Semi-classical limit of the scattering amplitude for trapping perturbations

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Abstract

We study the semi-classical asymptotic as $h \to 0$ of the scattering amplitude $f(\theta, \omega, \lambda, h)$ associated to a Schrödinger operator $H(h) = -\frac{1}{2}h^2\Delta + V(x)$ for short-range trapping perturbations. Our main result yields the leading term of the integral of $f(\theta, \omega, \lambda, h)$ for $\theta, \omega \in S^{n-1}$ fixed and $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, $\epsilon > 0$, $\lambda_0 > 0$ being a trapping energy level.

1 Introduction

Consider the Schrödinger operator $H(h) = -\frac{1}{2}h^2\Delta + V$, in $\mathbb{R}^n$, $n \geq 2$, $0 < h \leq 1$, and assume that the potential $V(x)$ satisfies the following condition.

Assumption $(\mathbf{V})_\rho$: V is a real C^∞-smooth function such that

$$\forall \alpha \in \mathbb{N}^n,\ \forall x \in \mathbb{R}^n,\ |\partial_x^\alpha V(x)| \leq C_\alpha < x >^{-\rho-|\alpha|},\ \rho > 1,$$

where $< x >= (1 + |x|^2)^{\frac{1}{2}}$.
The operator $H(h)$ with domain $D(H(h)) = H^2(\mathbb{R}^n)$ is self-adjoint in $L^2(\mathbb{R}^n)$. Moreover, we can define the scattering matrix $S(\lambda, h)$ related to $H_0(h) = -\frac{1}{2}h^2\Delta$ and $H(h)$, as an unitary operator:

$$S(\lambda, h) \ : \ L^2(S^{n-1}) \longrightarrow L^2(S^{n-1}),\ \lambda \in \mathbb{R}.$$

Next introduce the operator T by $S(\lambda, h) = Id - 2i\pi T(\lambda, h)$. It is known (see [9]) that $T(\lambda, h)$ has a kernel $T(\theta, \omega, \lambda, h)$, smooth in $(\theta, \omega) \in S^{n-1} \times S^{n-1} \setminus \{\theta = \omega\}$ and the scattering amplitude is given by

$$f(\theta, \omega, \lambda, h) = c(\lambda, h)T(\theta, \omega, \lambda, h),\ \lambda > 0,$$

with

$$c(\lambda, h) = -2\pi(2\lambda)^{-\frac{n-1}{4}}(2\pi h)^{\frac{n-1}{2}}e^{-i\frac{(n-3)\pi}{4}}.$$

D.Robert and H.Tamura [12] have studied the asymptotic behavior of f as $h \to 0$ for fixed $\theta, \omega \in S^{n-1}$, $\theta \neq \omega$, in the case where the energy λ is fixed in a non-trapping interval. More precisely, denote by $(q(t, y, \eta), p(t, y, \eta))$ the solution to the system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -\nabla_x V(q), \end{cases} \tag{1.1}$$

with initial data (y,η) at $t=0$, and let us recall the non-trapping condition.

(NT): We say that the energy λ is non-trapping, for the symbol $\frac{1}{2}|\xi|^2+V(x)$ if for any $R>0$ large enough, there exists $T=T(R)$ such that $|q(t,y,\eta)|>R$ for $|t|>T$ when $|y|<R$ and $\lambda=\frac{1}{2}|\eta|^2+V(y)$.

For ω and θ fixed in S^{n-1} and for λ satisfying the non-trapping condition, Robert and Tamura obtained for f an asymptotic

$$f(\theta,\omega,\lambda,h)=\sum_{j=1}^{l}\hat{\sigma}(z^j(\lambda),\lambda)^{-\frac{1}{2}}e^{ih^{-1}S_j-i\mu_j\frac{\pi}{2}}+O(h),$$

where $\hat{\sigma}$, z_j, S_j and μ_j will be define below. On the other hand, Yajima [18] obtained a such asymptotic without the non-trapping condition, but in an average norm. More precisely, under some hypotheses in [18] (in particular, he assumed $(V)_\rho$ with $\rho>\frac{n+1}{2}$), he established for ω and θ_0 fixed the estimate

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\int_U|f(\theta,\omega,\lambda,h)-\sum_{j=1}^{l}\hat{\sigma}(z^j(\lambda),\lambda)^{-\frac{1}{2}}e^{ih^{-1}S_j-i\mu_j\frac{\pi}{2}}|^2d\theta d\lambda=O(h),$$

where $\epsilon>0$ is small enough and U is a small open neighborhood of θ_0 in S^{n-1}.

Our goal is to generalize the results of Robert and Tamura for trapping perturbations and to obtain an asymptotic

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|f(\theta,\omega,\lambda,h)-\sum_{j=1}^{l}\hat{\sigma}(z^j(\lambda),\lambda)^{-\frac{1}{2}}e^{ih^{-1}S_j-i\mu_j\frac{\pi}{2}}|d\lambda=O(h),$$

where *we average only with respect to* λ provided ω and θ are fixed.

We recall some essential consequences of the non-trapping condition. First of all one can show that $\{\lambda>0\ :\ \lambda \text{ satisfies } \textbf{(NT)}\ \}$ is an open set, and that if λ is a non-trapping energy level for $\frac{1}{2}|\xi|^2+V(x)$, then every particle with energy λ will go to infinity as $t\to\pm\infty$. Let $R(z)=(H(h)-z)^{-1}$, $z\in\mathbb{C}\backslash\mathbb{R}$ be the resolvent of $H(h)$ and $R(\lambda\pm i0)=\lim_{\epsilon\to 0,\epsilon>0}R(\lambda\pm i\epsilon)$. Here we take the limit in the spaces of bounded operators $\mathcal{L}(L^2_\alpha,L^2_{-\alpha}),\alpha>\frac{1}{2}$, where $L^2_\alpha=\{f\ :<x>^\alpha f\in L^2(\mathbb{R}^n)\}$. One can establish (see [13]) the following estimate which is crucial in the approach of Robert and Tamura [12]

$$\|R(\lambda\pm i0)\|_{L^2_\alpha,L^2_{-\alpha}}=O(h^{-1}).$$

We begin by reviewing some results of classical mechanics, and for more details we refer to the book [11]. Let $(p(t),q(t))$ be a solution to (1.1) and assume that $|q(t)|\to+\infty$ as $|t|\to\pm\infty$. Then there exists $(r_\pm,v_\pm)\in\mathbb{R}^n\times\mathbb{R}^n\setminus 0$ such that

$$\lim_{t\to\pm\infty}|q(t)-v_\pm t-r_\pm|+|p(t)-v_\pm|=0.$$

From now up to the end of this paper, we will consider a fixed $\omega\in S^{n-1}$ and we denote by Λ_ω the plane orthogonal to ω and passing through 0. As ω is fixed, we can assume that $\omega=(0,\dots,0,1)$ and we can write the coordinates in Λ_ω as $z=(z_1,\dots,z_{n-1})$. We will also use the notation $\hat{z}=(z,0)$ for $z\in\Lambda_\omega$. Then there exists an unique solution $(q_\infty(t,z,\lambda,\omega),p_\infty(t,z,\lambda,\omega))$, of (1.1) such that

$$\begin{cases}\lim_{t\to-\infty}|p_\infty(t,z,\lambda,\omega)-\sqrt{2\lambda}\omega|=0,\\ \lim_{t\to-\infty}|q_\infty(t,z,\lambda,\omega)-\sqrt{2\lambda}\omega t-z|=0,\end{cases}\tag{1.2}$$

which depends smoothly on the parameter z.
Given $\lambda_0 > 0$, we will replace the condition **(NT)** by the following weaker one.

Assumption $(\mathbf{H}_{\omega,\lambda_0})$: There exist $\epsilon > 0$ and a neighborhood W of ω in S^{n-1} such that

$$\forall \lambda \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[,\ \forall \omega' \in W, \forall z \in \Lambda_{\omega'} \lim_{t \to +\infty} |q_\infty(t, z, \lambda, \omega')| = +\infty.$$

Let (q_∞, p_∞) be as above, take $\lambda_0 > 0$ and assume (H_{ω,λ_0}). Then if λ is close to λ_0, we have $\lim_{t\to+\infty} |q_\infty(t, z, \lambda, \omega)| = +\infty$. Thus there exist $\xi_\infty(z, \lambda) \in S^{n-1}$ and $r_\infty(z, \lambda) \in \mathbb{R}^n$ such that

$$\begin{cases} \lim_{t\to+\infty} |p_\infty(t, z, \lambda, \omega) - \sqrt{2\lambda}\xi_\infty(z, \lambda)| = 0, \\ \lim_{t\to+\infty} |q_\infty(t, z, \lambda, \omega) - \sqrt{2\lambda}\xi_\infty(z, \lambda)t - r_\infty(z, \lambda)| = 0. \end{cases} \tag{1.3}$$

Moreover, one can show that $z \to \xi_\infty(z, \lambda)$ is C^∞ (see [11]), and we may define

$$\hat{\sigma}(z, \lambda) = |det(\xi_\infty, \partial_{z_1}\xi_\infty, \dots, \partial_{z_n}\xi_\infty)|.$$

Definition 1.1 *We will say that $\theta \in S^{n-1}$ is regular for (λ, ω), if $\theta \neq \omega$ and $\forall z \in \Lambda_\omega$, $\xi_\infty(z, \lambda) = \theta \Longrightarrow \hat{\sigma}(z, \lambda) \neq 0$.*

Remark 1.1 *If θ is regular for (λ, ω), then there exists a finite set $\{z^1, \dots, z^l\}$ included in Λ_ω such that $\xi_\infty(z, \lambda) = \theta \Longrightarrow z \in \{z^1, \dots, z^l\}$.*

Indeed, by the implicit function theorem, the set $\Xi_{\lambda,\theta} = \{z \in \Lambda_\omega\ :\ \xi_\infty(z, \lambda) = \theta\}$ is a discrete one. Moreover, $\Xi_{\lambda,\theta}$ is bounded, since if this is not true there exists $C > 0$ such that (see [3], p 67)

$$\forall z \in \Xi_{\lambda,\theta},\ \forall t \in \mathbb{R},\ q_\infty(t, z, \lambda) \geq C(t + |z|)$$

and we deduce from this estimate, the inequality $\lim_{t\to+\infty} |p_\infty(t, z, \lambda) - \omega| \leq C < z >^{-\rho}$. As $\theta \neq \omega$, then for $|z|$ large enough, $\xi_\infty(z, \lambda) \neq \theta$.

We will show in Appendix (Proposition A.1) that for λ close enough to λ_0, the number l does not depend of λ and there exist smooth functions $z_1(\lambda), \dots, z_l(\lambda)$ such that $\xi_\infty(z, \lambda) = \theta \Longrightarrow z \in \{z^1(\lambda), \dots, z^l(\lambda)\}$.

To introduce the resonances by complex scaling, as in [14], [15] and [2] , we need an hypothesis of analyticity of V at infinity. More precisely we assume there exist $\theta_0 \in [0, \pi[$, $\epsilon > 0$ and $R > 0$ such that the potential V extends holomorphically to the domain

$$D_{\epsilon,R,\theta_0} = \{r\omega\ :\ \omega \in \mathbb{C}^n, dist(\omega, S^{n-1}) < \epsilon, r \in \mathbb{C},\ |r| > R,\ \arg r \in [-\epsilon, \theta_0 + \epsilon[\}$$

and

$$\exists \beta > 0, \exists C > 0,\ \forall x \in D_{\epsilon,R,\theta_o},\ |V(x)| \leq C|x|^{-\beta}.$$

Following [14], we define the resonances in the upper half-plane by complex scaling. Recall that the resonances coincide with the poles of the meromorphic continuation of the resolvent $(H(h) - z)^{-1}$: $L^2_{comp} \to D_{loc}(H(h))$ from the lower half-plane to a conic neighborhood of the positive half axis in the upper half-plane.

Denote by $Res(H(h))$ the set of resonances of $H(h)$, by $d(\lambda, h) = \min(h, dist(\lambda, Res(H(h))))$ and set $d(h) = \inf_{[\lambda_0-\epsilon,\lambda_0+\epsilon]} d(., h)$, where $\epsilon > 0$ will be fixed later. Under the previous assumptions, recently Burq showed (cf. [2]) that the resonances $\lambda \in Res(H(h))$ satisfy the estimate

$$|Im(\lambda)| \geq Cexp(-Ch^{-1}),\ C > 0$$

for $Re\lambda \in [\lambda_0-\epsilon, \lambda_0+\epsilon]$. One of the consequences of this result is that there exists $C>0$ such that $d(h) \geq Ce^{-Ch^{-1}}$. Indeed, we have $d(h) = min(h, \inf_{\xi \in Res(H(h))} dist(\xi, [\lambda_0-\epsilon, \lambda_0+\epsilon]))$, and, obviously, $h \geq Ce^{-Ch^{-1}}$ for $h>0$ small enough. Let $\xi \in Res(H(h))$, then $dist(\xi, [\lambda_0-\epsilon, \lambda_0+\epsilon]) \geq |Im\xi|$ and we know from [2] that $|Im(\xi)| \geq Ce^{-Ch^{-1}}$. Thus $\inf_{\xi \in Res(H(h))} dist(\xi, [\lambda_0-\epsilon, \lambda_0+\epsilon]) \geq Ce^{-Ch^{-1}}$.

Now we can state our main result.

Theorem 1.1 *Let $\omega \in S^{n-1}$, and $\lambda_0 > 0$. We assume the following conditions:*

1) $(V)_\rho$ with $\rho > 1$.

2) There exist $\theta_0 \in [0, \pi[$, $\epsilon > 0$ and $R > 0$ such that the potential V extends holomorphically to the domain D_{ϵ,R,θ_0} and $\exists \beta > 0, \exists C > 0,\ \forall x \in D_{\epsilon,R,\theta_o},\ |V(x)| \leq C|x|^{-\beta}$.

