Tunnel effect for semiclassical random walk

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Plan

Semiclassical random walk

Let $\phi \in C^\infty(\mathbb{R}^d)$ be a real function such that $d\mu_h = e^{-\phi(x)/h}dx$ is a probability measure. We are interrested in the random-walk operator defined on the space $\mathcal{C}_0$ of continuous function going to 0 at infinity by

$$T_h f(x) = \frac{1}{\mu_h(B_h(x))} \int_{B_h(x)} f(x')d\mu_h(x'),$$

where $B_h(x) = B(x,h)$. By duality, this defines an operator $T_h^\star$ on the set $\mathcal{M}_b$ of bounded Borel measures

$$\forall f \in \mathcal{C}_0, \forall \nu \in \mathcal{M}_b, \; T_h^\star(\nu)(f) = \nu(T_h f)$$

Stationnary distribution

Observe that if $d\nu$ has a density with respect to Lebesgue measure $d\nu = \rho(x)dx$,then

$$T_h^\star(d\nu) = \left(\int_{|x-y|<h} \frac{1}{\mu_h(B(x,h))}\rho(x)dx\right)e^{-\phi(y)/h}dy$$

As a consequence, the measure

$$d\nu_{h,\infty} = \frac{\mu_h(B_h(x))e^{-\phi(x)/h}}{Z_h}dx := m_h(x)dx$$

where Z_h is chosen so that $d\nu_{h,\infty}$ is a probability on $\mathbb{R}^d$ satisfies

$$T_h^\star(d\nu_{h,\infty}) = d\nu_{h,\infty}.$$

We say that $d\nu_{h,\infty}$ is stationnary for T_h.

Convergence to equilibrium

Question

For $d\nu \in \mathcal{M}_b$, what is the behavior of $(T_h^\star)^n(d\nu)$ when $n \to \infty$?

Under suitable assumptions on ϕ we can easily prove the following:

Theorem

For any probability measure $d\nu$, we have

$$\lim_{n \to +\infty} (T_h^\star)^n(d\nu) = d\nu_{h,\infty}$$

We are willing to compute the speed of convergence in the above limit. The answer is closely related to the spectral theory of T_h.

Some elementary properties

The operator T_h can be extended to $L^2(\mathbb{R}^d, d\nu_{h,\infty})$ by density. We denote T_h this extension. We have the following elementary properties:

Proposition

The following hold true:

- T_h is self-adjoint on $L^2(M, d\nu_{h,\infty})$.
- For all $p \in [1, \infty]$, $\| T_h \|_{L^p \to L^p} = 1$.
- 1 is an eigenvalue of T_h (Markov property) and $1 \notin \sigma_{ess}(T_h)$.

Assumptions on ϕ

We make the following assumptions on ϕ:

- there exists $c, R > 0$ and some constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^d$ such that:

$$\forall \alpha \in \mathbb{N}^d \setminus \{0\}, \forall x \in \mathbb{R}^d \, |\partial_x^\alpha \phi(x)| \leq C_\alpha$$

  and

$$\forall |x| \geq R, \, |\nabla \phi(x)| \geq c \text{ and } |\phi(x)| \geq c|x|.$$

- ϕ is a Morse function (i.e. ϕ the critical points of ϕ are non-degenerate).
- Denoting $\mathcal{U}^{(k)}$ the set of critical points of ϕ of index k, the values $\phi(U_j^{(1)}) - \phi(U_k^{(0)})$, $U_j^{(1)} \in \mathcal{U}^{(1)}$, $U_k^{(0)} \in \mathcal{U}^{(0)}$ are distincts.
  (recall that the index of a criticall point U is the number of negative eigenvalues of $Hess(\phi)(U)$).

Describtion of “small“ eigenvalues

Theorem [Bony-Hérau-Michel]

Suppose that the previous assumptions are fullfilled. Then

- There exists $\kappa_0 > 0$ such that:
  - $\sigma_{ess}(T_h) \cap [1-\kappa_0, 1] = \emptyset$
  - $\sigma(T_h) \cap [-1, -1+\kappa_0] = \emptyset$
- There are m_0 eigenvalues of T_h in the interval $[1-h^{3/2}, 1]$ and these eigenvalues enjoy the following asymptotics

$$\mu_{k,h} = 1 - h\theta_k e^{-S_k/h}(1 + \mathcal{O}(h))$$

where the coefficient θ_k, S_k are defined later.

