf{j}(\mathbf{m})$). Assume in addition that b and ν satisfy the relation (1.7), and hence that $b = J(\nabla V)$ for some smooth map J from $\mathbb{R}^d$ into the set of real antisymmetric matrices of size d, and that J is uniformly bounded on $\mathbb{R}^d$.
Let $\mathcal{O}(\underline{\mathbf{m}})$ be a smooth open connected set containing $\underline{\mathbf{m}}$ such that $\overline{\mathcal{O}(\underline{\mathbf{m}})} \subset \{V < \boldsymbol{\sigma}(\mathbf{m})\}$. Let then $(X_t)_{t\geq 0}$ be the solution to (1.1) such that $X_0 = \mathbf{m}$ and let

$$\tau_{\mathcal{O}(\underline{\mathbf{m}})} \;:=\; \inf\{t \geq 0\,,\; X_t \in \mathcal{O}(\underline{\mathbf{m}})\}$$

be the first hitting time of $\mathcal{O}(\underline{\mathbf{m}})$. The expectation of $\tau_{\mathcal{O}(\underline{\mathbf{m}})}$ and the non-zero small eigenvalue $\lambda(\mathbf{m},h)$ of $L_{V,b,\nu}$ are then related by the following formula in the limit $h\to 0$:

$$\mathbb{E}(\tau_{\mathcal{O}(\underline{\mathbf{m}})}) \;=\; \frac{1}{\lambda(\mathbf{m},h)}\Big(1+\mathcal{O}\big(\sqrt{h|\ln h|^3}\big)\Big)\,.$$

Let us mention here that the hypotheses of Corollary 1.12 are simply the minimal hypotheses permitting to apply at the same time Theorem 1.9 and [14, Theorem 5.2] in its refinement specified in [14, Remark 5.6].

2. General spectral estimates

2.1. **Proof of Proposition 1.1.** The unbounded operator $(P_\phi, \mathcal{C}_c^\infty(\mathbb{R}^d))$ is accretive, since, according to (1.10), one has:

$$\forall\, u \in \mathcal{C}_c^\infty(\mathbb{R}^d)\,,\;\; \mathrm{Re}\langle P_\phi u, u\rangle = \langle \Delta_\phi u, u\rangle = \|d_\phi u\|^2 \geq 0\,. \tag{2.1}$$

In order to prove that its closure is maximal accretive, it then suffices to show that $\mathrm{Ran}(P_\phi + 1)$ is dense in $L^2(\mathbb{R}^d)$ (see for example [7, Theorem 13.14]). The proof of this fact is rather standard but we give it for the sake of completeness (see in particular the proof of [9, Proposition 5.5] for a similar proof). Suppose that $f \in L^2(\mathbb{R}^d)$ is orthogonal to $\mathrm{Ran}(P_\phi + 1)$. It then holds $(P_\phi^* + 1)f = 0$ in the distribution sense and, since P_ϕ is real, one can assume that f is real. In particular, since $P_\phi^* = \Delta_\phi - b_h \cdot d_\phi$ is elliptic with smooth coefficients, f belongs to $\mathcal{C}^\infty(\mathbb{R}^d)$. Thus, for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R})$, one has

$$h^2\langle\nabla(\zeta f), \nabla(\zeta f)\rangle + \int \zeta^2(|\nabla\phi|^2 - h\Delta\phi + 1)f^2 = \langle(P_\phi^* + 1)\zeta f, \zeta f\rangle$$
$$= h^2\int |\nabla\zeta|^2 f^2 - h\int(b_h \cdot d\zeta)\zeta f^2.$$

Take now ζ such that $0 \le \zeta \le 1$, $\zeta = 1$ on $B(0,1)$ and $\operatorname{supp}\zeta \subset B(0,2)$, and define $\zeta_k := \zeta(\frac{\cdot}{k})$ for $k \in \mathbb{N}^*$. According to (1.12) and to the above relation, there exists $C > 0$ such that for every $k \in \mathbb{N}^*$, it holds

$$\begin{aligned}\int \zeta_k^2(|\nabla\phi|^2 - h\Delta\phi + 1)f^2 \ &\le\ C\frac{h^2}{k^2}\|f\|^2 + C\frac{h}{k}\|f\|\,\|(1 + |\nabla\phi|)\zeta_k f\| \\ &\le\ C(1 + \frac{1}{2\varepsilon})\frac{h^2}{k^2}\|f\|^2 + \frac{\varepsilon}{2}C\|(1 + |\nabla\phi|)\zeta_k f\|^2\,,\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Choosing $\varepsilon = \frac{1}{2C}$ and using (1.11), it follows that for every $h > 0$ small enough, it holds

$$\frac{1}{4}\|\zeta_k f\|^2 \ \le\ \int \zeta_k^2(\frac{1}{2}|\nabla\phi|^2 - h\Delta\phi + \frac{1}{2})f^2 \ \le\ \frac{4}{3}C(1 + \frac{1}{2\varepsilon})\frac{h^2}{k^2}\|f\|^2\,,$$

which implies, taking the limit $k \to +\infty$, that $f = 0$. Hence, the closure of P_ϕ, that we still denote by P_ϕ, is maximal accretive. Note moreover, that (2.1) implies that $D(P_\phi) \subset \{u \in L^2(\mathbb{R}^d),\ d_\phi u \in L^2(\mathbb{R}^d)\}$ and that $\mathrm{Re}\langle P_\phi u, u\rangle = \|d_\phi u\|^2$ for every $u \in D(P_\phi)$.

Let us now prove that $D(\Delta_\phi) \subset D(P_\phi)$, which amounts to show that for every $u \in D(\Delta_\phi)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $u_n \to u$ in $L^2(\mathbb{R}^d)$ and $(P_\phi u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $(\Delta_\phi, \mathcal{C}_c^\infty(\mathbb{R}^d))$ is essentially self-adjoint, for any such u, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $u_n \to u$ in $L^2(\mathbb{R}^d)$ and $(\Delta_\phi u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and it thus suffices to show that $(b_h \cdot d_\phi u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. For this purpose we introduce the exterior derivative d acting from 0-forms into 1-forms and the twisted semiclassical derivative $d_\phi = e^{-\phi/h} \circ hd \circ e^{\phi/h}$. Then we write with a slight abuse of notation $b_h \cdot d_\phi = b_h \cdot d_\phi$. Thanks to (1.11) and to (1.12), there exists $C > 0$ such that for every $h > 0$ small enough and every $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, one has

$$\begin{aligned}\|b_h \cdot d_\phi u\|^2 \le \int |b_h|^2|d_\phi u|^2 &\le C\langle|\nabla\phi|^2 d_\phi u, d_\phi u\rangle + C\|d_\phi u\|^2 \\ &\le 2C\langle\Delta_\phi^{(1)}\, d_\phi u, d_\phi u\rangle + 2C\|d_\phi u\|^2\,,\end{aligned}$$

where $\Delta_\phi^{(1)}$ denotes the Witten Laplacian acting on 1-forms, that is

$$\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes \mathrm{Id} + 2h\mathrm{Hess}\,\phi = (-h^2\Delta + |\nabla\phi|^2 - h\Delta\phi) \otimes \mathrm{Id} + 2h\mathrm{Hess}\,\phi.$$

Combined with the intertwining relation $\Delta_\phi^{(1)} d_\phi = d_\phi\Delta_\phi^{(0)}$, we get

$$\|b_h \cdot d_\phi u\|^2 \ \le\ 2C\big(\|\Delta_\phi^{(0)}u\|^2 + \|d_\phi u\|^2\big) \ \le\ 2C\|\Delta_\phi^{(0)}u\|\big(\|\Delta_\phi^{(0)}u\| + \|u\|\big) \tag{2.2}$$

for every $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. This implies that for any Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ such that $(\Delta_\phi u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, $(b_h \cdot d_\phi u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence, and thus that $D(\Delta_\phi) \subset D(P_\phi)$.

The statement about P_ϕ^* is then a straightforward consequence of the above analysis. Indeed, since $P_\phi^* = \Delta_\phi - b_h \cdot \nabla_\phi$ on $\mathcal{C}_c^\infty(\mathbb{R}^d)$, the above arguments imply that the closure of $(P_\phi^*, \mathcal{C}_c^\infty(\mathbb{R}^d))$ is maximal accretive and that its domain contains $D(\Delta_\phi)$. Moreover, P_ϕ^* is maximal accretive since P_ϕ is, and hence coincides with the closure of $(P_\phi^*, \mathcal{C}_c^\infty(\mathbb{R}^d))$.

Let us now prove the statement on the spectrum of P_ϕ. Throughout, we will denote $\mathbb{C}_+ = \{\mathrm{Re}(z) \geq 0\}$. It follows from (1.11) and from (1.12) that for every $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, it holds

$$|\langle b_h \cdot d_\phi u, u\rangle| \leq \|d_\phi u\|\|b_h u\| \leq C(\|d_\phi u\|^2 + \|u\|\|d_\phi u\|)\,. \tag{2.3}$$

Let us set $\Lambda_0 = 5C$ for some $C \geq 1$ satisfying (2.3), and let $z \in \mathbb{C}_+$ be such that $|\mathrm{Im}(z)| \geq \Lambda_0 \max(\mathrm{Re}(z), \sqrt{\mathrm{Re}(z)})$. Suppose first that $\mathrm{Re}(z)\|u\|^2 \geq \frac{1}{2}\|d_\phi u\|^2$. Then, thanks to the estimate (2.3), we have

$$\begin{aligned}
|\langle (b_h \cdot d_\phi - i\,\mathrm{Im}(z))u, u\rangle| &\geq |\mathrm{Im}(z)|\|u\|^2 - C(\|d_\phi u\|^2 + \sqrt{2\,\mathrm{Re}(z)}\|u\|^2) \\
&\geq \Big(|\mathrm{Im}(z)| - C\big(2\,\mathrm{Re}(z) + \sqrt{2\,\mathrm{Re}(z)}\big)\Big)\|u\|^2 \\
&\geq C\max(\mathrm{Re}(z), \sqrt{\mathrm{Re}(z)})\|u\|^2 \geq \mathrm{Re}(z)\|u\|^2.
\end{aligned}$$

Since $|\langle (b_h \cdot d_\phi - i\,\mathrm{Im}(z))u, u\rangle| \leq |\langle (P_\phi - z)u, u\rangle|$, this implies that

$$|\langle (P_\phi - z)u, u\rangle| \geq \mathrm{Re}(z)\|u\|^2. \tag{2.4}$$

Suppose now that $\mathrm{Re}(z)\|u\|^2 \leq \frac{1}{2}\|d_\phi u\|^2$. One then directly obtains

$$|\langle (P_\phi - z)u, u\rangle| \geq \langle (\Delta_\phi - \mathrm{Re}(z))u, u\rangle \geq \mathrm{Re}(z)\|u\|^2\,,$$

which, combined with (2.4), implies that

$$\|(P_\phi - z)u\| \geq \mathrm{Re}(z)\|u\| \tag{2.5}$$

for every $z \in \mathbb{C}_+ \setminus \Gamma_{\Lambda_0}$ and $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Since P_ϕ is closed, it follows that $P_\phi - z$ is injective with closed range, and hence semi-Fredholm, for every $z \in \mathbb{C} \setminus \Gamma_{\Lambda_0}$. Since the open set $\mathbb{C} \setminus \Gamma_{\Lambda_0}$ is connected, the index of $P_\phi - z$ is then constant on $\mathbb{C} \setminus \Gamma_{\Lambda_0}$ (see [13, Theorem 5.17 in Chap. 4]). But P_ϕ being maximal accretive, the index of $P_\phi - z$ is 0 on $\{\mathrm{Re}\, z < 0\}$. Hence, $(P_\phi - z)$ is invertible from $D(P_\phi)$ onto $L^2(\mathbb{R}^d)$ on $\mathbb{C} \setminus \Gamma_{\Lambda_0}$ and the resolvent estimate stated in Proposition 1.1 becomes a direct consequence of (2.5).

Let us now prove the fourth item of Proposition 1.1. Thanks to (1.11), there exist $c > 0$ and $R > 0$ such that

$$\forall |x| \geq R,\ |\nabla\phi(x)|^2 \geq c.$$

Take $c_1 \in (0, c)$ and let W be a nonnegative smooth function such that $\mathrm{supp}(W) \subset B(0, R)$ and $W(x) + |\nabla\phi(x)|^2 \geq \frac{c+c_1}{2}$ for all $x \in \mathbb{R}^d$. There exists consequently $h_0 > 0$ such that for all $h \in (0, h_0]$, one has

$$\tilde{W} \ :=\ W + |\nabla\phi|^2 - h\Delta\phi \ \geq\ c_1$$

on $\mathbb{R}^d$. Introduce the operator

$$\tilde{P}_\phi = P_\phi + W = -h^2\Delta + \tilde{W} + b_h d_\phi$$

with domain $D(P_\phi)$. Since P_ϕ is maximal accretive and $W \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$, $\tilde{P}_\phi$ is also maximal accretive (see for example [7, Theorem 13.25]). Moreover, for every $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and then for every $u \in D(P_\phi)$, one has

$$\mathrm{Re}\langle \tilde{P}_\phi u, u\rangle = \langle (-h^2\Delta + \tilde{W})u, u\rangle \geq c_1\|u\|^2 ,$$

which implies as above that for every z in $\{\mathrm{Re}(z) < c_1\}$, $\tilde{P}_\phi - z$ is invertible from $D(P_\phi)$ onto $L^2(\mathbb{R}^d)$. Hence, for every z in $\{\mathrm{Re}(z) < c_1\}$, we can write

$$P_\phi - z \;=\; \tilde{P}_\phi - z - W \;=\; (\mathrm{Id} - W(\tilde{P}_\phi - z)^{-1})(\tilde{P}_\phi - z).$$

Of course, $z \mapsto (\tilde{P}_\phi - z)^{-1}$ is holomorphic on $\{\mathrm{Re}\, z < c_1\}$ and thanks to the analytic Fredholm theorem, it then suffices to prove that

$$K(z) \;:=\; W(\tilde{P}_\phi - z)^{-1} \;:\; L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is compact for every z in $\{\mathrm{Re}(z) < c_1\}$. This follows from the compactness of the embedding $H^1_R \subset L^2(\mathbb{R}^d)$ and from the fact that for every $z \in \{\mathrm{Re}\, z < c_1\}$, $K(z)$ acts continuously from $L^2(\mathbb{R}^d)$ into H^1_R, where

$$H^1_R \;:=\; \{u \in H^1(\mathbb{R}^d),\ \mathrm{supp}(u) \subset B(0,R)\}\,.$$

Indeed, for any z in $\{\mathrm{Re}(z) < c_1\}$, the operator $d_\phi(\tilde{P}_\phi - z)^{-1} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous thanks to (2.1) and hence, since W is smooth and supported in $B(0,R)$, $K(z) : L^2(\mathbb{R}^d) \to H^1_R$ is also continuous.

To conclude, it remains to prove the last statement of Proposition 1.1. To this end, note first that $P_\phi e^{-\frac{\phi}{h}} = 0$ according to (1.14) and let us recall that, according to (1.11), $e^{-\frac{\phi}{h}} \in D(\Delta_\phi) \subset D(P_\phi)$. Thus, $\mathrm{Span}\{e^{-\frac{\phi}{h}}\} \subset \mathrm{Ker}\, P_\phi$ and 0 is an eigenvalue of P_ϕ. It has moreover finite algebraic multiplicity according to the preceding analysis. Conversely, the relation

$$\forall\, u \,\in\, D(P_\phi)\,, \quad \mathrm{Re}\langle P_\phi u, u\rangle \;=\; \|d_\phi u\|^2 \;=\; h^2\|e^{-\frac{\phi}{h}}\nabla(e^{\frac{\phi}{h}}u)\|^2$$

leads to $\mathrm{Ker}\, P_\phi \subset \mathrm{Span}\{e^{-\frac{\phi}{h}}\}$ and the same arguments also show that $\mathrm{Ker}\, P^*_\phi = \mathrm{Span}\{e^{-\frac{\phi}{h}}\}$. This implies that 0 is an eigenvalue of P_ϕ with algebraic multiplicity one. Indeed, if it was not the case, there would exist $u \in D(P_\phi)$ such that $u \notin \mathrm{Ker}\, P_\phi$ and $P_\phi u = e^{-\frac{\phi}{h}}$, and hence such that

$$0 \;<\; \langle P_\phi u, e^{-\frac{\phi}{h}}\rangle \;=\; \langle u, P^*_\phi e^{-\frac{\phi}{h}}\rangle \;=\; 0\,.$$

2.2. **Spectral analysis near the origin.** Let us denote by $(e^W_k)_{k\geq 1}$ the eigenfunctions of Δ_ϕ associated with the non-decreasing sequence of eigenvalue $(\lambda^W_k)_{k\geq 1}$. Let ϵ_0 and $h_0 > 0$ be given by Proposition 1.2. We recall that for every $h \in (0, h_0]$, it holds

$$\mathrm{card}\Big(\sigma(\Delta_\phi) \cap \{\mathrm{Re}\, z < \epsilon_0 h\}\Big) \;=\; n_0\,,$$

where n_0 is the number of local minima of ϕ. We define

$$\begin{array}{rccl} R_- \,:\, & \mathbb{C}^{n_0} & \longrightarrow & L^2(\mathbb{R}^d) \\ & (\alpha_k) & \longmapsto & \sum_{k=1}^{n_0} \alpha_k e^W_k \end{array}$$

and $R_+ := R_-^*$, i.e.

$$\begin{array}{rcl} R_+ \,:\, L^2(\mathbb{R}^d) & \longrightarrow & \mathbb{C}^{n_0} \\ u & \longmapsto & (\langle u, e_k^W\rangle)_{k=1,\dots,n_0}\,. \end{array}$$

Note in particular the relations

$$R_+R_- \;=\; \mathrm{Id}_{\mathbb{C}^{n_0}} \quad \text{and} \quad R_-R_+ \;=\; \Pi\,, \tag{2.6}$$

where Π denotes the orthogonal projection onto $\mathrm{Ran}(R_-) = \mathrm{Span}\big(e_k^W, k \in \{1,\dots,n_0\}\big)$. We also define the spectral projector

$$\hat\Pi \;:=\; 1 - \Pi\,.$$

For $z \in \mathbb{C}$, let us then consider on the Hilbert space $\hat E := \mathrm{Ran}(\hat\Pi)$ the following unbounded operator which will be useful in the rest of this section:

$$\hat P_{\phi,z} \;:=\; \hat\Pi(P_\phi - z)\hat\Pi \quad \text{with domain} \quad D(\hat P_{\phi,z}) \;:=\; \hat\Pi(D(P_\phi))\,. \tag{2.7}$$

Hence $D(\hat P_{\phi,z})$ is dense in $\hat E$ and, since $\mathrm{Ran}\,\Pi \subset D(\Delta_\phi) \subset D(P_\phi)$, it holds $\hat\Pi(D(P_\phi)) \subset D(P_\phi)$ and $\hat P_{\phi,z}$ is well and densely defined.

Lemma 2.1. *Let ϵ_0 and $h_0 > 0$ be given by Proposition 1.2. Then, for every $h \in (0, h_0]$, the operator $\hat P_{\phi,z} : D(\hat P_{\phi,z}) \to \hat E$ defined in* (2.7) *is invertible on $\{\mathrm{Re}\, z < \epsilon_0 h\}$. Moreover, for any $\epsilon_1 \in (0, \epsilon_0)$ it holds:*

$$\forall\, z \,\in\, \{\mathrm{Re}\, z < \epsilon_1 h\}\,, \quad \|\hat P_{\phi,z}^{-1}\|_{\hat E\to\hat E} \;=\; \mathcal{O}(h^{-1})\,,$$

uniformly with respect to z.

