# Semiclassical Random Walk on Manifolds

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# Plan

# General framework

Let

- $(M, g)$ be a Riemanian manifold, $d_g x$ be the volume form and $d_g(x, y)$ the associated distance.
- $\rho(x)$ be a measurable, bounded, strictly positive function such that $d\pi(x) = \rho(x) d_g x$ is a probability measure on $M$.

Let us define the semiclassical random walk operator on the space of bounded continuous function, by

$$T_h f(x) = \frac{1}{\pi(B_h(x))} \int_M \mathbf{1}_{B_h(x)}(y) f(y) d\pi(y)$$

where $h > 0$ is a small parameter and $B_h(x)$ is the geodesic ball centred in $x$ and with radius $h$.

Motivations: These operators appear in probabilistic framework since they are associated to natural random walk on $M$.

The kernel of $T_h$ is given by

$$t_h(x, dy) = \frac{\mathbf{1}_{\{d_g(x,y)\le h\}}}{\pi(B_h(x))} d\pi(y),\ \forall x \in \Omega.$$

This is a Markov kernel $(t_h(x, M) = T_h(1)(x) = 1,\ \forall x \in M)$.

**Definition**

Let $\nu_h$ be a probability measure on $M$. We say that $\nu_h$ is stationnary for $t_h(x, dy)$ if $T_h^t(\nu_h) = \nu_h$, where $T_h^t$ denotes the transpose operator of $T_h$ acting on Borel measure.

One can see easily that $t_h(x, dy)$ admits the following stationnary measure

$$d\nu_h = \frac{\pi(B_h(x))}{Z_h} d\pi(x)$$

where $Z_h$ is a normalizing constant.

# Convergence to stationnary measure

Given a Markov kernel $k(x, dy)$ on a metric space $(X, d)$ and $K$ the associated operator, we denote $k^n(x, dy)$ the kernel of the operator $K^n$.

**Theorem (cf Feller)**

Assume that $k(x, dy)$ is a *strictly positive and regular* Markov kernel and that $\pi$ is a stationnary measure for $k$. Then,

$$\forall x \in X, \forall B \in \mathcal{B}, \lim_{n \to \infty} k^n(x, B) = \pi(B)$$

$k$ *strictly positive* means that there exists $p \in \mathbb{N}$ such that $k^p(x, A) > 0$ for all open subset $A$. Think the regularity condition as, $k(x, dy)$ having a continuous density.

**Question**

What can we say about the convergence speed?

The answer is closely related to precise study of the spectral theory of $T_h$.

# Framework

Assume that $M$ is a compact manifold without boundary and that $\rho = 1/vol(M)$. We prove easily the following facts:

- $T_h$ is self-adjoint on $L^2(M, d\nu_h)$.
- $T_h$ is compact
- For all $p \in [1, \infty]$, $\| T_h \|_{L^p \rightarrow L^p} = 1$.

Hence, the spectrum of $T_h$ is made of eigenvalues and $\{0\}$ is the only possible accumulation point. We denote

$$1 = \mu_0(h) \geq \mu_1(h) \geq \mu_2(h) \geq \ldots \geq \mu_k(h) \ldots > 0$$

its positive eigenvalues, $(e_k^h)_{k \in \mathbb{N}}$ the associated $L^2$-normalized eigenfunctions.

# Reference operator

We use the following notations:

- $\Delta_g$ is the (negative) Laplace-Beltrami operator on $(M, g)$.
- $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ denotes the spectrum of the self adjoint operator $-\Delta_g$ on $L^2(M, d_g x)$.
- for $\xi \in \mathbb{R}^d$

$$G_d(\xi) := \frac{1}{\alpha_d} \int_{|y| \leq 1} e^{iy\xi} dy$$

where $\alpha_d$ = volume of the unit ball in $\mathbb{R}^d$.