Short heuristics

Let $f \in C_0^\infty(\mathbb{R}^d)$ be fixed, using the change of variable $y = x + hz$ and Taylor expansion, we show easily that

$$(1 - T_h)f(x) = -\frac{1}{2(d+2)}\Delta_{\phi,h}f(x) + \mathcal{O}(h^3)$$

where $-\Delta_{\phi,h} = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$ is the semiclassical Witten Lapacian.

Remark

- This expansion is not uniform with respect to f
- $-\Delta_{\phi,h}$ is known to have very small eigenvalues $\lambda \simeq e^{-\alpha/h}$ for some $\alpha > 0$
- The term $\mathcal{O}(h^3)$ is not an error term from a spectral point of view.

Description of small eigenvalues (I)

- Under the above assumptions, the spectrum of semiclassical Witten laplacian has been analyzed by many authors: Witten 85, Helffer-Sjöstrand 85, Cycon-Froese-Kirch-Simon 87, Bovier-Gayrard-Klein 04, Helffer-Klein-Nier 04.
- It is well known that $-\Delta_{\phi,h}$ has $m_0 := \sharp\mathcal{U}^{(0)}$ eigenvalues $0 = \lambda_1 \leq \ldots \leq \lambda_{m_0}$, in the interval $[0, h^{3/2}]$.
- The most accurate result in [HKN04] gives an approximation of these eigenvalues:

$$\lambda_k = (2d+4) b_k e^{-S_k/h}$$

with $b_k(h) = \sum_{j\geq 0} h^j \beta_{k,j}(x)$, $\beta_{k,1} = \theta_k$.

Description of small eigenvalues (II)

- The quantities, S_k, θ_k can be comptuted: there exists a labelling $\mathcal{U}^{(0)} = \{U_1^{(0)}, \ldots, U_{m_0}^{(0)}\}$ and $j : \{1, \ldots, m_0\} \to \{1, \ldots, m_1\}$ such that

$$S_k = 2(\phi(U_{j(k)}^{(1)}) - \phi(U_k^{(0)}))$$

and

$$\theta_k = \frac{|\hat{\lambda}_1(U_{j(k)}^{(1)})|}{\pi} \sqrt{\frac{\det(\text{Hess}\,\phi(U_k^{(0)}))}{\det(\text{Hess}\,\phi(U_{j(k)}^{(1)}))}}$$

where $\hat{\lambda}_1(U_{j(k)}^{(1)})$ is the negative eigenvalue of $\text{Hess}\,\phi(U_{j(k)}^{(1)})$.

- If $m_0 = 2$ then the above labelling and the function j are such that

$$S_2 = \min_{U^{(0)} \in \mathcal{U}^{(0)}, U^{(1)} \in \mathcal{U}^{(1)}} \phi(U^{(1)}) - \phi(U^{(0)}).$$

Interraction matrix

The strategy of Helffer-Klein-Nier is the following:

- Introduce
  - $F^{(0)}$ = eigenspace associated to the m_0 low lying eigenvalues on 0-forms
  - $\Pi^{(0)}$ = projector on $F^{(0)}$.
  - M = restriction of $\Delta_{\phi,h}$ to $F^{(0)}$.

  We have to compute the eigenvalues of M.
- We compute suitable BKW approximations $\psi_k^{(0)}$ of e_k, show that

$$\Pi^{(0)}\psi_k^{(0)} = \psi_k^{(0)} + error$$

  and compute the matrix of M in the base $\Pi^{(0)}\psi_k^{(0)}$.
- Doing that leads to error terms which are too big.
- In order to overcome this difficulty, use the super symmetric structure.

