



Scattering theory for the Schrödinger equation with repulsive potential

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Abstract

We consider the scattering theory for the Schrödinger equation with $-\Delta - |x|^\alpha$ as a reference Hamiltonian, for $0 < \alpha \leq 2$, in any space dimension. We prove that, when this Hamiltonian is perturbed by a potential, the usual short range/long range condition is weakened: the limiting decay for the potential depends on the value of α , and is related to the growth of classical trajectories in the unperturbed case. The existence of wave operators and their asymptotic completeness are established thanks to Mourre estimates relying on new conjugate operators. We construct the asymptotic velocity and describe its spectrum. Some results are generalized to the case where $-|x|^\alpha$ is replaced by a general second order polynomial.

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Résumé

Nous considérons la théorie de la diffusion pour l'équation de Schrödinger ayant $-\Delta - |x|^\alpha$ pour hamiltonien de référence, avec $0 < \alpha \leq 2$, en toute dimension d'espace. Nous démontrons que lorsque cet hamiltonien est perturbé par un potentiel, la notion habituelle de courte portée/longue

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portée est affaiblie : la décroissance limite de la perturbation dépend de la valeur de α , et est liée à la vitesse des trajectoires classiques dans le cas non perturbé. Nous établissons l’existence d’opérateurs d’ondes ainsi que leur complétude asymptotique grâce à des estimations de Mourre reposant sur de nouveaux opérateurs conjugués. En outre, nous construisons la vitesse asymptotique et nous décrivons son spectre. Enfin, nous généralisons certains résultats au cas où $-|x|^\alpha$ est remplacé par un polynôme du second degré.

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1. Introduction

The aim of this paper is to study the scattering theory for a large class of Hamiltonians with repulsive potential. We find optimal short range conditions for the perturbation, and prove asymptotic completeness under these conditions. The family of Hamiltonians is given by:

$$H_{\alpha,0} = -\Delta - |x|^\alpha, \quad 0 < \alpha \leq 2; \quad H_\alpha = H_{\alpha,0} + V_\alpha(x); \quad x \in \mathbb{R}^n, \quad n \geq 1. \quad (1.1)$$

The main new feature with respect to the usual free Schrödinger operator $H_{0,0} = -\Delta$ is the acceleration due to the potential $-|x|^\alpha$. The case $\alpha = 2$ is a borderline case: if $\alpha > 2$ classical trajectories reach infinite speed and $(H_{\alpha,0}, C_0^\infty(\mathbb{R}^n))$ is not essentially self-adjoint (see [12]).

The consequence of the acceleration is that the usual position variable increases faster than t along the evolution. Roughly speaking, the usual short range condition is:

$$|V_0(x)| \lesssim \langle x \rangle^{-1-\varepsilon}, \quad (1.2)$$

for some $\varepsilon > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. One expects it to be weakened in the case of H_α .

For the Stark Hamiltonian, associated to a constant electric field $E \in \mathbb{R}^n$ (see [8]), $-\Delta + E \cdot x$, it is well known that the short range condition (1.2) becomes $|V_s(x)| \lesssim \langle E \cdot x \rangle^{-1/2-\varepsilon}$. We refer to the papers by J.E. Avron and I.W. Herbst [2,18] for weaker conditions. The idea is that the drift caused by E (which may also model gravity, see, e.g., [33]) accelerates the particles in the direction of the electric field. This phenomenon has been observed for a larger class of Hamiltonians by M. Ben-Artzi [3,4]: generalizing the Stark Hamiltonian ($\alpha = 1$), let

$$\widehat{H}_{\alpha,0} = -\Delta - \operatorname{sgn}(x_1)|x_1|^\alpha, \quad 0 < \alpha \leq 2; \quad \widehat{H}_\alpha = \widehat{H}_{\alpha,0} + \widehat{V}_\alpha(x),$$

with $x = (x_1, x')$. In [4], asymptotic completeness is proved under the condition:

$$|\widehat{V}_\alpha| \lesssim M(x') \cdot \begin{cases} \langle x_1 \rangle^{\alpha-\varepsilon} & \text{for } x_1 \leq 0, \\ \langle x_1 \rangle^{-1+\alpha/2-\varepsilon} & \text{for } x_1 \geq 0, \end{cases} \quad (1.3)$$

with $\varepsilon > 0$ and $M(x') \rightarrow 0$ as $|x'| \rightarrow \infty$. In the one-dimensional case, we obtain similar results for $0 < \alpha < 2$, and a weaker condition for $\alpha = 2$. The proofs in [3,4] rely on some specific properties of one-dimensional Hamiltonians, and it seems they cannot be adapted to (1.1) when $n \geq 2$. Our approach is completely different, since it is based on Mourre estimates.