Proof. We begin by the following observation: the unbounded operator

$$\hat\Pi(P_\phi^* - z)\hat\Pi \quad \text{with domain} \quad \hat\Pi(D(P_\phi^*)) \;\subset\; D(P_\phi^*)$$

is well and densely defined on $\hat E$, and satisfies moreover

$$\hat\Pi(P_\phi^* - z)\hat\Pi \;=\; \hat P_{\phi,z}^*\,.$$

Indeed, the relation $\langle\hat\Pi(P_\phi - z)\hat\Pi v, w\rangle = \langle v, \hat\Pi(P_\phi^* - z)\hat\Pi w\rangle$, valid for every $v \in D(P_\phi)$ and $w \in D(P_\phi^*)$, implies that $\hat\Pi(P_\phi^* - z)\hat\Pi \subset \hat P_{\phi,z}^*$. Moreover, for every $v \in D(P_\phi)$ and $w \in D(\hat P_{\phi,z}^*)$, one has

$$\begin{aligned} \langle (P_\phi - z)v, w\rangle \;&=\; \langle (P_\phi - z)\Pi v, w\rangle + \langle (P_\phi - z)\hat\Pi v, w\rangle \\ &=\; \langle (P_\phi - z)\Pi v, w\rangle + \langle \hat\Pi v, \hat P_{\phi,z}^* w\rangle. \end{aligned}$$

Since $P_\phi\Pi$ is continuous, Π being continuous with finite rank, one has $|\langle P_\phi\Pi v, w\rangle| \le C\|v\|\|w\|$ for some $C > 0$ independent of (v, w), which implies that $w \in D(P_\phi^*)$. Hence $D(\hat P_{\phi,z}^*) \subset D(P_\phi^*)$ and since $\mathrm{Ran}(\Pi) \subset D(\Delta_\phi) \subset D(P_\phi^*)$, this implies $\hat\Pi(P_\phi^* - z)\hat\Pi = \hat P_{\phi,z}^*$.

Let now consider z in $\{\mathrm{Re}\, z < \epsilon_0 h\}$ and let us prove that $\hat P_{\phi,z}$ is invertible from $D(\hat P_{\phi,z})$ onto $\hat E$. First, according to Proposition 1.2, we have for every $u \in D(\Delta_\phi)$,

$$\begin{aligned} \mathrm{Re}\langle (P_\phi - z)\hat\Pi u, \hat\Pi u\rangle \;&=\; \langle (\Delta_\phi - \mathrm{Re}(z))\hat\Pi u, \hat\Pi u\rangle \\ &\ge\; (\epsilon_0 h - \mathrm{Re}\, z)\|\hat\Pi u\|^2\,, \end{aligned} \tag{2.8}$$

and the inequality (2.8) is also true when $u \in D(P_\phi)$. Indeed, for any $u \in D(P_\phi)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\Delta_\phi)$ such that $u_n \to u$ and $P_\phi u_n \to P_\phi u$ in $L^2(\mathbb{R}^d)$. Hence $\hat{\Pi} u_n \to \hat{\Pi} u$ and, since $P_\phi \Pi$ is continuous, it also holds $P_\phi \hat{\Pi} u_n \to P_\phi \hat{\Pi} u$. In particular, it follows that $\hat{P}_{\phi,z}$ is injective. Note that a similar analysis shows that $\hat{P}^*_{\phi,z}$ is also injective.

Second, let us show that $\hat{P}_{\phi,z}$ is closed, which will in particular imply that $\mathrm{Ran}(\hat{P}_{\phi,z})$ is closed according to (2.8). For shortness, we denote $\hat{P} = \hat{P}_{\phi,z}$ and $P = P_\phi$. Suppose that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $D(\hat{P}) \subset D(P)$ such that $u_n \to u$ and $\hat{P} u_n \to v$ in $\hat{E}$. Since $\mathrm{Ran}\,\Pi \subset D(\Delta_\phi) \subset D(P^*)$, it holds

$$\Pi P u_n \;=\; \sum_{k=1}^{n_0} \langle P u_n, e_k^W \rangle e_k^W \;=\; \sum_{k=1}^{n_0} \langle u_n, P^* e_k^W \rangle e_k^W \;\underset{n\to+\infty}{\longrightarrow}\; \sum_{k=1}^{n_0} \langle u, P^* e_k^W \rangle e_k^W\,,$$

and thus $(P - -z)u_n = \hat{P} u_n + \Pi(P - z) u_n$ converges. Since P is closed, this implies that $u \in D(P) \cap \mathrm{Ran}\hat{\Pi} = \hat{\Pi}(D(P))$ and that

$$(P - z)u \;=\; v + g \quad \text{with} \quad g \in \mathrm{Ran}\,\Pi\,.$$

Multiplying this relation by $\hat{\Pi}$, we get $v = \hat{P} u$, which proves that $\hat{P}$ is closed. To prove that $\hat{P}$ is invertible from $D(\hat{P})$ onto $\hat{E}$, it is thus enough to prove that $\mathrm{Ran}(\hat{P})$ is dense in $\hat{E}$. Let then $v \in \hat{E}$ be such that $\langle \hat{P} u, v \rangle = 0$ for all $u \in D(\hat{P})$. Then $v \in D(\hat{P}^*)$ and $\hat{P}^* v = 0$. By injectivity of $\hat{P}^*$, it thus holds $v = 0$, which proves the invertibility of $\hat{P} : D(\hat{P}_{\phi,z}) \to \hat{E}$.

The relation (2.8) then implies that for all $z \in \{\mathrm{Re}\, z \le \epsilon_1 h\}$, one has

$$\mathrm{Re}\langle (P_\phi - z)\hat{\Pi} u, \hat{\Pi} u \rangle \ge \delta h \|\hat{\Pi} u\|^2$$

with $\delta = \epsilon_0 - \epsilon_1 > 0$. Hence, for the operator norm on $\hat{E} \subset L^2(\mathbb{R}^d)$, one has

$$\hat{P}^{-1}_{\phi,z} = \mathcal{O}(h^{-1})\,,$$

uniformly with respect to $z \in \{\mathrm{Re}\, z < \epsilon_1 h\}$. □

For $z \in \mathbb{C}$, we now consider the Grushin operator $\mathcal{P}_\phi(z) : D(P_\phi) \times \mathbb{C}^{n_0} \to L^2(\mathbb{R}^d) \times \mathbb{C}^{n_0}$ defined by

$$\mathcal{P}_\phi(z) = \begin{pmatrix} P_\phi - z & R_- \\ R_+ & 0 \end{pmatrix}. \tag{2.9}$$

Lemma 2.2. *Let ϵ_0 and $h_0 > 0$ be given by Proposition 1.2. Then, the operator $\mathcal{P}_\phi(z)$ is invertible on $\{\mathrm{Re}\, z < \epsilon_0 h\}$. More precisely, for every $z \in \{\mathrm{Re}\, z < \epsilon_0 h\}$, $(u, u_-) \in D(P_\phi) \times \mathbb{C}^{n_0}$ and $(f, y) \in L^2(\mathbb{R}^d) \times \mathbb{C}^{n_0}$, it holds*

$$\mathcal{P}_\phi(z)(u, u_-) \;=\; (f, y)$$

if and only if

$$(u, u_-) \;=\; \big(R_- y + v\,,\, R_+ f - R_+(P_\phi - z) R_- y - R_+ P_\phi v \big)\,,$$

where

$$v \;:=\; \hat{P}^{-1}_{\phi,z} \hat{\Pi} f - \hat{P}^{-1}_{\phi,z} \hat{\Pi} P_\phi R_- y \;\in\; \hat{\Pi}(D(P_\phi))\,.$$

Proof. Let $(f,y)\in L^2(\mathbb{R}^d)\times\mathbb{C}^{n_0}$ and assume that $(u,u_-)\in D(P_\phi)\times\mathbb{C}^{n_0}$ satisfies

$$\left\{\begin{array}{l}(P_\phi - z)u + R_-u_- = f\\ R_+u = y.\end{array}\right. \tag{2.10}$$

Applying R_+ to the first equation and R_- to the second one, we get, according to (2.6):

$$u_- = R_+f - R_+(P_\phi - z)u \quad \text{and} \quad u = R_-y + v\,,$$

with $v\in \operatorname{Ran}\hat\Pi\cap D(P_\phi)=\hat\Pi(D(P_\phi))$ solution to

$$(P_\phi - z)R_-y + (P_\phi - z)v + R_-u_- = f.$$

Then, applying $\hat\Pi$ to the latter equation, we get, using $\hat\Pi R_- = 0$,

$$\hat\Pi(P_\phi - z)\hat\Pi v \;=\; \hat\Pi f - \hat\Pi(P_\phi - z)R_-y - \hat\Pi R_-u_- \;=\; \hat\Pi f - \hat\Pi P_\phi R_-y\,. \tag{2.11}$$

Conversely, note that if $v\in \operatorname{Ran}\hat\Pi\cap D(P_\phi)$ is solution to (2.11), then according to (2.6),

$$\Big(u = R_-y + v\,,\, u_- = R_+f - R_+(P_\phi - z)(R_-y + v)\Big) \;\in\; D(P_\phi)\times\mathbb{C}^{n_0}$$

is solution to (2.10).

Hence, the statement of Lemma 2.2 simply follows from Lemma 2.1 which implies that, for every $z\in\{\operatorname{Re} z<\epsilon_0 h\}$,

$$v \;=\; \hat P^{-1}_{\phi,z}\hat\Pi f - \hat P^{-1}_{\phi,z}\hat\Pi P_\phi R_-y \;\in\; \hat\Pi(D(P_\phi))$$

is the unique solution to (2.11). □

Proof of Theorem 1.3. Let ϵ_0 and h_0 be as in Lemmata 2.1 and 2.2, and take $\epsilon_1\in(0,\epsilon_0)$. For $z\in\{\operatorname{Re} z<\epsilon_0 h\}$, let $\mathcal{E}_\phi(z)=\mathcal{P}_\phi(z)^{-1}$. According to Lemma 2.2, it thus holds

$$\mathcal{E}_\phi(z) = \begin{pmatrix} E(z) & E_+(z)\\ E_-(z) & E_{-+}(z)\end{pmatrix},$$

where E, E_-, E_+, E_{-+} are holomorphic in $\{\operatorname{Re} z<\epsilon_0 h\}$ and satisfy the following formulas:

$$E_+(z) \;=\; R_- - \hat P^{-1}_{\phi,z}\hat\Pi P_\phi R_-\,,\quad E_-(z) \;=\; R_+ - R_+P_\phi\hat P^{-1}_{\phi,z}\hat\Pi\,, \tag{2.12}$$

$$E_{-+}(z) \;=\; -R_+(P_\phi - z)R_- + R_+P_\phi\hat P^{-1}_{\phi,z}\hat\Pi P_\phi R_- \tag{2.13}$$

and

$$E(z) \;=\; \hat P^{-1}_{\phi,z}\hat\Pi\,. \tag{2.14}$$

Moreover, $P_\phi - z$ is invertible if and only if $E_{-+}(z)$ is, in which case it holds

$$(P_\phi - z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z). \tag{2.15}$$

We refer in particular to [22] for more details in this connection.

We now want to use these formulas to compute the number of poles of $(P_\phi - z)^{-1}$. Thanks to (2.2), one has, for some $C>0$ and all $k\in\{1,\dots,n_0\}$,

$$\|b_h\cdot d_\phi e^W_k\| \le C\Big(\|\Delta_\phi e^W_k\| + \|d_\phi e^W_k\|\Big) \le C(\lambda^W_k + \sqrt{\lambda^W_k})\,.$$

Using the bound $\lambda_k^W \leq Che^{-2\frac{S}{h}}$ given by Proposition 1.2, this yields the existence of some $C > 0$ such that for every $k \in \{1, \dots, n_0\}$,

$$\|b_h \cdot d_\phi e_k^W\| \leq C\sqrt{h}e^{-\frac{S}{h}} \quad \text{and} \quad \|P_\phi e_k^W\| \leq C\sqrt{h}e^{-\frac{S}{h}}\,. \tag{2.16}$$

This shows that $R_+\Delta_\phi R_- = \mathcal{O}(he^{-2\frac{S}{h}})$ and $R_+ b\cdot d_\phi R_- = \mathcal{O}(\sqrt{h}e^{-\frac{S}{h}})$. Hence, for all $z \in \mathbb{C}$, it holds

$$\begin{aligned} R_+(P_\phi - z)R_- &= R_+P_\phi R_- - z\,\mathrm{Id}_{\mathbb{C}^{n_0}} \\ &= -z\,\mathrm{Id}_{\mathbb{C}^{n_0}} + \mathcal{O}(\sqrt{h}e^{-\frac{S}{h}}). \end{aligned} \tag{2.17}$$

On the other hand, we deduce from (2.16) and from the related relation

$$\langle P_\phi u, e_k^W\rangle = \langle u, \Delta_\phi e_k^W\rangle - \langle u, bd_\phi e_k^W\rangle = \mathcal{O}(he^{-2\frac{S}{h}} + \sqrt{h}e^{-\frac{S}{h}})\|u\|,$$

valid for any $u \in D(P_\phi)$ and $k \in \{1, \dots, n_0\}$, that

$$P_\phi R_- = \mathcal{O}(h^{\frac12}e^{-\frac{S}{h}}) \quad \text{and} \quad R_+P_\phi = \mathcal{O}(h^{\frac12}e^{-\frac{S}{h}})\,. \tag{2.18}$$

Moreover, we know from Lemma 2.1 that, uniformly on $\{\mathrm{Re}\, z < \epsilon_1 h\}$, it holds $\hat{P}_{\phi,z}^{-1} = \mathcal{O}(h^{-1})$. Therefore, injecting this estimate and (2.17), (2.18) into (2.13) and (2.12), we obtain respectively, uniformly on $\{\mathrm{Re}\, z < \epsilon_1 h\}$,

$$E_{-+}(z) \;=\; z\,\mathrm{Id}_{\mathbb{C}^{n_0}} + \mathcal{O}(h^{\frac12}e^{-\frac{S}{h}}) \tag{2.19}$$

and

$$E_+(z) \;=\; R_- + \mathcal{O}(h^{-\frac12}e^{-\frac{S}{h}}) \quad \text{and} \quad E_-(z) \;=\; R_+ + \mathcal{O}(h^{-\frac12}e^{-\frac{S}{h}})\,. \tag{2.20}$$

According to (2.19), $E_{-+}(z)$ is then invertible when $z \in \{\mathrm{Re}\, z < \epsilon_1 h\}$ satisfies $|z| \geq Ch^{\frac12}e^{-\frac{S}{h}}$ for C large enough and the spectrum of P_ϕ in $\{\mathrm{Re}\, z < \epsilon_1 h\}$ is then of order $\mathcal{O}(h^{\frac12}e^{-\frac{S}{h}})$. Moreover, for $|z| = \frac{\epsilon_1}{2}h$, it holds

$$E_{-+}(z) = z\Big(\mathrm{Id}_{\mathbb{C}^{n_0}} + \mathcal{O}(h^{-\frac12}e^{-\frac{S}{h}})\Big) \tag{2.21}$$

and injecting (2.21) and (2.20) into (2.15) shows that

$$(P_\phi - z)^{-1} = E(z) - \frac{1}{z}\Big(\Pi + \mathcal{O}(h^{-\frac12}e^{-\frac{S}{h}})\Big)\,.$$

Thus, the spectral projector on the open disc $D(0, \frac{\epsilon_1}{2}h)$ satisfies

$$\Pi_{D(0,\frac{\epsilon_1}{2}h)} := -\frac{1}{2\pi i}\int_{\partial D(0,\frac{\epsilon_1}{2}h)} (P_\phi - z)^{-1}dz = \Pi + \mathcal{O}(h^{-\frac12}e^{-\frac{S}{h}})\,,$$

where we recall that Π is a projector of rank n_0. This implies that for every $h > 0$ small enough, the rank of $\Pi_{D(0,\frac{\epsilon_1}{2}h)}$, which is the number of eigenvalues of P_ϕ in $D(0, \frac{\epsilon_1}{2}h)$ counted with algebraic multiplicity, is precisely n_0.

In order to achieve the proof of Theorem 1.3, it just remains to prove the resolvent estimate stated there. On the one hand, it follows easily from (2.14), (2.20), and Lemma 2.1 that

$$E(z) = \mathcal{O}(h^{-1})\,,\ E_-(z) = \mathcal{O}(1)\,, \ \text{ and } \ E_+(z) = \mathcal{O}(1)\,,$$

uniformly with respect to $z \in \{\mathrm{Re}\, z < \epsilon_1 h\}$. On the other hand, taking $\epsilon \in (0, \epsilon_1)$, it follows from (2.19) that $E_{-+}^{-1}(z) = \mathcal{O}(h^{-1})$, uniformly with respect to $z \in \{\mathrm{Re}\, z < \epsilon_1 h\} \cap \{|z| > \epsilon h\}$. Plugging all these estimates into (2.15), we obtain the announced result.

Eventually, since $\sigma(P_\phi^*) = \overline{\sigma(P_\phi)}$ and, for all $z \notin \sigma(P_\phi)$, $\|(P_\phi^* - \overline{z})^{-1}\| = \|(P_\phi - z)^{-1}\|$, it follows easily that the conclusions of Theorem 1.3 hold also true for P_ϕ^*. □

3. Geometric preparation

Let us begin this section by observing that the identity $b \cdot \nabla V = 0$ arising from (1.3) implies that $\mathcal{U} \subset \{x \in \mathbb{R}^d,\ b(x) = 0\}$, where we recall that $\mathcal{U}$ denotes the set of critical points of the Morse function V, as it can be easily proved using a Taylor expansion. Moreover, we have the following

Lemma 3.1. *Suppose that Assumptions 1 and 3 hold true and let $\mathbf{u} \in \mathcal{U}$ be a critical point of V. Then, there exists a smooth map $J_{\mathbf{u}} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ such that $J_{\mathbf{u}}(\mathbf{u})$ is antisymmetric and $b(x) = J_{\mathbf{u}}(x)\nabla V(x)$ for all x in some neighborhood of $\mathbf{u}$. Moreover, it holds*

$$J_{\mathbf{u}}(\mathbf{u}) \;=\; B(\mathbf{u})\mathrm{Hess}\, V(\mathbf{u})^{-1}\,,$$

where $B(\mathbf{u}) = \mathrm{Jac}_{\mathbf{u}} b$ is the Jacobian matrix of b at $\mathbf{u}$.