Using Supersymmetry (I)

- For $p = 0, \ldots, d-1$, denote $d^{(p)} : \Lambda^p \mathbb{R}^d \to \Lambda^{p+1}\mathbb{R}^d$ the exterior derivative and $d^{(p),*} : \Lambda^{p+1}\mathbb{R}^d \to \Lambda^p\mathbb{R}^d$ its formal adjoint. Then the Hodge Laplacian on p-form is defined by

$$\Delta^{(p)} = d^{(p),*} d^{(p)} + d^{(p-1)} d^{(p-1),*}.$$

- The semiclassical Witten Laplacian (Witten, 1985) on p-form is defined by introducing the twisted exterior derivatives $d^{(p)}_{\phi,h} = e^{-\phi/h}(hd^{(p)})e^{\phi/h}$ and $d^{(p),*}_{\phi,h}$ its adjoint and by setting

$$\Delta^{(p)}_{\phi,h} = d^{(p),*}_{\phi,h} d^{(p)}_{\phi,h} + d^{(p-1)}_{\phi,h} d^{(p-1),*}_{\phi,h}$$

- In particular, for $p = 0$, the Witten Laplacian on function is given by

$$\Delta_{\phi,h} = \Delta^{(0)}_{\phi,h} = d^{(0),*}_{\phi,h} d^{(0)}_{\phi,h} = h^2\Delta - |\nabla\phi|^2 + h\Delta\phi.$$

Using Supersymmetry (II)

The fondamental remarks are the following:

- $\Delta_{\phi,h}^{(p+1)} d_{\phi,h}^{(p)} = d_{\phi,h}^{(p)} \Delta_{\phi,h}^{(p)}$ and $d_{\phi,h}^{(p),*} \Delta_{\phi,h}^{(p+1)} = \Delta_{\phi,h}^{(p)} d_{\phi,h}^{(p),*}$
- Denote $F^{(1)}$ the eigenspace associated to low lying eigenvalues on 1 forms, then $d_{\phi,h}^{(0)}(F^{(0)}) \subset F^{(1)}$ and $d_{\phi,h}^{(0),*}(F^{(1)}) \subset F^{(0)}$. Hence

$$M = L^* L$$

where L is the matrix of $d_{\phi,h}^{(0)} : F^{(0)} \rightarrow F^{(1)}$.

- The matrix L is well approximated by

$$L \simeq (\langle d_{\phi,h}^{(0)} \psi_j^{(0)}, \psi_k^{(1)} \rangle)_{j=1,\dots,m_0, k=1,\dots,m_1}$$

where $\psi_k^{(1)}$ are BKW approximations of eigenfunctions on 1-form.

- We can conclude by computing the singular values of L.

First Reduction (I)

The operator T_h is self-adjoint on $L^2(\mathbb{R}^d, d\nu_{h,\infty})$.

- Using a unitary transformation, we are reduce to analyze the operator $\tilde{T}_h$ on $L^2(\mathbb{R}^d)$ which is given by

$$\tilde{T}_h f(x) = a_h(x)\frac{1}{\alpha_d h^d}\int_{|x-y|<h} a_h(y) f(y) dy$$

  where $a_h(x)^{-2} = \frac{1}{\alpha_d h^d}\int_{|x-y|<h} e^{(\phi(x)-\phi(y))/h} dy$.

- Observe that the operator $f \mapsto \frac{1}{\alpha_d h^d}\int_{|x-y|<h} f(y)dy$ is a fourier multiplier $G(hD_x)$ with

$$G(\xi) = \frac{1}{\alpha_d}\int_{|x|<1} e^{ix\cdot\xi} dx$$

  Here we use the notation $D_x = \frac{1}{i}\nabla_x$.

First Reduction (II)

- From the preceding observations we deduce:

$$\tilde{T}_h = a_h G(hD_x) a_h \quad \text{and} \quad a_h^{-2} = e^{\phi/h} G(hD_x)(e^{-\phi/h})$$

- Since we study the spectrum of $\tilde{T}_h$ near 1, we introduce

$$\widetilde{P}_h := 1 - \tilde{T}_h = a_h (V_h(x) - G(hD_x)) a_h$$

where $V_h(x) = a_h^{-2}(x) = e^{\phi/h} G(hD_x)(e^{-\phi/h})$.