- the function $G_d$ is radial and we let $\Gamma_d$ be such that $G_d(\xi) = \Gamma_d(|\xi|^2)$. Then, $\Gamma_d$ is analytic and near $s = 0$ we have

$$\Gamma_d(s) = 1 - \frac{s}{2(d+2)} + \mathcal{O}(s^2)$$

# An explicit example: the flat torus

- Suppose that $M = (\mathbb{R}/2\pi\mathbb{Z})^d$ is the flat $d$-dimensional torus endowed with the Euclidean metric. Then

$$T_h = \Gamma_d(-h^2\Delta_g).$$

Indeed, using Fourier expansion , it suffices to show that $T_h f_k = \Gamma_d(-h^2\Delta_g)f_k$ with $f_k(x) = e^{i\langle k,x\rangle}$, $k \in \mathbb{Z}^d$. Using the flatness of the metric, it comes

$$T_h f_k(x) = \frac{1}{c_d h^d}\int_{B(x,h)} e^{i\langle k,y\rangle}dy = \frac{e^{i\langle k,x\rangle}}{c_d}\int_{B(0,1)} e^{i\langle hk,u\rangle}du$$
$$= \Gamma_d(h^2|k|^2)e^{i\langle k,x\rangle} = \Gamma_d(-h^2\Delta_g)f_k(x)$$

- Using the Taylor expansion of $\Gamma_d$ in 0, this implies for all $k \in \mathbb{N}$:

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)}h^2 + O(h^4)$$

**Theorem [Lebeau-Michel, Annals of Probability, 2010]**

Assume that $M$ is compact without boundary. Let $h_0 > 0$ be small. There exist $\gamma < 1$ such that for any $h \in ]0, h_0]$ one has $Spec(T_h) \subset [-\gamma, 1]$ and 1 is a simple eigenvalue of $T_h$. Moreover, for any $k \in \mathbb{N}$,

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)} h^2 + O_k(h^4)$$

## Theorem (part 2)

Let $N(\tau,h)=card(Spec(T_h)\cap[1-\tau,1])$. For any $\delta\in]0,1[$, there exist $C>0$ s.t. for any $h\in]0,h_0]$ and any $\tau\in[0,\delta]$, we have

$$|N(\tau,h)-(2\pi h)^{-d}\int_{\Gamma_d(|\xi|_x^2)\in[1-\tau,1]}dxd\xi|\leq C(1+\tau h^{-2})^{\frac{d-1}{2}}$$

In particular, one has

$$N(\tau,h)\leq C(1+\tau h^{-2})^{d/2}$$

Suppose that $\mu_k(h)\in[\delta,1]$, then the associated eigenfuction $e_k^h$ satisfies

$$\|e_k^h\|_{L^\infty}\leq C\Big(1+\frac{1-\mu_k(h)}{h^2}\Big)^{d/4}\|e_k^h\|_{L^2}.$$

# Total variation estimate

The total variation distance between two probability measures $\mu, \nu$ is defined by

$$\|\mu-\nu\|_{TV} := \sup_{A \text{ measurable}} |\mu(A)-\nu(A)| = \frac{1}{2} \sup_{f\in L^\infty, \|f\|\leq 1} \left| \int f d\mu - \int f d\nu \right|$$

In particular,

$$\sup_{x\in M} \|t_h^n(x,dy) - d\nu_h(y)\|_{TV} = \frac{1}{2}\|T_h^n - \Pi_0\|_{L^\infty\to L^\infty}$$

where $\Pi_0$ denotes the othogonal projection on constant functions in $L^2(M, d\nu_h)$.

**Theorem [Lebeau-Michel, AOP, 2010]**

Let $h_0 > 0$ small. There exists $C > 0$ such that for all $h \in ]0, h_0]$ the following holds true :

$$e^{-\gamma'(h)nh^2} \leq 2 \sup_{x \in M} \|t_h^n(x, dy) - d\nu_h\|_{TV} \leq Ce^{-\gamma(h)nh^2} \quad \text{for all} \quad n$$

Here $\gamma(h), \gamma'(h)$ are positive functions s.t. $\gamma(h) \simeq \gamma'(h) \simeq \frac{\lambda_1}{2(d+2)}$ when $h \to 0$.