Proof. Let $\mathbf{u} \in \mathcal{U}$ that we assume to be 0 to lighten the notation. Thanks to the Taylor formula, there exists a smooth map $G : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ such that $b(x) = G(x)x$ for all $x \in \mathbb{R}^d$ and $G(0) = \mathrm{Jac}_0\, b$. The same construction works for ∇V and denoting by $\mathcal{S}_d$ the set of symmetric matrices, there exists a smooth map $A : \mathbb{R}^d \to \mathcal{S}_d$ such that $\nabla V(x) = A(x)x$ for all $x \in \mathbb{R}^d$ and $A(0) = \mathrm{Hess}\, V(0)$. The equation $\langle b(x), \nabla V(x)\rangle = 0$ for all $x \in \mathbb{R}^d$ then yields $\langle G(x)x, A(x)x\rangle = 0$ and hence, since $A(x)$ is symmetric, $\langle A(x)G(x)x, x\rangle = 0$ for all $x \in \mathbb{R}^d$. Expanding $A(x)G(x)$ in powers of x, this implies that

$$\forall\, x \,\in\, \mathbb{R}^d\,, \quad \langle A(0)G(0)x, x\rangle \;=\; 0\,.$$

Hence, the matrix $A(0)G(0)$ is antisymmetric. Since $A(0)$ is symmetric and invertible (since V is a Morse function), this implies that $G(0)A(0)^{-1}$ is antisymmetric. Moreover, $A(x)$ is then also invertible in a neighborhood $\mathcal{V}$ of 0 and we can thus define $J_0(x) = G(x)A(x)^{-1}$ on $\mathcal{V}$. One then has

$$J_0(x)\nabla V(x) \;=\; G(x)A(x)^{-1}A(x)x \;=\; b(x)$$

for all $x \in \mathcal{V}$ and $J_0(0) = G(0)A(0)^{-1}$ is antisymmetric thanks to the above analysis. □

Remark 3.2. *It is not clear from the above proof that the relation $b\cdot\nabla V = 0$ implies the existence of a smooth map $J : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ with antisymmetric matrices values such that $b = J(\nabla V)$. However, it follows from* (1.3) *that for such a map J, the vector fields of the form $b_h = J(\nabla V) + h\nu$ enter in our framework as soon as*

$$\mathrm{div}\,\nu \;=\; 0 \quad \textit{and} \quad \Big(\sum_{i=1}^{d} \partial_i J_{ij}\Big)_{j=1,\dots,d} \cdot \nabla V \;=\; \nu \cdot \nabla V\,. \tag{3.1}$$

This is for instance the case when $\nu = \big(\sum_{i=1}^{d} \partial_i J_{ij}\big)_{j=1,\dots,d}$, which is in particular satisfied when J appears to be constant. Moreover, when $\nu =$

$\big(\sum_{i=1}^d \partial_i J_{ij}\big)_{j=1,\dots,d}$, $L_{V,b,\nu}$ *(or equivalently P_ϕ) admits a supersymmetric structure according to (see indeed* (1.8)*)*

$$L_{V,b,\nu} \;=\; -h\,e^{\frac{V}{h}}\,\mathrm{div}\circ\Big(e^{-\frac{V}{h}}\big(I_d - J\big)\nabla\Big) \;=\; h\,\nabla^*\big(I_d - J\big)\nabla\,,$$

where the adjoint is considered with respect to m_h (or equivalently

$$P_\phi \;=\; \Delta_\phi + b_h\cdot d_\phi \;=\; d_\phi^*\big(I_d - J\big)d_\phi\,,$$

where the adjoint is now considered with respect to the Lebesgue measure). Using this structure, we may follow the general approach of [12] to analyse the spectrum of P_ϕ. Nevertheless, the operator P_ϕ still does not have any PT-symmetry and following this approach would again require to replace the use of the Fan inequalities by the one of Theorem A.4 in the final part of the analysis. We believe that this approach may yield complete asymptotic expansions of the small eigenvalues of P_ϕ (or $L_{V,b,\nu}$) in this setting.

However, when J has antisymmetric matrices values and (3.1) *holds but $\nu \neq \big(\sum_{i=1}^d \partial_i J_{ij}\big)_{j=1,\dots,d}$, the operator P_ϕ is not supersymmetric anymore (see [19] for related results).*

We are now in position to prove Lemma 1.8. Throughout the rest of this section, we denote

$$-\mu_1 < 0 < \mu_2 \le \dots \le \mu_d$$

the eigenvalues of $\mathrm{Hess}\,V(\mathbf{s})$ counted with multiplicity. For shortness, we will denote

$$B \;=\; B(\mathbf{s}) \;=\; \mathrm{Jac}_{\mathbf{s}} b \quad\text{and}\quad J \;=\; J(\mathbf{s}) \;=\; B(\mathbf{s})(\mathrm{Hess}\,V(\mathbf{s}))^{-1}\,.$$

We recall from Lemma 3.1 that J is antisymmetric.

Step 1 : Let us first prove that $\det(\mathrm{Hess}\,V(\mathbf{s}) + B^*) < 0$. Since the matrix $\mathrm{Hess}\,V(\mathbf{s}) + B^*$ is real, it thus admits at least one negative eigenvalue.

Since $\mathrm{Hess}\,V(\mathbf{s})$ is real and symmetric, there exists $P \in \mathcal{M}_d(\mathbb{R})$ such that

$$P^* = P^{-1} \quad\text{and}\quad \mathrm{Hess}\,V(\mathbf{s}) = P\,D\,P^{-1}\,,$$

where $D := \mathrm{Diag}(-\mu_1, \mu_2, \dots, \mu_d)$. It then holds:

$$\mathrm{Hess}\,V(\mathbf{s}) + B^* \;=\; \mathrm{Hess}\,V(\mathbf{s})\,(I_d - J) \;=\; P\,D\,(I_d - P^{-1}\,J\,P)\,P^{-1}\,. \tag{3.2}$$

Since $(P^{-1}\,J\,P)^* = -P^{-1}\,J\,P$, there exist moreover $p \in \{0, \dots, \lfloor \frac{d}{2} \rfloor\}$, $\eta_1, \dots, \eta_p > 0$, and $Q \in \mathcal{M}_d(\mathbb{R})$ satisfying $Q^* = Q^{-1}$ such that

$$Q^{-1}\,P^{-1}\,J\,P\,Q \;=\; \begin{bmatrix} A_1 & & (0) & \\ & \ddots & & \\ (0) & & A_p & \\ & & & (0) \end{bmatrix}$$

where, for every $k \in \{1, \dots, p\}$,

$$A_k = \begin{bmatrix} 0 & -\eta_k \\ \eta_k & 0 \end{bmatrix}\,.$$

Here, the rank of the matrix J is $2p$ and its nonzero eigenvalues are the $\pm i\eta_k$, $k \in \{1, \dots, p\}$. Therefore, it holds

$$Q^{-1}\,(I_d - P^{-1}\,J\,P)\,Q \;=\; \begin{bmatrix} B_1 & & (0) & \\ & \ddots & & \\ (0) & & B_p & \\ & & & I_{d-2p} \end{bmatrix} \tag{3.3}$$

where, for every $k \in \{1, \dots, p\}$,

$$B_k = \begin{bmatrix} 1 & \eta_k \\ -\eta_k & 1 \end{bmatrix}.$$

We then deduce from (3.2) and (3.3) that

$$\det(\operatorname{Hess} V(\mathbf{s}) + B^*) \;=\; -(\Pi_{k=1}^{d}\mu_k)\,(\Pi_{k=1}^{p}(1+\eta_k^2)) \;<\; 0\,,$$

which concludes this first step.

Step 2 : Let us denote by μ a negative eigenvalue of $\operatorname{Hess} V(\mathbf{s}) + B^*$ and let us show that μ is its only negative eigenvalue and has geometric multiplicity one.

Assume first by contradiction that μ has geometric multiplicity two and denote by ξ_1, ξ_2 two associated unitary eigenvectors such that $\langle \xi_1, \xi_2 \rangle = 0$. Let us also define $\xi_i' := P^{-1}\xi_i$ for $i \in \{1, 2\}$ so that ξ_1' and ξ_2' are orthogonal and unitary. According to (3.2), it holds moreover for $i \in \{1, 2\}$,

$$D\,(I_d - P^{-1}\,J\,P)\,\xi_i' \;=\; \mu\,\xi_i' \quad \text{and hence} \quad D^{-1}\xi_i' \;=\; \frac{1}{\mu}(I_d - P^{-1}\,J\,P)\,\xi_i'\,.$$

In particular, since $(P^{-1}\,J\,P)^* = -P^{-1}\,J\,P$, it holds for every $(a, b) \in \mathbb{R}^2$ satisfying $a^2 + b^2 = 1$:

$$\langle D^{-1}(a\xi_1' + b\xi_2'), a\xi_1' + b\xi_2' \rangle \;=\; \frac{1}{\mu}\,.$$

Applying the Max-Min principle to the symmetric matrix D^{-1}, this shows that the second eigenvalue $\mu_2(D^{-1})$ of the matrix D^{-1} satisfies $\mu_2(D^{-1}) \le \frac{1}{\mu} < 0$, contradicting $D^{-1} = \operatorname{Diag}(-\frac{1}{\mu_1}, \frac{1}{\mu_2}, \dots, \frac{1}{\mu_d})$.

Hence the negative eigenvalue μ has geometric multiplicity one and we have to show that it is the only negative eigenvalue of $\operatorname{Hess} V(\mathbf{s}) + B^*$. We reason again by contradiction, assuming that $\operatorname{Hess} V(\mathbf{s}) + B^*$ admits another negative eigenvalue that we denote by η. Note in particular that it follows from the relation (see indeed (3.2))

$$\operatorname{Hess} V(\mathbf{s})\,(I_d + J) \;=\; \operatorname{Hess} V(\mathbf{s})\big(\operatorname{Hess} V(\mathbf{s}) + B^*\big)^*(\operatorname{Hess} V(\mathbf{s}))^{-1}$$

that η is also an eigenvalue of $\operatorname{Hess} V(\mathbf{s}) - B^*(\mathbf{s}) = \operatorname{Hess} V(\mathbf{s})\,(I_d + J)$. Denote now by ξ_1 a unitary eigenvector of $\operatorname{Hess} V(\mathbf{s}) + B^*$ associated with μ and by ξ_2 a unitary eigenvector of $\operatorname{Hess} V(\mathbf{s}) - B^*$ associated with η. Defining again $\xi_i' := P^{-1}\xi_i$ for $i \in \{1, 2\}$, we have thus

$$D^{-1}\xi_1' \;=\; \frac{1}{\mu}(I_d - P^{-1}\,J\,P)\,\xi_1' \quad \text{and} \quad D^{-1}\xi_2' \;=\; \frac{1}{\eta}(I_d + P^{-1}\,J\,P)\,\xi_2'\,.$$

It follows that

$$\langle D^{-1}\xi_1', \xi_2'\rangle \;=\; 0\;, \quad \langle D^{-1}\xi_1', \xi_1'\rangle \;=\; \frac{1}{\mu} \quad \text{and} \quad \langle D^{-1}\xi_2', \xi_2'\rangle \;=\; \frac{1}{\eta}\,.$$

The vectors ξ_1' and ξ_2' are in particular linearly independent and it holds for some positive constant c and every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$\langle D^{-1}(a\xi_1' + b\xi_2'), a\xi_1' + b\xi_2'\rangle \;=\; \frac{a^2}{\mu} + \frac{b^2}{\eta} \;\le\; -c\,\|a\xi_1' + b\xi_2'\|^2$$

Applying again the Max-Min principle to the symmetric matrix D^{-1} leads to $\mu_2(D^{-1}) \le -c < 0$ and hence to a contradiction. This concludes the proof of the second step.

Step 3 : Let us now prove the relation

$$\det\Big(\operatorname{Hess} V(\mathbf{s}) + 2|\mu|\,\xi\,\xi^*\Big) \;=\; -\det \operatorname{Hess} V(\mathbf{s})\,, \tag{3.4}$$

which is equivalent to

$$\det\Big(I_d + 2\,|\mu| D^{-1}\,\xi'\,\xi'^*\Big) \;=\; -1\,, \tag{3.5}$$

where ξ denotes a unitary eigenvector of $\operatorname{Hess} V(\mathbf{s}) + B^*$ associated with μ and $\xi' := P^{-1}\xi$. To this end, note first that it obviously holds

$$\forall\, x \,\in\, (\xi')^{\perp}\,, \quad \Big(I_d + 2\,|\mu| D^{-1}\,\xi'\,\xi'^*\Big)x \;=\; x\,. \tag{3.6}$$

Moreover, since $D^{-1}\xi' = \frac{1}{\mu}(I_d - P^{-1}\,J\,P)\xi'$, it also holds

$$\begin{aligned}\Big(I_d + 2\,|\mu| D^{-1}\,\xi'\,\xi'^*\Big)\xi' \;&=\; \xi' + 2\,|\mu| D^{-1}\,\xi' \\ &=\; -\xi' + 2P^{-1}\,J\,P\xi'\,.\end{aligned} \tag{3.7}$$

Since $P^{-1}\,J\,P\xi'$ belongs to $(\xi')^{\perp}$, we deduce (3.5) and then (3.4) from (3.6) and (3.7).

Step 4 : To conclude the proof of the second item of Lemma 1.8, it only remains to show that the real symmetric matrix $M_V := \operatorname{Hess} V(\mathbf{s}) + 2|\mu|\,\xi\,\xi^*$ is positive definite, where we recall that ξ denotes a unitary eigenvector of $\operatorname{Hess} V(\mathbf{s}) + B^*$ associated with μ. This is an easy consequence of the Max-Min principle and of the relation $\det M_V = -\det D > 0$ obtained in the previous step. We have indeed, defining again $\xi' := P^{-1}\xi$,

$$\begin{aligned}\forall\, x \,\in\, \Big((1,0,\dots,0)^*\Big)^{\perp}\,, \quad \langle (D + 2|\mu|\,\xi'\,\xi'^*)x, x\rangle \;&=\; \langle Dx, x\rangle + 2|\mu|\,\langle \xi, x\rangle^2 \\ &\ge\; \mu_2\,\|x\|^2\,,\end{aligned}$$

which implies that the second eigenvalue of $D + 2|\mu|\,\xi'\,\xi'^*$, that is the second eigenvalue of M_V, is greater than or equal to μ_2, and hence positive. The first eigenvalue of M_V is then positive according to $\det M_V > 0$. This concludes this step of the proof.

Step 5 : We now prove the third item of Lemma 1.8. Since $\operatorname{Hess} V(\mathbf{s})(I_d - J)\xi = \mu\xi$ and $J^* = -J$, it first holds

$$(\operatorname{Hess} V(\mathbf{s}))^{-1}\,\xi \;=\; \frac{1}{\mu}\,(I_d - J)\xi \quad \text{and then} \quad \langle (\operatorname{Hess} V(\mathbf{s}))^{-1}\,\xi, \xi\rangle \;=\; \frac{1}{\mu}\,, \tag{3.8}$$

which proves the second part of the third item of Lemma 1.8. Defining again $\xi' := P^{-1}\xi$, this also means

$$-\frac{1}{\mu_1} + \sum_{k=2}^{d}(\frac{1}{\mu_k} + \frac{1}{\mu_1})\xi_k'^2 \;=\; -\frac{1}{\mu_1}\xi_1'^2 + \sum_{k=2}^{d}\frac{1}{\mu_k}\xi_k'^2 \;=\; \langle D^{-1}\,\xi', \xi'\rangle \;=\; \frac{1}{\mu}\,.$$

This implies that $\frac{1}{\mu} \geq -\frac{1}{\lambda_1}$, i.e. that $|\mu| \geq \mu_1$, with equality if and only if $\xi' = \pm(1,0,\dots,0)^*$, that is if and only if ξ is a unitary eigenvector of $(\operatorname{Hess} V(\mathbf{s}))^{-1}$ associated with $-\frac{1}{\mu_1}$, which is equivalent to the relation $J\xi = 0$ by (3.8), and hence to $B^*\xi = 0$ since $J = -(\operatorname{Hess} V(\mathbf{s}))^{-1}B^*$.

4. Spectral analysis in the case of Morse functions

4.1. **Construction of accurate quasimodes.** In the following, we assume that Assumption 4 is satisfied. Let us then consider some arbitrary $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, that is, according to Assumption 4, a local minimum of V which is not the global minimum $\underline{\mathbf{m}}$ of V. According to the labelling procedure of the introductory section leading to the definitions (1.15)–(1.17), it holds in particular $\mathbf{m} = \mathbf{m}_{i,j}$ and $\boldsymbol{\sigma}(\mathbf{m}) = \sigma_i$ for some $i \in \{2,\dots,N\}$ and $j \in \{1,\dots,N_i\}$. For every $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $\rho, \delta > 0$, where we recall that the mapping $\mathbf{j} : \mathcal{U}^{(0)} \to \mathcal{P}(\mathcal{V}^{(1)} \cup \{\mathbf{s}_1\})$ has been defined in (1.16) and that $V(\mathbf{s}) = \boldsymbol{\sigma}(\mathbf{m})$, we define the set

$$\mathcal{B}_{\mathbf{s},\rho,\delta} \;:=\; \{V \leq \boldsymbol{\sigma}(\mathbf{m}) + \delta\} \cap \Big\{x \in \mathbb{R}^d\,,\; |\xi(\mathbf{s}) \cdot (x - \mathbf{s})| \leq \rho\Big\}$$

and the set $\mathcal{C}_{\mathbf{s},\rho,\delta}$ by:

(4.1) $\qquad \mathcal{C}_{\mathbf{s},\rho,\delta}$ is the connected component of $\mathcal{B}_{\mathbf{s},\rho,\delta}$ containing $\mathbf{s}$,

where $\xi(\mathbf{s})$ has been defined in Lemma 1.8. We recall that $\xi(\mathbf{s})$ is an unitary eigenvector of the matrix $\operatorname{Hess} V(\mathbf{s}) + B^*(\mathbf{s})$ associated with its only negative eigenvalue $\mu(\mathbf{s})$ which has geometric multiplicity one. Let us also define

(4.2) $\qquad E_{\mathbf{m},\rho,\delta} \;:=\; \big(E_-(\mathbf{m}) \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + \delta\}\big) \setminus \cup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\mathcal{C}_{\mathbf{s},\rho,\delta}\,,$

where

(4.3) $\;E_-(\mathbf{m})$ is the connected component of $\{V < \sigma_{i-1}\}$ containing $\mathbf{m}$.

According to Assumption 4 and Remark 1.7, we recall that there is precisely one connected component $\widehat{E}(\mathbf{m}) \neq E(\mathbf{m})$ of $\{V < \boldsymbol{\sigma}(\mathbf{m})\}$ such that $\overline{E(\mathbf{m})} \cap \overline{\widehat{E}(\mathbf{m})} \neq \emptyset$. Moreover, it holds $\mathbf{j}(\mathbf{m}) = \partial\widehat{E}(\mathbf{m}) \cap \partial E(\mathbf{m})$ and the global minimum $\hat{\mathbf{m}}$ of $V|_{\widehat{E}(\mathbf{m})}$ satisfies $\boldsymbol{\sigma}(\hat{\mathbf{m}}) > \boldsymbol{\sigma}(\mathbf{m})$ and $V(\hat{\mathbf{m}}) < V(\mathbf{m})$ (see in this connection [20], where the notation $\widehat{E}(\mathbf{m})$ is introduced for an arbitrary Morse function).

According to the geometry of the Morse function V around $\partial E(\mathbf{m})$ and to Lemma 1.8, we have then the following result.

Lemma 4.1. *Assume that Assumption 4 is satisfied and let* $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, $\mathbf{s} \in \mathbf{j}(\mathbf{m})$*, and* $\xi(\mathbf{s})$ *be some unitary eigenvector of the matrix* $\operatorname{Hess} V(\mathbf{s}) + B^*(\mathbf{s})$ *associated with its unique negative eigenvalue (see Lemma 1.8). Then, there exists a neigborhood* $\mathcal{O}$ *of* $\mathbf{s}$ *such that:*

$$\forall\, x \in \mathcal{O} \setminus \{\mathbf{s}\},\quad \big(x - \mathbf{s} \in \xi(\mathbf{s})^{\perp} \implies V(x) > V(\mathbf{s})\big)\,.$$

It follows that there exist $\rho_0, \delta_0 > 0$ sufficiently small such that for all $\rho \in (0, \rho_0]$ and $\delta \in (0, \delta_0]$, the set $E_{\mathbf{m},3\rho,3\delta}$ defined in (4.2) *has exactly two connected components, $E^+_{\mathbf{m},3\rho_0,3\delta_0}$ and $E^-_{\mathbf{m},3\rho,3\delta}$, containing respectively $\mathbf{m}$ and $\hat{\mathbf{m}}$.*

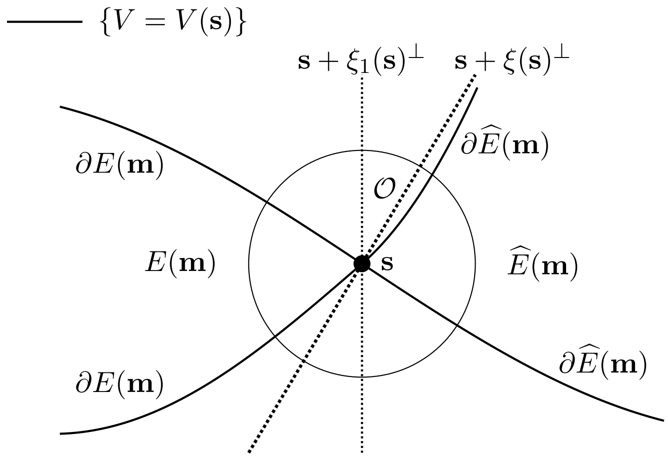

FIGURE 4.1. Representation of the Morse function V near $\mathbf{s} \in \mathbf{j}(\mathbf{m})$. Here, $\xi_1(\mathbf{s})$ denotes an eigenvector of $\operatorname{Hess} V(\mathbf{s})$ associated with its negative eigenvalue and $B^*(\mathbf{s})\xi(\mathbf{s}) \neq 0$. Note that according to the last item in Lemma 1.8, $\mathbf{s}+\xi_1(\mathbf{s})^\perp$ and $\mathbf{s} + \xi(\mathbf{s})^\perp$ coincide if and only if $B^*(\mathbf{s})\xi(\mathbf{s}) = 0$.