Notice that for $\alpha \in]0, 2]$, the Hamiltonian $H_{\alpha,0}$ shares an interesting difficulty with the Stark Hamiltonian: its symbol, $|\xi|^2 - |x|^\alpha$, is not signed, and can take arbitrarily large negative values.

The case $\alpha = 2$ is in some sense very instructive. A nonlinear scattering theory is already available in this case. In [7], the second author studied nonlinear perturbations of $H_{2,0}$ and showed that all the usual nonlinearities are short range. This is closely related to the fact that the classical trajectories can be computed explicitly:

$$x(t) = \frac{1}{2}(x_0 + \xi_0) e^{2t} + \frac{1}{2}(x_0 - \xi_0) e^{-2t}.$$

Thus $x(t)$ grows exponentially fast (in general). For $0 < \alpha < 2$, a formal computation shows that the classical trajectory $x(t)$ can go to infinity like $t^{1/(1-\alpha/2)}$; denoting $U_\alpha(x) = -|x|^\alpha$, the equations of motion imply:

$$0 = \ddot{x}(t) + \nabla U_\alpha(x(t)) = \ddot{x}(t) - \alpha|x(t)|^{\alpha-2}x(t).$$

Seeking a particular solution of the form $x(t) = t^\kappa y$, for a constant $y \in \mathbb{R}^n$, yields $\kappa - 2 = (\alpha - 1)\kappa$, hence $\kappa = 2/(2 - \alpha)$. We will prove that in general, $x(t)$ does go to infinity like $t^{1/(1-\alpha/2)}$. This shows that the acceleration caused by $-|x|^\alpha$ increases progressively as α ranges $]0, 2]$. For a small $\alpha > 0$, the particle moves hardly faster than in the free case $|x(t)| = \mathcal{O}(t)$. As α increases, the particle goes to infinity faster and faster, and reaches the maximal exponential growth for $\alpha = 2$. For $\alpha > 2$, it is known that particles can reach an infinite speed, which is the reason why $(H_{\alpha,0}, C_0^\infty(\mathbb{R}))$ is not essentially self-adjoint. This suggests to define as a new position variable,

$$p_\alpha(x) = \begin{cases} \ln\langle x \rangle & \text{for } \alpha = 2, \\ \langle x \rangle^{1-\alpha/2} & \text{for } 0 < \alpha < 2. \end{cases} \tag{1.4}$$

We assume that the multiplication potential $V_\alpha(x)$ is real-valued, and writes $V_\alpha(x) = V_\alpha^1(x) + V_\alpha^2(x)$, with:

$$V_\alpha^1 \text{ is a measurable real-valued function, compactly supported, and } \Delta\text{-compact,} \tag{1.5}$$

and $V_\alpha^2 \in L^\infty(\mathbb{R}^n; \mathbb{R})$ satisfies the *short range condition*:

$$|V_\alpha^2(x)| \lesssim p_\alpha(x)^{-1-\varepsilon}, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1.6}$$

for some $\varepsilon > 0$.

The operator H_α is essentially self-adjoint, with domain the domain of the harmonic oscillator, and we denote again H_α its self-adjoint extension. In Section 2, we prove that

H_α has no singular continuous spectrum and $\sigma(H_\alpha) = \mathbb{R}$. Under general assumptions (see Theorem 2.8), we also show that its point spectrum is empty. We can now state the main result of this paper:

Theorem 1.1 (Asymptotic completeness). *Let $0 < \alpha \leq 2$, and $H_{\alpha,0}, H_\alpha$ defined by (1.1). Assume that $V_\alpha = V_\alpha^1 + V_\alpha^2$ satisfies (1.5) and (1.6). Then the following limits exist:*