Let $\phi \in C_0^\infty(\mathbb{R})$ be a smooth function equal to 1 near 0.

## Lemma

The operator $A_h = h^{-2}(T_h - \Gamma_d(h^2\Delta_g))\phi(h^2\Delta_g)$ is an $h$-pseudodifferential operator whose principal symbol $a_0$ verifies

$$a_0(x,\xi) = \Big(\frac{S(x)}{3}|\xi|_x^2(\Gamma_d''(0) - \Gamma_d'(0)^2) + \frac{\Gamma_d''(0)}{3} Ric(x)(\xi,\xi)\Big)\phi(|\xi|_x^2) + O(\xi^3)$$

where $Ric(x)$ and $S(x)$ denotes the Ricci tensor and the scalar curvature at point $x$.

Using this Lemma, one can shows that for any $(\lambda_k, e_k)$ s.t. $-\Delta e_k = \lambda_k e_k$, one has

$$T_h e_k = (1 - \frac{h^2}{2(d+2)}\lambda_k + O(h^4))e_k.$$

Hence, to each eigenvalue $\lambda_k$ of $-\Delta$ corresponds at least one eigenvalue $\mu_l(h)$ of $T_h$ such that $\mu_l(h) = 1 - \frac{h^2}{2(d+2)}\lambda_k + O(h^4)$.

Using the preceding Lemma, on can show that for any $k$ and any $s>0$, there exists $h_0>0$ such that the eigenfunctions $(e_k^h)_{h\in]0,h_0]}$ (normalized in $L^2$), associated to the eigenvalue $\mu_k(h)$ satisfy

$$\|e_k^h\|_{H^s(M)} \leq C\Big(1+\frac{1-\mu_k(h)}{h^2}\Big)^{s/2}$$

Since $M$ is compact and $1-\mu_k(h)=O(h^2)$, this shows that the family $(e_k^h)_{h\in]0,h_0]}$ is compact in $H^2$. Hence, there exists $h_n\to 0$, $\lambda>0$ and $e_k\in H^2$ such that $e_k^{h_n}\to e_k$, $\mu_k(h_n)\to\lambda$ and

$$-\Delta e_k=\lambda e_k.$$

This shows that to each eigenvalue $\mu_l(h)$ of $T_h$ corresponds at least one eigenvalue $\lambda_k$ of $-\Delta$ such that $\mu_l(h)=1-\frac{h^2}{2(d+2)}\lambda_k+O(h^4)$.

# Semiclassical random walk on surfaces with hyperbolic ends

Consider a surface $(M,g)$ with finite volume and finitely many ends $E_1,\dots E_n$, with $E_i$ isometric to a hyperbolic cusp isometric to $\{(x,y)\in[x_i,\infty[\times(\mathbb{R}/\ell\mathbb{Z})\}$ endowed with the metric $g=\frac{dx^2+dy^2}{x^2}$ for some $x_i>0$.
Assume that $\rho=1$ and let $\pi=d_g x/vol(M)$. Since $M$ has finite volume, the kernel

$$t_h(x,dy)=\frac{1}{\pi(B_h(x))}\,\mathbb{1}_{d_g(x,y)<h}\,d\pi(y)$$

is a Markov kernel and the probability $d\nu_h=\frac{\pi(B_h(x))}{Z_h}d\pi(x)$ is well defined. Moreover, $d\nu_h$ is stationnary for $t_h(x,dy)$.