3) $(H)_{\omega,\lambda_0}$.

4) θ is regular for (λ_0, ω).

Then there exists $\epsilon > 0$ such that

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} |f(\theta,\omega,\lambda,h) - \sum_{j=1}^{l} \hat{\sigma}(z^j(\lambda),\lambda)^{-\frac{1}{2}} e^{ih^{-1}S_j(\lambda) - i\mu_j\frac{\pi}{2}}| d\lambda = O(h),$$

where

$$S_j(\lambda) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}|p_\infty(t,z^j(\lambda),\lambda,\omega)|^2 - V(q_\infty(t,z^j(\lambda),\lambda,\omega)) - \lambda\right) dt - \langle r_\infty(z^j(\lambda,\lambda), \sqrt{2\lambda}\theta\rangle \quad (1.4)$$

and $\mu_j \in \mathbb{Z}$ is the Maslov index of the trajectory $\big(q_\infty(t,z^j(\lambda),\lambda,\omega), p_\infty(t,z^j(\lambda),\lambda,\omega)\big)$ on the Lagrangian manifold

$$\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n\ :\ x = q_\infty(t,z,\lambda,\omega),\ \xi = p_\infty(t,z,\lambda,\omega),\ z \in \Lambda_\omega,\ t \in \mathbb{R}\}.$$

Remark 1.2 *If V has compact support, 1) and 2) are trivially satisfied, and 3) becomes*

$$\exists \epsilon > 0, \forall \lambda \in]\lambda_0-\epsilon, \lambda_0+\epsilon[\ , \forall z \in \Lambda_\omega \lim_{t\to+\infty} |q_\infty(t,z,\lambda,\omega)| = +\infty.$$

Example. There exist potentials satisfying our assumptions. For example for $n=2$, set $V_0(x_1,x_2) = 1 + x_1^2 - x_2^2$ and take $\rho \in C_0^\infty(\mathbb{R}^+)$ decreasing such that $\rho(t) = 1$ for $t \in [0,1]$ and $\rho(t) = 0$ for $t \geq 2$. Consider $V(x_1,x_2) = V_0(x_1,x_2)\rho(x_1^2+x_2^2)$. For $\lambda_0 > 1$ there exist trapped trajectories for the potential V. Indeed, for $0 < x_2 < 1$, the trajectory $q(t,\lambda_0,x_2)$ such that $q(0,\lambda_0,x_2) = (0,x_2)$ and $p(0,\lambda_0,x_2) = \dot{q}(0,\lambda_0,x_2) = (\sqrt{2(\lambda_0-1)}, -\sqrt{2}x_2)$ has the properties:

$$\forall t > 0,\ q(t,\lambda_0,x_2) = (\sqrt{(\lambda_0-1)}\sin\sqrt{2}t, x_2e^{-\sqrt{2}t}),$$
$$\lim_{t\to-\infty} |p(t,\lambda_0,x_2)| = +\infty.$$

Moreover, it is easy to show that V satisfies the hypotheses of the previous theorem with $\lambda_0 > 1$ close to 1 and $\omega = (1,0)$. In this case , using the work of Gérard and Sjöstrand [5], one can also show that there are no resonance in a box containing λ_0 and having size h^M.

2 Representation of the scattering matrix.

As in [12], we will use the Isozaki-Kitada formula for the representation of the scattering matrix. We begin with some notations. Given an open subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$, denote by $A^m_\infty(\Omega)$ the class of symbols $a \in C^\infty(\Omega)$ such that

$$\forall L > 1, \forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n,\ \exists C_{\alpha,\beta,L} > 0,\ \forall (x,\xi) \in \Omega,\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta,L} < x >^{m-|\alpha|} < \xi >^{-L} .$$

In the case when $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, we will write A^m_∞ instead of $A^m_\infty(\Omega)$. We use also the incoming and outgoing subsets of the phase space having the form:

$$\Gamma_\pm(R,d,\sigma) = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : |x| > R, d^{-1} < |\xi| < d, \pm cos(x,\xi) > \pm\sigma\}$$

for $R > 1,\ d > 1$ and $\sigma \in]-1,1[$, where $cos(x,\xi) = \frac{\langle x,\xi\rangle}{|x||\xi|}$.

The idea of Isozaki and Kitada was to approximate $e^{-ih^{-1}tH(h)}W_\pm(h),\ \pm t > 0$, by $I_h(c_\pm,\Phi_\pm)e^{-ih^{-1}tH_0(h)}$, where $I_h(c_\pm,\Phi_\pm)$ are Fourier Integral Operators with symbols $c_\pm$ and phases $\Phi_\pm$. Formally,

$$I_h(c_\pm,\Phi_\pm)(f)(x) = (2\pi h)^{-n} \int\int exp(ih^{-1}(\Phi_\pm(x,\xi) - \langle y,\xi\rangle))c_\pm(x,\xi)f(y)dyd\xi.$$

We may construct $c_\pm(x,\xi,h) \sim \sum_{j=0}^{\infty} c_{\pm j}(x,\xi)h^j$ and $\Phi_\pm(x,\xi)$ in such way that

$$\left(-\frac{1}{2}h^2\Delta + V(x) - \frac{1}{2}|\xi|^2\right)(e^{ih^{-1}\Phi_\pm}c_\pm) \sim 0.$$

Let $R_0 >> 1$, $1 < d_4 < d_3 < d_2 < d_1 < d_0$, and $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < \sigma_0 < 1$. Using the WKB method we find phases $\Phi_\pm$ and symbols $a_\pm,\ b_\pm,\ k_{\pm a}$ and $k_{\pm b}$ such that:

1) $\Phi_\pm(x,\xi)$ is a solution of the eikonal equation $\frac{1}{2}|\nabla_x\Phi_\pm(x,\xi)|^2 + V(x) = \frac{1}{2}|\xi|^2$, in $\Gamma_\pm(R_0,d_0,\pm\sigma_0)$.
2) $\Phi_\pm(x,\xi) - \langle x,\xi\rangle$ belongs to A^0_∞.
3) For all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $|\frac{\partial^2\Phi_\pm}{\partial_{x_j}\partial_{\xi_k}}(x,\xi) - \delta_{jk}| < \epsilon(R_0)$, δ_{jk} being the Kronecker symbols, where $\epsilon(R_0)$ can be made as small as we wish by taking R_0 large enough.
4) $a_\pm$ and $b_\pm$ belong to A^0_∞.
5) $supp(a_\pm) \subset \Gamma_\pm(3R_0,d_1,\mp\sigma_1)$ and $supp(b_\pm) \subset \Gamma_\pm(5R_0,d_3,\pm\sigma_4)$.
6) $k_{\pm a}$ and $k_{\pm b}$ belong to A^{-1}_∞.
7) $supp(k_{\pm a})) \subset \Gamma_\pm(3R_0,d_1,\mp\sigma_1)$ and $supp(k_{\pm b})) \subset \Gamma_\pm(5R_0,d_3,\mp\sigma_4)$.
8) $h^{-(N+2)}k_{\pm a} \in A^{-(N+2)}_\infty(\Gamma_\pm(4R_0,d_1,\mp\sigma_2))$ and $h^{-(N+2)}k_{\pm b} \in A^{-(N+2)}_\infty(\Gamma_\pm(6R_0,d_3,\pm\sigma_3))$, N being an integer as large as we want.

Moreover, by construction we have

$$k_{\pm a}(x,\xi,h) = e^{-ih^{-1}\Phi_\pm(x,\xi)}\left(H(h) - \frac{1}{2}|\xi|^2\right)\left(e^{ih^{-1}\Phi_\pm}a_\pm\right)(x,\xi), \qquad (2.1)$$

and

$$k_{\pm b}(x,\xi,h) = e^{-ih^{-1}\Phi_\pm(x,\xi)}\left(H(h) - \frac{1}{2}|\xi|^2\right)\left(e^{ih^{-1}\Phi_\pm}b_\pm\right)(x,\xi). \qquad (2.2)$$

Then we define

$$\begin{aligned} J_{\pm a} &= I_h(a_\pm, \Phi_\pm), \\ K_{\pm a}(h) &= H(h)J_{\pm a}(h) - J_{\pm a}(h)H_0(h) = I_h(k_{\pm a}, \Phi_\pm), \\ K_{\pm b}(h) &= H(h)J_{\pm b}(h) - J_{\pm b}(h)H_0(h) = I_h(k_{\pm b}, \Phi_\pm). \end{aligned}$$

Before going to the Isozaki-Kitada formula, let us recall some notations.

1) For $\alpha > 1$, we denote $L^2_\alpha = \{f : |f|_\alpha = | < x >^\alpha f|_{L^2} < \infty\}$.

2) For $\alpha > \frac{1}{2}$, one defines $F_0(\lambda, h) : L^2_\alpha(\mathbb{R}^n) \longrightarrow L^2(S^{n-1})$, by

$$(F_0(\lambda, h)f)(\omega) = (2\pi h)^{-\frac{n}{2}}(2\lambda)^{\frac{n-2}{4}} \int_{\mathbb{R}^n} e^{-ih^{-1}\sqrt{2\lambda}\langle x,\omega\rangle} f(x)dx, \ \lambda > 0.$$

3) We denote by $\| \ . \ \|_{\alpha,\beta}$ the norm of linear bounded operators in $\mathcal{L}(L^2_\alpha, L^2_\beta)$.

Now we recall the following.

Proposition 2.1 ***(Isozaki-Kitada [8]).*** *For $\lambda \in]\frac{d_4^{-2}}{2}, \frac{d_4^2}{2}[$, we have*

$$T(\lambda, h) = T_{+1}(\lambda, h) + T_{-1}(\lambda, h) - T_2(\lambda, h),$$

with

$$T_{\pm 1}(\lambda, h) = F_0(\lambda, h)J^*_{+a}(h)K_{\pm b}(h)F^*_0(\lambda, h)$$

and

$$T_2(\lambda, h) = F_0(\lambda, h)K^*_{+a}(h)R(\lambda + i0, H(h))(K_{+b}(h) + K_{-b}(h))F^*_0(\lambda, h).$$

Denote by $T_{\pm 1}(\theta, \omega, \lambda, h)$ the kernel of $T_{\pm 1}(\lambda, h)$ and by $T_2(\theta, \omega, \lambda, h)$ the kernel of $T_2(\lambda, h)$. We see easily that

$$T_{\pm 1}(\theta, \omega, \lambda, h) = c_0(\lambda, h)^2 \int e^{ih^{-1}\psi_\pm(x,\theta,\omega)} k_{\pm b}(x, \sqrt{2\lambda}\omega)\overline{a}(x, \sqrt{2\lambda}\theta)dx,$$

where $\psi_\pm(x, \theta, \omega) = \Phi_\pm(x, \sqrt{2\lambda}\omega) - \Phi_+(x, \sqrt{2\lambda}\theta)$. For $\theta \neq \omega$, one can integrate by parts, to get $T_{\pm 1}(\theta, \omega, \lambda, h) = O(h^\infty)$. In a such way, assuming $\theta \neq \omega$, we need only to study T_2.

3 Resolvent estimate.

Under the hypotheses of Theorem 1.1, we wish to prove the estimate

$$\int_{\lambda_0 - \epsilon}^{\lambda_0 + \epsilon} \| \ R(\lambda \pm i0) \ \|_{\alpha, -\alpha} \ d\lambda = O(h^{-\tilde{n}}), \tilde{n} \in \mathbb{N}.$$

According to the work of Bruneau and Petkov [1], we know that it suffices to show that a such estimate holds for the truncated resolvent $\chi R(\lambda \pm i0)\chi$, for suitable $\chi \in C_0^\infty(\mathbb{R}^n)$. Indeed we have the following proposition (see [1]).

Proposition 3.1 *Let $d > 0$. There exists $\rho > 0$ large enough such that for all $\chi \in C_0^\infty(R^n)$, $\chi = 1$ on $B(0, \rho)$ and for h small enough we have*

$$\forall \alpha > \frac{1}{2}, \ \| \ R(z) \ \|_{\alpha,-\alpha} \leq Ch^{-2} \left(1 + \| \ \chi R(z)\chi \ \|\right),$$

uniformly for $z \in \mathcal{B}_\pm = \{z \in \mathbb{C} : (Re \, z, \pm Im \, z) \in]d^{-1}, d[\times]0, 1]\}$.

Next we follow the argument of Stefanov [16] to obtain the following.

Proposition 3.2 *Assume hypotheses 1. and 2. of Theorem 1.1 fulfilled. Then there exists $\tilde{n} \geq n$ such that for ϵ and h small enough and $\chi \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \chi R(\lambda \pm i0)\chi \|_{\alpha,-\alpha} \, d\lambda = O(h^{-\tilde{n}}). \tag{3.1}$$

The proof is a λ-uniform modification of the semi-classical maximum principle of Tang and Zworski [17] with an adaptation of the tools used by Stefanov in the case of $H(h) = h^2 P$, where P is a compactly supported perturbation of the Laplacian [16]. Fix $\delta > 0$, $n^* \in \mathbb{N}^*$ and set $S(\lambda, h) = h^{3n^*+3\delta} d(\lambda, h)$ and $\omega(\lambda, h) = d(\lambda, h)h^\delta$.

Lemma 3.3 *Let $F(.,h)$ be a family of functions such that for all $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, $F(.,h)$ is holomorphic in $\Omega(h,\lambda) = [\lambda - 5\omega(h,\lambda), \lambda + 5\omega(h,\lambda)] + i[-S(h,\lambda), S(h,\lambda)h^{-2n^*-\delta}]$. Moreover, assume that F satisfies the estimates*

$$\begin{aligned} |F(z,h)| &\leq Aexp(Ah^{-n^* log \frac{1}{h}}) \text{ in } \Omega(h,\lambda), & (3.2)\\ |F(z,h)| &\leq \frac{C}{|Im(z)|} \text{ in } \Omega(h,\lambda) \cap \{Im(z) < 0\}. & (3.3) \end{aligned}$$

Then there exists $C > 0$, independent of $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, such that

$$\forall h << 1, \ \forall z \in \tilde{\Omega}(h,\lambda), \ |F(z,h| \leq \frac{C}{S(h,\lambda)}, \tag{3.4}$$

where $\tilde{\Omega}(h,\lambda) = [\lambda - \omega(h,\lambda), \lambda + \omega(h,\lambda)] + i[-S(h,\lambda), S(h,\lambda)]$.