Proof. For shortness, we denote $\xi = \xi(\mathbf{s})$. By a continuity argument, note that to prove the first part of Lemma 4.1, it is sufficient to prove that the linear hyperplane $\xi^\perp$ does not meet the cone $\{X \in \mathbb{R}^n\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle \leq 0\}$ outside the origin. The second part of the lemma then simply follows from the observation that the set $\mathcal{C}_{\mathbf{s},\rho,\delta}$ defined in (4.1) is thus an arbitrary small neighborhood of $\mathbf{s}$ when $\rho, \delta > 0$ tend to 0.

When $d \geq 3$, it is then enough to show that for any column vector $X \in \mathbb{R}^d \setminus \{0\}$ such that $\langle \operatorname{Hess} V(\mathbf{s})X, X\rangle = 0$, it holds $\operatorname{Span} X \oplus \xi^\perp = \mathbb{R}^d$, i.e. $\langle X, \xi\rangle \neq 0$. Indeed, when $d \geq 3$, any linear hyperplane meets $\{X \in \mathbb{R}^n\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle > 0\}$ and then meets $\{X \in \mathbb{R}^d \setminus \{0\}\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle = 0\}$ if and only if it meets $\{X \in \mathbb{R}^d \setminus \{0\}\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle \leq 0\}$. Let us then consider $X \in \mathbb{R}^d \setminus \{0\}$ such that $\langle \operatorname{Hess} V(\mathbf{s})X, X\rangle = 0$ and let us prove that $\langle X, \xi\rangle \neq 0$. To show this, let us work in orthonormal coordinates of $\mathbb{R}^d$ where $\operatorname{Hess} V(\mathbf{s})$ is diagonal, i.e. where $\operatorname{Hess} V(\mathbf{s}) = \operatorname{Diag}(-\mu_1, \mu_2, \ldots, \mu_d)$. It then follows from $\langle \operatorname{Hess} V(\mathbf{s})X, X\rangle = 0$ and from the third item of Lemma 1.8 that

$$\mu_1 X_1^2 \;=\; \sum_{k=2}^d \mu_k X_k^2 \quad \text{and} \quad \frac{1}{\mu_1}\xi_1^2 \;>\; \sum_{k=2}^d \frac{1}{\mu_k}\xi_k^2 \;\geq\; 0\,.$$

It holds in particular $X_1 \neq 0$ and thus, by multiplying the two above relations,

$$|\xi_1 X_1| \;>\; \Big(\sum_{k=2}^{d} \frac{1}{\mu_k}\xi_k^2\Big)^{\frac12}\Big(\sum_{k=2}^{d} \mu_k X_k^2\Big)^{\frac12} \;\geq\; |\sum_{k=2}^{d} \xi_k X_k|\,,$$

the last inequality resulting from the Cauchy-Schwarz inequality. The relation $\langle X, \xi\rangle \neq 0$ follows.

When $d=2$, the situation is slightly different since for any hyperplane H, either $H\setminus\{0\} \subset \{X \in \mathbb{R}^2\setminus\{0\}\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle \leq 0\}$ or $H\setminus\{0\} \subset \{X \in \mathbb{R}^2\setminus\{0\}\,;\ \langle \operatorname{Hess} V(\mathbf{s})X, X\rangle > 0\}$. Take again orthonormal coordinates where $\operatorname{Hess} V(\mathbf{s}) = \operatorname{Diag}(-\mu_1, \mu_2)$. We have then only to prove that the vector $\xi' := (-\xi_2, \xi_1)^*$, which spans $\xi^\perp$, satisfies

$$-\mu_1\xi_2^2 + \mu_2\xi_1^2 \;=\; \langle \operatorname{Hess} V(\mathbf{s})\xi', \xi'\rangle \;>\; 0\,.$$

This is obviously satisfied since equivalent to

$$0 \;>\; \frac{1}{\mu_2}\xi_2^2 - \frac{1}{\mu_2}\xi_1^2 \;=\; \langle (\operatorname{Hess} V(\mathbf{s}))^{-1}\xi, \xi\rangle\,,$$

which holds true thanks to iii) of Lemma 1.8. This concludes the proof of Lemma 4.1. ∎

Let us now define, for every $h \in (0,1]$ and for every $\rho_0, \delta_0 > 0$ small enough, the function $\kappa_{\mathbf{m},h}$ on the sublevel set $E_-(\mathbf{m}) \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + 3\delta_0\}$ (see (4.3)) as follows:

1. On the disjoint open sets $E^+_{\mathbf{m},3\rho_0,3\delta_0}$ and $E^-_{\mathbf{m},3\rho_0,3\delta_0}$ introduced Lemma 4.1,

$$\kappa_{\mathbf{m},h}(x) \;:=\; \begin{cases} +1 & \text{for } x \in E^+_{\mathbf{m},3\rho_0,3\delta_0} \\ -1 & \text{for } x \in E^-_{\mathbf{m},3\rho_0,3\delta_0} \end{cases}. \tag{4.4}$$

2. For every $\mathbf{s} \in \mathbf{j}(\mathbf{m})$ and $x \in \mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0} \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + 3\delta_0\}$ (see (4.1)),

$$\kappa_{\mathbf{m},h}(x) \;:=\; C_{\mathbf{s},h}^{-1} \int_0^{\xi(\mathbf{s})\cdot(x-\mathbf{s})} \chi(\rho_0^{-1}\eta)\, e^{-\frac{|\mu(\mathbf{s})|\eta^2}{2h}}\, d\eta\,, \tag{4.5}$$

where the orientation of $\xi(\mathbf{s})$ is chosen in such a way that there exists a neighborhood $\mathcal{O}$ of $\mathbf{s}$ such that $E(\mathbf{m}) \cap \mathcal{O}$ is included in the half-plane $\{\xi(\mathbf{s})\cdot(x-\mathbf{s}) > 0\}$ (see Lemma 4.1 and Figures 4.1 and 4.2), $\chi \in C^\infty(\mathbb{R};[0,1])$ is even and satisfies $\chi \equiv 1$ on $[-1,1]$, $\chi(\eta) = 0$ for $|\eta| \geq 2$, and

$$C_{\mathbf{s},h} \;:=\; \frac12 \int_{-\infty}^{+\infty} \chi(\rho_0^{-1}\eta)\, e^{-\frac{|\mu(\mathbf{s})|\eta^2}{2h}}\, d\eta\,.$$

Note in particular that

$$\exists \gamma > 0 \ \text{ s.t. } \ C_{\mathbf{s},h}^{-1} \;=\; \sqrt{\frac{2|\mu(\mathbf{s})|}{\pi h}}\, \Big(1 + \mathcal{O}(e^{-\frac{\gamma}{h}})\Big)\,. \tag{4.6}$$

Note also that for every $\rho_0, \delta_0 > 0$ small enough, thanks to the definitions (4.4) and (4.5), and since the sets $E^+_{\mathbf{m},3\rho_0,3\delta_0}$, $E^-_{\mathbf{m},3\rho_0,3\delta_0}$, and $\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}$'s, $\mathbf{s} \in \mathbf{j}(\mathbf{m})$, are two by two disjoint (see Lemma 4.1), $\kappa_{\mathbf{m},h}$ is well defined and is $\mathcal{C}^\infty$ on $E_-(\mathbf{m}) \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + 3\delta_0\}$.

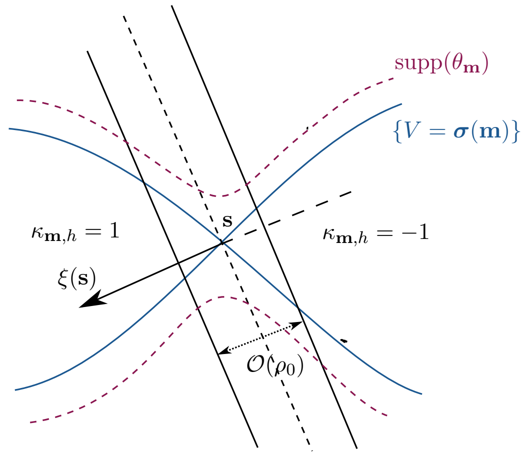

FIGURE 4.2. The support of the function $\kappa_{\mathbf{m},h}$

Consider now a smooth function $\theta_{\mathbf{m}}$ such that

$$\theta_{\mathbf{m}}(x) \;:=\; \begin{cases} 1 & \text{for } x \in \{V \le \boldsymbol{\sigma}(\mathbf{m}) + \frac{3}{2}\delta_0\} \cap E_-(\mathbf{m}) \\ 0 & \text{for } x \in \mathbb{R}^d \setminus \big(\{V < \boldsymbol{\sigma}(\mathbf{m}) + 2\delta_0\} \cap E_-(\mathbf{m})\big) \end{cases} . \tag{4.7}$$

The function $\theta_{\mathbf{m}}\kappa_{\mathbf{m},h}$ then belongs to $\mathcal{C}_c^\infty(\mathbb{R}^d;[-1,1])$ and

$$\operatorname{supp}\theta_{\mathbf{m}}\kappa_{\mathbf{m},h} \;\subset\; E_-(\mathbf{m}) \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + 2\delta_0\}\,.$$

Definition 4.2. *For any $\mathbf{m} \in \mathcal{U}^{(0)}$ let us define the function $\psi_{\mathbf{m},h}$ by*

$$\psi_{\mathbf{m},h}(x) \;:=\; \theta_{\mathbf{m}}(x)\left(\kappa_{\mathbf{m},h}(x) + 1\right)$$

when $\mathbf{m} \neq \underline{\mathbf{m}}$ and, when $\mathbf{m} = \underline{\mathbf{m}}$, $\psi_{\mathbf{m},h}(x) := 1$. We then define, for any $\mathbf{m} \in \mathcal{U}^{(0)}$, the quasimode $\varphi_{\mathbf{m},h}$ by

$$\varphi_{\mathbf{m},h}(x) \;:=\; \frac{\psi_{\mathbf{m},h}(x)}{\|\psi_{\mathbf{m},h}\|_{L^2(m_h)}}\,.$$

Note that, for every $h \in (0,1]$, it holds $L_{V,b,\nu}\varphi_{\underline{\mathbf{m}},h} = 0$ and for every $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, the quasimodes $\psi_{\mathbf{m},h}$ and $\varphi_{\mathbf{m},h}$ belong to $\mathcal{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^+)$ with supports included in $E_-(\mathbf{m}) \cap \{V < \boldsymbol{\sigma}(\mathbf{m}) + 2\delta_0\}$. We have more precisely the following lemma resulting from the previous construction.

Lemma 4.3. *Assume that Assumption 4 is satisfied. For every $\mathbf{m} \in \mathcal{U}^{(0)}$ and every small $\epsilon > 0$ fixed, there exist $\rho_0, \delta_0 > 0$ small enough such that for every $h \in (0,1]$ one has:*

i) *It holds*

$$\operatorname{supp}\psi_{\mathbf{m},h} \;\subset\; \overline{E(\mathbf{m})} + B(0,\epsilon)\,.$$

ii) *When $\mathbf{m} \neq \underline{\mathbf{m}}$, there exists a neighborhood $\mathcal{O}_{\rho_0,\delta_0}$ of $\overline{E(\mathbf{m})}$ such that:*

$$\mathcal{O}_{\rho_0,\delta_0} \setminus \cup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0} \;\subset\; \{\theta_{\mathbf{m}}\,\kappa_{\mathbf{m},h} = 1\}\,.$$

In particular, it holds

$$\operatorname{argmin}_{\operatorname{supp}\psi_{\mathbf{m},h}} V \;=\; \operatorname{argmin}_{\{\theta_{\mathbf{m}}\,\kappa_{\mathbf{m},h}=1\}} V \;=\; \operatorname{argmin}_{E(\mathbf{m})} V \;=\; \{\mathbf{m}\}\,.$$

iii) *When* $\mathbf{m} \neq \underline{\mathbf{m}}$, *it holds:*

$$\forall x \in \operatorname{supp} \nabla\psi_{\mathbf{m},h}\,,\ \Big(V(x) < \boldsymbol{\sigma}(\mathbf{m}) + \frac{3}{2}\delta_0 \implies x \in \cup_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}\Big)\,.$$

Let moreover $\mathbf{m}'$ *belong to* $\mathcal{U}^{(0)}$ *with* $\mathbf{m} \neq \mathbf{m}'$. *The following then hold true for every* $\rho_0, \delta_0 > 0$ *small enough and every* $h \in (0, 1]$*:*

iv) *if* $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$, *then* $\operatorname{supp}(\psi_{\mathbf{m},h}) \cap \operatorname{supp}(\psi_{\mathbf{m}',h}) = \emptyset$,

v) *if* $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$, *then*
- *either* $\operatorname{supp}(\psi_{\mathbf{m},h}) \cap \operatorname{supp}(\psi_{\mathbf{m}',h}) = \emptyset$,
- *or* $\psi_{\mathbf{m},h} = 2$ *on* $\operatorname{supp}(\psi_{\mathbf{m}',h})$ *and* $V(\mathbf{m}') > V(\mathbf{m})$.

Proof. The first part of Lemma 4.3 follows from Assumption 4 and from the construction of the quasimodes $\varphi_{\mathbf{m},h}$ defined in Definition 4.2 for $\mathbf{m} \in \mathcal{U}^{(0)}$, see indeed (4.4), (4.5), and (4.7). Let us then prove the second part of Lemma 4.3.

When $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}')$ and $\mathbf{m} \neq \mathbf{m}'$, note first that $\mathbf{m}$ and $\mathbf{m}'$ differ from $\underline{\mathbf{m}}$ since $\boldsymbol{\sigma}(\mathbf{m}) = +\infty$ if and only if $\mathbf{m} = \underline{\mathbf{m}}$. When moreover $\mathbf{m}' \notin E_-(\mathbf{m})$, it holds $E_-(\mathbf{m}) \neq E_-(\mathbf{m}')$ and hence $E_-(\mathbf{m}) \cap E_-(\mathbf{m}') = \emptyset$, implying $\operatorname{supp}(\psi_{\mathbf{m},h}) \cap \operatorname{supp}(\psi_{\mathbf{m}',h}) = \emptyset$. In the case when $\mathbf{m}' \in E_-(\mathbf{m})$, the statement of Lemma 4.3 follows from ii) of Assumption 4 and of Remark 1.7, which indeed imply that $\overline{E(\mathbf{m})} \cap \overline{E(\mathbf{m}')} = \emptyset$ (see the first item of Lemma 4.3).

When $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$ and $\mathbf{m}' \notin E(\mathbf{m})$, it holds $\overline{E(\mathbf{m})} \cap \overline{E(\mathbf{m}')} = \emptyset$, and again, according to the first item of Lemma 4.3, it holds $\operatorname{supp}(\psi_{\mathbf{m},h}) \cap \operatorname{supp}(\psi_{\mathbf{m}',h}) = \emptyset$ for every $\rho_0, \delta_0 > 0$ small enough. Lastly, when $\boldsymbol{\sigma}(\mathbf{m}) > \boldsymbol{\sigma}(\mathbf{m}')$ and $\mathbf{m}' \in E(\mathbf{m})$, it holds $\overline{E(\mathbf{m}')} \subset E_-(\mathbf{m}') \subset E(\mathbf{m})$ and then, according to the second item of Lemma 4.3, $\psi_{\mathbf{m},h} = 2$ on $\operatorname{supp}(\psi_{\mathbf{m}',h})$ for every $\rho_0, \delta_0 > 0$ small enough. Besides, the relation $V(\mathbf{m}') > V(\mathbf{m})$ follows from $\mathbf{m}' \in E(\mathbf{m})$ and from the first item of Assumption 4. □

4.2. **Quasimodal estimates.** We write in the sequel $a \sim b$ and $a \lesssim b$ to mean, in the limit $h \to 0$, equality/inequality up to a multiplicative factor $1 + \mathcal{O}(h)$. Moreover, we define for shortness, for any critical point $\mathbf{u}$ of V:

$$D_{\mathbf{u}} \ :=\ \sqrt{|\det \operatorname{Hess} V(\mathbf{u})|}\ >\ 0.$$

Proposition 4.4. *Assume that Assumption 4 is satisfied and consider the families* $\big(\psi_{\mathbf{m},h}, \mathbf{m} \in \mathcal{U}^{(0)}\big)$ *and* $\big(\varphi_{\mathbf{m},h}, \mathbf{m} \in \mathcal{U}^{(0)}\big)$ *of Definition 4.2. Then, for every* $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$ *and* $\rho_0, \delta_0 > 0$ *small enough, it holds in the limit* $h \to 0$*:*

$$\|\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} \ \sim\ 4\,\frac{D_{\underline{\mathbf{m}}}}{D_{\mathbf{m}}}\, e^{-\frac{V(\mathbf{m})-V(\underline{\mathbf{m}})}{h}}\,. \tag{4.8}$$

Moreover, there exists $C > 0$ *such that for every* $\mathbf{m}, \mathbf{m}' \in \mathcal{U}^{(0)}$, *it holds in the limit* $h \to 0$*:*

$$\langle \varphi_{\mathbf{m},h}, \varphi_{\mathbf{m}',h}\rangle \ =\ \delta_{\mathbf{m},\mathbf{m}'} + \mathcal{O}(e^{-\frac{c}{h}}). \tag{4.9}$$

Proof. To prove the relation (4.8), write, according to Definition 4.2,

$$\|\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} \ =\ Z_h^{-1} \int \Big(\theta_{\mathbf{m}}(\kappa_{\mathbf{m},h} + 1)\Big)^2 e^{-\frac{V(x)}{h}}\, dx\,,$$

where Z_h is the normalizing constant defined by (1.9). Hence, according to Lemma 4.3 and standard tail estimates and Laplace asymptotics, we get, in the limit $h \to 0$,

$$Z_h \;\sim\; (2\pi h)^{\frac{d}{2}}\, D_{\underline{\mathbf{m}}}^{-1}\, e^{-\frac{V(\underline{\mathbf{m}})}{h}}$$

as well as

$$\int \big(\theta_{\mathbf{m}}(\kappa_{\mathbf{m},h}+1)\big)^2\, e^{-\frac{V(x)}{h}}\, dx \;\sim\; 4\,(2\pi h)^{\frac{d}{2}}\, D_{\mathbf{m}}^{-1}\, e^{-\frac{V(\mathbf{m})}{h}}\,.$$

The estimate (4.8) then follows easily.