# Basic facts

Let $T_h$ be the operator with kernel $t_h(x, dy)$:

$$T_h f(x) = \frac{1}{\pi(B_h(x))} \int_M \mathbf{1}_{B_h(x)}(y) f(y) d\pi(y)$$

One can see easily that

- $T_h$ is self-adjoint on $L^2(M, d\nu_h)$
- $T_h$ acts on any $L^p$, $p \in [1, \infty]$ with norm 1.
- $T_h$ is not compact on $L^2$ anymore (since $M$ is unbounded).

## Theorem [Christianson-Guillarmou-Michel, *Ann. H. Poincaré*, 2011]

There exists $h_0 > 0$ and $c, \delta > 0$ such that the following hold true:

i) For any $h \in ]0, h_0]$, the essential spectrum of $T_h$ is given by the interval
$$I_h = [\frac{h}{\sinh(h)} A, \frac{h}{\sinh(h)}]$$
where $A = \min_{x>0} \frac{\sin(x)}{x} > -1$.

ii) For any $h \in ]0, h_0]$, $Spec(T_h) \cap [-1, -1+\delta] = \emptyset$.

iii) For any $h \in ]0, h_0]$, 1 is a simple eigenvalue of $K_h$ and the spectral gap $g(h) := \mathrm{dist}(\mathrm{Spec}(T_h) \setminus \{1\}, 1)$ enjoys
$$ch^2 \leq g(h) \leq \min\left(\frac{(\lambda_1 + \alpha(h))h^2}{8}, 1 - \frac{\sinh(h)}{h}\right)$$
where $\lambda_1$ is the smallest non-zero $L^2$ eigenvalue of $\Delta_g$ on $M$ and $\alpha(h)$ a function tending to 0 as $h \to 0$.

- Assume to simplify that there is only one cusp. The surface $M$ can be written $M = M_0 \cup E_0$ with $M_0$ compact and $E_0$ isometric to $\{(x, y) \in [x_0, \infty[ \times (\mathbb{R}/\ell\mathbb{Z})\}$ endowed with the metric $g = \frac{dx^2+dy^2}{x^2}$.
- Setting $x = e^t$, $E_0 = \{(t, y) \in ]t_0, \infty[ \times (\mathbb{R}/\ell\mathbb{Z})\}$ is endowed with the metric $g = dt^2 + e^{-2t} dy^2$.
- The function $m \in E_0 \mapsto t(m)$ can be extended to a smooth function on the whole suface $M$ such that $0 < t(m) < t_0$ for all $m \in M_0$.

# Structure of geodesic balls in the cusp

In $\mathbb{H}^2$, the hyperbolic ball centered in $(e^t, y)$ and with radius $h > 0$ is the Euclidean ball centered in $(e^t cosh(h), y)$ and with radius $e^t \sinh(h)$.

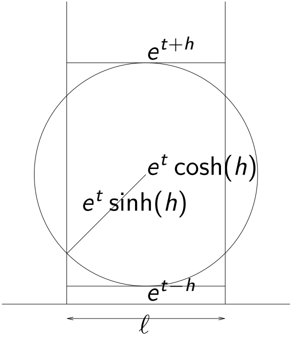


Figure: The hyperbolic ball of radius $h$ is tangent to itself when the center is at $t = \log(\ell/2\sinh(h))$. For $t > \log(\ell/2\sinh(h)$ the ball overlaps on itself.

# Consequence:

- For $f$ smooth supported in $t < t_h := \log(\ell/2\sinh(h))$, $T_h f$ is well approximated by $\Gamma_d(h^2\Delta_g)f$, but it is not the case for $f$ supported in $t \geq t_h$.
- The function $(e^t, y) \mapsto |B_h(e^t, y)|$ is singular at $t = t_h$. This prevents to describe $T_h$ as a pseudo with regular symbol.
- We use instead a variational approach.