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} e^{-itH_{\alpha,0}}, \tag{1.7}$$

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} e^{-itH_\alpha} \mathbf{1}^c(H_\alpha), \tag{1.8}$$

where $\mathbf{1}^c(H_\alpha)$ is the projection on the continuous spectrum of H_α . If we denote (1.7) by Ω^+ , then (1.8) is equal to $(\Omega^+)^*$, and we have:

$$(\Omega^+)^* \Omega^+ = \mathbf{1} \quad \text{and} \quad \Omega^+ (\Omega^+)^* = \mathbf{1}^c(H_\alpha). \tag{1.9}$$

In the case $\alpha = 2$, Korotyaev [25] has shown, with a different approach, the asymptotic completeness under the hypothesis $|V_\alpha^2(x)| \lesssim \langle x \rangle^{-\varepsilon}$, with $\varepsilon > 0$.

To prove this result, we establish Mourre estimates, relying on new conjugate operators, adapted to the repulsive potential $-|x|^\alpha$ (as a matter of fact, we work with the smoother repulsive potential $-\langle x \rangle^\alpha$; see below). To give an idea of the difficulty at this stage, consider the one-dimensional case. For $\alpha = 2$, the natural idea for a conjugate operator is to consider the generator of dilations $(xD + Dx)/2$:

$$i[H_{2,0}, (xD + Dx)/2] = -2\Delta + 2x^2.$$

This is the harmonic oscillator, which is of course positive. This seems an encouraging point. Nevertheless, it is not $H_{2,0}$ -bounded; we must find another conjugate operator. Therefore, we look for a pseudo-differential operator A_2 with symbol $a_2(x, \xi)$, and try to solve:

$$\{\xi^2 - x^2, a_2(x, \xi)\} = 4, \quad \text{on the energy level } \{(x, \xi); \xi^2 - x^2 = E\}.$$

A solution to this equation is given by: $a_2(x, \xi) = \ln(\xi + x) - \ln(\xi - x)$. Now consider the case $0 < \alpha < 2$. For $x > 0$, we try to solve:

$$\{\xi^2 - x^\alpha, a_\alpha(x, \xi)\} = 2 - \alpha, \quad \text{on } \{(x, \xi) \in \mathbb{R}_+^* \times \mathbb{R}; \xi^2 - x^\alpha = E\}.$$

Plugging $a_\alpha(x, \xi) = \xi x^{1-\alpha}$ into this equation, we get:

$$\{\xi^2 - x^\alpha, a_\alpha(x, \xi)\} = 2 - \alpha + 2E(1 - \alpha)x^{-\alpha}, \quad \text{for } \xi^2 - x^\alpha = E.$$

The term in $x^{-\alpha}$ should not matter for the Mourre estimate, since it is compact on the energy level. This formal discussion is the foundation for the constructions of Section 3.3.

To apply Mourre’s method, truncations in energy are needed, of the form $\chi(-\Delta - |x|^\alpha)$. However, $H_{\alpha,0}$ is not elliptic, so it is not clear that this defines a good pseudo-differential operator. These difficulties are solved in Section 3, where we consider the general case.

Theorem 1.1 shows how the short range condition (1.6) takes the acceleration caused by the potential into account. Let us note that the condition (1.6) is not necessarily the weakest one, but the decay $|V(x)| \lesssim p_\alpha(x)^{-1}$ at infinity is expected to be the borderline case between long range and short range scattering, because the position variable increases exactly like t along the evolution (compare with Theorem 1.2). For the case of the Stark Hamiltonian, it is well-known that the case $\varepsilon = 0$ in (1.3) is the limiting case, which involves long range effects (see [27]).

