Semiclassical analysis of a random walk on a manifold

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General framework

Let (M, g) be a smooth, compact, connected Riemannian manifold of dimension d, equipped with its canonical volume form $d_g x$. We denote

- $d_g(x, y)$ the Riemannian distance on $M \times M$.
- for $x \in M$ and $h > 0$, $B(x, h) = \{y, d_g(x, y) \leq h\}$
- $|B(x, h)| = \int_{B(x,h)} d_g y$

For any given $h > 0$, let T_h be the operator acting on continuous functions on M

$$(T_h f)(x) = \frac{1}{|B(x, h)|} \int_{B(x,h)} f(y) d_g y$$

The Markov kernel

We denote by K_h the kernel of T_h, which is given by

$$K_h(x,y)d_gy = \frac{\mathbf{1}_{\{d_g(x,y)\leq h\}}}{|B(x,h)|}d_gy$$

- for any $x \in M$, $K_h(x,y)d_gy$ is a probability measure on M (hence, K_h is a Markov kernel)
- K_h is associated to the following natural random walk (X_n) on M: if the walk is at x, then it moves to a point $y \in B(x,h)$ with a probability given by $K_h(x,y)d_gy$. For $f \in C^0(M)$, $T_h(f)(X_n) = E(f(X_{n+1})|X_n)$. Or equivalently, for A, B measurable,

$$P(X_{n+1} \in A \text{ and } X_n \in B) = E(T_h(1_A)(X_n)1_B(X_n)).$$

Basic properties

We define $d\nu_h = \frac{|B(x,h)|}{h^d Z_h} d_g x$ where Z_h is chosen so that $d\nu_h$ is a probability measure. We have the following facts:

- T_h is self-adjoint on $L^2(M, d\nu_h)$.
- T_h is compact
- For all $p \in [1, \infty]$, $\|T_h\|_{L^p \to L^p} = 1$.

Hence, the spectrum of T_h is made of eigenvalues and $\{0\}$ is the only possible accumulation point. We denote

$$1 = \mu_0(h) \geq \mu_1(h) \geq \mu_2(h) \geq ... \geq \mu_k(h) ... > 0$$

the positive eigenvalues, $(e_k^h)_k$ the associated normalized eigenfunctions.

Reference operator

We use the following notations:

- Δ_g is the (negative) Laplace-Beltrami operator on (M, g).
- $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ denotes the spectrum of the self adjoint operator $-\Delta_g$ on $L^2(M, d_g x)$.
- for $\xi \in \mathbb{R}^d$

$$G_d(\xi) = \frac{1}{c_d} \int_{|y| \leq 1} e^{iy\xi} dy$$

  where c_d = volume of the unit ball in$\mathbb{R}^d$.
- the function G_d is radial and we let Γ_d be such that $G_d(\xi) = \Gamma_d(|\xi|^2)$. Then, Γ_d is analytic and near $s = 0$ we have

$$\Gamma_d(s) = 1 - \frac{s}{2(d+2)} + \mathcal{O}(s^2)$$

- We denote $\Gamma_{d,h} = \Gamma_d(-h^2 \Delta_g)$.

Theorem 1 (part 1)

Let $h_0 > 0$ be small. There exist $\gamma < 1$ such that for any $h \in]0, h_0]$ one has $Spec(T_h) \subset [-\gamma, 1]$ and 1 is a simple eigenvalue of T_h. For any given $L > 0$, there exists C such that for all $h \in]0, h_0]$ and all $k \leq L$, one has

$$|\frac{1 - \mu_k(h)}{h^2} - \frac{\lambda_k}{2(d+2)}| \leq Ch^2$$

Theorem 1 (part 2)

Let $N(a,h) = card(Spec(T_h) \cap [a,1])$. For any $\delta \in]0,1[$, there exist $C > 0$ s.t. for any $h \in]0, h_0]$ and any $\tau \in [0, (1-\delta)h^{-2}]$, we have

$$|N(1-\tau h^2, h) - (2\pi h)^{-d} \int_{\Gamma_d(|\xi|_x^2) \in [1-\tau h^2, 1]} dx d\xi| \leq C(1+\tau)^{\frac{d-1}{2}}$$