# Variational approach

Let us define the variance

$$\mathcal{V}_h(f) = \|f\|^2_{L^2(M,d\nu_h)} - \langle f, 1\rangle^2_{L^2(M,d\nu_h)}.$$

and the Dirichlet form assiocated to our problem

$$\mathcal{E}_h(f) = \langle (1 - T_h)f, f\rangle_{L^2(M,d\nu_h)}$$

The spectral gap $g(h)$ is the the best constant such that the following inequality holds true:

$$\mathcal{V}_h(f) \leq \frac{1}{g(h)}\mathcal{E}_h(f)$$

Observe also that

$$\mathcal{E}_h(f) = \frac{1}{2Z_h}\int_{M\times M} 1\!\!1_{d_g(m,m')<h}(f(m) - f(m'))^2 dv_g(m)xdv_g(m')$$

$$\mathcal{V}_h(f) = \frac{1}{2}\int_{M\times M}(f(m) - f(m'))^2 d\nu_h(m)d\nu_h(m')$$

# Difficulties due to unbounded domain

In the case where the state space $M$ is bounded we can use a standard "path method" to prove that there exists a constant $C > 0$ such that

$$\mathcal{V}_h(f) \leq \frac{C}{h^2}\mathcal{E}_h(f).$$

Indeed we can decompose any geodesic curve $\gamma$ from $m$ to $m'$ into a **finite** reunion of small curves of length $h$. Using some change of variable we can prove the lower bound for the spectral gap.
Here, the surface $M$ is unbounded and this argument fails.

# Estimate gluing

For $0 \leq a < c < b \leq \infty$, define

$$\mathcal{V}_h^{[a,b]}(f) = \frac{1}{2}\int_{t(m),t(m')\in[a,b]} (f(m) - f(m'))^2 d\nu_h(m) d\nu_h(m'),$$

$$\mathcal{E}_h^{[a,b]}(f) = \frac{1}{2Z_h}\int_{t(m'),t(m)\in[a,b],d(m,m')<h} (f(m) - f(m'))^2 dv_g(m) dv_g(m'),$$

$$\mathcal{I}_h^c(f) = \frac{1}{2}\int_{t(m)\in[a,c],t(m')\in[c,b]} (f(m) - f(m'))^2 d\nu_h(m) d\nu_h(m')$$

Then, we have

$$\mathcal{V}_h^{[a,b]}(f) = \mathcal{V}_h^{[a,c]}(f) + \mathcal{V}_h^{[c,b]}(f) + 2\mathcal{I}_h^c(f)$$

and we want to control $\mathcal{I}_h$ with respect to the variance.

Using the fact that

$$\mathcal{I}_h^c(f) = \frac{1}{\nu_h(C_c)} \int_{m'' \in C_c} \mathcal{I}_h^c(f) d\nu_h(m'')$$

where $C_c := \{m \in M; c-1 < t(m) < c+1\}$, we get

$$\mathcal{I}_h^c(f) \leq \frac{2\nu_h(t(m) \in [c,b])}{\nu_h(C_c)} \mathcal{V}_h^{[a,c+1]}(f) + \frac{2\nu_h(t(m) \in [a,c])}{\nu_h(C_c)} \mathcal{V}_h^{[c-1,b]}(f)$$

for any $a+1 \leq c \leq b-1$. Taking $a=0$, $c=t_0$ and $b=+\infty$, we obtain

$$\mathcal{V}_h(f) \leq C\left(\mathcal{V}_h^{[0,t_0+1]}(f) + e^{t_0} \mathcal{V}_h^{[t_0-1,\infty]}(f)\right)$$

On the other hand, we have clearly

$$\mathcal{E}_h(f) \geq \frac{1}{2}\Big(\mathcal{E}_h^{[0,t_0+1]}(f) + \mathcal{E}_h^{[t_0-1,\infty]}(f)\Big).$$

Hence, we are reduced to prove that

- $\mathcal{E}_h^{[0,t_0+1]}(f) \geq Ch^2\mathcal{V}_h^{[0,t_0+1]}(f)$
- $\mathcal{E}_h^{[t_0-1,\infty]}(f) \geq Ch^2 e^{t_0}\mathcal{V}_h^{[t_0-1,\infty]}(f).$

We will concentrate on the second inequality (which corresponds to the cusp part).