Let us now prove the relation (4.9). According to Definition 4.2, note first that $\langle \varphi_{\mathbf{m},h}, \varphi_{\mathbf{m},h}\rangle = 1$ for every $\mathbf{m} \in \mathcal{U}^{(0)}$. Moreover, when $\mathbf{m}, \mathbf{m}' \in \mathcal{U}^{(0)}$ and $\mathbf{m} \neq \mathbf{m}'$, it follows from Lemma 4.3 that, up to switching $\mathbf{m}$ and $\mathbf{m}'$, we are in one of the two following cases:

– either $\operatorname{supp}(\varphi_{\mathbf{m},h}) \cap \operatorname{supp}(\varphi_{\mathbf{m}',h}) = \emptyset$, and then

$$\langle \varphi_{\mathbf{m},h}, \varphi_{\mathbf{m}',h}\rangle \;=\; 0\,,$$

– or $\psi_{\mathbf{m},h} = 2$ on $\operatorname{supp}(\psi_{\mathbf{m}',h})$ and $V(\mathbf{m}') > V(\mathbf{m})$, and then, using the preceding estimates,

$$\begin{aligned}\langle \varphi_{\mathbf{m},h}, \varphi_{\mathbf{m}',h}\rangle \;&=\; \frac{2}{\|\psi_{\mathbf{m},h}\|_{L^2(m_h)}} \int_{\operatorname{supp}\psi_{\mathbf{m}',h}} \frac{\psi_{\mathbf{m}',h}}{\|\psi_{\mathbf{m}',h}\|_{L^2(m_h)}} \frac{e^{-\frac{V(x)}{h}}}{Z_h}\, dx \\ &=\; \frac{1}{\|\psi_{\mathbf{m},h}\|_{L^2(m_h)} \|\psi_{\mathbf{m}',h}\|_{L^2(m_h)}} \mathcal{O}\big(e^{-\frac{V(\mathbf{m}')-V(\underline{\mathbf{m}})}{h}}\big) \;=\; \mathcal{O}\big(e^{-\frac{C}{h}}\big)\,,\end{aligned}$$

where $C = \frac{V(\mathbf{m}')-V(\mathbf{m})}{2} > 0$.

This leads to (4.9). ☐

Proposition 4.5. *For every $\mathbf{m} \in \underline{\mathcal{U}}^{(0)}$ and $\rho_0, \delta_0 > 0$ small enough, it holds in the limit $h \to 0$:*

$$\langle L_{V,b,\nu}\psi_{\mathbf{m},h}, \psi_{\mathbf{m},h}\rangle_{L^2(m_h)} \;\sim\; \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \frac{2|\mu(\mathbf{s})|}{\pi} \frac{D_{\underline{\mathbf{m}}}}{D_{\mathbf{s}}}\, e^{-\frac{V(\mathbf{s})-V(\underline{\mathbf{m}})}{h}} \tag{4.10}$$

and then

$$\langle L_{V,b,\nu}\varphi_{\mathbf{m},h}, \varphi_{\mathbf{m},h}\rangle_{L^2(m_h)} \;\sim\; \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \frac{|\mu(\mathbf{s})|}{2\pi} \frac{D_{\mathbf{m}}}{D_{\mathbf{s}}}\, e^{-\frac{V(\mathbf{s})-V(\mathbf{m})}{h}}\,. \tag{4.11}$$

Proof. Note first that thanks to (1.3), one has $\operatorname{div}(b_h m_h) = 0$ and hence:

$$\forall\, u, v \,\in\, \mathcal{C}_c^\infty(\mathbb{R}^d)\,,\quad \langle b_h \cdot \nabla u, u\rangle_{L^2(m_h)} \;=\; -\frac{1}{2}\int u^2 \operatorname{div}(b_h m_h) dx \;=\; 0\,.$$

Using this relation together with (1.4), (4.4)–(4.7), Definition 4.2, and Lemma 4.3, we get, in the limit $h\to 0$,

$$\begin{aligned}\langle L_{V,b,\nu}\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)} &= \langle(-h\Delta+\nabla V\cdot\nabla)\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)}\\ &= Z_h^{-1}h\int|\nabla\big(\theta_{\mathbf{m}}(\kappa_{\mathbf{m},h}+1)\big)|^2\,e^{-\frac{V}{h}}\,dx\\ &= Z_h^{-1}h\int\theta_{\mathbf{m}}^2|\nabla\kappa_{\mathbf{m},h}|^2\,e^{-\frac{V}{h}}\,dx\,+\,Z_h^{-1}\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+\delta_0}{h}})\\ &= Z_h^{-1}\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+\delta_0}{h}})\end{aligned}$$

$$+Z_h^{-1}\sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}C_{\mathbf{s},h}^{-2}h\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}}\theta_{\mathbf{m}}^2(x)\chi^2(\rho_0^{-1}\xi\cdot(x-\mathbf{s}))e^{-\frac{|\mu|(\xi\cdot(x-\mathbf{s}))^2}{h}}\,e^{-\frac{V}{h}}\,dx\,, \tag{4.12}$$

where for short we denote $\xi=\xi(\mathbf{s})$ and $\mu=\mu(\mathbf{s})$. From the second item in Lemma 1.8 and the Taylor expansion of $V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2$ around $\mathbf{s}\in\mathbf{j}(\mathbf{m})$,

$$\begin{aligned}V(x)+|\mu|(\xi\cdot(x-\mathbf{s}))^2 &= V(\mathbf{s})+\frac12\langle\operatorname{Hess}V(\mathbf{s})\,(x-\mathbf{s}),x-\mathbf{s}\rangle\\ &\quad+|\mu|\langle\xi\xi^*(x-\mathbf{s}),x-\mathbf{s}\rangle+\mathcal{O}(|x-\mathbf{s}|^3)\,,\end{aligned}$$

it is clear that for ρ_0 and δ_0 small enough, $V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2$ uniquely attains its minimal value in $\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}$ at $\mathbf{s}$ since:

$$\nabla\big(\,V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2\,\big)(\mathbf{s})=0\quad\text{and}\quad\operatorname{Hess}\big(\,V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2\,\big)(\mathbf{s})=M_V\,.$$

Moreover, using again the second item in Lemma 1.8 and a standard Laplace method, it holds in the limit $h\to0$, for every $\mathbf{s}\in\mathbf{j}(\mathbf{m})$,

$$\begin{aligned}C_{\mathbf{s},h}^{-2}\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}}\theta_{\mathbf{m}}^2\chi^2(\rho_0^{-1}\langle\xi,\cdot-\mathbf{s}\rangle)\,e^{-\frac{|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2}{h}}\,e^{-\frac{V}{h}}\,dx &\sim \frac{(2\pi h)^{\frac d2}}{C_{\mathbf{s},h}^2D_{\mathbf{s}}}\,e^{-\frac{V(\mathbf{s})}{h}}\\ &\sim \frac{2\,(2\pi h)^{\frac d2}\,|\mu|}{\pi\,h\,D_{\mathbf{s}}}\,e^{-\frac{V(\mathbf{s})}{h}}\,,\end{aligned}\tag{4.13}$$

where we also used (4.6) at the last line. The statement of Proposition 4.5 then follows from (4.12) and (4.13), using also $Z_h\sim(2\pi h)^{\frac d2}D_{\underline{\mathbf{m}}}^{-1}\,e^{-\frac{V(\underline{\mathbf{m}})}{h}}$. □

Proposition 4.6. *Let $\mathbf{m}\in\underline{\mathcal{U}}^{(0)}$. For ρ_0 and δ_0 sufficiently small, it holds in the limit $h\to0$:*

$$\|L_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} = \langle L_{V,b,\nu}\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)}\,\mathcal{O}(h)\,. \tag{4.14}$$

and

$$\|L^*_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} = \langle L_{V,b,\nu}\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)}\,\mathcal{O}(1)\,. \tag{4.15}$$

Proof. Let $\mathbf{s}\in\mathbf{j}(\mathbf{m})$ and denote for short $\xi=\xi(\mathbf{s})$ and $\mu=\mu(\mathbf{s})$. We first recall the Taylor expansion of $V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2$ around $\mathbf{s}$,

$$V(x)+|\mu|(\xi\cdot(x-\mathbf{s}))^2 = V(\mathbf{s})+\frac12\langle M_V\,(x-\mathbf{s}),x-\mathbf{s}\rangle+\mathcal{O}(|x-\mathbf{s}|^3)\,,$$

which implies, according to the second item of Lemma 1.8, that for ρ_0 and δ_0 small enough:

– $\nabla\big(\,V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2\,\big)(\mathbf{s})=0,$

– $V+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2$ uniquely attains its minimal value in $\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}$ at $\mathbf{s}$.

Note now that according to (1.4), it holds

$$L_{V,b,\nu}\,\psi_{\mathbf{m},h} \;=\; \theta_{\mathbf{m}}\,L_{V,h}\,\kappa_{\mathbf{m},h}+\big(1+\kappa_{\mathbf{m},h}\big)\,L_{V,h}\,\theta_{\mathbf{m}}-2h\nabla\kappa_{\mathbf{m},h}\cdot\nabla\theta_{\mathbf{m}}\,,$$

with on $\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}$, for every $\mathbf{s}\in\mathbf{j}(\mathbf{m})$, according to (4.5),

$$\begin{aligned}L_{V,b,\nu}\,\kappa_{\mathbf{m},h} \;&=\; -h\Delta\kappa_{\mathbf{m},h}+\nabla V\cdot\nabla\kappa_{\mathbf{m},h}+b_h\cdot\nabla\kappa_{\mathbf{m},h}\\ &=\; C_{\mathbf{s},h}^{-1}\chi(\rho_0^{-1}\langle\xi,\cdot-\mathbf{s}\rangle)\,e^{-\frac{|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2}{2h}}\,\Big(\nabla V\cdot\xi+b_h\cdot\xi+|\mu|\langle\xi,\cdot-\mathbf{s}\rangle\Big)\\ &\quad-h\,C_{\mathbf{s},h}^{-1}\,\mathrm{div}\,\Big(\chi(\rho_0^{-1}\langle\xi,\cdot-\mathbf{s}\rangle)\,\xi\Big)\,e^{-\frac{|\mu|\langle\xi,\cdot-\mathbf{s}\rangle^2}{2h}}\,,\end{aligned}$$

where we recall that $b_h=b+h\nu$. It then follows from (4.4)–(4.7) that in the limit $h\to 0$,

$$\begin{aligned}\|L_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} \;&=\; \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\|\mathbf{1}_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}}L_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)}+\frac{\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+\delta_0}{h}})}{Z_h}\\ &=\; \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\frac{C_{\mathbf{s},h}^{-2}}{Z_h}\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}}\chi^2(\rho_0^{-1}\xi\cdot(x-\mathbf{s}))\,e^{-\frac{V+|\mu|(\xi\cdot(x-\mathbf{s}))^2}{h}}\\ &\qquad\times\Big(\nabla V\cdot\xi+b\cdot\xi+|\mu|\xi\cdot(x-\mathbf{s})\;+h\nu\cdot\xi\Big)^2\,dx\\ &\quad+\;\frac{\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+c}{h}})}{Z_h}\end{aligned}$$

for some real constant $c\in(0,\delta_0)$. Moreover, using $b(\mathbf{s})=0$ and the first item of Lemma 1.8, the Taylor expansion of $\nabla V+b$ around $\mathbf{s}$ satisfies

$$\begin{aligned}(\nabla V+b)\cdot\xi+|\mu|\xi\cdot(x-\mathbf{s}) \;&=\; \langle(\mathrm{Hess}\,V(\mathbf{s})+B)(x-\mathbf{s}),\xi\rangle+|\mu|\xi\cdot(x-\mathbf{s})\\ &\qquad+\mathcal{O}((x-\mathbf{s})^2)\\ &=\; \mu\xi\cdot(x-\mathbf{s})+|\mu|\xi\cdot(x-\mathbf{s})+\mathcal{O}((x-\mathbf{s})^2)\\ &=\; \mathcal{O}((x-\mathbf{s})^2)\,.\end{aligned}$$

It then follows from Proposition 4.5, standard tail estimates, and Laplace asymptotics, that in the limit $h\to 0$,

$$\begin{aligned}\|L_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} \;&=\; \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})}\frac{C_{\mathbf{s},h}^{-2}}{Z_h}\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}}\mathcal{O}((x-\mathbf{s})^4+h^2)\,e^{-\frac{V+|\mu|(\xi\cdot(x-\mathbf{s}))^2}{h}}\,dx\\ &\qquad+\;\frac{\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+c}{h}})}{Z_h}\\ &=\; \langle L_{V,b,\nu}\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)}\,\mathcal{O}(h)\,,\end{aligned}$$

which proves (4.14).

To prove (4.15), we observe that since $L^*_{V,b,\nu} = L_{V,-b,-\nu}$, the same computation as above shows that in the limit $h \to 0$,

$$\begin{aligned}\|L^*_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} &= \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \frac{C^{-2}_{\mathbf{s},h}}{Z_h}\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}} \chi^2(\rho_0^{-1}\xi\cdot(x-\mathbf{s}))\, e^{-\frac{V+|\mu|(\xi\cdot(x-\mathbf{s}))^2}{h}} \\ &\quad \times \big(\nabla V\cdot\xi - b\cdot\xi + |\mu|\xi\cdot(x-\mathbf{s}) - h\nu\cdot\xi\big)^2\,dx \\ &\quad + \frac{\mathcal{O}(e^{-\frac{\boldsymbol{\sigma}(\mathbf{m})+c}{h}})}{Z_h}.\end{aligned}$$

However, contrary to the preceding case, one has here only

$$\nabla V\cdot\xi - b\cdot\xi + |\mu|\xi\cdot(x-\mathbf{s}) = \mathcal{O}(x-\mathbf{s})\,,$$

which implies, in the limit $h \to 0$,

$$\begin{aligned}\|L^*_{V,b,\nu}\psi_{\mathbf{m},h}\|^2_{L^2(m_h)} &= \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m})} \frac{C^{-2}_{\mathbf{s},h}}{Z_h}\int_{\mathcal{C}_{\mathbf{s},3\rho_0,3\delta_0}} \mathcal{O}((x-\mathbf{s})^2+h^2)\, e^{-\frac{V+|\mu|(\xi\cdot(x-\mathbf{s}))^2}{h}}\,dx \\ &\quad + \frac{\mathcal{O}(e^{-\frac{V(\mathbf{s})+c}{h}})}{Z_h} \\ &= \langle L_{V,b,\nu}\psi_{\mathbf{m},h},\psi_{\mathbf{m},h}\rangle_{L^2(m_h)}\mathcal{O}(1)\,,\end{aligned}$$

which is exactly (4.15). □

4.3. **Proof of Theorem 1.9.** Throughout this section, we denote for shortness

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(m_h)}\,,\quad \|\cdot\| = \|\cdot\|_{L^2(m_h)}\,,\quad L_{V,b,\nu} = L_V\,,$$

and we label the local minima $\mathbf{m}_1,\dots,\mathbf{m}_{n_0}$ of V in so that $(S(\mathbf{m}_j))_{j\in\{1,\dots,n_0\}}$ is non-increasing (see (1.17)):

$$S(\mathbf{m}_1) = +\infty \ \text{ and, for all } j\in\{2,\dots,n_0\},\ S(\mathbf{m}_{j+1}) \le S(\mathbf{m}_j) < +\infty\,.$$

For all $j \in \{1,\dots,n_0\}$, we will also denote for shortness

$$S_j := S(\mathbf{m}_j)\,,\ \varphi_j := \varphi_{\mathbf{m}_j,h}\,,\ \text{ and } \tilde\lambda_j(h) := \langle L_V\varphi_j,\varphi_j\rangle\,.$$

From Proposition 4.5, one knows that for all $j \in \{2,\dots,n_0\}$, one has

$$\tilde\lambda_j(h) = \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_j)} \frac{|\mu(\mathbf{s})|}{2\pi}\frac{D_{\mathbf{m}_j}}{D_{\mathbf{s}}}\, e^{-\frac{S_j}{h}}\big(1+\mathcal{O}(h)\big). \tag{4.16}$$

Moreover, since $(S_j)_{j\in\{1,\dots,n_0\}}$ is non-increasing, we deduce from this estimate that there exists $h_0 > 0$ and $C > 0$ such that for all $h \in (0,h_0]$ and all $i,j \in \{1,\dots,n_0\}$, one has

$$i \le j \implies \lambda_i(h) \le C\lambda_j(h). \tag{4.17}$$

The two following lemmata are straightforward consequence of the previous analysis.

Lemma 4.7. *For every $j,k \in \{1,\dots,n_0\}$ and $h \in (0,1]$, one has*

$$\langle L_V\varphi_j,\varphi_k\rangle = \delta_{jk}\,\tilde\lambda_j(h)\,.$$

Proof. When $j = k$, the statement if obvious. When $j \neq k$, then it follows from Lemma 4.3 that we are in one of the three following cases:

– either $\mathrm{supp}(\varphi_j) \cap \mathrm{supp}(\varphi_k) = \emptyset$ and the conclusion is obvious,
– either there exists $c_h > 0$ such that $\varphi_j = c_h$ on $\mathrm{supp}(\varphi_k)$ and

$$\langle L_V \varphi_j, \varphi_k \rangle \; = \; \langle L_V(c_h), \varphi_k \rangle \; = \; 0\,,$$

– or there exists $c_h > 0$ such that $\varphi_k = c_h$ on $\mathrm{supp}(\varphi_j)$ and

$$\langle L_V \varphi_j, \varphi_k \rangle \; = \; \langle \varphi_j, L_V^* \varphi_k \rangle \; = \; \langle \varphi_j, L_V^*(c_h) \rangle \; = \; 0\,.$$

□

Lemma 4.8. *For ρ_0, δ_0 sufficiently small and every $j \in \{1, \dots, n_0\}$, it holds in the limit $h \to 0$,*

$$\|L_V\, \varphi_j\| = \mathcal{O}(\sqrt{h\tilde{\lambda}_j(h)})\,. \tag{4.18}$$

and

$$\|L_V^*\, \varphi_j\| = \mathcal{O}(\sqrt{\tilde{\lambda}_j(h)})\,. \tag{4.19}$$

Proof. This is a simple rewriting of Proposition 4.6, using the fact that for every $\mathbf{m} \in \mathcal{U}^{(0)}$ and $h \in (0,1]$, $\varphi_{\mathbf{m},h} = \frac{\psi_{\mathbf{m},h}}{\|\psi_{\mathbf{m},h}\|}$. □

We now introduce, for every $h > 0$ small enough, the spectral projector Π_h associated with the n_0 smallest eigenvalues of L_V as described in Theorem 1.3. Let then ϵ_0 be given by Theorem 1.3. According to Theorem 1.3, for every $h > 0$ small enough, Π_h satisfies

$$\Pi_h \; := \frac{1}{2i\pi} \int_{z \in \partial D(0, \frac{\epsilon_0}{2})} (z - L_V)^{-1} dz \tag{4.20}$$

and in particular:

$$\Pi_h \; = \; \mathcal{O}(1)\,. \tag{4.21}$$

Lemma 4.9. *For all $j \in \{1, \dots, n_0\}$, we have, in the limit $h \to 0$,*

$$\|(1 - \Pi_h)\varphi_j\| = \mathcal{O}(\sqrt{h\tilde{\lambda}_j(h)}) \tag{4.22}$$

and

$$\|(1 - \Pi_h^*)\varphi_j\| = \mathcal{O}(\sqrt{\tilde{\lambda}_j(h)}) \tag{4.23}$$

Proof. Thanks to the resolvent identity, one has

$$\begin{aligned}(1 - \Pi_h)\varphi_j &= \frac{1}{2i\pi} \int_{z \in \partial D_{\epsilon_0}} (z^{-1} - (z - L_V)^{-1})\varphi_j \, dz \\ &= \frac{-1}{2i\pi} \int_{z \in \partial D_{\epsilon_0}} z^{-1}(z - L_V)^{-1} L_V \varphi_j \, dz.\end{aligned}$$

Moreover, it follows from Theorem 1.3 and from (1.13) that for any $z \in \partial D_{\epsilon_0}$,

$$\|(z - L_V)^{-1}\|_{L^2(m_h) \to L^2(m_h)} \; = \; \mathcal{O}(1).$$

Combined with (4.18), this proves (4.22). On the other hand, on has similarly

$$(1-\Pi_h^*)\varphi_j \;=\; \frac{-1}{2i\pi}\int_{z\in\partial D_{\epsilon_0}} z^{-1}(z-L_V^*)^{-1}L_V^*\varphi_j\,dz$$

and $\|(z-L_V^*)^{-1}\|_{L^2(m_h)\to L^2(m_h)}=\mathcal{O}(1)$. Then, (4.23) follows immediately from (4.19). □

Proposition 4.10. *For every $j\in\{1,\dots,n_0\}$ and $h>0$ small enough, let us define $v_j:=\Pi_h\varphi_j$. Then, there exists $c>0$ such that for all $j,k\in\{1,\dots,n_0\}$, one has in the limit $h\to 0$,*

$$\langle v_j,v_k\rangle \;=\; \delta_{jk}+\mathcal{O}(e^{-\frac{c}{h}}) \tag{4.24}$$

and

$$\langle L_V v_j,v_k\rangle \;=\; \delta_{jk}\tilde\lambda_j(h)+\mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)})\,. \tag{4.25}$$

In particular, it follows from (4.24) that for every $h>0$ small enough, the family $(v_1,\dots,v_{n_0})$ is a basis of $\operatorname{Ran}\Pi_h$.