Proof: It is just a repetition of Tang-Zworski's proof [17] with control on λ. Set $\alpha = S(h,\lambda)h^{-2n^*-\delta}$, and $f(z,h,\lambda) = (\pi\alpha^2)^{-\frac{1}{2}} \int_{\lambda-3\omega}^{\lambda+3\omega} exp(-\frac{(x-z)^2}{\alpha^2})dx$, where $\omega = \omega(h,\lambda)$.
Obviously, the function $f(.,h,\lambda)$ is analytic in $\Omega(h,\lambda)$. It is clear that, as in [17], for h small enough, we have

$$\begin{aligned} |f(z,h,\lambda)| &\leq e \text{ in } \Omega(h,\lambda), & (3.5)\\ |f(z,h,\lambda)| &\geq \frac{1}{2} \text{ for } z \in [\lambda - \omega(h,\lambda), \lambda + \omega(h,\lambda)] + i[-S(h,\lambda), \alpha]. & (3.6) \end{aligned}$$

Moreover, there exists $C > 0$ independent of $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ such that

$$\forall z \in \Omega(h,\lambda) \cap \{|Re(z) - \lambda| \geq 4\omega(h,\lambda)\}, \ |f(z,h,\lambda)| \leq Ce^{-h^{-2n^*-2\delta}}. \tag{3.7}$$

Let $z = u + iv \in \Omega(h,\lambda) \cap \{|Re(z) - \lambda| \geq 4\omega(h,\lambda)\}$, then

$$\begin{aligned} |f(z,h,\lambda)| &\leq (\pi\alpha^2)^{-\frac{1}{2}} e^{\frac{v^2}{\alpha^2}} \int_{\lambda-3\omega}^{\lambda+3\omega} exp(-\frac{(x-u)^2}{\alpha^2})dx, \\ &\leq (\pi\alpha^2)^{-\frac{1}{2}} e^{\frac{v^2}{\alpha^2}} \int_{\lambda-3\omega}^{\lambda+3\omega} exp(-\frac{\omega^2}{\alpha^2})dx \leq 6\alpha\omega\pi^{-\frac{1}{2}} e^{\frac{v^2}{\alpha^2}} e^{-\frac{\omega^2}{\alpha^2}}. \end{aligned}$$

On the other hand, $v \leq \alpha$, $\frac{\omega}{\alpha} = h^{-n^*-\delta}$, $\alpha\omega = h^{n^*+3\delta} d(\lambda,h)^2$ and $d(\lambda,h)^2 \leq h^2$. Hence we get

$$|f(z,h,\lambda)| \leq 6\pi^{-\frac{1}{2}} h^{n^*+3\delta+2} e^{-h^{-2n^*-2\delta}}.$$

Define $G(z,h,\lambda)=e^{iz\sigma}f(z,h,\lambda)F(z,h)$, where

$$\sigma=\frac{log(S(h,\lambda)Ae^{Ah^{-n^*}log(\frac{1}{h})})}{S(h,\lambda)h^{-n^*-\delta}}.$$

It is easy to see that there exists $C>0$ such that

$$\forall h<<1,\ \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ |\sigma(h,\lambda)S(h,\lambda)|\leq C. \tag{3.8}$$

Indeed,

$$\sigma(h,\lambda)S(h,\lambda)=h^{n^*+\delta}log\left(S(h,\lambda)Ae^{Ah^{-n^*}log(\frac{1}{h})}\right),$$

and we conclude with the following inequality

$$\begin{aligned}log\left(S(h,\lambda)Ae^{Ah^{-n^*}log(\frac{1}{h})}\right) &\leq log\left(Ah^{3n^*+3\delta+2}e^{Ah^{-n^*-\frac{\delta}{2}}}\right)\\ &\leq log(Ah^{3n^*+3\delta+2})+h^{-n^*-\frac{\delta}{2}}\leq Ch^{-n^*-\delta}.\end{aligned}$$

$G(.,h,\lambda)$ being holomorphic in $\Omega(h,\lambda)$. We can find a constant C independent of λ such that $|G(z,h,\lambda)|\leq\frac{C}{S(h,\lambda)}$, $\forall z\in\Omega(h,\lambda)$. By the maximum principle, it suffices to obtain this estimate on the boundary of $\Omega(h,\lambda)$.

1) According to (3.2) and (3.5), on $[\lambda-5\omega(\lambda,h),\lambda+5\omega(\lambda,h)]+iS(h,\lambda)h^{-2n^*-\delta}$, we get

$$\begin{aligned}|G(z,h,\lambda)| &\leq e^{-\sigma S(h,\lambda)h^{-2n^*-\delta}}eAexp(Ah^{-n^*}log\frac{1}{h})\\ &\leq e^{-h^{-n^*}\log(S(h,\lambda)Ae^{h^{-n^*}\log(\frac{1}{h})})}eAexp(Ah^{-n^*}log\frac{1}{h})\\ &\leq \frac{1}{S(h,\lambda)}exp(-Ah^{-n^*}log\frac{1}{h})eAexp(Ah^{-n^*}log\frac{1}{h})\leq\frac{eA}{S(h,\lambda)},\end{aligned}$$

since $h^{-n^*}>1$ and $\log(S(h,\lambda)Ae^{h^{n^*}\log(\frac{1}{h})})>0$.

2) On $[\lambda-5\omega(\lambda,h),\lambda+5\omega(\lambda,h)]-iS(h,\lambda)$, we deduce from (3.3), (3.5) and (3.8) the estimate

$$|G(z,h,\lambda)|\leq e^{\sigma S(h,\lambda)}\frac{e}{S(h,\lambda)}\leq\frac{e^{C+1}}{S(h,\lambda)}.$$

3) On $\lambda\pm5\omega(\lambda,h)+i[-S(h,\lambda),S(h,\lambda)h^{-2n^*-\delta}]$, from the estimates (3.2) and (3.7) we get

$$\begin{aligned}|G(z,h,\lambda)| &\leq \sup_{x\in^*[-S(h,\lambda),S(h,\lambda)h^{-2n^*-\delta}]}(e^{-x\sigma})e^{-h^{-2n^*-2\delta}}Aexp(Ah^{-n^*}log\frac{1}{h}),\\ &\leq e^{S(h,\lambda)\sigma}e^{-h^{-n^*-\delta}}\leq e^{C}e^{-h^{-n^*-\delta}}\leq\frac{e^C}{S(h,\lambda)}.\end{aligned}$$

Finally, $|G(z,h,\lambda)|\leq\frac{C}{S(h,\lambda)}$ on $\partial\Omega(h,\lambda)$, and hence the same is true on $\Omega(h,\lambda)$. Taking $z\in\tilde{\Omega}(h,\lambda)$ and using (3.6) and (3.8), we obtain

$$\begin{aligned}|G(z,h,\lambda)| &\geq \frac{1}{2}|F(z,h,\lambda)|\inf_{x\in[-S(h,\lambda),S(h,\lambda)]}(e^{-x\sigma})\\ &\geq \frac{1}{2}|F(z,h,\lambda)|e^{-\sigma S(h,\lambda)}\geq\frac{1}{2}|F(z,h,\lambda)|e^{-C}\geq\frac{C'}{2}|F(z,h,\lambda)|.\end{aligned}$$

By a minoration and a majoration of G, we deduce the result. ∎

Proof of Proposition 3.2: We apply the above lemma to $F(z,h) = R_\chi(z,h) := \chi R(z,h)\chi$ with $n^* = n+1$. By construction $F(.,h,\lambda)$ is holomorphic in each $\Omega(h,\lambda)$, $\lambda \in [\lambda_0-\epsilon, \lambda_0+\epsilon]$. Moreover, it is well-known that $R_\chi(z,h)$ satisfies $R_\chi(z,h) \leq \frac{1}{|Im(z)|}$ in the lower half-plane. Hence, it suffices to check (3.2). For potentials satisfying assumption ii) of Theorem 1.1 we use the following.

Lemma 3.4 ***(Tang-Zworski [17]).*** *Let U be a compact neighbourhood of λ_0 in $Re(z) > 0$, and let $g(h) << 1$. Then there exists $C > 0$ such that for $h << 1$ we have*

$$\| R_\chi(z,h) \| \leq Ce^{Ch^{-n}\log\frac{1}{g(h)}}, \ \textit{for } z \in U \setminus \bigcup_{\xi\in Res(H(h))} B(\xi, g(h)). \tag{3.9}$$

Set $g(h) = \frac{d(h)}{2} = \frac{1}{2}\inf\{d(h,\lambda),\ \lambda \in [\lambda_0-\epsilon, \lambda_0+\epsilon]\}$, and observe that for $h << 1$, we have

$$\Omega(h,\lambda) \bigcap \left(\bigcup_{\xi\in Res(H(h))} B(\xi, g(h)) \right) = \emptyset.$$

Then we have $\| R_\chi(z,h) \| \leq Ce^{Ch^{-n}\log\frac{2}{d(h)}}$, $\forall z \in \Omega(h,\lambda)$.

Recall that there exists $C > 0$ such that $d(h) \geq Ce^{-Ch^{-1}}$. Therefore from $\| R_\chi(z,h) \| \leq Ce^{Ch^{-n}\log\frac{2}{d(h)}}$ and $d(h) \geq Ce^{-Ch^{-1}}$ we deduce

$$\| R_\chi(z,h) \| \leq Ce^{Ch^{-n-1}} \leq Ce^{Ch^{-n^*}\log\frac{1}{h}}, \ \forall z \in \Omega(h,\lambda).$$

Consequently, we can apply Lemma 3.3, to obtain

$$\| R_\chi(z,h) \| \leq \frac{C}{S(h,\lambda)}, \ \forall z \in \tilde{\Omega}(h,\lambda).$$

In particular, for $\lambda \in [\lambda_0-\epsilon, \lambda_0+\epsilon]$ and $\beta \in [-S(h,\lambda), S(h,\lambda)h^{-n^*-\delta}]$, we have $\| R_\chi(\lambda \pm i\beta, h) \| \leq \frac{C}{S(h,\lambda)}$.
In this inequality, taking the limit $\beta \to 0$, we obtain immediately

$$\| R_\chi(\lambda \pm i0, h) \| \leq \frac{C}{S(h,\lambda)}, \ \forall \lambda \in [\lambda_0-\epsilon, \lambda_0+\epsilon]. \tag{3.10}$$

An integration of this estimate yields

$$\begin{aligned}
\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R_\chi(\lambda \pm i0, h) \| \, d\lambda \ &\leq \ \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{C}{S(h,\lambda)} d\lambda \leq \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{Ch^{-3n^*-3\delta}}{d(h,\lambda)} d\lambda, \\
&\leq \ Ch^{-3n^*-3\delta} \sum_{\xi\in K} \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{1}{|\lambda-\xi|} + h^{-1} d\lambda,
\end{aligned}$$

where $K = Res(H(h)) \bigcap \left([\lambda_0-2\epsilon, \lambda_0+2\epsilon] + i[-\epsilon^2, \epsilon^2]\right)$.
Next note that K is a finite set and $card(K) = O((\frac{\lambda_0}{h})^n)$. Moreover, for $\xi = \alpha + i\beta \in K$ and

$\epsilon^2 > \beta > 0$ close to 0, we have

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{1}{|\lambda-\xi|}d\lambda &= \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{1}{\sqrt{(\lambda-\alpha)^2+\beta^2}}d\lambda = \int_{\frac{\lambda_0-\epsilon-\alpha}{\beta}}^{\frac{\lambda_0+\epsilon-\alpha}{\beta}} \frac{1}{\sqrt{1+\lambda^2}}d\lambda \\ &\leq \int_{\frac{-3\epsilon}{\beta}}^{\frac{3\epsilon}{\beta}} \frac{1}{\sqrt{1+\lambda^2}}d\lambda = 2\left(\int_0^1 \frac{1}{\sqrt{1+\lambda^2}}d\lambda + \int_1^{\frac{3\epsilon}{\beta}} \frac{1}{\sqrt{1+\lambda^2}}d\lambda\right) \\ &\leq 2 + 2log(\frac{3\epsilon}{\beta}) \leq 3\log(\frac{3\epsilon}{\beta}).\end{aligned}$$

On the other hand, for $\xi \in K$ we have $|Im(\xi)| \geq Ae^{-Ah^{-1}}$. Then

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \frac{1}{|\lambda-\xi|}d\lambda \leq \log(\frac{3\epsilon}{A}e^{Ah^{-1}}) \leq Ch^{-1},$$

which implies

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R_\chi(\lambda \pm i0, h) \| \, d\lambda &\leq Ch^{-3n^*-3\delta}\left(\frac{\lambda_0}{h}\right)^n (h^{-1}+h^{-1}) \\ &\leq Ch^{-4n-3\delta-4}\end{aligned}$$

and the proof is complete. ∎

From Propositions 3.1 and 3.2, we deduce for ϵ small enough, the estimate

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda+i0) \|_{\alpha,-\alpha} \, d\lambda = O(h^{-\tilde{n}}). \tag{3.11}$$

Now we can prove the following.