In particular, one has

$$N(1-\tau h^2, h) \leq C(1+\tau)^{d/2}$$

Denote e_k^h the eigenfunction of T_h associated to $\mu_k(h) \in [\delta, 1]$, and set $\tau_k(h) = h^{-2}(1-\mu_k(h))$, then

$$\|e_k^h\|_{L^\infty} \leq C(1+\tau_k(h))^{d/4} \|e_k^h\|_{L^2}.$$

Let $|\Delta_h|$ be the positive, bounded, self adjoint operator on $L^2(M, d\nu_h)$ defined by

$$1 - T_h = \frac{h^2}{2(d+2)}|\Delta_h|$$

Let F_1 and F_2 be the two closed subset of $\mathbb{C}$,

$$F_1 = \{z, dist(z, spec(-\Delta_g)) \leq \varepsilon\}$$
$$F_2 = \{z, Re(z) \geq A, |Im(z)| \leq \varepsilon Re(z)\}$$

with $\varepsilon > 0$ small and $A > 0$ large. Let $F = F_1 \cup F_2$ and $U = \mathbb{C} \setminus F$.

Theorem 2

There exists $C, h_0 > 0$ such that for all $h \in]0, h_0]$, and all $z \in U$

$$\|(z - |\Delta_h|)^{-1} - (z + \Delta_g)^{-1}\|_{L^2 \to L^2} \leq Ch^2$$

Convergence to stationary measure

- As $T_h 1 = 1$ and T_h self-adjoint on $L^2(M, d\nu_h)$ then $d\nu_h$ is a stationary measure for $K_h(x,y)d_g y$, ie: for all $f \in C(M)$

$$\int_M f(x)d\nu_h(x) = \int_M T_h f(x) d\nu_h(x)$$

- it follows from general theory of Markov chains that for all $x \in M$, $K_h^n(x,y)d_g y \to d\nu_h(y)$, when $n \to \infty$, where $K_h^n(x,y)d_g y$ denotes the kernel of T_h^n.

Question: how fast does it converge?

Total variation estimate

The total variation distance between two probability measures μ, ν is defined by

$$\|\mu - \nu\|_{TV} = \sup_{A \text{ measurable}} |\mu(A) - \nu(A)|$$

Theorem 3

Let $h_0 > 0$ small. There exists $C > 0$ such that for all $h \in]0, h_0]$ the following holds true :

$$e^{-\gamma'(h)nh^2} \leq 2 \sup_{x \in M} \|K_h^n(x,y)d_g y - d\nu_h\|_{TV} \leq Ce^{-\gamma(h)nh^2} \quad \text{for all } n$$

Here $\gamma(h), \gamma'(h)$ are positive functions s.t. $\gamma(h) \simeq \gamma'(h) \simeq \frac{\lambda_1}{2(d+2)}$ when $h \to 0$.

The Metropolized operator

If one is interested in sampling the probability measure $d\mu_M = d_g x / Vol(M)$, we can use a kernel modified according to the Metropolis strategy. Remark that $d\mu_M = \rho_h(x) d\nu_h$ with $\rho_h(x) = \frac{h^d Z_h}{Vol(M)|B(x,h)|}$. We define

$$M_h(x, dy) = m_h(x)\delta_{y=x} + \mathcal{K}_h(x, y) d_g y$$

where the functions m_h and $\mathcal{K}_h$ are defined by

$$\mathcal{K}_h(x, y) := K_h(x, y) min\Big(\frac{\rho_h(y)}{\rho_h(x)}, 1\Big) = \frac{1_{d_g(x,y)\leq h}}{|B(x, h)|} min\Big(\frac{|B(x, h)|}{|B(y, h)|}, 1\Big)$$

$$m_h(x) = 1 - \int_M \mathcal{K}_h(x, y) d_g y$$

Then, $M_h(x, dy)$ is still a Markov kernel and the operator

$$M_h(f)(x) = \int_M f(y) M_h(x, dy)$$

is self-adjoint on the space $L^2(M, d_g x)$, and therefore $d\mu_M$ is invariant for M_h.

Theorem 4

Let $h_0 > 0$ small. There exists $C > 0$ such that for all $h \in]0, h_0]$ the following holds true :

$$e^{-\gamma'(h)nh^2} \leq 2 \sup_{x \in M} \|M_h^n(x, dy) - d\mu_M\|_{TV} \leq Ce^{-\gamma(h)nh^2} \quad \text{for all } n$$

Here $\gamma(h), \gamma'(h)$ are two positive functions such that $\gamma(h) \simeq \gamma'(h) \simeq \frac{\lambda_1}{2(d+2)}$ when $h \to 0$.