Proof. Since, for some $c>0$, every $j\in\{1,\dots,n_0\}$, and every $h>0$ small enough, it holds $\tilde\lambda_j(h)=\mathcal{O}(e^{-\frac{c}{h}})$, the first identity follows directly from (4.9), (4.22), and from the relation

$$\langle v_j,v_k\rangle \;=\; \langle\varphi_j,\varphi_k\rangle+\langle\varphi_j,v_k-\varphi_k\rangle+\langle v_j-\varphi_j,v_k\rangle\,.$$

To prove the second estimate, observe that

$$\begin{aligned}\langle L_V v_j,v_k\rangle &= \langle L_V\Pi_h\varphi_j,\Pi_h\varphi_k\rangle\\ &= \langle L_V\varphi_j,\varphi_k\rangle+\langle L_V(\Pi_h-1)\varphi_j,\varphi_k\rangle+\langle \Pi_h L_V\varphi_j,(\Pi_h-1)\varphi_k\rangle\\ &= \langle L_V\varphi_j,\varphi_k\rangle+\langle (\Pi_h-1)\varphi_j,L_V^*\varphi_k\rangle+\langle \Pi_h L_V\varphi_j,(\Pi_h-1)\varphi_k\rangle\,.\end{aligned}$$

Moreover, thanks to Lemma 4.8, (4.21), and Lemma 4.9, one has

$$|\langle(\Pi_h-1)\varphi_j,L_V^*\varphi_k\rangle| \;\le\; \|(\Pi_h-1)\varphi_j\|\|L_V^*\varphi_k\| \;=\; \mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)})$$

and

$$|\langle\Pi_h L_V\varphi_j,(\Pi_h-1)\varphi_k\rangle| \;\le\; \|\Pi_h\|\,\|L_V\varphi_j\|\|(\Pi_h-1)\varphi_k\| \;=\; \mathcal{O}(\sqrt{h^2\tilde\lambda_j(h)\tilde\lambda_k(h)})\,.$$

Gathering these two estimates and using Lemma 4.7, we obtain (4.25). □

We now orthonormalize the basis $(v_1,\dots,v_{n_0})$ of $\operatorname{Ran}\Pi_h$ by a Graam-Schmidt procedure: for all $j\in\{1,\dots,n_0\}$, let us define by induction

$$\tilde e_j \;=\; v_j-\sum_{k=1}^{j-1}\frac{\langle v_j,\tilde e_k\rangle}{\|\tilde e_k\|^2}\,\tilde e_k \quad\text{and then}\quad e_j \;=\; \frac{\tilde e_j}{\|\tilde e_j\|}\,. \tag{4.26}$$

Lemma 4.11. *There exists $c>0$ such that for all $j\in\{1,\dots,n_0\}$, one has in the limit $h\to 0$:*

$$\tilde e_j \;=\; v_j+\sum_{k=1}^{j-1}\alpha_{j,k}v_k$$

with $\alpha_{jk} = \mathcal{O}(e^{-\frac{c}{h}})$. *In particular, it holds:*

$$\forall\, j\ \in\ \{1,\dots,n_0\}\,,\quad \|\tilde e_j\|\ =\ 1+\mathcal{O}(e^{-\frac{c}{h}}).$$

Proof. One proceeds by induction on j. For $j=1$, one has $\tilde e_1 = v_1 = \varphi_1 = 1$ and there is nothing to prove. Suppose now that the above formula is true for all $\tilde e_l$ with $1\le l\le j<n_0$. Then $\tilde e_{j+1} = v_{j+1} - r_{j+1}$ with

$$r_{j+1}\ =\ \sum_{k=1}^{j}\frac{\langle v_{j+1},\tilde e_k\rangle}{\|\tilde e_k\|^2}\,\tilde e_k\,.$$

Since by induction, $\|\tilde e_k\| = 1+\mathcal{O}(e^{-\frac{c}{h}})$ for all $k\in\{1,\dots,j\}$, it follows that

$$r_{j+1}\ =\ (1+\mathcal{O}(e^{-\frac{c}{h}}))\sum_{k=1}^{j}\langle v_{j+1},\tilde e_k\rangle\tilde e_k$$

Moreover, for all $k\in\{1,\dots,j\}$, one also has by induction

$$\tilde e_k\ =\ v_k+\sum_{l=1}^{k-1}\alpha_{k,l}v_l\ =\ \sum_{l=1}^{k}\beta_{k,l}v_k$$

with $\beta_{k,l}=\mathcal{O}(1)$ for any $l\in\{1,\dots,k\}$ (and actually $\beta_{k,l}=\mathcal{O}(e^{-\frac{c}{h}})$ when $l<k$), which implies

$$r_{j+1} = (1+\mathcal{O}(e^{-\frac{c}{h}}))\sum_{k=1}^{j}\sum_{l,m=1}^{k}\beta_{k,l}\beta_{k,m}\langle v_{j+1},v_l\rangle v_m.$$

Since, thanks to Proposition 4.10, it holds $\langle v_{j+1},v_l\rangle = \mathcal{O}(e^{-\frac{c}{h}})$ for all $l,m\le k<j+1$, then

$$r_{j+1}\ =\ \sum_{m=1}^{j}\gamma_{j,m}v_m\,,$$

where $\gamma_{j,m}=\mathcal{O}(e^{-\frac{c}{h}})$ for all $m\in\{1,\dots,j\}$. This proves the first part of the lemma. The second one is obvious. ☐

Proposition 4.12. *For all* $j,k\in\{1,\dots,n_0\}$, *one has in the limit* $h\to 0$*:*

$$\langle L_V e_j,e_k\rangle\ =\ \delta_{jk}\tilde\lambda_j(h)+\mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)})\,.$$

Proof. Thanks to Lemma 4.11, one has for all $j,k\in\{1,\dots,n_0\}$,

$$\langle L_V\tilde e_j,\tilde e_k\rangle = \langle L_V v_j,v_k\rangle+\sum_{p=1}^{j-1}\sum_{q=1}^{k-1}\alpha'_{p,q}\langle L_V v_p,v_q\rangle\,,$$

where, for all p,q, it holds $\alpha'_{p,q}\ =\ \alpha_{j,p}\alpha_{k,q}\ =\ \mathcal{O}(e^{-\frac{c}{h}})$. Combined with Proposition 4.10, this implies

$$\langle L_V\tilde e_j,\tilde e_k\rangle = \delta_{jk}\tilde\lambda_j(h)+\mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)})+\sum_{p=1}^{j-1}\sum_{q=1}^{k-1}\alpha'_{p,q}\langle L_V v_p,v_q\rangle\,. \tag{4.27}$$

On the other hand, thanks to Proposition 4.10 and (4.17), one has in the limit $h \to 0$, for all $1 \le p < j$ and $1 \le q < k$,

$$\begin{aligned}\langle L_V v_p, v_q\rangle &= \delta_{pq}\tilde\lambda_p(h) + \mathcal{O}(\sqrt{h\tilde\lambda_p(h)\tilde\lambda_q(h)}) = \mathcal{O}(\sqrt{\tilde\lambda_p(h)\tilde\lambda_q(h)})\\ &= \mathcal{O}(\sqrt{\tilde\lambda_j(h)\tilde\lambda_k(h)})\,.\end{aligned}$$

Combined with (4.27) and using the fact that $\alpha'_{p,q} = \mathcal{O}(e^{-\frac{c}{h}}) = \mathcal{O}(\sqrt{h})$, this shows that

$$\langle L_V \tilde e_j, \tilde e_k\rangle = \delta_{jk}\tilde\lambda_j(h) + \mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)}).$$

Eventually, since $e_k = (1+\mathcal{O}(e^{-\frac{c}{h}}))\tilde e_k$ according to Lemma 4.11, we obtain

$$\langle L_V e_j, e_k\rangle = (1+\mathcal{O}(e^{-\frac{c}{h}}))\langle L_V \tilde e_j, \tilde e_k\rangle = \delta_{jk}\tilde\lambda_j(h) + \mathcal{O}(\sqrt{h\tilde\lambda_j(h)\tilde\lambda_k(h)})\,,$$

which completes the proof. □

We are now in position to prove Theorem 1.9. We recall that $(e_1,\dots,e_{n_0})$ is an orthonormal basis of $\operatorname{Ran}\Pi_h$ and that $L_V|_{\operatorname{Ran}\Pi_h} : \operatorname{Ran}\Pi_h \to \operatorname{Ran}\Pi_h$ has exactly n_0 eigenvalues $\lambda_1,\dots,\lambda_{n_0}$, with $\lambda_j = 0$ iff $j = 1$, counted with algebraic multiplicity. Let us denote $\hat e_j = e_{n_0+1-j}$ and let $\mathcal{M}$ denote the matrix of L_V in the basis $(\hat e_1,\dots,\hat e_{n_0})$. Since this basis is orthonormal, it holds

$$\mathcal{M} = \big(\langle L_V \hat e_k, \hat e_j\rangle\big)_{j,k\in\{1,\dots,n_0\}}\,.$$

Moreover, since

$$L_V(\hat e_{n_0}) = L_V(e_1) = 0 \quad\text{and}\quad L_V^*(\hat e_{n_0}) = 0\,,$$

then $\mathcal{M}$ has the form

$$\mathcal{M} = \begin{pmatrix}\mathcal{M}' & 0\\ 0 & 0\end{pmatrix} \quad\text{with}\quad \mathcal{M}' := \big(\langle L_V \hat e_k, \hat e_j\rangle\big)_{j,k\in\{1,\dots n_0-1\}}\,.$$

On the other hand, denoting $\hat\lambda_j(h) := \tilde\lambda_{n_0+1-j}(h)$ for $j \in \{1,\dots,n_0-1\}$, one deduces from Proposition 4.12 that for every $j,k \in \{1,\dots n_0-1\}$, it holds in the limit $h \to 0$,

$$\langle L_V \hat e_k, \hat e_j\rangle = \langle L_V e_{n_0-k}, e_{n_0-j}\rangle = \delta_{jk}\hat\lambda_j(h) + \mathcal{O}(\sqrt{h\hat\lambda_j(h)\hat\lambda_k(h)})\,,$$

that is

$$\langle L_V \hat e_k, \hat e_j\rangle = \sqrt{\hat\lambda_j(h)\hat\lambda_k(h)}\big(\delta_{jk} + \mathcal{O}(\sqrt{h})\big). \tag{4.28}$$

For all $j \in \{1,\dots,n_0-1\}$, let us now define

$$\begin{aligned}\hat S_j := S_{n_0+1-j} \quad\text{and}\quad \nu_j := \zeta(\mathbf{m}_{n_0+1-j}) &= \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_{n_0+1-j})} \frac{|\mu(\mathbf{s})|}{2\pi}\,\frac{D_{\mathbf{m}_{n_0+1-j}}}{D_{\mathbf{s}}}\\ &= e^{\frac{\hat S_j}{h}}\,\hat\lambda_j(h)\big(1+\mathcal{O}(h)\big)\,,\end{aligned}$$

where $\zeta(\mathbf{m})$, $\mathbf{m} \in \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, is defined in (1.19), and the last estimate follows from (4.16). Since the sequence $(S_j)_{j\in\{2,\dots,n_0\}}$ is non-increasing, there exists a partition $J_1\sqcup\ldots\sqcup J_p$ of $\{1,\dots,n_0-1\}$ such that for all $k \in \{1,\dots,p\}$, there exists $\iota(k) \in \{1,\dots,n_0-1\}$ such that

$$\forall j \in J_k\,,\ \hat S_j = \hat S_{\iota(k)} \quad\text{and}\quad \forall 1 \le k < k' \le p\,,\ \hat S_{\iota(k)} < \hat S_{\iota(k')}. \tag{4.29}$$

Hence, we deduce from (4.28) that

$$\mathcal{M}' \;=\; \widehat{\Omega}\left(\mathcal{J}+\mathcal{O}(\sqrt{h})\right)\widehat{\Omega}$$

with

$$\mathcal{J} \;=\; \operatorname{diag}\,(\nu_j,\ j=1,\dots,n_0-1)$$

and

$$\widehat{\Omega} \;=\; \operatorname{diag}\,(e^{-\frac{\hat{S}_j}{2h}},\ j=1,\dots,n_0-1) \;=\; \operatorname{diag}\,(e^{-\frac{\hat{S}_{\iota(k)}}{2h}}I_{r_k},\ k=1,\dots,p)\,,$$

where, for every $k\in\{1,\dots,p\}$, $r_k=\operatorname{card}(J_k)$. Factorizing by $e^{-\frac{\hat{S}_{\iota(1)}}{h}}$, we get

$$\mathcal{M}' \;=\; e^{-\frac{\hat{S}_{\iota(1)}}{h}}\Omega\left(\mathcal{J}+\mathcal{O}(\sqrt{h})\right)\Omega$$

with

$$\Omega \;=\; \operatorname{diag}\left(e^{\frac{\hat{S}_{\iota(1)}-\hat{S}_{\iota(k)}}{2h}}\,I_{r_k}, k=1,\dots,p\right).$$

Denoting $\tau_1=1$ and, for $k\in\{2,\dots,p\}$, $\tau_k=e^{\frac{\hat{S}_{\iota(k-1)}-\hat{S}_{\iota(k)}}{2h}}$, we observe that, thanks to (4.29), τ_k is exponentiallys small when $h\to 0$. Moreover, with this notation, one has

$$\Omega \;=\; \operatorname{diag}\left(\tau_1 I_{r_1},\tau_1\tau_2 I_{r_2},\dots,(\Pi_{j=1}^{p}\tau_j)I_{r_p}\right).$$

This shows that $e^{-\frac{\hat{S}_{\iota(1)}}{h}}\mathcal{M}'$ is a graded matrix in the sense of Definition A.1. Hence, we can apply Theorem A.4 and we get that in the limit $h\to 0$,

$$\sigma(\mathcal{M}') \;\subset\; \bigsqcup_{k=1}^{p} e^{-\frac{\hat{S}_{\iota(1)}}{h}}\varepsilon_k^2\left(\sigma(M_k)+\mathcal{O}(\sqrt{h})\right),$$

where for every $k\in\{1,\dots,p\}$, $\varepsilon_k=\prod_{l=1}^{k}\tau_l$ and $M_k=\operatorname{diag}\,(\nu_j,\,j\in J_k)$. Moreover, still according to Theorem A.4, $\mathcal{M}'$ admits in the limit $h\to 0$, for every $k\in\{1,\dots,p\}$ and every eigenvalue λ of M_k with multiplicity r_k', exactly r_k' eigenvalues counted with multiplicity of order $e^{-\frac{\hat{S}_{\iota(1)}}{h}}\varepsilon_k^2\big(\lambda+\mathcal{O}(\sqrt{h})\big)$.

Going back to the initial parameters, one has, for every $k\in\{1,\dots,p\}$,

$$e^{-\frac{\hat{S}_{\iota(1)}}{h}}\varepsilon_k^2=e^{-\frac{\hat{S}_{\iota(k)}}{h}} \quad\text{and}\quad \sigma(M_k)=\{\nu_j,\,j\in J_k\}\,.$$

Hence, the eigenvalues of $\mathcal{M}'$ satisfy:

$$\forall\, j\,\in\,\{1,\dots,n_0-1\}\,,\quad \lambda_{n_0+1-j}(h) \;=\; e^{-\frac{\hat{S}_j}{h}}\left(\nu_j+\mathcal{O}(\sqrt{h})\right),$$

which is exactly the announced result.

4.4. **Proof of Theorem 1.11.** As in the preceding subsection, we denote for shortness

$$\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{L^2(m_h)}\,,\quad \|\cdot\|=\|\cdot\|_{L^2(m_h)}\,,\quad L_{V,b,\nu}=L_V\,,$$

and we label the local minima $\mathbf{m}_1,\dots,\mathbf{m}_{n_0}$ of V so that $(S(\mathbf{m}_j))_{j\in\{1,\dots,n_0\}}$ is non-increasing (see (1.17)):

$$S(\mathbf{m}_1)=+\infty\ \text{ and, for all } j\in\{2,\dots,n_0\},\ \ S(\mathbf{m}_{j+1})\le S(\mathbf{m}_j)<+\infty\,.$$

Let moreover $\mathbf{m}^*\in\mathcal{U}^{(0)}\setminus\{\underline{\mathbf{m}}\}$ be such that

$$S(\mathbf{m}^*)\;=\;S(\mathbf{m}_2)\quad\text{and}\quad \zeta(\mathbf{m}^*)\;=\;\min_{\mathbf{m}\in S^{-1}(S(\mathbf{m}_2))}\zeta(\mathbf{m})\,, \tag{4.30}$$

where the prefactors $\zeta(\mathbf{m})$, $\mathbf{m}\in\mathcal{U}^{(0)}\setminus\{\underline{\mathbf{m}}\}$, are defined in (1.19), and let us define, for any $h>0$,

$$\lambda(h)\;:=\;\zeta(\mathbf{m}^*)\,e^{-\frac{S(\mathbf{m}^*)}{h}}\,.$$

According to the unitary equivalence (see (1.13))

$$L_V\;=\;\frac{1}{h}\Omega^*\,P_\phi\,\Omega\,,$$

and to the localization of the spectrum of P_ϕ stated in Proposition 1.1 and in Theorem 1.3, it holds for every $h>0$ small enough, taking ϵ_0 as in the statement of Theorem 1.3,

$$\|\,e^{-tL_V}-\Pi_0\,\|\;\le\;\|\,e^{-tL_V}\,\Pi_h-\Pi_0\,\|+\|\,e^{-tL_V}\,(\mathrm{Id}-\Pi_h)\|\,, \tag{4.31}$$

where, as in the preceding subsection,

$$\Pi_h\;:=\;\frac{1}{2i\pi}\int_{z\in\partial D(0,\frac{\epsilon_0}{2})}(z-L_V)^{-1}dz.$$

Moreover, it follows from Proposition 1.1 that $\sigma(P_\phi)\subset\Gamma_{\Lambda_0}\subset\tilde\Gamma_{\Lambda_0}$ with $\tilde\Gamma_{\Lambda_0}=\{z\in\mathbb{C},\ |\,\mathrm{Im}(z)|\le\Lambda_0(\mathrm{Re}(z)+1)\}$. Hence, for every $t>0$, the operator $e^{-tL_V}(I-\Pi_h)$ can be written as the complex integral

$$e^{-tL_V}(\mathrm{Id}-\Pi_h)\;=\;-\int_{\Gamma_0\cup\Gamma_\pm}e^{-tz}(z-L_V)^{-1}dz\,,$$

where

$$\Gamma_0\;=\;\Big\{\frac{\epsilon_0}{2}+i\Lambda_0x\,,\ x\in[-\frac{\epsilon_0}{2}-\frac{1}{h},\frac{\epsilon_0}{2}+\frac{1}{h}]\Big\}$$

and

$$\Gamma_\pm\;=\;\Big\{x\pm i\Lambda_0(x+\frac{1}{h})\,,\ x\in[\frac{\epsilon_0}{2},+\infty)\Big\}\,.$$

From the resolvent estimates proven in Theorem 1.3, it holds $(z-L_V)^{-1}=\mathcal{O}(1)$ uniformly on Γ_0, and then, for every $t>0$,

$$\int_{\Gamma_0}e^{-tz}(z-L_V)^{-1}dz\;=\;e^{-t\frac{\epsilon_0}{2}}\mathcal{O}(\frac{1}{h})\,.$$