# Reduction to the study of a random walk on euclidean space

Assume that $t_0 = 1$ and consider the surface $W := \mathbb{R}_t \times (\mathbb{R}/\ell\mathbb{Z})_y$. Then $E_0$ can be seen as the subset $\{t > 1\}$ of $W$ and the metric $g$ can be extended to $W$ by $g := dt^2 + e^{-2\mu(t)}dy^2$ where $\mu(t)$ is a smooth function on $\mathbb{R}$ equal to $|t|$ on $\mathbb{R} \setminus [-1,1]$ and such that $e^{-\mu(t)} \geq c_0$ when $t \in [-1,1]$ for some $c_0 > 0$.
Let us introduce the random walk operator associated to $W$:

$$T_h^W f(m) = \frac{1}{|B_h(m)|} \int_{B_h(m)} f(m')dv_g(m')$$

and the associated functionals:

$$\mathcal{E}_h^W(f) = \langle (1 - T_h^W)f, f\rangle_{L^2(W, d\nu_h^W)}$$

and

$$\mathcal{V}_h^W(f) = \|f\|^2_{L^2(W, d\nu_h^W)} - \langle f, 1\rangle^2_{L^2(W, d\nu_h^W)}.$$

For any $f \in L^2(E_0)$ we denote $f^s$ the function obtained by extending $f$ by symmetry $t \mapsto 1-t$. One can see easily that

$$\mathcal{V}_h^{[0,\infty)}(f) \leq \mathcal{V}_h^W(f^s)$$

and

$$\mathcal{E}_{\frac{h}{2}}^W(f^s) \leq C\mathcal{E}_h^{[0,\infty)}(f).$$

Hence, it remains to show the following

**Proposition**

There exists $C>0$ and $h_0>0$ such that for all $f \in L^2(W)$ and for all $h \in ]0,h_0]$, we have:

$$Ch^2\mathcal{V}_h^W(f) \leq \mathcal{E}_h^W(f)$$

Let us decompose the operator $T_h^W$ in fourier series. First, observe that

$$B_h(t,y) := \{(t',y'); |t-t'| \leq h, |y-y'| \leq \alpha_h(t,t')\}$$

for a certain function $\alpha_h(t,t')$ satisfying

- $\alpha_h(t,t') \leq \ell/2$ for all $t, t', h$.
- $\alpha_h(t,t') \geq \epsilon h$ for some $\epsilon > 0$ and for $|t| < 1$ and $|t-t'| < h/2$.
- For $t \geq t_0 + 1$,

$$\alpha_h(t,t') = \min\left(e^t\sqrt{\sinh(h)^2 - (\cosh(h) - e^{t'-t})^2}, \ell/2\right)$$

For $f(t,y)=\sum_{k\in\mathbb{Z}} f_k(t)e^{2ik\pi y/\ell}$, we have

$$T_h^W f = \sum_{k\in\mathbb{Z}} (T_{h,k}^W f_k)(t) e^{2i\pi k y/\ell}$$

with

$$T_{h,k}^W f_k(t) = \frac{2}{|B_h(t)|}\int_{t-h}^{t+h} f_k(t')\frac{\sin(2\pi k\alpha_h(t,t')/\ell)}{2\pi k\alpha_h(t,t')/\ell}\alpha_h(t,t')e^{-\mu(t')}dt'$$

and

$$T_{h,0}^W f_0(t) = \frac{2}{|B_h(t)|}\int_{t-h}^{t+h} \alpha_h(t,t') f_0(t') e^{-\mu(t')}dt'$$