Lemma 3.5 *Assume the hypotheses of Theorem 1.1 fulfilled. Let* $\omega_\pm \in A^0_\infty$ *be such that* $supp(\omega_\pm) \subset \Gamma_\pm(R, d, \sigma_\pm)$. *Then for* $\alpha > 1$ *and* $\epsilon > 0$ *small enough we have the following assertions:*

i)

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda \pm i0)\omega_\pm(x, hD_x) \|_{-\alpha+\delta,-\alpha} \, d\lambda = O(h^{-1}), \ \forall \delta > 1.$$

ii) If $\sigma_+ > \sigma_-$, *then*

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_\mp(x, hD_x)R(\lambda \pm i0)\omega_\pm(x, hD_x) \|_{-\alpha,\alpha} \, d\lambda = O(h^\infty).$$

iii) If $\omega \in A^0_\infty$ *and* $supp(\omega) \subset \{|x| < \frac{9}{10}R\}$, *then*

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega(x, hD_x)R(\lambda \pm i0)\omega_\pm(x, hD_x) \|_{-\alpha,\alpha} \, d\lambda = O(h^\infty).$$

Proof: We treat ω_+ only, since the analysis of ω_- is completely similar. We use the construction of Robert and Tamura [12] for an approximation of $R(\lambda + i0)\omega_+(x, hD_x)$. Denote

$$U(t, h) = e^{-ih^{-1}tH(h)}\omega_+(x, hD_x).$$

Then there exist $\omega_N(x,hD_x)\in A_\infty^{-N}$, $U_N(t,h)\in\mathcal{L}(L^2_{-\alpha+\delta},L^2_{-\alpha})$ and $R_N(t,h)\in\mathcal{L}(L^2_{-\alpha+\delta},L^2_{\alpha})$ such that

$$U(t,h)=U_N(t,h)+h^Ne^{-ih^{-1}tH(h)}\omega_N(x,hD_x)+ih^{-1}\int_0^t e^{-ih^{-1}(t-s)H(h)}R_N(s,h)ds,$$

with

$$\forall\delta>1,\ \| U_N(t,h)\|_{-\alpha+\delta,-\alpha}=O(<t>^{-\delta}),\tag{3.12}$$
$$\forall\delta>1,\ \| R_N(t,h)\|_{-\alpha+\delta,\alpha}=O(h^N<t>^{-\delta}),\tag{3.13}$$
$$\omega_N(x,hD_x)\in A_\infty^{-N}\text{uniformly with respect to } h.\tag{3.14}$$

Hence

$$R(\lambda+i0)\omega_+(x,hD_x)=ih^{-1}\int_0^{+\infty}e^{ih^{-1}t\lambda}U(t,h)dt=\sum_{k=1}^3Q_k(\lambda,h),\tag{3.15}$$

where

$$\begin{aligned}Q_1&=ih^{-1}\int_0^{+\infty}e^{ih^{-1}t\lambda}U_N(t,h)dt,\\ Q_2&=h^NR(\lambda+i0)\omega_N(x,hD_x),\\ Q_3&=ih^{-1}\int_0^{+\infty}R(\lambda+i0)e^{ih^{-1}s\lambda}R_N(s,h)ds.\end{aligned}$$

Proof of i). By using (3.15), we get

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\|R(\lambda+i0)\omega_+(x,hD_x)\|_{-\alpha+\delta,-\alpha}d\lambda\le\sum_{k=1}^3\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\|Q_k(\lambda,h)\|_{-\alpha+\delta,-\alpha}d\lambda.$$

As $\delta>1$, we have

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| Q_1\|_{-\alpha+\delta,-\alpha}\,d\lambda&\le h^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\int_0^{+\infty}\| U_N(t,h)\|_{-\alpha+\delta,-\alpha}\,dtd\lambda\\ &\le Ch^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\int_0^{+\infty}<t>^{-\delta}\,dtd\lambda\le 2C'\epsilon h^{-1}.\end{aligned}$$

Therefore,

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| Q_1(\lambda,h)\|_{-\alpha+\delta,-\alpha}\,d\lambda=O(h^{-1}).$$

Moreover, $\omega_N\in A_{-N}$, and then for $N\ge 2\alpha-\delta$ we have $\|\omega_N(x,hD_x))\|_{-\alpha+\delta,\alpha}\le C$. Choosing $N\ge\max(2\alpha-\delta,\tilde n-1)$, we obtain

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| Q_2\|_{-\alpha+\delta,-\alpha}\,d\lambda&\le h^N\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| R(\lambda+i0)\|_{\alpha,-\alpha}\|\omega_N(x,hD_x))\|_{-\alpha+\delta,\alpha}\,d\lambda\\ &\le Ch^N\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| R(\lambda+i0)\|_{\alpha,-\alpha}\,d\lambda\le Ch^{N-\tilde n}\le Ch^{-1}\end{aligned}$$

and this yields

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\| Q_2(\lambda,h)\|_{-\alpha+\delta,-\alpha}\,d\lambda=O(h^{-1}).$$

Finally, we have

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| Q_3(\lambda,h) \|_{-\alpha+\delta,-\alpha}\, d\lambda \leq h^{-1} \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \int_0^{+\infty} \| R(\lambda+i0) \|_{\alpha,-\alpha} \| R_N(s,h) \|_{-\alpha+\delta,-\alpha}\, ds d\lambda$$
$$\leq Ch^{-1} \int_0^{+\infty} h^N < s >^{-\delta} ds \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda+i0) \|_{\alpha,-\alpha}\, d\lambda \leq Ch^{N-\tilde{n}-1}.$$

Assuming $N \geq \tilde{n}$, we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| Q_3(\lambda,h) \|_{-\alpha+\delta,-\alpha}\, d\lambda = O(h^{-1})$$

and the proof of i) is complete.

Proof of ii): We estimate

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)R(\lambda+i0)\omega_+(x,hD_x) \|_{-\alpha,\alpha}\, d\lambda \leq \sum_{k=1}^{3} \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \|\omega_-(x,hD_x)Q_k(\lambda,h)\|_{-\alpha+\delta,-\alpha} d\lambda.$$

First of all we have $\omega_-(x,hD_x)Q_2 = h^N\omega_-(x,hD_x)R(\lambda+i0)\omega_N(x,hD_x)$. Consequently,

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_2 \|_{-\alpha,\alpha}\, d\lambda \leq h^N \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)R(\lambda+i0) \|_{\alpha+\delta,\alpha} \| \omega_N(x,hD_x) \|_{-\alpha,\alpha+\delta}\, d\lambda$$
$$\leq h^N \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_-^*(x,hD_x) \|_{-\alpha,-\alpha-\delta} \| \omega_N(x,hD_x) \|_{-\alpha,\alpha+\delta}\, d\lambda.$$

Assuming $N \geq 2\alpha+\delta$, we get $\| \omega_N(x,hD_x) \|_{-\alpha,\alpha+\delta} = O(1)$ and

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_2 \|_{-\alpha,\alpha}\, d\lambda \leq Ch^N \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_-^*(x,hD_x) \|_{-\alpha,-\alpha-\delta}\, d\lambda. \tag{3.16}$$

Moreover, from $\omega_-^* \sim \sum_\alpha \frac{h^{|\alpha|}}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{\omega}_-$ we deduce $\omega_-^* = \tilde{\omega}_{-l} + \omega_{-l}$, with $supp(\tilde{\omega}_{-l}) \subset supp(\omega_-)$ and $h^{-l}\omega_{-l} \in A_{-l}$ uniformly with respect to h. Therefore

$$\| R(\lambda-i0)\omega_-^*(x,hD_x) \|_{-\alpha,-\alpha-\delta} \leq \| R(\lambda-i0)\tilde{\omega}_{-l}(x,hD_x) \|_{-\alpha,-\alpha-\delta} + \| R(\lambda-i0)\omega_{-l}(x,hD_x) \|_{-\alpha,-\alpha-\delta} .$$

According to i), for any l large enough, we get

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\tilde{\omega}_{-l}(x,hD_x) \|_{-\alpha,-\alpha-\delta}\, d\lambda \leq Ch^{-1}. \tag{3.17}$$

Moreover,

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_{-l}(x,hD_x) \|_{-\alpha,-\alpha-\delta}\, d\lambda \leq \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0) \|_{\alpha+\delta,-\alpha-\delta}\, d\lambda \| \omega_{-l}(x,hD_x) \|_{-\alpha,\alpha+\delta}$$
$$\leq Ch^{-\tilde{n}} \| \omega_{-l}(x,hD_x) \|_{-\alpha,\alpha+\delta} .$$

For $l \geq 2\alpha+\delta$ we have $\| \omega_{-l}(x,hD_x) \|_{-\alpha,\alpha+\delta} \leq Ch^l$. Next we suppose also that $l \geq \tilde{n}-1$, to conclude

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_{-l}(x,hD_x) \|_{-\alpha,-\alpha-\delta} \, d\lambda = O(h^{-1}). \tag{3.18}$$

From (3.17) and (3.18) we deduce

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_-^*(x,hD_x) \|_{-\alpha,-\alpha-\delta} \, d\lambda = O(h^{-1}). \tag{3.19}$$

Combining (3.19) and (3.16), we have

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_2(\lambda,h) \|_{-\alpha,-\alpha-\delta} \, d\lambda = O(h^{N-1}). \tag{3.20}$$

Now we will study $\| \omega_-(x,hD_x)Q_3 \|_{-\alpha,\alpha}$. First we have

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_3 \|_{-\alpha,\alpha} \, d\lambda &\leq h^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\int_0^\infty \| \omega_-(x,hD_x)R(\lambda+i0)R_N(t,h) \|_{-\alpha,\alpha} \, dt d\lambda \\ &\leq h^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| R(\lambda-i0)\omega_-^*(x,hD_x) \|_{-\alpha,-\alpha-\delta} \, d\lambda \int_0^\infty \| R_N(t,h) \|_{-\alpha,\alpha+\delta} \, dt.\end{aligned}$$

According to (3.13) and (3.19), we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_3 \|_{-\alpha,\alpha} \, d\lambda \leq h^{-1}Ch^{-1}\int_0^\infty h^N < t >^{-\delta} dt,$$

and for $\delta > 1$ we get

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_3 \|_{-\alpha,\alpha} \, d\lambda = O(h^{N-2}). \tag{3.21}$$

Finally, consider the estimate of $\| \omega_-(x,hD_x)Q_1(\lambda,h) \|_{-\alpha,\alpha}$. We have

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_1(\lambda,h) \|_{-\alpha,\alpha} \, d\lambda &\leq Ch^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \int_0^{+\infty} e^{ih^{-1}t\lambda}\omega_-(x,hD_x)U_N(t,h)dt \|_{-\alpha,\alpha} \, d\lambda \\ &\leq Ch^{-1}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\int_0^{+\infty} \| \omega_-(x,hD_x)U_N(t,h) \|_{-\alpha,\alpha} \, dt d\lambda \\ &\leq 2\epsilon Ch^{-1}\int_0^{+\infty} \| \omega_-(x,hD_x)U_N(t,h) \|_{-\alpha,\alpha} \, dt.\end{aligned}$$

From the result of Robert and Tamura [12], Lemma 2.1, we know that

$$\int_0^{+\infty} \| \omega_-(x,hD_x)U_N(t,h) \|_{-\alpha,\alpha} \, dt = O(h^\infty)$$

which yields

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)Q_1 \|_{-\alpha,\alpha} \, d\lambda = O(h^\infty). \tag{3.22}$$

Finally, using (3.20), (3.21) and (3.22), we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| \omega_-(x,hD_x)R(\lambda+i0)\omega_+(x,hD_x) \|_{-\alpha,\alpha} \, d\lambda = O(h^{N-2}).$$

As N can be chosen arbitrary large, ii) is proved. The proof of iii) is identical, and we leave it to the reader. ■

Take $\chi(.,R) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ for $|x| \leq R$ and $\chi = 0$ for $|x| \geq R+1$, and set

$$\chi_a(x) = \chi(x, 20R_0),\ \chi_b(x) = \chi(x, 10R_0).$$

From Lemma 3.5, as in [12], we deduce the following.

Lemma 3.6 *Assume the hypothesis of Theorem 1.1 fulfilled. Then for $\alpha > \frac{n}{2}$ close to $\frac{n}{2}$, we have the following assertions:*

$$i)\ \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| K_{+a}^*(h)R(\lambda+i0)K_{+b}(h) \|_{-\alpha,\alpha}\, d\lambda = O(h^{\frac{N}{2}}),$$

$$ii)\ \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| K_{+a}^*(h)R(\lambda+i0)(1-\chi_b)K_{-b}(h) \|_{-\alpha,\alpha}\, d\lambda = O(h^{\frac{N}{2}}),$$

$$iii)\ \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \| ((1-\chi_a)K_{+a})^*(h)R(\lambda+i0)\chi_b K_{-b}(h) \|_{-\alpha,\alpha}\, d\lambda = O(h^{\frac{N}{2}}).$$

Denote $\Phi_+(x) = \Phi_+(x, \sqrt{2\lambda}\theta)$ and $\Phi_-(x) = \Phi_-(x, \sqrt{2\lambda}\omega)$, hence

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} |f(\theta,\omega,\lambda,h) - c_1(\lambda,h)G_0(\theta,\omega,\lambda,h)|d\lambda = O(h^{\frac{N}{3}}) \tag{3.23}$$

with

$$G_0(\theta,\omega,\lambda,h) = \langle R(\lambda+i0)g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\rangle \tag{3.24}$$

$$c_1(\lambda,h) = 2\pi(2\lambda)^{(n-3)/4}(2\pi h)^{-(n+1)/2}e^{-i\frac{(n-3)\pi}{4}} \tag{3.25}$$

and

$$g_{-b} = e^{ih^{-1}\Phi_-}[\chi_b, H_0(h)]\left(b_-(., \sqrt{2\lambda}\omega)e^{ih^{-1}\Phi_-}\right), \tag{3.26}$$

$$g_{-a} = e^{ih^{-1}\Phi_+}[\chi_a, H_0(h)]\left(a_+(., \sqrt{2\lambda}\theta)e^{ih^{-1}\Phi_+}\right). \tag{3.27}$$

4 Short time localization.

4.1 First localization.

We start with the well-known formula

$$R(\lambda+i0) = ih^{-1}\int_0^T e^{ih^{-1}t\lambda}e^{-ih^{-1}tH(h)}dt + e^{ih^{-1}T\lambda}R(\lambda+i0)e^{-ih^{-1}TH(h)}. \tag{4.1}$$

Our goal is to show that the remainder $e^{ih^{-1}T\lambda}R(\lambda+i0)e^{-ih^{-1}TH(h)}$is bounded by $O(h^\infty)$. To do this, we will prove that for $T>0$ large enough, we have

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \langle R(\lambda+i0)e^{-ih^{-1}TH(h)}g_{-b}e^{ih^{-1}\Phi_-}, g_{+a}e^{ih^{-1}\Phi_+}\rangle d\lambda = O(h^N). \tag{4.2}$$

The proof is based on an judicious application of Egorov's lemma. For this purpose we need to study the Hamiltonian flow $\Phi_t(x,\eta) = (q(t,x,\eta), p(t,x,\eta))$ associated to $H(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$.