- The important operator in the sequel is

$$P_h = V_h(x) - G(hD_x) = e^{\phi/h} G(hD_x)(e^{-\phi/h}) - G(hD_x)$$

- The Witten Laplacian on functions has the same form:

$$-\Delta_{\phi,h} = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi = -h^2\Delta + e^{\phi/h} h^2 \Delta (e^{-\phi/h})$$

We have seen the factorization $-\Delta_{\phi,h} = d^*_{\phi,h} d_{\phi,h}$.

Question

Can we generalize such factorization to pseudodifferential operator P_h s.t. $P_h(e^{-\phi/h}) = 0$? In particular to $P_h = G(hD) - V_h(x)$?

More precise question

Let P_h be a self-adjoint pseudodifferential operator with symbol p, such that $P_h(e^{-\phi/h}) = 0$. What assumption do we need on p so that there exists a pseudodifferential operator Q s.t.

$$P_h = (d_{\phi,h} Q)^* Q d_{\phi,h}.$$

Recall on pseudodifferential operators

- Let $\tau > 0$, we say that a symbol $p \in C^\infty(\mathbb{R}^{2d}, \mathbb{C})$ belongs to the class $\mathcal{S}^0_\tau(1)$ if
  - for all $x \in \mathbb{R}^d$, $\xi \mapsto p(x, \xi)$ is analtytic with respect to $\xi \in B_\tau = \{\xi \in \mathbb{C}^d, \ |\operatorname{Im} \xi| < \tau\}$
  - $\forall (x, \xi) \in \mathbb{R}^d \times B_\tau, \ |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha,\beta}$.
- We say that $p \in \mathcal{S}^0_\infty(1)$ if $p \in \mathcal{S}^0_\tau(1)$ for all $\tau > 0$.
- For $p \in \mathcal{S}^0_\tau(1)$, $\tau \in [0, \infty]$ we define the Weyl-quantization of p:

$$\operatorname{Op}_h^w(p)u(x) = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi/h} p(\frac{x+y}{2}, \xi) u(y) dy d\xi$$

for any $u \in \mathcal{S}(\mathbb{R}^d)$.

Let ϕ be as before. Let $p \in \mathcal{S}^0_\infty(1)$ and $P_h = \mathrm{Op}_h^w(p)$. Assume that the following assumptions hold true:

- p is real-valued (and hence P_h is self-adjoint).
- $P_h(e^{-\phi/h}) = 0$
- For all $x \in \mathbb{R}^d$, the function $\xi \in \mathbb{R}^d \mapsto p(x,\xi)$ is even.
- Near any critical points $U \in \mathcal{U}$ we have

$$p(x,\xi) = |\xi|^2 + |\nabla\phi(x)|^2 + \mathcal{O}(h + |(x-U,\xi)|^4).$$

- $\forall \delta > 0, \exists \alpha > 0, \forall (x,\xi) \in T^*\mathbb{R}^d, (d(x,\mathcal{U})^2 + |\xi|^2 \geq \delta \Longrightarrow p(x,\xi) \geq \alpha)$

Remark

The operator $P_h = G(hD) - V_h(x)$ satisfies the above assumptions since G is the fourier transform of $\mathbb{1}_{|z|<1}$.

Let us introduce the operator $\mathrm{D}_{\phi,\mathrm{h}} = -i(h\nabla_x + \nabla\phi(x))$ and $\mathcal{A} : T^*\mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ given by $\mathcal{A}_{i,j}(x,\xi) = (\langle\xi_i\rangle\langle\xi_j\rangle)^{-1}$.

Theorem (Bony-Hérau-Michel)

Under the above assumptions, there exists $\tau > 0$ and a real valued symbol $q \in \mathcal{S}^0_\tau(T^*\mathbb{R}^d, \mathcal{A})$ such that

$$P_h = \mathrm{D}^*_{\phi,\mathrm{h}}\, Q^* Q\, \mathrm{D}_{\phi,\mathrm{h}}$$

with $Q = \mathrm{Op}^w_h(q)$. Moreover, the principal symbol q^0 of Q satisfies $q^0(x,\xi) = Id + \mathcal{O}((x-U,\xi)^2)$ near $(U,0)$ for any critical point $U \in \mathcal{U}$.