We obtain more precise informations by constructing the asymptotic velocity (see, e.g., [10]). We note $C_\infty(\mathbb{R}^n)$ the set of continuous functions which go to 0 at ∞ . Let $B^m := (B_1^m, \dots, B_n^m)$ be a sequence of commuting self-adjoint operators on a Hilbert space \mathcal{H} . Suppose that for every $g \in C_\infty(\mathbb{R}^n)$, there exists

$$s\text{-}\lim_{m \rightarrow \infty} g(B^m). \tag{1.10}$$

Then by [10, Proposition B.2.1], there exists a unique vector $B = (B_1, \dots, B_n)$ of commuting self-adjoint operators such that (1.10) equals $g(B)$. B is densely defined if, for some $g \in C_\infty(\mathbb{R}^n)$ such that $g(0) = 1$,

$$s\text{-}\lim_{R \rightarrow \infty} \left(s\text{-}\lim_{m \rightarrow \infty} (g(R^{-1} B_m)) \right) = \mathbf{1}.$$

We denote $B := s\text{-}C_\infty\text{-}\lim_{m \rightarrow \infty} B^m$.

Theorem 1.2 (Asymptotic velocity). *Let σ_α be given by:*

$$\sigma_\alpha = \begin{cases} 2 - \alpha & \text{if } 0 < \alpha < 2, \\ 2 & \text{if } \alpha = 2. \end{cases}$$

There exists a bounded self-adjoint operator P_α^+ , which commutes with H_α , such that

- (i) $P_\alpha^+ = s\text{-}C_\infty\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} \frac{p_\alpha(x)}{t} e^{-itH_\alpha}$.
- (ii) *The operator P_α^+ satisfies $P_\alpha^+ = \sigma_\alpha \mathbf{1}^c(H_\alpha)$.*
- (iii) *For any $J \in C_\infty(\mathbb{R})$, we have:*

$$J(P_\alpha^+) \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+) = s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} J(\nabla_\alpha) e^{-itH_\alpha} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+),$$

where $\nabla_\alpha := [iH_\alpha, p_\alpha(x)]$ is the local velocity.

Let us note that the limits we stated in Theorems 1.1 and 1.2 are for $t \rightarrow +\infty$; analogous results obviously hold for $t \rightarrow -\infty$.

Notice that computing the asymptotic velocity is all the more interesting that the free dynamics, $e^{-itH_{\alpha,0}}$, is not known in the case $0 < \alpha < 2$. On the other hand, it is very well understood in the case $\alpha = 2$, since a generalized Mehler’s formula is available (see [21] and Section 2.2 below). For $\alpha = 2$, we also consider more general Hamiltonians:

$$\begin{aligned}
 H_0 &= -\Delta - \sum_{k=1}^{n_-} \omega_k^2 x_k^2 + \sum_{k=n_-+1}^{n_-+n_+} \omega_k^2 x_k^2 + \sum_{k=n_-+n_++1}^{n_-+n_++n_E} E_k x_k; \\
 H &= H_0 + V(x),
 \end{aligned}
 \tag{1.11}$$

with $\omega_j > 0$ and $E_j \neq 0$. We prove the existence of wave operators in this more general case, under weaker conditions than (1.6) (see Section 6.1). The proof is based on an explicit formula for the dynamics e^{-itH_0} (Mehler’s formula, see Section 2.2).

Asymptotic completeness is shown if $n_- + n_+ = n$, under a condition similar to (1.6) in Section 6.2. In that case, we also construct asymptotic velocities in each space direction. The asymptotic velocity, given by Theorem 1.2, also exists and is equal to P_ℓ^+ , where $\omega_\ell = \max_{1 \leq j \leq n_-} \omega_j$, and P_ℓ^+ is the asymptotic velocity in the direction x_ℓ .

To our knowledge, there is very little motivation from a physical point of view to study the above Hamiltonians: in general, electromagnetic fields have saddle points, like the potential in H_0 , but the above model should then be valid only *locally*, in a neighborhood of the saddle point. In (1.1), the potentials $-|x|^\alpha$ are unbounded from below; this does not seem physically relevant (notice however that the Stark potential $E \cdot x$ is also unbounded).

On the other hand, we believe that these models are mathematically interesting. The dependence on $\alpha \in]0, 2]$ is somehow well understood, in particular thanks to the definition of the position variable p_α (1.4) and to the study of the asymptotic velocity. We also introduce new conjugate operators in order to obtain Mourre estimates (see (3.13) and (3.17)). Here again, the dependence of the analysis upon α seems to be interesting (in particular the limiting case $\alpha = 2$ is better understood than in [4]).