Using in addition the resolvent estimates proven in Proposition 1.1, it holds $\|(z-L_V)^{-1}\|\le\frac{1}{\mathrm{Re}\,z}\le\frac{2}{\epsilon_0}$ on $\Gamma_\pm$, and then

$$\int_{\Gamma_\pm}e^{-tz}(z-L_V)^{-1}dz\;=\;\mathcal{O}(1)\int_{\frac{\epsilon_0}{2}}^{+\infty}e^{-tx}dx\;=\;\frac{e^{-t\frac{\epsilon_0}{2}}}{t}\mathcal{O}(1)\,.$$

It follows that for every $t > 0$, it holds

$$\| e^{-tL_V} (\mathrm{Id} - \Pi_h)\| \;=\; e^{-t\frac{\epsilon_0}{2}}\, \mathcal{O}\Big(\frac{1}{t} + \frac{1}{h}\Big)\,.$$

Moreover, $e^{-tL_V} (\mathrm{Id} - \Pi_h) = \mathcal{O}(1)$ since $\Pi_h = \mathcal{O}(1)$ (see (4.21)) and $e^{-tL_V} = \mathcal{O}(1)$ (by maximal accretivity of L_V). Hence, there exists $C > 0$ such that for every $t \geq 0$ and $h > 0$ small enough, it holds

$$\| e^{-tL_V} (I - \Pi_h)\| \;\leq\; C \min\{1, \frac{e^{-t\frac{\epsilon_0}{2}}}{h}\} \;\leq\; 2Ce^{-\lambda(h)t}\,.$$

Thus, according to (4.31), it just remains to show that

$$\exists\, C \,>\, 0\,,\; \| e^{-tL_V}\, \Pi_h - \Pi_0 \,\| \;\leq\; C\, e^{-(\lambda(h) - C\sqrt{h})t}\,. \tag{4.32}$$

To this end, let us first recall from Proposition 1.1 that the spectral projector $\Pi_{\{0\}}$ associated with the eigenvalue 0 of L_V has rank 1 and is actually the orthogonal projector Π_0 on $\mathrm{Span}\{1\}$ according to the relations

$$\mathrm{Span}\{1\} \;=\; \mathrm{Im}\, \Pi_{\{0\}} \;=\; \mathrm{Im}\, \Pi^*_{\{0\}} \;=\; (\,\mathrm{Ker}\, \Pi_{\{0\}})^{\perp}\,.$$

It follows that

$$e^{-tL_V}\, \Pi_h - \Pi_0 \;=\; e^{-tL_V} \left(\Pi_h - \Pi_{\{0\}}\right).$$

Since moreover $\Pi_h - \Pi_{\{0\}} = \mathcal{O}(1)$ (thanks to the resolvent estimate of Theorem 1.3), it suffices to show that

$$\exists\, C \,>\, 0\,,\; \| e^{-tL_V} \left(\Pi_h - \Pi_{\{0\}}\right)|_{\mathrm{Ran}(\Pi_h - \Pi_{\{0\}})} \,\| \;\leq\; C\, e^{-(\lambda(h) - C\sqrt{h})t}\,.$$

Using the notation of the preceding subsection, this means proving that the matrix $\mathcal{M}'$ of L_V in the orthonormal basis $(\hat{e}_1, \ldots, \hat{e}_{n_0 - 1})$ of $\mathrm{Ran}(\Pi_h - \Pi_0)$ satisfies

$$\exists\, C \,>\, 0\,,\; \| e^{-t\mathcal{M}'} \,\| \;\leq\; C\, e^{-(\lambda(h) - C\sqrt{h})t}\,.$$

Let us now consider a subset $\mathcal{V}^{(0)}$ (in general non unique) of $\mathcal{U}^{(0)} \setminus \{0\}$ such that

$$\mathbf{m} \in \mathcal{V}^{(0)} \mapsto (\zeta(\mathbf{m}), S(\mathbf{m})) \in \{(\zeta(\mathbf{m}), S(\mathbf{m})), \mathbf{m} \in \mathcal{U}^{(0)} \setminus \{0\}\} \quad \text{is a bijection.}$$

Then, for any $K > 0$ and for every $h > 0$ small enough, the closed discs of the complex plane

$$D_{\mathbf{m},K} \;:=\; D\Big(\zeta(\mathbf{m})e^{-\frac{S(\mathbf{m})}{h}}, K\sqrt{h}e^{-\frac{S(\mathbf{m})}{h}}\Big)\,,\quad \mathbf{m} \in \mathcal{V}^{(0)}\,,$$

are included in $\{\mathrm{Re}\, z > 0\}$ and two by two disjoint. Moreover, according to Theorem 1.9, $K > 0$ can be chosen large enough so that when $h > 0$ is small enough, the $n_0 - 1$ non zero small eigenvalues of L_V are included in

$$\cup_{\mathbf{m} \in \mathcal{V}^{(0)}} D\Big(\zeta(\mathbf{m})e^{-\frac{S(\mathbf{m})}{h}}, \frac{K}{2}\sqrt{h}e^{-\frac{S(\mathbf{m})}{h}}\Big)\,.$$

In particular, for every $t \geq 0$ and for every $h > 0$ small enough, it holds

$$e^{-t\mathcal{M}'} \;=\; \sum_{\mathbf{m} \in \mathcal{V}^{(0)}} \frac{1}{2i\pi} \int_{z \in \partial D_{\mathbf{m},K}} e^{-tz}(z - \mathcal{M}')^{-1} dz.$$

Using now the specific form of $\mathcal{M}'$ exhibited in the preceding section and Theorem A.4, it holds for every $\mathbf{m}\in\mathcal{V}^{(0)}$, in the limit $h\to 0$,

$$\frac{1}{2i\pi}\int_{z\in\partial D_{\mathbf{m},K}} e^{-tz}(z-\mathcal{M}')^{-1}dz \;=\; \mathcal{O}\Big(e^{-t\zeta(\mathbf{m})e^{-\frac{S(\mathbf{m})}{h}}(1-K\sqrt{h})}\Big)\,.$$

Indeed, the resolvent estimate of Theorem A.4 implies

(4.33)
$$\forall z\in\partial D_{\mathbf{m},K}\,,\ \|(\mathcal{M}'-z)^{-1}\|=\mathcal{O}\Big(\operatorname{dist}(z,\sigma(\mathcal{M}'))^{-1}\Big)=\mathcal{O}(\frac{1}{\sqrt{h}}e^{\frac{S(\mathbf{m})}{h}}).$$

The relation (4.32) follows easily, which concludes the first part of Theorem 1.11.

Finally, let us assume that the element $\mathbf{m}^*$ satisfying (4.30) is unique. In this case, $\mathbf{m}^*$ necessarily belongs to $\mathcal{V}^{(0)}$ and the associated eigenvalue $\lambda(\mathbf{m}^*,h)$ (see (1.18)) is then real and simple for every $h>0$ small enough. In particular, it holds

$$\frac{1}{2i\pi}\int_{z\in\partial D_{\mathbf{m}^*,K}} e^{-tz}(z-\mathcal{M}')^{-1}dz \;=\; e^{-t\lambda(\mathbf{m}^*,h)}\Pi_{\{\lambda(\mathbf{m}^*,h)\}},$$

where $\Pi_{\{\lambda(\mathbf{m}^*,h)\}}$ is the spectral projector (whose rank is one)

$$\Pi_{\{\lambda(\mathbf{m}^*,h)\}} \;=\; \frac{1}{2i\pi}\int_{z\in\partial D_{\mathbf{m}^*,K}}(z-\mathcal{M}')^{-1}dz.$$

Moreover, the resolvent estimate (4.33) shows that $\Pi_{\{\lambda(\mathbf{m}^*,h)\}}=\mathcal{O}(1)$. Since in addition, it holds in this case (see (1.18))

$$\begin{aligned}\forall\mathbf{m}\in\mathcal{V}^{(0)}\setminus\{\mathbf{m}^*\}\,,\quad \lambda(\mathbf{m}^*,h) \;&=\; \zeta(\mathbf{m}^*)e^{-\frac{S(\mathbf{m}^*)}{h}}(1+\mathcal{O}(\sqrt{h}))\\ &\geq\; \zeta(\mathbf{m})e^{-\frac{S(\mathbf{m})}{h}}(1-K\sqrt{h})\end{aligned}$$

for every $K>0$ and for every $h>0$ small enough, we obtain that in the limit $h\to 0$,

$$e^{-t\mathcal{M}'} \;=\; \mathcal{O}(e^{-t\lambda(\mathbf{m}^*,h)})\,,$$

and thus the relation (4.32) remains valid if ones replaces $\lambda(h)-C\sqrt{h}$ there by $\lambda(\mathbf{m}^*,h)$. This concludes the proof of Theorem 1.11.

Appendix A. Some results in linear algebra

Let us start with notations.
Given any matrix $M\in\mathcal{M}_d(\mathbb{C})$ and $\lambda\in\sigma(M)$ we denote by $m(\lambda)$ the multiplicity of λ, $m(\lambda)=\dim\operatorname{Ker}(M-\lambda)^d$. We recall that for every $r>0$ small enough,

(A.1)
$$m(\lambda) \;=\; \operatorname{rank}\Big(\Pi_{D(\lambda,r)}(M)\Big) \;=:\; n(D(\lambda,r);M)\,,$$

where

$$\Pi_{D(\lambda,r)}(M) \;=\; \frac{1}{2i\pi}\int_{\partial D(\lambda,r)}(M-z)^{-1}dz\,.$$

We denote by $\mathscr{D}_0(E)$ the set of complex matrices on a vector space E which are diagonalizable and invertible.
Given two subset $A,B\subset\mathbb{C}$ we say that $A\subset B+\mathcal{O}(h)$ if there exists $C>0$ such that $A\subset B+B(0,Ch)$.

Definition A.1. *Let $\mathscr{E} = (E_j)_{j=1,\dots,p}$ be a sequence of finite dimensional vector spaces E_j of dimension $r_j > 0$, let $E = \oplus_{j=1,\dots,p} E_j$ and let $\tau = (\tau_2, \dots, \tau_p) \in (\mathbb{R}_+^*)^{p-1}$. Suppose that $(h, \tau) \mapsto \mathcal{M}_h(\tau)$ is a map from $(0,1] \times (\mathbb{R}_+^*)^{p-1}$ into the set of complex matrices on E.*

We say that $\mathcal{M}_h(\tau)$ is an $(\mathscr{E}, \tau, h)$-graded matrix if there exists $\mathcal{M}' \in \mathscr{D}_0(E)$ independent of h, τ such that $\mathcal{M}_h(\tau) = \Omega(\tau)(\mathcal{M}' + \mathcal{O}(h))\Omega(\tau)$ with $\Omega(\tau)$ and $\mathcal{M}'$ such that

- $\mathcal{M}' = \operatorname{diag}(M_j,\ j = 1, \dots, p)$ *with* $M_j \in \mathscr{D}_0(E_j)$
- $\Omega(\tau) = \operatorname{diag}(\varepsilon_j(\tau) I_{r_j},\ j = 1, \dots, p)$ *with* $\varepsilon_1(\tau) = 1$ *and* $\varepsilon_j(\tau) = (\prod_{k=2}^{j} \tau_k)$ *for all* $j \geq 2$.

Throughout, we denote by $\mathscr{G}(\mathscr{E}, \tau, h)$ the set of $(\mathscr{E}, \tau, h)$-graded matrices.

Lemma A.2. *Suppose that $\mathcal{M}_h(\tau)$ is a family of $(\mathscr{E}, \tau, h)$-graded matrices and that $p \geq 2$. Then one has*

$$\mathcal{M}_h(\tau) = \begin{pmatrix} J(h) & \tau_2 B_h(\tau')^* \\ \tau_2 B_h(\tau') & \tau_2^2 \mathcal{N}_h(\tau') \end{pmatrix} \tag{A.2}$$

with

- $J(h) = M_1 + \mathcal{O}(h)$ *with* $M_1 \in \mathscr{D}_0(E_1)$
- $\mathcal{N}_h(\tau') \in \mathscr{G}(\mathscr{E}', \tau', h)$ *with* $\tau' = (\tau_3, \dots, \tau_p)$ *and* $\mathscr{E}' = (E_j)_{j=2,\dots,p}$.
- $B_h(\tau') \in \mathscr{M}(E_1, \oplus_{j=2}^{p} E_j)$ *satisfies*

$$B_h(\tau')^* = (b_2(h)^*, \tau_3 b_3(h)^*, \tau_3\tau_4 b_4(h)^*, \dots, \tau_3 \dots \tau_p b_p(h)^*)$$

with $b_j(h) : E_1 \to E_j$ independent of τ and $b_j(h) = \mathcal{O}(h)$.

Moreover, the matrix $\mathcal{N}_h(\tau') - B_h(\tau') J(h)^{-1} B_h(\tau')^*$ belongs to $\mathscr{G}(\mathscr{E}', \tau', h)$.

Proof. Assume that $\mathcal{M}_h(\tau) = \Omega(\tau)(\mathcal{M}' + \mathcal{O}(h))\Omega(\tau)$ with $\Omega(\tau)$ and $\mathcal{M}'$ as in Definition A.1. First observe that

$$\Omega(\tau) = \begin{pmatrix} I_{r_p} & 0 \\ 0 & \tau_2 \Omega'(\tau') \end{pmatrix}$$

with

$$\Omega'(\tau') = \begin{pmatrix} I_{r_{p-1}} & 0 & \dots & \dots & 0 \\ 0 & \tau_3 I_{r_{p-2}} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \tau_3\tau_4 \dots \tau_p I_{r_1} \end{pmatrix}.$$

On the other hand, we can write

$$\mathcal{M}' + \mathcal{O}(h) = \begin{pmatrix} J(h) & B'(h)^* \\ B'(h) & \mathcal{N}'(h) \end{pmatrix}$$

with $J(h) = M_1 + \mathcal{O}(h)$ for some $M_1 \in \mathscr{D}_0(E_1)$, $B'(h) = \mathcal{O}(h)$ and $\mathcal{N}'(h) = \mathcal{N}'_0 + \mathcal{O}(h)$ with $\mathcal{N}'_0 = \operatorname{diag}(M_j,\ j = 2, \dots, p)$. Therefore,

$$\Omega(\tau)\mathcal{M}'_h\Omega(\tau) = \begin{pmatrix} J(h) & \tau_2 B'(h)^* \Omega'(\tau') \\ \tau_2 \Omega'(\tau') B'(h) & \tau_2^2 \Omega'(\tau')\mathcal{N}'(h)\Omega'(\tau') \end{pmatrix}$$

which has exactly the form (A.2) with $B_h(\tau') = \Omega'(\tau')B'(h)$ and $\mathcal{N}_h(\tau') = \Omega'(\tau')\mathcal{N}'(h)\Omega'(\tau')$. By construction, $\mathcal{N}_h(\tau')$ belongs to $\mathscr{G}(\mathscr{E}', \tau', h)$ and $B_h(\tau')$

has the required form. □

Lemma A.3. *Let M be a complex diagonalizable matrix. Then there exists $C>0$ such that*

$$\forall \lambda \notin \sigma(M),\ \|(M-\lambda)^{-1}\| \leq C\,\mathrm{dist}\,(\lambda, \sigma(M))^{-1}$$

Proof. Let P be an invertible matrix such that $PMP^{-1} = D$ is diagonal. Then

$$\|(M-\lambda)^{-1}\| = \|P(D-\lambda)^{-1}P^{-1}\| \leq C\|(D-\lambda)^{-1}\| = C\,\mathrm{dist}\,(\lambda, \sigma(M))^{-1}.$$

□

Theorem A.4. *Suppose that $\mathcal{M}_h(\tau)$ is $(\mathscr{E}, \tau, h)$-graded. Then, there exists $\tilde\tau_0, h_0 > 0$ such that for all $0 < \tau_j \leq \tilde\tau_0$ and all $h \in (0, h_0]$, one has*

$$\sigma(\mathcal{M}_h(\tau)) \subset \bigsqcup_{j=1}^{p} \varepsilon_j(\tau)^2(\sigma(M_j) + \mathcal{O}(h)).$$

Moreover, for any eigenvalue λ of M_j with multiplicity $m_j(\lambda)$, there exists $K>0$ such that denoting $D_j := \{z \in \mathbb{C},\ |z - \varepsilon_j(\tau)^2\lambda_j| < K\varepsilon_j(\tau)^2 h\}$, one has

$$n(D_j; \mathcal{M}_h(\tau)) = m_j(\lambda), \tag{A.3}$$

where $n(D_j; \mathcal{M}_h)$ is defined by (A.1)*. Moreover, there exists $C>0$ such that*

$$\|(\mathcal{M}(\tau, h) - z)^{-1}\| \leq C\,\mathrm{dist}\,(z, \sigma(\mathcal{M}(\tau, h)))^{-1}$$

for all $z \in \mathbb{C} \setminus \cup_{j=1}^{p} \cup_{\lambda\in\sigma(M_j)} B(\varepsilon_j(\tau)^2\lambda, \varepsilon_j(\tau)^2 Kh)$.

Proof. We prove the theorem by induction on p. Throughout the proof the notation $\mathcal{O}(\cdot)$ is uniform with respect to the parameters h and τ. For $p=1$, one has $\mathcal{M}_h(\tau) = M_1 + \mathcal{O}(h)$ with $M_1 \in \mathcal{M}_{r_1}(\mathbb{R})$ independent of h, diagonalizable and invertible. Let us denote λ_j^1, $j = 1, \dots, n_1$ its eigenvalues and $m_j = m(\lambda_j^1)$ the corresponding multiplicities. The function $z \mapsto (\mathcal{M}_h - z)^{-1}$ is meromorphic on $\mathbb{C}$ with poles in $\sigma(\mathcal{M}_h)$. Moreover, Lemma A.3 and the identity

$$\mathcal{M}_h - z = (M_1 - z)(\mathrm{Id} + (M_1 - z)^{-1}\mathcal{O}(h)),\ \forall z \notin \sigma(M_1)$$

show that for any $C>0$ large enough, $(\mathcal{M}_h - z)$ is invertible on $\mathbb{C} \setminus \cup_{j=1}^{n_1} D(\lambda_j^1, Ch)$ with $\|(M_1 - z)^{-1}\| = \mathcal{O}(\frac{1}{Ch})$ and

$$(\mathcal{M}_h - z)^{-1} = (\mathrm{Id} + (M_1 - z)^{-1}\mathcal{O}(h))^{-1}(M_1 - z)^{-1}. \tag{A.4}$$

Hence, for every $C>0$ large enough, the associated spectral projector writes

$$\Pi_{D(\lambda_j^1, Ch)}(\mathcal{M}_h) = \frac{1}{2i\pi}\int_{\partial D(\lambda_j^1, Ch)} (\mathrm{Id} + \mathcal{O}(\frac{1}{C}))^{-1}(M_1 - z)^{-1} dz.$$

This implies that for $C>0$ large enough,

$$\mathrm{rank}\,(\Pi_{D(\lambda_j^1, Ch)}(\mathcal{M}_h)) = \mathrm{rank}\,(\Pi_{D(\lambda_j^1, Ch)}(M_1)) = m_j,$$

which is exactly (A.3). As a consequence

$$\sum_{j=1}^{n_1} \operatorname{rank}(\Pi_{D(\lambda_j^1,Ch)}(\mathcal{M}_h)) = \sum_{j=1}^{n_1} m_j = r_1$$

is maximal and hence $\sigma(\mathcal{M}_h) \subset \cup_{j=1}^{n_1} D(\lambda_j^1, Ch)$. Eventually, (A.4) shows that for any $z \in \mathbb{C} \setminus \cup_{j=1}^{n_1} D(\lambda_j^1, Ch)$, one has

$$\|(\mathcal{M}_h - z)^{-1}\| \leq C' \|(M_1 - z)^{-1}\|$$

for some constant $C' > 0$. Using Lemma A.3 we get

$$\|(\mathcal{M}_h - z)^{-1}\| \leq C' \operatorname{dist}(z, \sigma(M_1))^{-1} \leq C'' \operatorname{dist}(z, \sigma(\mathcal{M}_h))^{-1}$$

for all $z \in \mathbb{C} \setminus \cup_{j=1}^{n_1} D(\lambda_j^1, 2Ch)$. This completes the initialization step.