Structure of the operator

Let $x = (x_1, \dots, x_d)$ be a local system of coordinates and $g_{i,j}(x)$ the metric g. In these coordinates, the geodesic ball of radius r centred at x is given by

$$B(x,r) = \{x+u, \sum k_{i,j}(x,u)u_i u_j \leq r^2\}$$

where $(k_{i,j}(x,u))$ is a symmetric matrix smooth w.r.t. (x,u) such that $k_{i,j}(x,0) = g_{i,j}(x)$. In these local coordinates, we have

$$T_h f(x) = \frac{1}{|B(x,h)|} \int_{{}^t u k(x,u) u \leq h^2} f(x+u) \sqrt{det(g(x+u))} du$$
$$= \frac{h^d}{|B(x,h)|} \int_{|v|\leq 1} f(x+hm(x,hv)v)\rho(x,hv)dv$$

where $m(x,w)$ is a symmetric and positive matrix s.t. $m(x,0) = g^{-1/2}(x)$ and $\rho(x,0) = 1$.

Computation of the symbol

We define the symbol of T_h by $\sigma(T_h)(x,\xi,h) = e^{-ix\xi/h}T_h(e^{ix\xi/h})$. Then,

$$\sigma(T_h)(x,\xi,h) = \frac{h^d}{|B(x,h)|}\int_{|v|\leq 1} e^{i^t\xi.m(x,hv)v}\rho(x,hv)dv$$

In particular, since $m(x,0) = g^{-1/2}(x)$ and $\rho(x,0) = 1$, one has

$$\sigma(T_h)(x,\xi,0) = \Gamma_d(|\xi|_x^2)$$

Remark

$\forall\alpha,\beta$, $\exists C_{\alpha,\beta}$ independent of h such that

$$|\partial_x^\alpha\partial_\xi^\beta\sigma(T_h)(x,\xi,h)| \leq C_{\alpha,\beta}(1+|\xi|)^{|\alpha|}$$

Hence, $\sigma(T_h)(x,\xi,h)$ is not in a good symbol class.

Class of operator

Definitions

- $a(x,\xi,h) \in S^{-\infty}$ if $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)| \leq C_k(1+|\xi|)^{-k}, \forall k \in \mathbb{N}$.
- $S_{cl}^{-\infty} = \{a \in S^{-\infty}$, a has an expansion in powers of $h\}$.
- A family of operators $(R_h)_{h\in]0,1]}$ is smoothing if for all s,t,N we have
$$\|R_h\|_{H^s\to H^t} \leq C_{s,t,N} h^N,\ \forall h \in]0,1]$$
- $A_h \in \mathcal{E}_{cl}^{-\infty}$ if $A_h = a(x,hD_x,h) + R_h$ with $a \in S_{cl}^{-\infty}$ and R_h smoothing.
- A family of operators $(C_h)_{h\in]0,1]}$ on $\mathcal{D}'(M)$, belongs to $\widetilde{\mathcal{E}}_{cl}^0$ if C_h is bounded uniformly in h on $L^2(M)$ and for any $\Phi_0 \in C_0^\infty$:
$$\Phi_0(-h^2\Delta_g)C_h \quad \text{and} \quad C_h\Phi_0(-h^2\Delta_g) \quad \text{belongs to} \quad \mathcal{E}_{cl}^{-\infty}$$

Lemma 1

Let h_0 small. For $h \in]0, h_0]$, the operator T_h belongs to the class $\widetilde{\mathcal{E}}^0_{cl}$.

Proof. Let $\phi_0 \in C_0^\infty$.

- Observe that $\phi_0(-h^2\Delta_g) \in \mathcal{E}_{cl}^{-\infty}$. As T_h is bounded on $C^k(M)$ for all k, it suffices to show that for all $a \in S_{cl}^{-\infty}$, $T_h a(x, hD_x) \in \mathcal{E}_{cl}^{-\infty}$.
- In local coordinates, we have $T_h a(x, hD_x) = b(x, hD_x)$ with

$$b(x, \xi, h) = \frac{h^d}{|B(x,h)|} \int_{|v|\leq 1} e^{i^t\xi . m(x,hv)v} a(x + hm(x,hv)v, \xi, h)\rho(x, hv)dv$$