As mentioned above, in our analysis, we replace $H_{\alpha,0}$ and H_α by:

$$H_{\alpha,0} = -\Delta - \langle x \rangle^\alpha \quad \text{and} \quad H_\alpha = H_{\alpha,0} + V_\alpha(x).
 \tag{1.12}$$

This does not affect the results, since for large $|x|$, $|\langle x \rangle^\alpha - |x|^\alpha|$ is estimated by $\langle x \rangle^{\alpha-2}$, which is a short range perturbation for $0 < \alpha < 2$ (no smoothness is required for the perturbative potentials). We therefore prove Theorems 1.1 and 1.2 with $H_{\alpha,0}$ (respectively, H_α) replaced by $H_{\alpha,0}$ (respectively, H_α).

The paper is organized as follows.

- In Section 2 we show some elementary properties of the Hamiltonians. In particular, we recall Mehler’s formula for $\alpha = 2$, and prove the absence of eigenvalues for H_α in many cases.
- Section 3 is devoted to the Mourre estimate. In Section 3.2, we treat some rather technical features. For example, $\chi(\xi^2 - \langle x \rangle^\alpha)$ is not a good symbol, and we need some preparations before being able to use the pseudo-differential calculus (see Proposition 3.5). We give the conjugate operator A_α , which is a pseudo-differential operator, in Section 3.3. Section 3.4 is devoted to the regularity results and the Mourre estimate is established in Section 3.5.
- In Section 4 we prove asymptotic completeness. The Mourre estimate yields a minimal velocity estimate for A_α . We obtain a minimal velocity estimate for the observable $p_\alpha(x)$ using a lemma due to C. Gérard and F. Nier (see [15]).

- In Section 5, we construct the asymptotic velocity and describe its spectrum.
- In Section 6, we generalize our results in the case of the Hamiltonians defined in (1.11).

The main results of this paper were announced in [5].

2. Elementary properties

2.1. Domain and spectrum

We begin with some properties on the spectrum of the operator H_α . For $0 < \alpha \leq 2$, introduce $N_\alpha = -\Delta + \langle x \rangle^\alpha$. It is self-adjoint, with domain

$$D(N_\alpha) = \{u \in H^2(\mathbb{R}^n); \langle x \rangle^\alpha u \in L^2(\mathbb{R}^n)\}.$$

It can be viewed as the “confining” counterpart of $H_{\alpha,0}$ (the repulsive potential $-\langle x \rangle^\alpha$ is replaced by the confining one $+\langle x \rangle^\alpha$). Since it is not easy to know the domains of $H_{\alpha,0}$, we work on a core for these operators, $D(N_2)$, the domain of the harmonic oscillator. We recall an extension of Nelson’s theorem due to C. Gérard and I. Łaba [14, Lemma 1.2.5]:

Theorem 2.1 (Nelson’s theorem). *Let \mathcal{H} be a Hilbert space, $N \geq 1$ a self-adjoint operator on \mathcal{H} , H a symmetric operator such that $D(N) \subset D(H)$, and*

$$\|Hu\| \lesssim \|Nu\|, \quad u \in D(N), \tag{2.1}$$

$$|(Hu, Nu) - (Nu, Hu)| \lesssim \|N^{1/2}u\|^2, \quad u \in D(N). \tag{2.2}$$

Then H is essentially self-adjoint on $D(N)$, and we denote \overline{H} its extension. If $u \in D(\overline{H})$, then $(1 + i\varepsilon N)^{-1}u$ converges to u in the graph topology of $D(\overline{H})$ as $\varepsilon \rightarrow 0$.

From this theorem, we deduce the following:

Lemma 2.2. *For any $\alpha \in]0, 2]$, the operator $H_{\alpha,0}$ is essentially self-adjoint on $D(N_2)$.*

Proof. For $u \in D(N_2)$, we have:

$$\|H_{\alpha,0}u\| \leq \|\langle x \rangle^\alpha u\| + \|-\Delta u\| \lesssim \|N_2u\|,$$

which proves (2.1). Now, let us prove (2.2). A straightforward computation shows that

$$[H_{\alpha,0}, N_2] = [\langle x \rangle^2 + \langle x \rangle^\alpha, \Delta].$$