Lemma 4.1 *Take $d > 1$, $\sigma \geq 0$ and assume $[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \subset]\frac{d^{-2}}{2}, \frac{d^2}{2}[$. Then there exists $R_0 > 0$ such that*

$$\forall R > R_0,\ \forall (x,\eta) \in \Gamma_+(R,d,\sigma),\ \frac{1}{2}|\eta|^2 + V(x) \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \Longrightarrow \forall t > 0,\ \Phi_t(x,\eta) \in \Gamma_+(R,d,0).$$

Proof: Denoting $\phi_t(x,\xi) = (q(t), p(t))$ and following Lemma 1 in [1], we get

$$\begin{cases} (|q|^2)' = 2\langle q, \partial_\xi H(q,p)\rangle, \\ \langle q, \partial_\xi H(q,p)\rangle' = \{H, \langle x, \partial_\xi H\rangle\}(q,p). \end{cases} \tag{4.3}$$

Then $\langle q, \partial_\xi H(q,p)\rangle' = 2\lambda + (\{H, \langle x, \partial_\xi H\rangle\}(q,p) - 2H(q,p))$. Set $\mu = \epsilon\lambda$. A simple calculation yields $\{H, \langle x, \partial_\xi H\rangle\}(x,\xi) - 2H(x,\xi)) = -\langle x, \nabla_x V(x)\rangle - V(x)$, hence there exists $R_0 > 0$ such that

$$\forall |x| > R > R_0,\ \forall \xi \in \mathbb{R}^n\ |\{H, \langle x, \partial_\xi H\rangle\}(x,\xi) - 2H(x,\xi))| < 2\mu.$$

Chose $(x,\eta) \in \Gamma_+(R,d,\sigma)$ and $R > R_0$ such that $H(x,\eta) = \lambda$ and denote by $[T_m, T_M]$ the maximal interval containing 0 such that for $t \in [T_m, T_M]$ we have

$$|\{H, \langle x, \partial_\xi H\rangle\}(q(t,x,\eta), p(t,x,\eta)) - 2H(q(t,x,\eta), p(t,x,\eta)))| < 2\mu.$$

For $|x| > R$ we have $T_M > 0$ and

$$\langle q, \partial_\xi H(q,p)\rangle' \geq 2(\lambda - \mu),\ \forall t \in [0, T_M].$$

Thus

$$\langle q, \partial_\xi H(q,p)\rangle \geq \langle x, \eta\rangle + 2(\lambda - \mu)t,\ \forall t \in [0, T_M]. \tag{4.4}$$

Integrating once more, we obtain

$$\forall t \in [0, T_M],\ |q(t)|^2 \geq |x|^2 + 2\langle x, \eta\rangle t + 2(\lambda - \mu)t^2 \geq |x|^2 + 2\sigma|\eta||x|t + 2(\lambda - \mu)t^2 \geq |x|^2 + 2(\lambda - \mu)t^2$$

From the last inequality we deduce $T_M = +\infty$ and hence

$$|q(t,x,\eta)|^2 \geq |x|^2 + 2(\lambda - \mu)t^2,\ \forall t > 0.$$

In particular, we get $|q(t,x,\eta)| > R,\ \forall t > 0$. As a direct consequence of (4.4), we obtain

$$\forall t > 0,\ \langle q(t), p(t)\rangle \geq \sigma|x||\eta| + 2(\lambda - \mu)t > 0,$$

and this yields $\cos(q(t), p(t)) > 0$, for $t > 0$.

By the equality

$$\frac{1}{2}|p(t)|^2 + V(q(t)) = \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \subset]\frac{d^{-2}}{2}, \frac{d^2}{2}[,\ \forall t > 0,$$

combined with $|V(x)| \leq C < x >^{-1}$ and $|q(t)| \geq R > R_0$, R_0 large enough, we get

$$|p(t)| \in]d^{-1}, d[,\ \forall t > 0.$$

■

From the argument of the above proof we get the following.

Corollary 4.1 *Let $0 < \sigma < 1$ and $\epsilon > 0$. Then there exists $R_1 > 0$ such that for $t > 0$ and $(x, \eta) \in \Gamma_+(R_1, d, -\sigma) \cap H^{-1}([\lambda_0 - \epsilon, \lambda_0 + \epsilon])$, we have*

$$|q(t, x, \eta)|^2 \geq |x|^2 - \sigma|x||\eta|t + 2(\lambda_0 - 2\epsilon)t^2.$$

Proof: As in the above lemma, choose R_0 so that

$$|\{H, \langle x, \partial_\xi H\rangle\}(x, \xi) - 2H(x, \xi))| < 2\epsilon, \ \forall |x| > R_0, \ \forall \xi \in \mathbb{R}^n.$$

Next choose $R > R_0$ and set $R_1 = NR$, $\mu = \epsilon$. Then we can find N large enough such that

$$\begin{cases} \frac{1}{N^2} < 1 - \sigma^2, \\ |x| \geq NR_0 \Longrightarrow |V(x)| \leq \epsilon. \end{cases} \tag{4.5}$$

For $(x, \eta) \in \Gamma_+(R_1, d, -\sigma) \cap H^{-1}([\lambda_0 - \epsilon, \lambda_0 + \epsilon])$, let $[T_m, T_M]$ be the maximal interval containing 0 such that

$$|\{H, \langle x, \partial_\xi H\rangle\}(q(t, x, \eta), p(t, x, \eta)) - 2H(q(t, x, \eta), p(t, x, \eta)))| < 2\epsilon.$$

In the same way as in Lemma 4.1, we have

$$\begin{aligned} \forall t \in [0, T_M], \ |q(t, x, \eta)|^2 &\geq |x|^2 + 2\langle x, \eta\rangle t + 2(\lambda - \mu)t^2 \\ &\geq |x|^2 - 2\sigma|x||\eta|t + 2(\lambda - \epsilon)t^2. \end{aligned}$$

Then $|q(t, x, \eta)| \geq R, \ \forall t \in [0, T_M]$. Indeed,

$$\Delta = 4\sigma^2|x|^2|\eta|^2 - 8(\lambda - \epsilon)(|x|^2 - R^2) = 8|x|^2\left(\frac{1}{2}|\eta|^2\sigma^2 - (\lambda - \epsilon)(1 - \frac{R^2}{|x|^2})\right).$$

From $|x| \geq NR$ we deduce $|V(x)| \leq \epsilon$, and this implies $\frac{1}{2}|\eta|^2 \leq \lambda + \epsilon$. Hence

$$\begin{aligned} \Delta &\leq 8|x|^2\left((\lambda + \epsilon)\sigma^2 - (\lambda - \epsilon)(1 - \frac{1}{N^2})\right) \\ &\leq 8|x|^2\left(\lambda(\sigma^2 - 1 + \frac{1}{N^2}) + \epsilon(\sigma^2 + 1 - \frac{1}{N^2})\right). \end{aligned}$$

As $\sigma^2 - 1 + \frac{1}{N^2} < 0$, we have

$$\Delta \leq 8|x|^2\left((\lambda_0 - \epsilon)(\sigma^2 + 1 - \frac{1}{N^2}) + \epsilon(\sigma^2 - 1 + \frac{1}{N^2})\right) < 0$$

for $\epsilon > 0$ small enough and we conclude that $T_M = +\infty$. Otherwise, $|q(T_M)| \geq R > R_0$, and this leads to the existence of $\delta > 0$ such that

$$\forall t \in [0, T_M + \delta], \ |\{H, \langle x, \partial_\xi H\rangle\}(q(t, x, \eta), p(t, x, \eta)) - 2H(q(t, x, \eta), p(t, x, \eta)))| < 2\epsilon$$

which contradicts the definition of T_M. The proof is complete. ∎

Before going further, we need to reformulate the assumption (H_{ω,λ_0}) with the following.

Lemma 4.2 *The condition (H_{ω,λ_0}) is equivalent to the following assertion.*
There exist $R > 0$, $\epsilon > 0$, and a neighbourhood W of ω in S^{n-1} such that

$$\forall \eta \in \mathbb{R}^n, \ \forall |x| > R, \ \frac{1}{2}|\eta|^2 + V(x) \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \ \text{and} \ \frac{\eta}{|\eta|} \in W \ \text{imply} \ \lim_{t \to \pm\infty} |q(t, x, \eta)| = +\infty. \tag{4.6}$$

Proof: First assume (4.6) and fix an energy $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$. Let V be an open subset of S^{n-1} such that $\omega \in V \Subset W$ and let $\omega' \in V$ and $z \in \mathbb{R}^n$. By definition, there exists $T_0 > 0$ such that

$$\forall t \leq -T_0,\ p_\infty(t, z, \lambda, \omega') \in W.$$

Then

$$\lim_{s\to+\infty} |q(s, q_\infty(t, z, \lambda, \omega'), p_\infty(t, z, \lambda, \omega'))| = +\infty.$$

As

$$q(s, q_\infty(t, z, \lambda, \omega'), p_\infty(t, z, \lambda, \omega')) = q_\infty(t + s, z, \lambda, \omega'),$$

we obtain

$$\lim_{t\to+\infty} |q_\infty(t, z, \lambda, \omega')| = +\infty.$$

Conversely, assume $(\mathbf{H}_{\omega,\lambda_0})$ satisfied and suppose there exists a sequence (x_n, ω_n) such that

$$\lim_{n\to+\infty} |x_n| = +\infty,\ \lim_{n\to+\infty} |\omega_n| = \omega, \text{ and } \forall n \in \mathbb{N}, q(., x_n, \omega_n) \text{ remains bounded on } \mathbb{R}^+.$$

According to Corollary 4.1, we can suppose that there exists $\sigma > 0$ such that $(x_n, \omega_n) \in \Gamma_-(R, d, -\sigma)$. Hence it suffices to show that

$$\forall \epsilon > 0, \exists R > 0, \forall (x_0, \xi_0) \in \Gamma_-(R, d, -\sigma), \forall t < 0, |p(t, x_0, \xi_0) - \xi_0| < \epsilon. \tag{4.7}$$

Indeed, this implies that for n large enough and $t < 0$, we have $|p(t, x_n, \omega_n) - \omega| < 2\epsilon$. Choosing ϵ small enough, we can assume that $\lim_{t\to-\infty} p(t, x_n, \omega_n) = \omega' \in W$. Then $(q, p)(., x_n, \omega_n) = (q_\infty, p_\infty)(., z, \omega')$, with $\omega' \in W$. According to (4.6), we deduce $\lim_{t\to+\infty} |q(t, x_n, \omega_n)| = +\infty$, which is a contradiction.
To establish (4.7) we observe that

$$|p(t) - \xi_0| \leq \int_0^t |\nabla_x V(q(s))| ds \leq \int_0^t < q(s) >^{-1-\rho} ds.$$

According to the proof of Lemma 4.1 (see section 4), we have $q(s) \geq x_0^2 + \lambda s^2$. Hence

$$|p(t) - \xi_0| \leq C \int_0^{-\infty} (1 + x_0^2 + \lambda s^2)^{-\frac{1+\rho}{2}} ds \leq C < x_0 >^{-\rho} \leq \epsilon$$

for $|x_0| > R$ large enough and the proof of Lemma 4.2 is complete. ∎

Lemma 4.3 *Assume* $(\mathbf{H}_{\omega,\lambda_0})$ *and choose* $\lambda_0 \in]d^{-1}, d[$. *Then there exist* $R_0 > 0$ *large enough and* $[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \subset]d^{-1}, d[$ *such that for all* $R > 0$ *there exists* $T_0(R) > 0$ *satisfying*

$$\forall\ 5R_0 < |x| < 6R_0,\ \forall \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon],\ \forall \eta \in \mathbb{R}^n,$$

$$\frac{\eta}{|\eta|} \in W \text{ and } \frac{1}{2}|\eta|^2 + V(x) = \lambda \Longrightarrow \forall t > T_0(R),\ \Phi_t(x, \eta) \in \Gamma_+(R, d, 0).$$

Proof: From $(\mathbf{H}_{\omega,\lambda_0})$ we deduce the existence of $R_0 > 0$, $\epsilon > 0$ and a neighbourhood W of ω in S^{n-1} such that

$$\forall \eta \in \mathbb{R}^n, \forall |x| > R_0,\ \frac{1}{2}|\eta|^2 + V(x) \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \text{ and } \frac{\eta}{|\eta|} \in W \Longrightarrow \lim_{t\to\pm\infty} |q(t, x, \eta)| = +\infty.$$

Therefore,

$$\forall R_1>0,\ \forall x\in\mathcal{C}(R_0),\ \forall\omega'\in W,\ \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],$$
$$\exists T(x,\omega')>0 \text{ such that } \forall t>T(x,\omega')\ \Phi_t(x,e(\lambda,x)\omega')\in\Gamma_+(R_1,d,0),$$

where $\frac{1}{2}e(\lambda,x)^2+V(x)=\lambda$ and $\mathcal{C}(R_0)=\{x\in\mathbb{R}^n\ :\ 5R_0<|x|<6R_0\}$.