thanks to the cut off function a, it is clear that $b \in S_{cl}^{-\infty}$. □

Lemma 2

Let $\Phi_0 \in C_0^\infty([0,\infty[)$, then $(T_h - \Gamma_{d,h})\Phi_0(-h^2\Delta_g) = h^2 A_h$ with $A_h \in \mathcal{E}_{cl}^{-\infty}$. Its principal symbol, $\sigma_0(A_h)$, satisfies near $\xi = 0$

$$\sigma_0(A_h)(x,\xi) = \Big(\frac{S(x)}{3}|\xi|_x^2(\Gamma_d''(0) - \Gamma_d'(0)^2) + \frac{\Gamma_d''(0)}{3} Ric(x)(\xi,\xi)\Big)\Phi_0(|\xi|_x^2) + \mathcal{O}(\xi^3)$$

where $Ric(x)$ and $S(x)$ are the Ricci tensor and the scalar curvature at x. Moreover, denoting $\sum h^k a_k(x,\xi)$ the symbol of A_h, we have $a_k(x,0) = 0$ for all $k \geq 0$.

In particular for $f \in H^2(M)$ we have $\|(T_h - \Gamma_{d,h})f\|_{L^2} \leq Ch^4\|f\|_{H^2}$. It is one of the main ingredient in the proof of Theorem 2.

Proof. Let $m_0 \in M$ be fixed. We use the geodesic coordinates x, centred at m_0. Thanks to the formula giving the principal symbol of a pseudo conjugate by a change of variable, it suffices to compute $\sigma(T_h)$ in $x = 0$. In the geodesic coordinates we have

- $\sqrt{det(g)(y)} = 1 - \frac{1}{6}Ric(y, y) + O(y^3)$,
- $m(0, v) = Id$, $\rho(0, v) = 1$

Consequently, $|B(0, h)| = h^d c_d(1 + \frac{\Gamma_d'(0)}{3}S(0)h^2) + O(h^3)$ and

$$\sigma(T_h)(0, \xi, h) = \frac{h^d}{|B(0, h)|}\int_{|v|\leq 1} e^{i\xi . v}\sqrt{det(g)(hv)}dv$$

$$= \Gamma_d(|\xi|^2) + h^2\Big(- \Gamma_d(|\xi|^2)\frac{\Gamma_d'(0)}{3}S + \frac{1}{6}\sum Ric_{j,k}\frac{\partial^2 G_d}{\partial\xi_j\partial\xi_k}(\xi)\Big) + \mathcal{O}(h^3)$$

□

Control of high frequencies

Lemma 3

Let $\chi \in C_0^\infty(\mathbb{R})$ be equal to 1 near 0. There exists $h_0 > 0$, $C > 0$ such that

$$\| T_h(1-\chi)(\frac{-h^2\Delta_g}{s})\|_{L^2\to L^2} \leq \frac{C}{\sqrt{s}}$$

for all $h \in]0, h_0]$ and all $s \geq 1$.

Proof. Thanks to the support properties of χ we have

$$\|(\frac{-h^2}{s}\Delta_g)^{-1/2}(1-\chi)(\frac{-h^2}{s}\Delta_g)\|_{L^2\to L^2} = O(1)$$

Hence, it suffices to prove that

$$\|T_h(-h^2\Delta_g)^{1/2}\|_{L^2\to L^2} = O(1)$$

On the other hand, in local coordinates we have

$$T_h f(x) = Ch^{-d}\int e^{ih^{-1}(x-y)\xi} b(x,\xi,h) f(y) dy$$

with

$$b(x,\xi,h) = \int_{|v|\le 1} e^{ih^{-1}\,{}^t\xi m(x,hv)v}\rho(x,hv)dv,$$

$m(x,0) = g(x)^{-1/2}$ and $\rho(x,0) = 1$.

Using the stationary phase method, we can compute the symbol $b(x, \xi, h)$:

$$b(x, \xi, h) = e^{i\Phi_+(x,\xi,h)}\tau_+(x, \xi, h) + e^{i\Phi_-(x,\xi,h)}\tau_-(x, \xi, h) + n(x, \xi, h)$$

where $n \in S^{-\infty}$, $\tau_\pm$ are two symbols of degree $-(d+1)/2$ and the phase $\phi_\pm$ satisfy

$$\phi_\pm(x, \xi, h) = |\xi|_x + O(h)$$

Therefore, T_h is locally the sum of two quantized canonical transformation with symbol of degree $-(d+1)/2$ and phases close to $(x-y)\xi + h|\xi|_x$. In particular , as $(d+1)/2 \geq 1$ we gain one h-derivative. $\square$

Estimate on eigenfunctions(I)

Let $\delta \in]0,1[$ and for $h > 0$, $e^h \in L^2$ and $z_h \in [\delta, 1]$ be such that $\|e^h\|_{L^2} = 1$ and $(T_h - z_h)e^h = 0$.