Hence, it suffices to show that $\|[\langle x \rangle^\alpha, \Delta]\| \lesssim \|N_2^{1/2}u\|$ for $0 < \alpha \leq 2$. But we have:

$$[\langle x \rangle^\alpha, \Delta] = -2i\alpha \langle x \rangle^{\alpha-2} xD - n\alpha \langle x \rangle^{\alpha-2} - \alpha(\alpha - 2)\langle x \rangle^{\alpha-4} x^2,$$

which is clearly bounded by $N_2^{1/2}$ for $0 < \alpha \leq 2$. \square

Before going further, let us notice the following characterization of H_α -compactness:

Lemma 2.3. *Let $V_\alpha(x) = V_\alpha^1(x) + V_\alpha^2(x)$, where V_α^1 is a compactly supported measurable function, and $V_\alpha^2 \in L^\infty(\mathbb{R}^n)$ with $V_\alpha^2(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then, V_α is $H_{\alpha,0}$ -compact if and only if V_α^1 is Δ -compact.*

Proof. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi_2 = 1$ near the support of χ_1 . We have:

$$\begin{aligned} \chi_1(x)(H_{\alpha,0} + i)^{-1} &= (\Delta - i)^{-1}(\Delta - i)\chi_1(H_{\alpha,0} + i)^{-1} \\ &= (\Delta - i)^{-1}\chi_1(\Delta - i)(H_{\alpha,0} + i)^{-1} + (\Delta - i)^{-1}[\Delta, \chi_1](H_{\alpha,0} + i)^{-1} \\ &= -(\Delta - i)^{-1}\chi_1 - (\Delta - i)^{-1}\chi_1 \langle x \rangle^\alpha (H_{\alpha,0} + i)^{-1} \\ &\quad + (\Delta - i)^{-1}[\Delta, \chi_1](\Delta - i)^{-1}(\Delta - i)\chi_2(H_{\alpha,0} + i)^{-1} \\ &= (\Delta - i)^{-1}\mathcal{O}(1) + (\Delta - i)^{-1}[\Delta, \chi_1](\Delta - i)^{-1} \\ &\quad \times \chi_2(-1 - \langle x \rangle^\alpha (H_{\alpha,0} + i)^{-1}) \\ &\quad + (\Delta - i)^{-1}[\Delta, \chi_1](\Delta - i)^{-1}[\Delta, \chi_2](H_{\alpha,0} + i)^{-1} \\ &= (\Delta - i)^{-1}\mathcal{O}(1), \end{aligned} \tag{2.3}$$

since $[\Delta, \chi_1](\Delta - i)^{-1}$ and $[\Delta, \chi_1](\Delta - i)^{-1}[\Delta, \chi_2]$ are bounded. On the other hand, we have:

$$\begin{aligned} \chi_1(x)(\Delta - i)^{-1} &= (H_{\alpha,0} + i)^{-1}(H_{\alpha,0} + i)\chi_1(\Delta - i)^{-1} \\ &= (H_{\alpha,0} + i)^{-1}\chi_1(x)(H_{\alpha,0} + i)(\Delta - i)^{-1} \\ &\quad - (H_{\alpha,0} + i)^{-1}[\Delta, \chi_1](\Delta - i)^{-1} \\ &= (H_{\alpha,0} + i)^{-1}\mathcal{O}(1). \end{aligned} \tag{2.4}$$

Since $V_\alpha^2(x) \rightarrow 0$ as $x \rightarrow \infty$, we get:

$$\mathbf{1}_{|x|>R} V_\alpha(x)(H_{\alpha,0} + i)^{-1} = \mathbf{1}_{|x|>R} V_\alpha^2(x)(H_{\alpha,0} + i)^{-1} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

and $\mathbf{1}_{|x|>R} V_\alpha(x)(\Delta - i)^{-1} \rightarrow 0$ for the norm topology as $R \rightarrow +\infty$. So, from (2.3) and (2.4),

$$\begin{aligned} V_\alpha \text{ is } H_{\alpha,0}\text{-compact} &\Leftrightarrow \mathbf{1}_{|x|<R} V_\alpha \text{ is } H_{\alpha,0}\text{-compact} \Leftrightarrow \\ V_\alpha^1 \text{ is } H_{\alpha,0}\text{-compact} &\Leftrightarrow V_\alpha^1 \text{ is } \Delta\text{-compact.} \quad \square \end{aligned}$$