By continuity with respect to the initial data, we get

$$\forall R_1>0,\ \forall x\in\mathcal{C}(R_0),\ \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ \forall\omega'\in W,\ \exists r(x,\lambda,\omega')>0,\ \exists T(x,\lambda,\omega')>0$$
$$\text{such that } \forall(y,\lambda',\omega'')\in B((x,\lambda,\omega'),r(x,\lambda,\omega')),\ \Phi_{T(x)}(y,e(\lambda',y)\omega'')\in\Gamma_+(R_1,d,0).$$

By using Lemma 4.1, we can choose $R_1>0$ so that

$$\forall t>0,\ \forall(x,\eta)\in\Gamma_+(R_1,d,0),\ \Phi_t(x,\eta)\in\Gamma_+(R_1,d,0).$$

Hence

$$\forall x\in\mathcal{C}(R_0),\ \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ \forall\omega'\in W,\ \exists r(x,\lambda,\omega')>0,\ \exists T(x,\lambda,\omega')>0$$
$$\text{such that } \forall(y,\lambda',\omega'')\in B((x,\lambda,\omega'),r(x,\lambda,\omega')),\ \forall t>T(x,\lambda),\ \Phi_t(y,e(\lambda',y)\omega''))\in\Gamma_+(R_1,d,0).$$

The balls $B((x,\lambda,\omega'),r(x,\lambda,\omega'))$ recover the compact set $\mathcal{C}(R_0)\times[\lambda_0-\epsilon,\lambda_0+\epsilon]\times W$, and we can find a finite recoverment

$$\mathcal{C}(R_0)\times[\lambda_0-\epsilon,\lambda_0+\epsilon]\times W\subset\bigcup_{j=1}^{q}B((x_j,\lambda_j,\omega_j'),r_j).$$

We set $T_0=\max_{j=1,\dots,q}(T(x_j))$ and for $R_1>R$ we have

$$\forall x\in\mathcal{C}(R_0),\ \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ \forall\omega'\in W,\ \forall t>T_0,\ \Phi_t(x,e(\lambda',x)\omega'))\in\Gamma_+(R_1,d,0)\subset\Gamma_+(R,d,0)$$

which completes the proof. ■

Now we are able to prove the following.

Proposition 4.1 *There exist $\epsilon>0$ and $T_0>0$ such that $\forall T>T_0$, we have*

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\left|\langle R(\lambda+i0)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle-\int_0^T e^{ih^{-1}t\lambda}F(t,\theta,\omega,h)dt\right|d\lambda=O(h^N),$$

with $F(t,\theta,\omega,h)=\langle e^{-ith^{-1}H(h)}g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle$.

Proof: Set $U=\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,\ x\in C(R_0),\ |\xi|\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ \frac{\xi}{|\xi|}\in W\}$ and introduce $\omega\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ so that $\omega=1$ in U, and $\omega=0$ in the complementary of a neighbourhood V of U. Setting $\omega_b=\chi_b\omega$, we have

$$\begin{aligned}\langle R(\lambda+i0)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle &= \langle R(\lambda+i0)\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle\\ &+ \langle R(\lambda+i0)(1-\omega_b)(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle\\ &= I(\lambda,h)+J(\lambda,h).\end{aligned}$$

Clearly,

$$(1-\omega_b)(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-} = \sum_{|\alpha|\leq M} h^{|\alpha|}(1-\omega_b)^{(\alpha)}(x,\nabla_x\Phi_-)g_\alpha(x)+R_M(x),$$

with $supp(g_\alpha)\subset supp(g_{-b})$, and $|R_M(x)|\leq h^M<x>^{-M}$. On the one hand, without a loss of generality, we can assume

$$|x|\geq R_0 \Longrightarrow \forall\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ \frac{\nabla_x\Phi_-}{|\nabla_x\Phi_-|}\in W.$$

Then $\forall x\in\mathcal{C}(R_0)$, $(x,\nabla_x\Phi_-)\in U$. On the other hand, we have $supp(g_{-b})\subset C(R_0)$, and consequently $(1-\omega)^{(\alpha)}(x,\nabla_x\Phi_-(x))g_\alpha(x)=0$, $\forall\alpha$. Moreover, for M large enough, we have $\| R_M\|_\alpha=O(h^M)$ uniformly with respect to λ. Hence

$$\begin{aligned}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|J(\lambda,h)|d\lambda &= \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|\langle R(\lambda+i0)(1-\omega_b)(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle|d\lambda\\ &= \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|\langle R(\lambda+i0)R_M,g_{+a}e^{ih^{-1}\Phi_+}\rangle|d\lambda\\ &\leq \|R_M\|_\alpha\|g_{+a}\|_\alpha\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\|R(\lambda+i0)\|_{\alpha,-\alpha}d\lambda\leq Ch^{M-\tilde{n}}.\end{aligned}$$

As we can take M as large as we wish, we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|J(\lambda,h)|d\lambda=O(h^\infty).$$

It remains to estimate $I(\lambda,h)$. For this purpose we write

$$\begin{aligned}I(\lambda,h) &= \int_0^T e^{ih^{-1}t\lambda}\langle e^{-ih^{-1}tH(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle dt\\ &+ \langle e^{-ih^{-1}T\lambda}R(\lambda+i0)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle\\ &= I_1(\lambda,T,h)+I_2(\lambda,T,h).\end{aligned}$$

From the estimate $|(1-\omega_b)(x,hD_x)(g_{-b}e^{ih^{-1}\Phi_-})|\leq C_Mh^M<x>^{-M}$ for every M, we deduce

$$I_1(\lambda,T,h)=\int_0^T e^{ih^{-1}t\lambda}\langle e^{-ih^{-1}tH(h)}g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle dt+O(h^\infty),$$

where $O(h^\infty)$ is independent of λ in any compact set. Therefore,

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\left|I_1(\lambda,T,h)-\int_0^T e^{ih^{-1}t\lambda}\langle e^{-ih^{-1}tH(h)}g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle dt\right|d\lambda=O(h^\infty)$$

and it suffices to show that $\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|I_2(\lambda,T,h)|d\lambda=O(h^N)$.
Let $\beta\in C^\infty$ be such that

$$\beta(x,\xi)=\begin{cases}1 \text{ in } \Gamma_+(R,d,0),\\ 0 \text{ in } \Gamma_+(R,d,-\sigma)^c\end{cases}$$

and write

$$\begin{aligned}
I_2(\lambda,T,h) &= \langle e^{-ih^{-1}T\lambda}R(\lambda+i0)\beta(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle\\
&+ \langle e^{-ih^{-1}T\lambda}R(\lambda+i0)(1-\beta)(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle\\
&= I_{+2}(\lambda,T,h)+I_{-2}(\lambda,T,h).
\end{aligned}$$

Taking $R\geq 20R_0+1$, we deduce from Lemma 3.5 that

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\parallel \chi_a R(\lambda+i0)\beta(x,hD_x)\parallel_{-\alpha,\alpha}d\lambda=O(h^\infty).$$

Moreover, applying Lemma B.1, we deduce

$$\parallel e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)(g_{-b}e^{ih^{-1}\Phi_-})\parallel_{L^2_\alpha}=O(1)$$

uniformly in λ. Hence

$$\begin{aligned}
\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|I_{+2}(\lambda,T,h)|\,d\lambda &= \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|\langle e^{-ih^{-1}t\lambda}R(\lambda+i0)\beta(x,hD_x)e^{-ih^{-1}tH(h)}\omega_b(x,hD_x)g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle|d\lambda\\
&\leq \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\parallel \chi_a R(\lambda+i0)\beta(x,hD_x)\parallel_{-\alpha,\alpha}d\lambda\times\\
&\qquad \sup_{\lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon]}\|e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)(g_{-b}e^{ih^{-1}\Phi_-})\|_{L^2_\alpha}\times\|g_{+a}e^{ih^{-1}\Phi_+}\|_{L^2_\alpha}\\
&\leq C_N h^N,\ \forall N.
\end{aligned}$$

Therefore it remains to estimate $I_{-2}(\lambda,T,h)$. Recall that we have the estimate

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\parallel R(\lambda+i0)\|_{\alpha,-\alpha}d\lambda=O(h^{-\tilde n}).$$

Moreover,

$$\begin{aligned}
\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}|I_{-2}(\lambda)|\,d\lambda &\leq \int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\|R(\lambda+i0)\|_{\alpha,-\alpha}\|g_{+a}\|_{L^2_\alpha}\|(1-\beta)(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)\|_{-\alpha,\alpha}\|g_{-b}\|_{L^2_{-\alpha}}d\lambda\\
&\leq C\|(1-\beta)(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)\|_{-\alpha,\alpha}\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\parallel R(\lambda+i0)\|_{\alpha,-\alpha}d\lambda\\
&\leq Ch^{-\tilde n}\|(1-\beta)(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)\|_{-\alpha,\alpha}
\end{aligned}$$

and it suffices to show that

$$\parallel (1-\beta)(x,hD_x)e^{-ih^{-1}TH(h)}\omega_b(x,hD_x)\parallel_{-\alpha,\alpha}=O(h^\infty). \tag{4.8}$$

By the construction of ω and from Lemma 4.3 we deduce the existence of $T_0>0$ such that

$$\forall T>T_0,\ \forall(x,\xi)\in supp(\omega_b),\ \Phi_T(x,\xi)\in\Gamma_+(R,d,0).$$

Then

$$\forall T>T_0,\ \forall(x,\xi)\in supp(\omega_b),\ \beta(\Phi_T(x,\xi))=1,$$

and for $T>T_0$, Lemma B.2 implies (4.8). ■

From the above proposition we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\left|G_0(\theta,\omega,\lambda,h)-ih^{-1}\int_0^{T_0}e^{ih^{-1}t\lambda}F(t,\theta,\omega,h)dt\right|d\lambda=O(h^\infty)$$

with

$$F(t,\theta,\omega,h)=\langle e^{ih^{-1}tH(h)}g_{-b}e^{ih^{-1}\Phi_-},g_{+a}e^{ih^{-1}\Phi_+}\rangle.$$

4.2 Second localization.

In this subsection we follow exactly the same construction as that of [12]. Introduce

$$Z_j(\lambda)=\{z\in\Lambda_\omega:|z-z_j(\lambda)|<\epsilon\},\ \ 1\le j\le l,$$

for $\epsilon>0$ small enough, and set

$$Y_j=\{y\in supp(g_{-b}):y=q_\infty(s,z_j(\lambda),\lambda,\omega),\ s<0,\ \lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon]\},$$

where $z_j(\lambda)=z_j(\lambda,\theta)$ is given by Propostion A.1. For R_0 large enough, one can find $S_1>S_0>>1$ such that

$$Y_j\subset\Pi_{-j}=\{y:y=q_\infty(s,z,\lambda,\omega),\ -S_1<s<-S_0,\ \lambda\in[\lambda_0-\epsilon,\lambda_0+\epsilon],\ z\in Z_j(\lambda)\}.$$

Put $\Sigma_-=\{x\in\mathbb{R}^n\ :\ \cos(x,\omega)<-\sigma_4\}$ and $\Sigma_+=\{x\in\mathbb{R}^n\ :\ |x|\ge R_1$ and $\cos(x,\theta)>-\sigma_2\}$ with $R_1>0$. For R_1 large enough independent of R_0,and for $y\in\Sigma_-$ and $x\in\Sigma_+$ we have

$$\lim_{t\to-\infty}p(t,y,\nabla_x\Phi_-(y))=\sqrt{2\lambda}\omega\text{ and }\lim_{t\to+\infty}p(t,x,\nabla_x\Phi_+(x))=\sqrt{2\lambda}\theta.$$

Hence from Hamilton-Jacobi theory we conclude that the set

$$\mathcal{E}=\{y\in\Sigma_-|\exists t\in\mathbb{R}\text{ such that }q(t,y,\nabla_x\Phi_-(y))=x\in\Sigma_+\text{ and }p(t,y,\nabla_x\Phi_-(y))=\nabla_x\Phi_+(x)\}$$

is included in $\bigcup_{j=1}^{l}\{q_\infty(s,z_j,\lambda,\omega),s\le S_0'\}$ for some $S_0'\in\mathbb{R}$. Set

$$\mathcal{F}=\{y\in supp(g_{-b})|\exists t\in\mathbb{R}\text{ such that }q(t,y,\nabla_x\Phi_-(y))=x\in supp(g_{+a})\text{ and }p(t,y,\nabla_x\Phi_-(y))=\nabla_x\Phi_+(x)\}.$$

As $\bigcup_{j=1}^{l}\{q_\infty(s,z_j,\lambda,\omega),-S_0\le s\le S_0'\}$ is a compact set, and $\mathcal{F}\subset\mathcal{E}\cap supp(g_{-b})\subset\mathcal{E}\cap\{|x|\ge 5R_0\}$, choosing R_0 large enough, we obtain

$$\mathcal{F}\subset\bigcup_{j=1}^{l}\{q_\infty(s,z_j(\lambda),\lambda,\omega),s<-S_0\}.$$

Moreover, $\mathcal{F}$ is bounded and for S_1 large enough we deduce

$$\mathcal{F}\subset\bigcup_{j=1}^{l}\{q_\infty(s,z_j(\lambda),\lambda,\omega),-S_1<s<-S_0\}.$$