From the inequality $\alpha_h(t,t') \geq \epsilon h$ for $|t| < 1$ and $|t-t'| < h/2$, we deduce easily that for any $k \neq 0$ one has

$$||T^W_{h,k} f||_{L^2(\mathbb{R},|B_h(t)|e^{-\mu(t)}dt)} \leq (1-\epsilon h^2)||f||_{L^2(\mathbb{R},|B_h(t)|e^{-\mu(t)}dt)}.$$

Hence, it remains to show a spectral gap for $T^W_{h,0}$ acting on $L^2(\mathbb{R}, e^{-\mu(t)}dt)$:

$$T^W_{h,0} f_0(t) = \frac{2}{|B_h(t)|}\int_{t-h}^{t+h} \alpha_h(t,t') f_0(t') e^{-\mu(t')} dt'$$

Let $\rho(t) = e^{-\mu(t)}$ and denote introduce the associated probability measure $\pi = \rho(x)dx/(\int_{\mathbb{R}} \rho(y)dy)$. Consider the random walk operator on $\mathbb{R}$ defined by

$$K_h^{\rho}(f)(t) = \frac{1}{\pi(B_h(t))} \int_{t-h}^{t+h} f(t)d\pi(t).$$

Then, using the structure of $\alpha_h$ and $|B_h(t)|$ we can show that

$$T_{h,0}^{W} f_0 \simeq K_h^{\rho}(f_0).$$

To complete the proof, it suffices to use the fact that operators of type $K_h^{\rho}$ have spectral gap $g(h) \simeq ch^2$. This is proved in the next section.

# Semiclassical random walk on Euclidean space

Let $\rho \in C^1(\mathbb{R}^d)$ be a strictly positive bounded function such that $d\pi = \rho(x)dx$ is a probability measure. Consider the random-walk operator defined by

$$T_h f(x) = \frac{1}{\pi(B_h(x))} \int_{B_h(x)} f(x')d\pi(x').$$

and its stationnary measure

$$d\nu_h = \frac{\pi(B_h(x))\rho(x)}{Z_h} dx$$

where $Z_h$ is chosen so that $d\nu_h$ is a probability on $\mathbb{R}^d$.

## Definition

We say that a density $\rho$ is *smooth tempered of exponential type (STE)* if $\rho$ is smooth and if there are some positive numbers $(C_\alpha)_{\alpha\in\mathbb{N}^d}$, $R>0, \kappa_0>0$, such that

$$\forall |x|\geq R,\ |\partial_x^\alpha \rho(x)|\leq C_\alpha \rho(x)$$

and

$$\forall |x|\geq R,\ \Delta\rho(x)\geq \kappa_0 \rho(x).$$

## Definition

We say that a density $\rho$ is gaussian if $\rho(x)=(\frac{\alpha}{\pi})^{\frac{d}{2}}e^{-\alpha|x|^2}$ for some $\alpha>0$.

In order to describe the eigenvalues of $T_h$, let us introduce the operator

$$L_\rho = -\Delta + V(x)$$

with $V(x) := \frac{\Delta\rho(x)}{\rho(x)}$. Observe that :

- $L_\rho$ is non-negative on $L^2(\mathbb{R}^d)$ and 0 is a simple eigenvalue associated to $\rho \in L^1 \cap L^\infty \subset L^2$.
- $\rho$ gaussian $\Longrightarrow \sigma_{ess}(L_\rho) = \emptyset$.
- $\rho$ STE $\Longrightarrow \sigma_{ess}(L_\rho) = [\kappa, +\infty[$ with $\kappa = \liminf_{|x|\to\infty} \frac{\Delta\rho(x)}{\rho(x)}$.

In the following, we will denote $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \ldots$ the $L^2(\mathbb{R}^d, dx)$ eigenvalues of $L_\rho$ and
$1 = \mu_0(h) > \mu_1(h) \geq \ldots \mu_k(h) \geq \ldots$ those of $T_h$.