Then Lemma 2.2 and the Kato–Rellich theorem [30, Theorem X.12] imply:

Lemma 2.4. *Let $0 < \alpha \leq 2$. For $V_\alpha = V_\alpha^1 + V_\alpha^2$ satisfying (1.5) and (1.6), the operator H_α is self-adjoint on $D(H_{\alpha,0})$, and essentially self-adjoint on $D(N_2)$.*

Notice that $H_{2,0} + 1 = -\Delta - x^2$ is conjugated to the generator of dilations. This is obvious from a glance at the symbols: $\xi^2 - x^2$ can be written as $(\xi + x) \cdot (\xi - x) = y \cdot \eta$, with suitable new variables corresponding to a rotation of angle $\pi/4$ in the phase space. To make this argument precise, we set, for $u \in \mathcal{S}'(\mathbb{R}^n)$:

$$Uu(x) = \frac{1}{(\sqrt{2}\pi)^{n/2}} e^{-ix^2/2} \int e^{i\sqrt{2}x \cdot y} e^{-iy^2/2} u(y) dy.$$

The operator U is an isometry on $L^2(\mathbb{R}^n)$. We have:

$$xUu = U\left(\frac{y - D_y}{\sqrt{2}}u\right) \quad \text{and} \quad D_xUu(x) = U\left(\frac{D_y + y}{\sqrt{2}}u\right). \tag{2.5}$$

Using these relations, it is easy to see that

$$N_2U = UN_2 \quad \text{and} \quad H_{2,0}U = U\tilde{H}_{2,0}, \tag{2.6}$$

where

$$\tilde{H}_{2,0} = D_x x + x D_x - 1. \tag{2.7}$$

Then, from [28, Proposition 6.2] on the spectrum of $Dx + xD$ and the Weyl’s essential spectrum theorem [31, Theorem XIII.14], we obtain:

Proposition 2.5. *The spectrum of $H_{2,0}$ is purely absolutely continuous, and $\sigma(H_2) = \mathbb{R}$ if V_2 is an $H_{2,0}$ -compact real-valued potential.*

For $0 < \alpha < 2$, we have the following proposition:

Proposition 2.6. *Let V_α be a $H_{\alpha,0}$ -compact potential with $0 < \alpha < 2$. Then*

$$\sigma(H_\alpha) = \mathbb{R}. \tag{2.8}$$

Proof. It is enough to show that $\sigma(H_{\alpha,0}) = \mathbb{R}$. In that case, we have $\sigma_{\text{ess}}(H_{\alpha,0}) = \mathbb{R}$ and then, by the Weyl’s theorem [31, Theorem XIII.14], $\sigma_{\text{ess}}(H_\alpha) = \mathbb{R}$. Since H_α is self-adjoint, we get the proposition.

Let $\varphi \in C_0^\infty([0, +\infty[; [0, 1])$ so that $\varphi = 1$ near 1. For $E \in \mathbb{R}$, we set:

$$u(x_1) = e^{ix_1^{1+\alpha/2}/(1+\alpha/2)} e^{iEx_1^{1-\alpha/2}/(2-\alpha)},$$

$$u_{\varepsilon,\delta}(x) = u(x_1)\sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|),$$

where $x = (x_1, x')$. We first note that $\|u_{\varepsilon,\delta}\|_{L^2(\mathbb{R}^n)}$ does not depend on ε and δ . We have:

$$\Delta u_{\varepsilon,\delta} = \partial_{x_1}^2(u)\sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|) + 2\partial_{x_1}(u)\partial_{x_1}(\sqrt{\varepsilon}\varphi(\varepsilon x_1))\delta^{(n-1)/2}\varphi(\delta|x'|)$$

$$+ u\partial_{x_1}^2(\sqrt{\varepsilon}\varphi(\varepsilon x_1))\delta^{(n-1)/2}\varphi(\delta|x'|) + u\sqrt{\varepsilon}\varphi(\varepsilon x_1)\partial_{x'}^2(\delta^{(n-1)/2}\varphi(\delta|x'|)).$$