Next we can find $\pi_{-j} \in C_0^\infty(\Pi_{-j})$, such that $0 \leq \pi_{-j} \leq 1$, and $\pi_{-j} = 1$ in Y_j. From the discussion above we see that for $\epsilon > 0$ small enough, the compact sets:

$$K_{\lambda_0} = \{\Phi_t(x, \nabla_x \Phi_-(x, \sqrt{2\lambda}\omega)),\ x \in supp(g_{-b}) \cap supp(1 - \sum_j \pi_{-j}),\ t \in [0, T_0],\ \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]\}$$

and

$$F_{\lambda_0} = \{(x, \nabla_x \Phi_+(x, \sqrt{2\lambda}\theta)),\ x \in supp(g_{+a})\}$$

are disjoint. Then Lemma B.2, gives the estimate

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \left| G_0(\theta, \omega, \lambda, h) - ih^{-1} \sum_{j=1}^{l} \int_0^{T_0} e^{ih^{-1}t\lambda} F_{-j}(t, \theta, \omega, h) dt \right| d\lambda = O(h^\infty)$$

with

$$F_{-j}(t, \theta, \omega, h) = \langle e^{ih^{-1}tH(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-}, g_{+a} e^{ih^{-1}\Phi_+} \rangle.$$

Similarly, we define

$$X_j = \{x \in supp(g_{+a}) : x = q(t, y, \nabla_x \Phi_-(y)),\ y \in Y_j,\ t > 0,\ \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]\}$$

and there exists $T_0 > T_1 >> 1$ such that

$$X_j \subset \Pi_{+j} = \{x : x = q(t, y, \nabla_x \Phi_+(y)),\ y \in \Pi_{-j},\ T_1 < t < T_0,\ \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]\}.$$

Thus we may construct $\pi_{+j} \in C_0^\infty(\Pi_{+j})$, such that $0 \leq \pi_{+j} \leq 1$ and $\pi_{+j} = 1$ in X_j. Repeating the above argument, we obtain

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon} \left| G_0(\theta, \omega, \lambda, h) - ih^{-1} \sum_{j=1}^{l} \int_0^{T_0} e^{ih^{-1}t\lambda} F_j(t, \theta, \omega, h) dt \right| d\lambda = O(h^\infty)$$

with

$$F_j(t, \theta, \omega, h) = \langle e^{ih^{-1}tH(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-}, \pi_{+j} g_{+a} e^{ih^{-1}\Phi_+} \rangle.$$

4.3 Approximation of the unitary group $\mathbf{e^{ih^{-1}tH(h)}}$.

For our analysis we need an approximation for $\psi_j(t, x, h) = e^{ih^{-1}tH(h)} \pi_{-j} g_{-b} e^{ih^{-1}\Phi_-}$, $x \in \Pi_{+j}$, $T_1 < t < T_0$. For the reader's convenience, we recall some results proved in [12].

Lemma 4.4 *The point $x = q(t, y, \nabla_x \Phi_-(y)) \in \Pi_{+j}$, with $y \in \Pi_{-j}$, is non-focal, that is*

$$D(t, y) = det(\frac{\partial}{\partial y} q(t, y, \nabla_x \Phi_-(y))) \neq 0,\ T_1 < t < T_0.$$

Using (3.26), we can write g_{-b} in the form

$$g_{-b}(x, \lambda, h) = ihg_{0b}(x, \lambda) + h^2 g_b(x, \lambda, h),$$

where

$$g_{0b}(x, \lambda) = \langle \nabla_x \Phi_-(x, \sqrt{2\lambda}\omega), \nabla \chi_b(x) \rangle b_{-0}(x, \sqrt{2\lambda}\omega), \text{ and } supp(g_b) \subset supp(\chi_b). \tag{4.9}$$

Similarly, (3.27) yields

$$g_{+a}(x, \lambda, h) = ihg_{0a}(x, \lambda) + h^2 g_a(x, \lambda, h)$$

with

$$g_{0a}(x,\lambda) = \langle \nabla_x \Phi_+(x, \sqrt{2\lambda}\theta), \nabla\chi_a(x)\rangle a_{+0}(x, \sqrt{2\lambda}\theta), \text{ and } supp(g_a) \subset supp(\chi_a). \tag{4.10}$$

From Lemma 4.4 we deduce the following representation of $\psi_j(t,x,\lambda,h)$, $T_1 < t < T_0$, $x \in \Pi_{+j}$ which is uniform with respect to $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ (cf. [10], sect.12) :

$$\psi_j(t,x,\lambda,h) = e^{ih^{-1}S_j(t,y,\lambda) - i\mu_j\frac{\pi}{2}} |D(t,y)|^{-\frac{1}{2}} ih\pi_{-j}(y) g_{0b}(y,\lambda) + O(h), \tag{4.11}$$

for $x = q(t,y,\nabla_x\Phi_-(y))$, $y \in \Pi_{-j}$. Here S_j is the action along the trajectory joining the points x and y, that is

$$S_j(t,y,\lambda) = \Phi_-(y,\sqrt{2\lambda}\omega) + \int_0^t \left(\frac{1}{2}|p(\tau,y,\nabla_x\Phi_-)|^2 - V(q(\tau,y,\nabla_x\Phi_-)) \right) d\tau, \tag{4.12}$$

and $\mu_j \in \mathbb{Z}$ is the Maslov index of this trajectory.

Notice that μ_j is locally constant, so for $\epsilon > 0$ small enough, μ_j is independent of $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$. We insert (4.11) into the expression of F_j, and after a change of variables $x = q(t,y,\nabla_x\Phi_-) \to y$, we get

$$F_j(t,\theta,\omega,\lambda,h) = h^2 \int e^{ih^{-1}\phi_j(t,y) - i\mu_j\frac{\pi}{2}} M_j(t,y) |D(t,y)|^{\frac{1}{2}} dy + O(h^3), \tag{4.13}$$

where

$$\phi_j(t,y,\lambda) = S_j(t,y,\lambda) - \Phi_+(q(t,y,\nabla_x\Phi_-), \sqrt{2\lambda}\theta)$$

and

$$M_j(t,y,\lambda) = \pi_{-j}(y) g_{0b}(y,\lambda) \pi_{+j}(q(t,y,\nabla_x\Phi_-)) g_{0a}(q(t,y,\nabla_x\Phi_-),\lambda).$$

Thus the proof of the theorem is reduced to the study on the asymptotic behavior of the integral

$$L_j(t,\theta,\omega,\lambda,h) = h^2 \int e^{ih^{-1}\phi_j(t,y) - i\mu_j\frac{\pi}{2}} M_j(t,y) |D(t,y)|^{\frac{1}{2}} dy.$$

5 Stationary phase method.

We will study the asymptotics of the integral

$$N_j(\theta,\omega,\lambda,h) = \int_{T_1}^{T_0} e^{ih^{-1}t\lambda} L_j(t,\theta,\omega,\lambda,h) dt, \ 1 \le j \le l.$$

The direction θ is regular for (λ_0,ω), so for every $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ we can make a change of variables

$$(z,s) \in Z_j \times] - S_1, -S_0[\longrightarrow y = q_\infty(s,z,\lambda,\omega) \in \Pi_{-j}.$$

We obtain

$$N_j = h^2 \int_{T_1}^{T_0} \int_{-S_1}^{-S_0} e^{ih^{-1}t\lambda - i\mu_j\frac{\pi}{2}} I_j(t,s,\theta,\omega,\lambda,h) ds dt,$$

with

$$I_j(t,s,\theta,\omega,\lambda,h) = \int e^{ih^{-1}\Phi_j(t,s,z)} f_j(t,s,z) |D_\infty(t+s,z)|^{\frac{1}{2}} |D_\infty(s,z)|^{\frac{1}{2}} dz,$$

$$\Phi_j(t,s,z,\lambda) = S_j(t, q_\infty(s,z,\lambda,\omega),\lambda) - \Phi_+(q_\infty(t+s,z,\lambda,\omega), \sqrt{2\lambda}\theta),$$

$$f_j(t,s,z,\lambda) = \pi_{-j}(q_\infty(s,z,\lambda,\omega)) g_{0b}(q_\infty(s,z,\lambda,\omega),\lambda) \pi_{+j}(q_\infty(t+s,z,\lambda,\omega)) g_{0a}(q_\infty(t+s,z,\lambda,\omega),\lambda).$$

Now let us apply the stationary phase method to the integral I_j. As in [12], for (t,s,λ) fixed, the only stationary point of the phase $\Phi_j(t,s,\lambda,z)$ is $z_j(\lambda)$. We refer to Theorem 7.7.6 in [7] for the stationary phase method for integral depending on parameters. We apply this theorem at each $(z_j(\lambda_0),\lambda_0)$ and we use the following.

Lemma 5.1 *For every $j=1,\dots,N$, we have*

1) $sgn(\frac{\partial^2\Phi_j}{\partial z^2})(z_j(\lambda))=n-1$,
2) $|det(\frac{\partial^2\Phi_j}{\partial z^2})(z_j(\lambda))|=(2\lambda)^{\frac{n-2}{2}}\hat{\sigma}_j(z_j(\lambda),\lambda)D_\infty(t+s,z_j(\lambda))E_0(t,s)$,
where $E_0(t,s)=exp\left(\int_{t+s}^{\infty}\Delta_x\Phi_+(q_\infty(\tau,z_j(\lambda),\lambda,\omega),\sqrt{2\lambda}\theta)d\tau\right)$.

Finally, we get

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\left|I_j(t,s,\theta,\omega,\lambda,h)-c_3(\lambda,h)e^{ih^{-1}(S_j-t\lambda)}\chi_0(t,s)\right|d\lambda=O(h),$$

with

$$\int_{T_1}^{T_0}\int_{-S_1}^{-S_0}\chi_0(t,s)dtds=-1,\text{ et } c_3(\lambda,h)=(2\lambda)^{-\frac{n-3}{4}}(2\pi h)^{\frac{n-1}{2}}e^{i(n-1)\frac{\pi}{4}}$$

and

$$\int_{\lambda_0-\epsilon}^{\lambda_0+\epsilon}\left|N_j(\theta,\omega,\lambda,h)-c_2(\lambda,h)e^{ih^{-1}S_j-i\mu_j\frac{\pi}{2}}\hat{\sigma}_j(z_j(\lambda),\lambda)^{-\frac{1}{2}}\right|d\lambda=O(h),$$

with

$$c_2(\lambda,h)=-(2\lambda)^{-\frac{n-3}{4}}(2\pi h)^{\frac{n-1}{2}}h^2e^{i(n-1)\frac{\pi}{4}}.$$

This equality combined with (3.23) and (4.2), completes the proof of Theorem 1.1.

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Appendix.

A A regularity result.

The following Proposition, announced in Introduction, examines the properties of $z\in\Xi_{\lambda,\theta}$ when λ and θ vary .

Proposition A.1 *Let $\lambda_0>0$, and $(\omega,\theta)\in S^{n-1}\times S^{n-1}$. Assume $(\mathbf{H}_{\omega,\lambda_0})$, and let θ be regular for (λ_0,ω). Then there exist a neighbourhood $]\lambda_0-\epsilon,\lambda_0+\epsilon[$ of λ_0, a neighbourhood U of θ, and functions $z^j\in C^\infty(]\lambda_0-\epsilon,\lambda_0+\epsilon[\times U\,,\,\Lambda_\omega)$, $j=1,\dots,l$ such that we have:*

1) $\forall\lambda\in]\lambda_0-\epsilon,\lambda_0+\epsilon[$, $\forall\theta'\in U$, *θ' is regular for* (λ,ω).

2) $\forall j\in\{1,\dots,d\}$, $\forall\lambda\in]\lambda_0-\epsilon,\lambda_0+\epsilon[$, *$z^j(\lambda,.)$ is a diffeomorphism from U onto its image $U_j(\lambda)$, and $U_i(\lambda)\cap U_j(\lambda)=\emptyset$, for $i\neq j$.*

3) $\forall\theta'\in U$, $\forall\lambda\in]\lambda_0-\epsilon,\lambda_0+\epsilon[$, $\forall z\in\Lambda_\omega$,

$$\xi_\infty(z,\lambda)=\theta'\Longleftrightarrow\exists j\in\{1,\dots,l\},\ z\in U_j(\lambda),\ \text{and}\ \ z=z^j(\lambda,\theta')$$

.

Proof: According to Remark 1.1, there exist $z_1^0, \dots, z_l^0$ such that $\xi_\infty(z_j^0, \lambda_0) = \theta$. Let

$$\begin{aligned}\Phi : \mathbb{R} \times \Lambda_\omega &\longrightarrow \mathbb{R} \times S^{n-1},\\ (\lambda, z) &\longmapsto (\lambda, \xi_\infty(z, \lambda)),\end{aligned}$$

and let θ be regular for (λ_0, ω). Then $\forall j \in \{1, \dots, l\}$, the map $D_{(\lambda_0, z_j^0)}\Phi$ is invertible. From the local inversion theorem, we deduce that there exist $\epsilon > 0$, an open neighbourhood U of θ in S^{n-1}, and disjoint open sets $\tilde{U}_j$ containing (λ_0, z_j^0) such that

$$\forall j \in \{1, \dots, l\},\ \Phi_{|\tilde{U}_j} \text{ is a diffeomorphism from } \tilde{U}_j \text{ onto }]\lambda_0 - \epsilon, \lambda_0 + \epsilon[\times U.$$

Denote $\pi_x : \mathbb{R} \times \Lambda_\omega \longrightarrow \Lambda_\omega$, the canonical projection and set $U_j = \pi_x(\tilde{U}_j)$. Then for ϵ and U small enough, we can assume that

$$\forall (\lambda, \theta') \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[\times U,\ \xi_\infty(z, \lambda) = \theta' \Longrightarrow z \in \bigcup_{j=1}^{l} U_j.$$

Otherwise, there will exist some sequences $(\lambda_k)_{k\in\mathbb{N}}$,$(\theta_k)_{k\in\mathbb{N}}$, and $(y_k)_{k\in\mathbb{N}}$ such that

$$\begin{aligned}&\lambda_k \in \mathbb{R} \text{ and } \lim_{k\to+\infty} \lambda_k = \lambda_0,\\ &\theta_k \in S^{n-1} \text{ and } \lim_{k\to+\infty} \theta_k = \theta,\\ &y_k \notin \bigcup_{j=1}^{l} U_j,\ \xi_\infty(y_k, \lambda_k) = \theta_k.\end{aligned}$$

As in Remark 1.1, we conclude that $(y_k)_{k\in\mathbb{N}}$ is bounded. Consequently, we may suppose that (y_k) converges to $y \notin \bigcup_{j=1}^{l} U_j$. This implies immediately $\xi_\infty(y, \lambda) = \theta$, with $y \notin \{z_j^0, j = 1, \dots, l\}$, which is not possible.
This completes the proof of **3)**.