Lemma 4

There exists $h_0 > 0$, and for all $j \in \mathbb{N}$ there exists $C_j > 0$, such that, the following inequality holds true

$$\sup_{h \in]0,h_0]} \|(-h^2\Delta_g + 1)^j e^h\|_{L^2} \leq C_j$$

Proof. We localize as before. As $e^h = \frac{1}{z_h} T_h e^h$ and T_h is the sum of two quantized canonical transformation of degree $-(1+d)/2 \leq -1$, we gain one h-derivative, and we can iterate this argument. □

Estimate on eigenfunctions(II)

Lemma 5

Let s_1 be such that $|\Gamma_d(s)| \leq \delta/2$ for $s \geq s_1 - 1$ and let $\chi_1 \in C_0^\infty(\mathbb{R}^+)$ be equal to 1 on $[0, s_1]$. Then,

$$(1 - \chi_1(-h^2\Delta_g))e^h = O_{C^\infty}(h^\infty)$$

Proof. Recall that $z_h \in [\delta, 1]$. We write $1 - \chi_1 = \chi_2 + \chi_3$ with χ_3 supported in $[s_2, \infty[$, $s_2 >> 1$. For $j = 2, 3$ we have

$$\chi_j(-h^2\Delta_g)(z_h - T_h)\chi_j(-h^2\Delta_g) \geq \frac{\delta}{3}\chi_j(-h^2\Delta_g)$$

(apply Lemma 2 in the case j=2 and Lemma 3 in the case j=3). Hence, $\|\chi_j(-h^2\Delta_g)e^h\|_{L^2} = O(h^\infty)$. To gain regularity, use interpolation and the preceding Lemma. □

Estimate on eigenfunctions(III)

Lemma 6

For all $j \in \mathbb{N}$, there exists C_j such that for all $h \in]0, h_0]$, we have

$$\|e^h\|_{H^j(M)} \leq C_j(1 + h^{-2}(1 - z_h))^{j/2}$$

Proof. Let $\tilde{e}^h = \chi_1(-h^2\Delta_g)e^h$. From Lemma 5 and Lemma 2 we have

$$O(h^\infty) = (T_h - z_h)\tilde{e}^h = ((\Gamma_d - 1)(-h^2\Delta_g) + O(h^2) + (1 - z_h))\tilde{e}^h$$

Moreover, there exists a smooth function F_d non vanishing on $[0, s_1 + 2]$ s.t. $(\Gamma_d - 1)(s) = sF_d(s)$. Hence,

$$-\Delta_g F_d(-h^2\Delta_g)\tilde{e}^h = (O(1) + \frac{1 - z_h}{h^2})\tilde{e}^h$$

This shows that $\|\tilde{e}^h\|_{H^j} \leq C(1 + h^{-2}(1 - z_h))^{j/2}\|\tilde{e}^h\|_{L^2}$. □

Localization of the spectrum

Recall that $(-\Delta_g - \lambda_k)e_k = 0$, $(T_h - \mu_k(h))e_k^h = 0$ and

$$1 - T_h = \frac{h^2}{2(d+2)}|\Delta_h|$$

Let $k \in \mathbb{N}$ be fixed and $\chi \in C_0^\infty(\mathbb{R}^+)$. Then

- for $h > 0$ small enough, $e_k = \chi(-h^2\Delta_g)e_k$
- Lemma 2$\Longrightarrow (T_h - \Gamma_d(-h^2\Delta_g))e_k = O(h^4)$.

As $\Gamma_d(-h^2\Delta_g)e_k = (1 + h^2\Gamma_d'(0)\lambda_k)e_k + O(h^4)e_k$ and T_h is selfadjoint on $L^2(M, d\nu_h)$ it follows that for $h > 0$ small enough

$$card\Big(Spec(|\Delta_h|) \cap [\lambda_k - C_0h^2, \lambda_k + C_0h^2]\Big) \geq m_k.$$

where m_k is the multiplicity of λ_k.