Suppose now that $p \geq 2$ and let $\mathcal{M}_h(\tau) \in \mathscr{G}(\mathscr{E}, \tau, h)$. We have

$$\mathcal{M}_h(\tau) = \begin{pmatrix} J(h) & \tau_2 B_h(\tau')^* \\ \tau_2 B_h(\tau') & \tau_2^2 \mathcal{N}_h(\tau') \end{pmatrix}$$

with $J(h), B_h(\tau')$ and $\mathcal{N}_h(\tau')$ as in Lemma A.2. In order to lighten the notation we will drop the variable τ, τ' in the proof below. For $\lambda \in \mathbb{C}$, let

$$\mathcal{P}(\lambda) := \mathcal{M}_h(\tau) - \lambda = \begin{pmatrix} J(h) - \lambda & \tau_2 B_h^* \\ \tau_2 B_h & \tau_2^2 \mathcal{N}_h - \lambda \end{pmatrix}. \tag{A.5}$$

This is an holomorphic function, and since it is non trivial, its inverse is well defined excepted for a finite number of values of λ which are exactly the spectral values of $\mathcal{M}_h$.

We first study the part of the spectrum of $\mathcal{M}_h$ which is of largest modulus. Let λ_n^1, $n = 1, \ldots, n_1$, denote the eigenvalues of the matrix M_1. Since $J(h) = M_1 + \mathcal{O}(h)$ and $M_1 \in \mathscr{D}_0(E_1)$, then the initialization step shows that there exists $C > 0$ such that $\sigma(J(h)) \subset \cup_{n=1}^{n_1} D(\lambda_n^1, Ch)$. Moreover, since M_1 is invertible, there exists $c_1, d_1 > 0$ and $h_0 > 0$ such that for all $n = 1, \ldots, n_1$, one has $\lambda_n^1 \in K(c_1, d_1)$ where $K(c_1, d_1) = \{z \in \mathbb{C},\ c_1 \leq |z| \leq d_1\}$. Let $n \in \{1, \ldots, n_1\}$ be fixed and consider $D_n = D_n(h) = \{z \in \mathbb{C},\ |z - \lambda_n^1| \leq Mh\}$ for some $M > C > 0$ and $\tilde{D}_n = \{z \in \mathbb{C},\ |z - \lambda_n^1| \leq 2Mh\}$. Observe that for $h > 0$ small enough, the disks $\tilde{D}_n$ are disjoint. By definition, one has $\mathcal{N}_h(\tau') = \mathcal{O}(1)$ and since $|\lambda| \geq c_1 - \mathcal{O}(h) \geq c_1/2$, this implies that for $\tau_2 > 0$ small enough with respect to c_1 and $\lambda \in \tilde{D}_n$, the matrix $\tau_2^2 \mathcal{N}_h(\tau') - \lambda$ is invertible, and $(\tau_2^2 \mathcal{N}_h(\tau') - \lambda)^{-1} = \mathcal{O}(1)$. Moreover, it follows from the initialization step that for $\lambda \in \tilde{D}_n \setminus D_n$, $J(h) - \lambda$ is invertible and

$$\|(J(h) - \lambda)^{-1}\| = \mathcal{O}(\operatorname{dist}(\lambda, \sigma(J(h))^{-1}) = \mathcal{O}(h^{-1}).$$

Combined with the fact that $B_h = \mathcal{O}(h)$, this implies that for $h > 0$ small enough and $\lambda \in \tilde{D}_n \setminus D_n$, $J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h$ is invertible with

$$\begin{aligned} &\Big(J(h) - \lambda - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h\Big)^{-1} \\ &\qquad = (J(h) - \lambda)^{-1} \Big(I - \tau_2^2 B_h^* (\tau_2^2 \mathcal{N}_h - \lambda)^{-1} B_h (J(h) - \lambda)^{-1}\Big)^{-1} \\ &\qquad = (J(h) - \lambda)^{-1} (I + \mathcal{O}(h)). \end{aligned} \tag{A.6}$$

Hence, the standard Schur complement procedure shows that for $\lambda \in \tilde{D}_n \setminus D_n$, $\mathcal{P}(\lambda)$ is invertible with inverse $\mathcal{E}(\lambda)$ given by

$$\mathcal{E}(\lambda) = \begin{pmatrix} E(\lambda) & -\tau_2 E(\lambda) B_h^*(\tau_2^2 \mathcal{N}_h - \lambda)^{-1} \\ -\tau_2(\tau_2^2\mathcal{N}_h - \lambda)^{-1} B_h E(\lambda) & E_0(\lambda) \end{pmatrix} \tag{A.7}$$

with

$$E(\lambda) = \Big(J(h) - \lambda - \tau_2^2 B_h^*(\tau_2^2\mathcal{N}_h - \lambda)^{-1} B_h\Big)^{-1}$$

and

$$E_0(\lambda) = (\tau_2^2\mathcal{N}_h - \lambda)^{-1} + \tau_2^2(\tau_2^2\mathcal{N}_h - \lambda)^{-1} B_h E(\lambda) B_h^*(\tau_2^2\mathcal{N}_h - \lambda)^{-1}\,.$$

Let us now consider the spectral projector $\Pi_{D_n}(\mathcal{M}_h)$. Then,

$$\operatorname{rank}(\Pi_{D_n}(\mathcal{M}_h)) \ \geq \ \operatorname{rank}(\tilde{\Pi}_n)\,,$$

where we defined

$$\tilde{\Pi}_n = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \Pi_{D_n}(\mathcal{M}_h) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, an elementary computation shows that

$$\tilde{\Pi}_n = \frac{1}{2i\pi}\int_{\partial D_n} \begin{pmatrix} E(\lambda) & 0 \\ 0 & 0 \end{pmatrix} d\lambda = \begin{pmatrix} E_n & 0 \\ 0 & 0 \end{pmatrix}$$

with

$$\begin{aligned} E_n &= \frac{1}{2i\pi}\int_{\partial D_n} \Big(J(h) - \lambda - \tau_2^2 B_h^*(\tau_2^2\mathcal{N}_h - \lambda)^{-1} B_h\Big)^{-1} d\lambda \\ &= \frac{1}{2i\pi}\int_{\partial D_n} (J(h) - \lambda)^{-1}(I + \mathcal{O}(h)) d\lambda\,, \end{aligned}$$

where the last equality follows from (A.6). It follows that for $h > 0$ small enough, the rank of E_n is bounded from below by the multiplicity $m(\lambda_n^1)$ of λ_n^1 and hence

$$\operatorname{rank}(\Pi_{D_n}(\mathcal{M}_h)) \geq m(\lambda_n^1) \tag{A.8}$$

for all $n = 1, \dots, n_1$.

Let us now study the part of the spectrum of order smaller than τ_2^2. Thanks to the last part of Lemma A.2, the matrix $\mathcal{Z}_h(\tau') := \mathcal{N}_h - B_h J(h)^{-1} B_h^*$ is classical $(\mathscr{E}', \tau')$-graded. Hence, it follows from the induction hypothesis that uniformly with respect to h, one has

$$\sigma(\mathcal{Z}_h(\tau')) \subset \bigsqcup_{j=2}^{p} \tilde{\varepsilon}_j^2(\sigma(M_j) + \mathcal{O}(h)) \tag{A.9}$$

with $\tilde{\varepsilon}_j = \tau_2^{-1}\varepsilon_j = \prod_{l=3}^{j}\tau_l$ for $j \geq 3$ and $\tilde{\varepsilon}_2 = 1$. One also knows that for all $j = 2, \dots, p$ and all $\lambda \in \sigma(M_j)$, one has

$$\operatorname{rank}\Pi_{D_j}(\mathcal{Z}_h) = m_j(\lambda)$$

where $D_j = D(\lambda\tilde{\varepsilon}_j^2, Kh\tilde{\varepsilon}_j^2)$ for some $K > 0$. Moreover, one has for all $z \notin \cup_{j=2}^{p} \cup_{\lambda\in\sigma(M_j)} D(\lambda\tilde{\varepsilon}_j^2, Kh\tilde{\varepsilon}_j^2)$ the resolvent estimate

$$(\mathcal{Z}_h(\tau') - z)^{-1} = \mathcal{O}(\operatorname{dist}(z, \sigma(\mathcal{Z}_h(\tau'))^{-1}). \tag{A.10}$$

For $j = 2, \dots, p$, let $\lambda_1^j, \dots, \lambda_{n_j}^j$ denote the eigenvalues of the matrix $M_j \in \mathcal{D}_0$. As above, there exists $c_j, d_j > 0$ such that $\lambda_n^j \in K(c_j, d_j)$ for all $n = 1, \dots, n_j$. Suppose now that $j \in \{2, \dots, p\}$ and $n \in \{1, \dots, n_j\}$ are fixed and consider, for $M > K$,

$$D'_{j,n} = \{z \in \mathbb{C},\ |z - \varepsilon_j^2 \lambda_n^j| \leq M h \varepsilon_j^2\} = \tau_2^{-2}\{z' \in \mathbb{C},\ |z' - \tilde{\varepsilon}_j^2 \lambda_n^j| \leq M h \tilde{\varepsilon}_j^2\}\,.$$

Since M_1 is invertible, $J(h) - \lambda$ is invertible and $(J(h) - \lambda)^{-1} = \mathcal{O}(1)$ for λ in $D'_{j,n}$ and h, τ_2 small enough. Moreover, for any $\lambda \in \partial D'_{j,n}$, it holds, noting $\lambda' = \tau_2^{-2}\lambda$,

$$\begin{aligned}
\tau_2^2 \mathcal{N}_h - \lambda - \tau_2^2 B_h (J(h) - \lambda)^{-1} B_h^* &= \tau_2^2(\mathcal{N}_h - \lambda' - B_h(J(h) - \lambda)^{-1} B_h^*)\\
&= \tau_2^2\big(\mathcal{Z}_h - \lambda' - B_h\big((J(h) - \lambda)^{-1} - J(h)^{-1}\big) B_h^*\big)\\
&= \tau_2^2(\mathcal{Z}_h - \lambda')(I + \mathcal{O}(h^2 |\lambda| \|(\mathcal{Z}_h - \lambda')^{-1})\|).
\end{aligned}$$

Hence, according to the relations (A.9), (A.10), and to $\varepsilon_j = \tau_2 \tilde{\varepsilon}_j$, it holds

$$\begin{aligned}
\tau_2^2 \mathcal{N}_h - \lambda - \tau_2^2 B_h (J(h) - \lambda)^{-1} B_h^* &= \tau_2^2(\mathcal{Z}_h - \lambda')(I + \mathcal{O}(h^2 \varepsilon_j^2 \|(\mathcal{Z}_h - \lambda')^{-1})\|)\\
&= \tau_2^2(\mathcal{Z}_h - \lambda')(I + \mathcal{O}(h \frac{\varepsilon_j^2}{\tilde{\varepsilon}_j^2}))\\
&= \tau_2^2(\mathcal{Z}_h - \lambda')(I + \mathcal{O}(h \tau_2^2)). \qquad \text{(A.11)}
\end{aligned}$$

The latter operator is then invertible around $\partial D'_{j,n}$ for h, τ_2 small enough, and the Schur complement formula then permits to write the inverse of $\mathcal{P}(\lambda)$ as

$$\text{(A.12)} \quad \mathcal{E}(\lambda) = \begin{pmatrix} E_0(\lambda) & -\tau_2 (J(h) - \lambda)^{-1} B_h^* E(\lambda) \\ -\tau_2 E(\lambda) B_h (J(h) - \lambda)^{-1} & E(\lambda) \end{pmatrix}$$

with

$$E(\lambda) = \Big(\tau_2^2 \mathcal{N}_h - \lambda - \tau_2^2 B_h (J(h) - \lambda)^{-1} B_h^*\Big)^{-1}$$

and

$$E_0(\lambda) = (J(h) - \lambda)^{-1} + \tau_2^2 (J(h) - \lambda)^{-1} B_h^* E(\lambda) B_h (J(h) - \lambda)^{-1}.$$

As above, let us consider the corresponding projector $\Pi_{D'_{j,n}}(\mathcal{M}_h)$. From $\lambda = \tau_2^2 \lambda'$, we get

$$\Pi_{D'_{j,n}}(\mathcal{M}_h) = \frac{\tau_2^2}{2i\pi} \int_{\partial \hat{D}'_{j,n}} \mathcal{E}(\tau_2^2 \lambda') d\lambda'$$

with $\hat{D}'_{j,n} = \{z' \in \mathbb{C},\ |z' - \tilde{\varepsilon}_j^2 \lambda_n^j| \leq M h \tilde{\varepsilon}_j^2\}$. It follows moreover from (A.11) that for every $\lambda' \in \partial \hat{D}'_{j,n}$ and h, τ_2 small enough,

$$\text{(A.13)} \qquad E(\tau_2^2 \lambda') = \tau_2^{-2} (\mathcal{Z}_h - \lambda')^{-1} (I + \mathcal{O}(h)),$$

and the same argument as above shows that $\operatorname{rank}(\Pi_{D'_{j,n}}(\mathcal{M}_h)) \geq \operatorname{rank}(E'_n)$ with

$$E'_n = \frac{\tau_2^2}{2i\pi} \int_{\partial \hat{D}'_{j,n}} E(\tau_2^2 z) dz = \frac{1}{2i\pi} \int_{\partial \hat{D}'_{j,n}} (\mathcal{Z}_h - z)^{-1} (I + \mathcal{O}(h))^{-1} dz.$$

By the induction hypothesis, this shows that for h small enough, the rank of E'_n is exactly the multiplicity of λ_n^j and hence

$$\operatorname{rank}(\Pi_{D'_{j,n}}(\mathcal{M}_h)) \geq m(\lambda_n^j)$$

for all $j = 2, \ldots, p$ and $n = 1, \ldots, n_j$. Combined with (A.8), this shows that for all $j = 1, \ldots, p$ and $n = 1, \ldots, n_j$, one has

$$\operatorname{rank}(\Pi_{D_{j,n}}(\mathcal{M}_h)) \geq m(\lambda_n^j)$$

with $D_{j,n} = \varepsilon_j^2 D(\lambda_n^j, Mh)$. Since $\sum_{j,n} m(\lambda_n^j)$ is equal to the total dimension of the space, this implies that

$$\operatorname{rank}(\Pi_{D_{j,n}}(\mathcal{M}_h)) = m(\lambda_n^j) \tag{A.14}$$

which proves the localization of the spectrum and (A.3).

It remains to prove the resolvent estimate. Suppose that $\lambda \in \mathbb{C}$ is such that $\lambda \notin \cup_{j=1}^p \cup_{\mu \in \sigma(M_j)} D(\varepsilon_j^2(\tau)\mu, \varepsilon_j^2(\tau)Kh)$. We suppose first that $|\lambda| \geq c_0$ for $c_0 > 0$ such that $|\lambda_n^1| \geq 2c_0$ for all $n = 1, \ldots, n_1$. Then $\mathcal{P}(\lambda) = \mathcal{M}_h(\tau) - \lambda$ is invertible with inverse $\mathcal{E}(\lambda)$ given by (A.7). Using (A.6) it is clear that $E(\lambda) = \mathcal{O}(h^{-1}) = \mathcal{O}(\operatorname{dist}(\lambda, \sigma(\mathcal{M}_h(\tau))^{-1})$. On the other hand, since $(\tau_2^2 \mathcal{N}_h - \lambda)^{-1} = \mathcal{O}(1)$ and $B_h = \mathcal{O}(h)$ we have also $E_0(\lambda) = \mathcal{O}(1)$ and then $\mathcal{E}(\lambda) = \mathcal{O}(\operatorname{dist}(\lambda, \sigma(\mathcal{M}_h(\tau)))^{-1})$.

Suppose now that $|\lambda| \leq c_0$. Then $\mathcal{P}(\lambda) = \mathcal{M}_h(\tau) - \lambda$ is invertible with inverse $\mathcal{E}(\lambda)$ given by (A.12). Setting $\lambda' = \tau_2^{-2}\lambda$ one deduces from (A.13) and from (A.9),(A.10) that

$$E(\lambda) = \mathcal{O}(\tau_2^{-2} \operatorname{dist}(\lambda', \sigma(\mathcal{Z}_h))^{-1}) = \mathcal{O}(\operatorname{dist}(\lambda, \sigma(\mathcal{M}_h(\tau))^{-1}).$$

This completes the proof. □

References

1. F. Bouchet and J. Reygner, *Generalisation of the Eyring-Kramers transition rate formula to irreversible diffusion processes*, Ann. Henri Poincaré **17** (2016), no. 12, 3499–3532.
2. A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein, *Metastability in reversible diffusion processes. I. Sharp asymptotics for capacities and exit times*, J. Eur. Math. Soc. **6** (2004), no. 4, 399–424. MR 2094397
3. A. Bovier, V. Gayrard, and M. Klein, *Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues*, J. Eur. Math. Soc. **7** (2005), no. 1, 69–99.
4. G. Di Gesù and D. Le Peutrec, *Small noise spectral gap asymptotics for a large system of nonlinear diffusions*, J. Spectr. Theory **7** (2017), no. 4, 939–984.
5. G. Di Gesù, T. Lelièvre, D. Le Peutrec, and B. Nectoux, *The exit from a metastable state: concentration of the exit point distribution on the low energy saddle points, part 1*, Journal de Mathématiques Pures et Appliquées (2019), In press: https://doi.org/10.1016/j.matpur.2019.06.003.
6. B. Helffer, *Semi-classical analysis for the Schrödinger operator and applications*, Lecture Notes in Mathematics, vol. 1336, Springer-Verlag, 1988.
7. ———, *Spectral theory and its applications*, vol. 139, Cambridge University Press, 2013.
8. B. Helffer, M. Klein, and F. Nier, *Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach*, Mat. Contemp. **26** (2004), 41–85.
9. B. Helffer and F. Nier, *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, vol. 1862, Springer-Verlag, Berlin, 2005.

10. ———, *Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary*, Mém. Soc. Math. Fr. (N.S.) (2006), no. 105.
11. B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten*, Comm. Partial Differential Equations **10** (1985), no. 3, 245–340.
12. F. Hérau, M. Hitrik, and J. Sjöstrand, *Tunnel effect and symmetries for Kramers-Fokker-Planck type operators*, J. Inst. Math. Jussieu **10** (2011), no. 3, 567–634.
13. T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452
14. C. Landim, M. Mariani, and I. Seo, *Dirichlet's and Thomson's Principles for Non-selfadjoint Elliptic Operators with Application to Non-reversible Metastable Diffusion Processes*, Arch. Ration. Mech. Anal. **231** (2019), no. 2.
15. C. Landim and I. Seo, *Metastability of nonreversible random walks in a potential field and the Eyring-Kramers transition rate formula*, Comm. Pure Appl. Math. **71** (2018), no. 2, 203–266.
16. D. Le Peutrec, *Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian*, Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), no. 3-4, 735–809.
17. D. Le Peutrec and B. Nectoux, *Small eigenvalues of the Witten Laplacian with Dirichlet boundary conditions: the case with critical points on the boundary*, 2019, Preprint available on `https://arxiv.org/abs/1907.07517`, 61 pages.
18. G. Menz and A. Schlichting, *Poincaré and logarithmic Sobolev inequalities by decomposition of the energy landscape*, Ann. Probab. **42** (2014), no. 5, 1809–1884.
19. L. Michel, *Around supersymmetry for semiclassical second order differential operators*, Proc. Amer. Math. Soc. **144** (2016), no. 10, 4487–4500.
20. ———, *About small eigenvalues of the Witten Laplacian*, Pure Appl. Anal. **1** (2019), no. 2, 149–206.
21. L. Michel and M. Zworski, *A semiclassical approach to the Kramers-Smoluchowski equation*, SIAM J. Math. Anal. **50** (2018), no. 5, 5362–5379.
22. J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 7, 2095–2141.
23. E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).