Set $z_j = \pi_x \circ \Phi^{-1}_{|\tilde{U}_j}$. These functions are clearly C^∞, and for λ in $]\lambda_0 - \epsilon, \lambda_0 + \epsilon[$, $z_j(\lambda, .)$ is a diffeomorphism from U onto its image. For ϵ and U small enough, we can assume that

$$\forall (\lambda, \mu) \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[^2,\ \forall (i, j) \in \{1, \dots, l\}^2 \text{ avec } i \neq j,\ z_i(\lambda, U) \cap z_j(\mu, U) = \emptyset$$

and this implies **2)**.

Take $\lambda \in]\lambda_0 - \epsilon, \lambda_0 + \epsilon[$ and $\theta' \in U$. Let z be such that $\xi_\infty(z, \lambda) = \theta'$. Hence for ϵ and U small enough, z is close to one of the points z_j^0 and, as $\hat{\sigma}(\lambda_0, z_j^0) \neq 0$, by continuity we obtain $\hat{\sigma}(\lambda, z) \neq 0$, and θ' will be regular for (λ, ω). ■

B Egorov lemma for operators in weighted spaces.

For k and l in $\mathbb{N}$, consider the weighted Sobolev spaces

$$H_k^l(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) | \forall \alpha \in \{0, \dots, l\},\ < x >^k \partial_x^\alpha f \in L^2(\mathbb{R}^n)\}.$$

Our aim is to study the action of the propagator $e^{-ith^{-1}H(h)}$ associated to the self-adjoint operator $H(h) = -\frac{1}{2}h^2\delta + V(x)$ with domain $H^2(\mathbb{R}^n)$. Denote by $||.||$ the L^2 norm. We have the following.

Lemma B.1 *Let* $f \in H_k^{l+k}(\mathbb{R}^n)$, $l, k \in \mathbb{N}$. *Then* $\forall t \in \mathbb{R}$, $e^{-ith^{-1}H(h)}f \in H_k^l(\mathbb{R}^n)$, *and*

$$\forall \alpha \in \{0, \dots, l\},\ \| < x >^k D_x^\alpha e^{-ith^{-1}H(h)} f \| \leq C < t >^k \sum_{|\beta| \leq |l|+k} \| < x >^k D_x^\beta f \|$$

with C *independent of* $h \in [0,1]$.

Proof : We will proceed by an induction on $k \in \mathbb{N}$.
1) The case $k = 0$ is a consequence of the fact that $e^{-ith^{-1}H(h)}$ is an isometry on $H^l(\mathbb{R}^n)$.
2) Assume the property true for k and let f be in some H_{k+1}^{l+k+1}, where l is in $\mathbb{N}^n$.
We have

$$\begin{aligned} < x >^{k+1} D_x^l e^{-ith^{-1}H(h)} f &= D_x^l < x >^{k+1} e^{-ith^{-1}H(h)} f + [< x >^{k+1}, D_x^l] e^{-ith^{-1}H(h)} f \\ &= D_x^l e^{-ith^{-1}H(h)} < x >^{k+1} f + ihD_x^l \int_0^t e^{-ish^{-1}H(h)} [\Delta, < x >^{k+1}] e^{i(s-t)h^{-1}H(h)} f ds \\ &\quad - \sum_{0 \neq \beta \leq l} \partial_x^\beta (< x >^{k+1}) \partial_x^{l-\beta} e^{-ith^{-1}H(h)} f. \end{aligned}$$

From the case $k = 0$ we know that

$$\begin{aligned} \| D_x^l e^{-ith^{-1}H(h)} < x >^{k+1} f \| &\leq \| D_x^l < x >^{k+1} f \| \leq \sum_{|\beta| \leq |l|+k+1} \binom{\alpha}{\beta} \| \partial_x^\beta (< x >^{k+1}) \partial_x^{\alpha-\beta} f \| \\ &\leq \sum_{|\beta| \leq |l|+k+1} C_{\alpha,\beta} \| < x >^{k+1-|\beta|} \partial_x^{\alpha-\beta} f \| \leq C_{k+1} \sum_{|\beta| \leq |l|+k+1} \| < x >^{k+1} \partial_x^\beta f \| \end{aligned} \tag{B.1}$$

Moreover, for $|\beta| \neq 0$, we have $|\partial_x^\beta (< x >^{k+1})| \leq C_\beta < x >^{k+1-|\beta|} \leq C_\beta < x >^k$, and we obtain the following estimate

$$\| \sum_{0 \neq \beta \leq l} \partial_x^\beta (< x >^{k+1}) \partial_x^{\alpha-\beta} e^{-ith^{-1}H(h)} f \| \leq \sum_{0 \neq \beta \leq l} C_\beta \| < x >^k \partial_x^{\alpha-\beta} e^{-ith^{-1}H(h)} f \|.$$

By induction we deduce

$$\| \sum_{0 \neq \beta \leq l} \partial_x^\beta (< x >^{k+1}) \partial_x^{l-\beta} e^{-ith^{-1}H(h)} f \| \leq C_{k+1} \sum_{0 \neq \beta \leq l} \| < x >^k \partial_x^\beta f \|. \tag{B.2}$$

To complete the proof, we need to estimate the term $D_x^l \int_0^t e^{-ish^{-1}H(h)} [\Delta, < x >^{k+1}] e^{i(s-t)h^{-1}H(h)} f ds$.
Observe that $[< x >^{k+1}, \partial_{x_j}^2] = -\partial_{x_j}^2 (< x >^{k+1}) - 2\partial_{x_j} (< x >^{k+1}) \partial_{x_j}$. Thus, using $|\partial_{x_j}^2 (< x >^{k+1}$ $)| \leq C < x >^{k-1}$, and $|\partial_{x_j} (< x >^{k+1})| \leq C < x >^k$, we obtain

$$\begin{aligned} \| D_x^l e^{-ish^{-1}H(h)} [\Delta, < x >^{k+1}] e^{i(s-t)h^{-1}H(h)} f \| &\leq C \| \partial_x^l < x >^{k-1} e^{i(s-t)h^{-1}H(h)} f \| \\ &\quad + C \sup_j \| \partial_x^l < x >^k \partial_{x_j} e^{i(s-t)h^{-1}H(h)} f \|. \end{aligned}$$

We write once more

$$\partial_x^l < x >^{k-1} = \sum_{\beta \leq l} \binom{l}{\beta} \partial_x^\beta (< x >^{k-1}) \partial_x^{l-\beta}$$

and

$$\partial_x^l < x >^k \partial_{x_j} = \sum_{\beta \leq l} \binom{l}{\beta} \partial_x^\beta (< x >^k) \partial_x^{l-\beta} \partial_{x_j}.$$

As $|\partial_x^\beta(< x >^{k-1})| \leq C < x >^{k-1-|\beta|} \leq C < x >^{k-1}$, then, by induction, we may apply the hypothesis to the first term to get

$$\|\partial_x^l < x >^{k-1} e^{i(s-t)h^{-1}H(h)} f\| \leq C < s-t >^{k-1} \sum_{|\beta|\leq |l|+k-1} \|< x >^{k-1} D_x^\beta f\| . \tag{B.3}$$

Similarly, we have

$$\|\partial_x^l < x >^k \partial_{x_j} e^{i(s-t)h^{-1}H(h)} f\| \leq C < s-t >^k \sum_{|\beta|\leq |l|+k+1} \|< x >^k D_x^\beta f\| . \tag{B.4}$$

Summing (B.3) and (B.4), and integrating the sum, we deduce

$$\|D_x^l \int_0^t e^{-ish^{-1}H(h)}[\Delta, < x >^{k+1}] e^{i(s-t)h^{-1}H(h)} f ds\| \leq C < t >^{k+1} \sum_{|\beta|\leq |l|+k+1} \|< x >^k D_x^\beta f\| . \tag{B.5}$$

Finally, using (B.1), (B.2) and (B.5), we obtain

$$\|< x >^{k+1} D_x^l e^{-ith^{-1}H(h)} f \| \leq C < t >^{k+1} \sum_{|\beta|\leq |l|+k+1} \|< x >^{k+1} D_x^\beta f \|$$

and the proof of the lemma is complete. ■

Now we introduce a class of pseudo-differential operators

$$A_u^m(\mathbb{R}^n \times \mathbb{R}^n) = \{p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \, : \, \forall \alpha, \beta \in \mathbb{N}^n, \, |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C < x >^{m-|\alpha|} < \xi >^{u-|\beta|}\}.$$

Remark that A_u^m is included in the Hormander's class $S_{1,0}^u$, [7]. Then it is easy to prove the following propositions.

Proposition B.1 *Let (m_1, m_2) and (u_1, u_2) be in $\mathbb{N} \times \mathbb{N}$. Let $a \in A_{u_1}^{m_1}$ and $b \in A_{u_2}^{m_2}$. Then*

$$a(x, hD_x) \circ b(x, hD_x) = c(x, hD_x), \text{ with } c \in A_{u_1+u_2}^{m_1+m_2}.$$

Proposition B.2 *Let m and u be in $\mathbb{N}$ and let $p \in A_u^m$. Then*

$$p(x, hD_x) \, : \, H_k^l \to H_{k-u}^{l-m}$$

is continuous.

After this preparation we are able to prove our weighted Egorov Lemma.

Lemma B.2 *Let $w \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $p \in A_u^m(\mathbb{R}^n \times \mathbb{R}^n)$, $(m, u) \in \mathbb{R} \times \mathbb{R}$. Assume that for $t \geq 0$, $p(x,\xi)$ vanishes on $\Phi_t(supp(\omega))$. Then we have*

$$\forall \alpha \in \mathbb{N}^n, \, \| p(x, hD_x) e^{-ith^{-1}H(h)} \omega(x, hD_x) \|_{-\alpha,\alpha} = O(h^\infty),$$

where $\| A \|_{\alpha,\beta}$ denotes the norm of operators $A \, : \, L_\alpha^2 \to L_\beta^2$.

Proof : Using Lemma B.1, we reduce the proof to the case $\alpha = 0$. Indeed,

$$\|p(x, hD_x) e^{-ith^{-1}H(h)} \omega(x, hD_x)\|_{-\alpha,\alpha} = \| < x >^\alpha p(x, hD_x) e^{-ith^{-1}H(h)} \omega(x, hD_x) < x >^\alpha \|,$$

where $\|.\| = \|.\|_{0,0} = \|.\|_{L^2,L^2}$
On the other hand,

$$< x >^\alpha p(x, hD_x) = \tilde{p}(x, hD_x), \text{ with } \tilde{p} \in A_u^{m+\alpha}$$

and

$$\omega(x, hD_x) < x >^\alpha = \omega_N(x, hD_x) + h^N r_N(x, hD_x),$$

where $\omega_N \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and $r_N \in A_{-N}^{-N}$. For N large enough, we deduce from Lemma B.1 that

$$\|\tilde{p}(x, hD_x)e^{-ith^{-1}H(h)}r_N(x, hD_x)\| = O(1).$$

So in the following we can assume $\alpha = 0$. We may also suppose that $m, u \in \mathbb{N}$. From the calculus in part 11 of [4], we know that

$$e^{-ith^{-1}H(h)}\omega(x, hD_x)e^{ith^{-1}H(h)} = \omega_N(x, hD_x) + h^N R_{N,t}$$

with $supp(\omega_N) \subset \Phi_t(supp(\omega))$ and

$$R_{N,t} = \int_0^t e^{i(t-\tau)h^{-1}H(h)}r_N(x, hD_x)e^{-i(t-\tau)h^{-1}H(h)}d\tau,$$

where $r_N \in A_{-N}^{-N}$. Therefore

$$\begin{aligned}\|p(x, hD_x)e^{-ith^{-1}H(h)}\omega(x, hD_x)\| &= \|p(x, hD_x)e^{-ith^{-1}H(h)}\omega(x, hD_x)e^{-ith^{-1}H(h)}\| \\ &\leq \|p(x, hD_x)\omega_N(x, hD_x)\| + h^N\|p(x, hD_x)R_{N,t}\|.\end{aligned}$$

The property $supp(p) \cap supp(\omega_N) = \emptyset$, implies $\|p(x, hD_x)\omega_N(x, hD_x)\| = O(h^\infty)$, and it remains to show that $\|p(x, hD_x)R_{N,t}\| = O(1)$. Next we have

$$\|p(x, hD_x)e^{i(t-\tau)h^{-1}H(h)}r_N(x, hD_x)\| \leq \|p(x, hD_x)\|_{H_u^m,L^2}\|e^{i(t-\tau)h^{-1}H(h)}\|_{H_u^{m+u},H_u^m}\|r_N(x, hD_x)\|_{L^2,H_u^{m+u}}.$$

As $p \in A_u^m$ and $r_N \in A_{-N}^N$,we get $\|p(x, hD_x)\|_{H_u^m,L^2} = O(1)$ and $\|r_N(x, hD_x)\|_{L^2,H_u^{m+u}} = O(1)$. Moreover, we deduce from Lemma B.1 the estimate $\|e^{i(t-\tau)h^{-1}H(h)}\|_{H_u^{m+u},H_u^m} \leq C < t - \tau >^m$. Combining the last estimates, we obtain

$$\|p(x, hD_x)e^{i(t-\tau)h^{-1}H(h)}r_N(x, hD_x)e^{-i(t-\tau)h^{-1}H(h)}\| \leq C < t - \tau >^m$$

and by integration we conclude that $\|R_{N,t}\| = O(1)$. This completes the proof of Lemma B.2. ■

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