Localization of the spectrum

On the other hand, if $e^h \in L^2(M)$ satisfies $|\Delta_h| e^h = \tau^h e^h$ with τ^h bounded, we have

- $e^h = \chi_1(-h^2\Delta_g)e^h + O_{C^\infty}(h^\infty)$ (thanks to Lemma 5).
- hence, Lemma 2 $\Longrightarrow (T_h - \Gamma_d(-h^2\Delta_g))e^h = O(h^4)$ and $dist(\tau^h, Spec(-\Delta_g)) = O(h^2)$

Hence, it remains to show that for $h > 0$ small we have

$$card\Big(Spec(|\Delta_h|) \cap [\lambda_k - C_0 h^2, \lambda_k + C_0 h^2]\Big) \leq m_k.$$

Multiplicity of the eigenvalues

Let $p \geq m_k$ and $e_1^h, \dots, e_p^h$ satisfy $(|\Delta_h| - \tau_l(h))e_l^h = 0$ with $\tau_l(h) \in [\lambda_k - C_0 h^2, \lambda_k + C_0 h^2]$, (e_l^h) orthonormal for the scalar product $\langle ., . \rangle_{L^2(M, d\nu_h)}$.
By Lemma 6, we have

$$\|e_l^h\|_{H^j} \leq C_j (1 + \tau_l(h))^{j/2}, \forall j$$

and we can suppose that e_l^h converges in H^2 when $h \to 0$.
Denoting f_l its limit we get $-\Delta_g f_l = \lambda_k f_l$ for all $l = 1, \dots, p$ and the functions f_l are orthogonal for the scalar product $\langle ., . \rangle_{L^2(M, d_g x)}$.
Consequently, $p \leq m_k$.

Proof of Weyl type estimate

- Use Lemma 2 and Lemma 3 to approximate the number of eigenvalues of T_h in an interval by those of $\Gamma_d(-h^2\Delta_g)$.
- Use wellknow Weyl estimates on pseudodifferential operators on a manifold.

From Theorem 1 to Theorem 3

We can suppose that $n \geq h^{-2}$. Recall that the total variation between two probability μ, ν on M is given by

$$\|\mu - \nu\|_{TV} := \sup_{A \subset M} |\mu(A) - \nu(A)| = \frac{1}{2} \sup_{\|f\|_{L^\infty} \leq 1} |\mu(f) - \nu(f)|$$

Consequently,

$$\sup_{x \in M} \| T_h^n(x, dy) - d\nu_h\|_{TV} = \frac{1}{2} \| T_h^n - \Pi_0^h\|_{L^\infty \to L^\infty}$$

where Π_0^h is the orthogonal projector in $L^2(M, d\nu_h)$ on the space of constant functions.

From Theorem 1 to Theorem 3

Recall that e_k^h denotes the normalized eigenfunction of T_h associated to the eigenvalue $\mu_k(h)$ and let Π_k^h be the orthogonal projector on $span(e_k^h)$. Then,

$$(T_h^n - \Pi_0^h)e_1^h = \mu_1(h)^n e_1^h = (1 - h^2\frac{\lambda_1}{2(d+2)} + O(h^4))^n e_1^h$$

and the left inequality in Theorem 3 is straightforward.
To prove the right inequality, let $\delta \in]0,1[$ and denote

$$T_h - \Pi_0(h) = T_{h,1} + T_{h,2}$$

with $T_{h,1} = \sum_{1-\delta \le \mu_k \le \mu_1} \mu_k(h)\Pi_k(h)$.

From Theorem 1 to Theorem 3

- Denote $\tau_k(h) = h^{-2}(1-\mu_k(h))$, thanks to the eigenfunction estimate, we have

$$\|T_{h,1}^n\|_{L^\infty\to L^\infty} \leq \sum_{\tau_1\leq\tau_k\leq h^{-2}(1-\delta)} (1-h^2\tau_k)^n(1+\tau_k)^\alpha$$

Thanks to Weyl law, we get for $nh^2 \geq 1$,

$$\|T_{h,1}^n\|_{L^\infty\to L^\infty} \leq C\int_{\tau_1}^{+\infty} e^{-nh^2x}(1+x)^\beta dx \leq Ce^{-nh^2\tau_1}$$

- Finally, we observe that

$$\begin{aligned}\|T_{h,2}^n\|_{L^2,L^\infty} &\leq \|T_{h,2}^{n-1}\|_{L^2,L^2}\|T_{h,2}\|_{L^2,L^\infty}\\ &\leq C(1-\delta)^n h^{-d/2} << e^{-nh^2\gamma(h)}\end{aligned}$$