The proof goes in several steps:

- **Step 1:** Show that there exists $\widehat{Q}_\phi$ s.t. $P_h = \mathrm{D}^*_{\phi,\mathrm{h}}\,\widehat{Q}_\phi\,\mathrm{D}_{\phi,\mathrm{h}}$
- **Step 2:** Show that we can modify $\widehat{Q}_\phi$ in order that it has a pseudodifferential squareroot $\widehat{Q}_\phi = \check{Q}^*\check{Q}$
- **Step 3:** Arrange things so that $\check{Q}$ has analytic symbol in a small strip

- **Step 1: Show that there exists $\widehat{Q}_\phi$ s.t. $P_h = \mathrm{D}^*_{\phi,h}\,\widehat{Q}_\phi\,\mathrm{D}_{\phi,h}$.** Let $P_{\phi,h} = e^{\phi/h} P_h e^{-\phi/h}$. Since P_h has a symbol which is analytic w.r.t. ξ, $P_{\phi,h}$ is a pseudo. Moreover, $P_{\phi,h}(1) = 0$. Hence, we can factorize

$$P_{\phi,h} = \tilde{Q}_\phi h D_x.$$

Going back to P_h, we get $P_h = \overline{Q}_\phi\,\mathrm{D}_{\phi,h}$. Moreover, we have an exact expression for $\overline{Q}_\phi$.

It remains to factorize $\overline{Q}_\phi$ by $\mathrm{D}^*_{\phi,\mathrm{h}}$ on the left.

- This is equivalent to show that

$$\check{Q}_\phi := e^{-\phi/h}\overline{Q}_\phi e^{\phi/h} = e^{-2\phi/h}\tilde{Q}_\phi e^{2\phi/h}$$

  can be factorized by *div* on the left.
- We introduce the symbol $\check{q}_\phi$ of the left-quantization of $\check{Q}_\phi$. Since $\xi \mapsto p(x,\xi)$ is an even function for all $x \in \mathbb{R}^d$, exact computations shows that $\check{q}_\phi(y,0) = 0$ for all y.
- Going back to the original operator by conjugation by $e^{\phi/h}$, we get the first step.

- **Second Step: Show that you can choose $\widehat{Q}_\phi$ non negative and construct its squareroot**

  To simplify, assume we work on $\mathbb{R}^2$. Then

$$P_h = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} & \widehat{Q}_{12} \\ \widehat{Q}_{12}^* & \widehat{Q}_{22} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

  The key point is that $[D_{1,\phi}, D_{2,\phi}] = 0$ so that for any bounded operators A, B, we can rewrite P as

$$P_h = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} + BD_{2,\phi} + D_{2,\phi}^* B^* & \widehat{Q}_{12} - BD_{1,\phi} \\ \widehat{Q}_{12}^* - D_{1,\phi}^* B^* & \widehat{Q}_{22} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

  or

$$P_h = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \begin{pmatrix} \widehat{Q}_{11} + D_{2,\phi}^* A D_{2,\phi} & \widehat{Q}_{12} \\ \widehat{Q}_{12}^* & \widehat{Q}_{22} - D_{1,\phi}^* A D_{1,\phi} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

To simplify, asume that ϕ has only one critical point in $x = 0$.
Denote $p(x, \xi) = p(x_1, x_2, \xi_1, \xi_2)$ the symbol of P.
Given $\delta > 0$, we have to deal with 3 microlocal regions:
$\Omega_0 = \{|\xi|^2 + |x|^2 \leq 2\delta\}$, $\Omega_1 = \{|\xi_1|^2 + |x_1|^2 \geq \delta\}$,
$\Omega_2 = \{|\xi_2|^2 + |x_2|^2 \geq \delta\}$.