# Spectral analysis in the Gaussian case

**Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)**

Suppose that $\rho$ is gaussian, then the operator $T_h$ is compact and for any $k \in \mathbb{N}$ fixed,

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)} h^2 + O_k(h^4).$$

Moreover, there exists $\tau_0 > 0$ such that for any $\tau \in [0, \tau_0]$, the number $N(\tau, h)$ of eigenvalues of $T_h$ in $[1-\tau, 1]$ satisfies

$$N(\tau, h) \leq C(1 + \tau h^{-2})^d.$$

# Spectral analysis in the tempered case

**Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)**

Suppose that $\rho$ is STE, then:

- the essential spectrum of $T_h$ on $L^2(\mathbb{R}^d, d\nu_h)$ is contained in $[M, A_h]$ where $M > -1$ and $A_h = 1 - \frac{\kappa}{2(d+2)}h^2 + O(h^4)$.
- for all $\alpha \in ]0,1[$, if $\lambda_k \in [0, \alpha\kappa]$, then

$$\mu_k(h) = 1 - \frac{\lambda_k}{2(d+2)}h^2 + O_{k,\alpha}(h^4).$$

# Sketch of proof

- The operator $T_h$ acting on $L^2(\mathbb{R}^d, d\pi)$ is unitarily conjugated to $\tilde{T}_h : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dx)$ defined by

$$\tilde{T}_h = a_h(x)\Gamma_d(h^2\Delta)a_h(x)$$

with $a_h(x) = (\alpha_d h^d \rho(x)/\pi(B_h(x)))^{1/2}$.

- The function $a_h$ enjoys nice estimates. For instance, in the gaussian case

$$\exists C, R > 0, \forall |x| \geq R, \ \frac{1}{a_h^2(x)} \geq \max(1 + Ch^2|x|^2, Ce^{h|x|})$$

- Using this inequality and properties of the function $\Gamma_d$, one can show some spatial-decay estimate of the eigenfuntions of $T_h$. This allow to over come the lack of compactness of $\mathbb{R}^d$

# Total variation estimates: Upper bound

**Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)**

There exist $C>0$ and $h_0>0$ such that for all $n\in\mathbb{N}$, $h\in]0,h_0]$ and $\tau>0$,

$$\sup_{|x|<\tau}\|t_h^n(x,dy)-d\nu_h\|_{TV}\leq Cq(\tau,h)e^{-ng(h)}$$

where $q(\tau,h)=e^{\alpha\tau(\tau+3h)}$ if $\rho=(\frac{\alpha}{\pi})^{\frac{d}{2}}e^{-\alpha|x|^2}$ is gaussian and $q(\tau,h)=h^{-\frac{d}{2}}\sup_{|x|<\tau}\frac{1}{\rho(x)}$ if $\rho$ is STE.

# Total variation estimates: Lower bound

The following theorem shows that contrary to compact case, convergence (for the total variation distance) can not be uniform with respect to the starting point $x$.

**Theorem (Guillarmou-Michel, Math. Res. Letters, 2011)**

There exists $C > 0$ such that for all $n \in \mathbb{N}$, $h \in ]0,1]$, $\tau > 0$, we have

$$\inf_{|x| \geq \tau + (n+1)h} \| t_h^n(x, dy) - d\nu_h \|_{TV} \geq 1 - Cp(\tau)$$

where $p(\tau) = e^{-2\alpha\tau(\tau - h)}$ if $\rho = (\frac{\alpha}{\pi})^{\frac{d}{2}} e^{-\alpha|x|^2}$ is gaussian and $p(\tau) = \int_{|y| \geq \tau} \rho(y)^2 dy$ if $\rho$ is STE.

**Proof:** Compute $(T_h^n - \Pi_{0,h}) f_\tau$ with

$$f_\tau(x) = \mathbb{1}_{[\tau, +\infty[}(|x|) - \mathbb{1}_{[0,\tau[}(|x|)$$