- On Ω_0, since

$$p(x, \xi) = |\xi|^2 + |\nabla\phi(x)|^2 + \mathcal{O}(|(x, \xi)|^3),$$

  it is easy to prove that $\widehat{Q}_{ij} = \delta_{ij} + \mathcal{O}(h + \epsilon)$.
- Ω_1 and Ω_2 are treated in a similar way, using the preceding remark. Let us study Ω_1.
  The idea is to chose A and B in order to kill the antidiagonal terms and get a positive lower bound for diagonal terms.

- Killing $\widehat{Q}_{12}$ is done by choosing $B = \widehat{Q}_{12}/D_{1,\phi}$. This is possible since on Ω_1, $D_{1,\phi}^* D_{1,\phi} \geq \epsilon > 0$.
- Assume now that $\widehat{Q}_{12} \simeq 0$. We want to insure that $\widehat{Q}_{11}$ and $\widehat{Q}_{22}$ are positive. The fondamental point is that there exists $\alpha > 0$ such that

$$\forall (x,\xi) \in \Omega_1,\ p(x,\xi) \geq 2\alpha.$$

On the other hand,

$$p(x,\xi) = (|\xi_1|^2 + |\partial_1\phi|^2)\widehat{q}_{11}(x,\xi) + (|\xi_2|^2 + |\partial_2\phi|^2)\widehat{q}_{22}(x,\xi) + \mathcal{O}(h)$$

As a consequence

$$\widehat{q}_{11}(x,\xi) + (|\xi_2|^2 + |\partial_2\phi|^2)\frac{\widehat{q}_{22}(x,\xi) - \frac{\alpha}{(1+|\xi_2|^2+|\partial_2\phi|^2)}}{(|\xi_1|^2 + |\partial_1\phi|^2)} \geq \frac{\alpha}{|\xi_1|^2 + |\partial_1\phi|^2}$$

and we can take $A = Op_h\left(\frac{\widehat{q}_{22}(x,\xi) - \frac{\alpha}{1+(|\xi_2|^2+|\partial_2\phi|^2)}}{(|\xi_1|^2+|\partial_1\phi|^2)}\right)$.

- Doing that we get a new factorisation on Ω_1:

$$P = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \mathrm{Op}_h^w \begin{pmatrix} q_{11} & o(a(\xi)) \\ o(a(\xi)) & \frac{\alpha}{1+(|\xi_2|^2+|\partial_2\phi|^2)} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

with $q_{11} \geq \frac{\alpha}{|\xi_1|^2+|\partial_1\phi|^2}$ on Ω_1 and $a(\xi) = \langle \xi_1 \rangle^{-1} \langle \xi_2 \rangle^{-1}$.

- Gluing all microlocal region we get a final prefactorisation:

$$P = \begin{pmatrix} D_{1,\phi}^* \\ D_{2,\phi}^* \end{pmatrix} \cdot \mathrm{Op}_h^w \begin{pmatrix} q_{11} & o(a(\xi)) \\ o(a(\xi)) & q_{22} \end{pmatrix} \begin{pmatrix} D_{1,\phi} \\ D_{2,\phi} \end{pmatrix}$$

with $q_{11}, q_{22} \geq \frac{\alpha}{|\xi|^2+|\nabla\phi|^2}$

- Finally, operators such that $\mathrm{Op}_h^w \begin{pmatrix} q_{11} & o(a(\xi)) \\ o(a(\xi)) & q_{22} \end{pmatrix}$ can be written as square of pseudo by standard arguments.

Back to random walk

- The factorization theorem applies to $P_h = G(hD_x) - V_h(x)$. This shows that

$$P_h^{(0)} := 1 - \tilde{T}_h = L_\phi^* L_\phi$$

with $L_\phi = Q_\phi D_\phi a_h$ and $Q_\phi = Op_h^w(q_\phi)$

- We define an operator on 1-form:

$$P_h^{(1)} = L_\phi L_\phi^* + (Q_\phi^*)^{-1} D_\phi^* \Omega D_\phi Q_\phi^{-1}$$

where Ω is an operator acting on 1-form such that $P_h^{(1)}$ is elliptic.

- Observe that $P_h^{(1)} L_\phi = L_\phi P_h^{(0)}$
- Using this structure we can follow the strategy of proof of [Helffer-Klein-Nier] to get the announced result.