The second term is equal to $\mathcal{O}(x_1^{\alpha/2})\varepsilon^{3/2}|\varphi'|(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|)$ which is $\mathcal{O}(\varepsilon^{1-\alpha/2})$ in L^2 -norm. The third and fourth terms are $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\delta^2)$ respectively in L^2 norm. Then

$$H_{\alpha,0}u_{\varepsilon,\delta} = \sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|)H_{\alpha,0}(u) + o(1),$$

as $\varepsilon, \delta \rightarrow 0$. Since

$$\langle x \rangle^\alpha - \langle x_1 \rangle^\alpha = \mathcal{O}(1)\langle x_1 \rangle^{\alpha-2}\langle x' \rangle^2,$$

we get:

$$\sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|)(\langle x \rangle^\alpha - \langle x_1 \rangle^\alpha)u = \mathcal{O}(\varepsilon^{2-\alpha}\delta^{-2}).$$

If $\varepsilon^{2-\alpha}\delta^{-2} \rightarrow 0$, we have

$$H_{\alpha,0}u_{\varepsilon,\delta} = \sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|)(-\partial_{x_1}^2 - \langle x_1 \rangle^\alpha)u(x_1) + o(1).$$

But we have:

$$\partial_{x_1}^2 u(x_1) = (-x_1^\alpha - E - E^2 x_1^{-\alpha} / 4 + i\alpha x_1^{\alpha/2-1} / 2 - E\alpha x_1^{-\alpha/2-1} / 4)u(x_1),$$

and then, there is a $\mu > 0$ so that

$$(H_{\alpha,0} - E)u_{\varepsilon,\delta} = \sqrt{\varepsilon}\varphi(\varepsilon x_1)\delta^{(n-1)/2}\varphi(\delta|x'|)\mathcal{O}(x_1^{-\mu}) + o(1) = \mathcal{O}(\varepsilon^\mu) + o(1) = o(1).$$

By the Weyl’s criterion [29, Theorem VII.12], E is in $\sigma(H_{\alpha,0})$. \square

2.2. Generalized Mehler’s formula

In this section, we restrict our attention to the case $\alpha = 2$ and drop the index 2. We consider a more general Hamiltonian on $L^2(\mathbb{R}^n)$,

$$H_0 = -\Delta - \sum_{k=1}^{n_-} \omega_k^2 x_k^2 + \sum_{k=n_-+1}^{n_-+n_+} \omega_k^2 x_k^2 + \sum_{k=n_-+n_++1}^{n_-+n_++n_E} E_k x_k, \tag{2.9}$$

with $n_- + n_+ + n_E \leq n$, $\omega_k > 0$ and $E_k \neq 0$ if $n_E \neq 0$. By convention, $\sum_{j=a}^b = 0$ if $b < a$.

In this case, H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ from Faris–Lavine theorem [30, Theorem X.38]. The kernel of e^{-itH_0} is known explicitly, through a generalized Mehler’s formula (see, e.g., [21]):

$$e^{-itH_0} f = \prod_{k=1}^n \left(\frac{1}{2i\pi g_k(2t)} \right)^{1/2} \int_{\mathbb{R}^n} e^{iS(t,x,y)} f(y) dy, \tag{2.10}$$

where

$$S(t, x, y) = \sum_{k=1}^n \frac{1}{g_k(2t)} \left(\frac{x_k^2 + y_k^2}{2} h_k(2t) - x_k y_k \right) - \sum_{k=n_+ + n_+ + 1}^{n_+ + n_+ + n_E} \left(\frac{E_k}{2} (x_k + y_k)t + \frac{E_k^2}{12} t^3 \right),$$

and the functions g_k and h_k , related to the classical trajectories, are given by:

$$g_k(t) = \begin{cases} \frac{\sinh(\omega_k t)}{\omega_k}, & \text{for } 1 \leq k \leq n_-, \\ \frac{\sin(\omega_k t)}{\omega_k}, & \text{for } n_- + 1 \leq k \leq n_- + n_+, \\ t, & \text{for } k > n_- + n_+, \end{cases} \tag{2.11}$$

$$h_k(t) = \begin{cases} \cosh(\omega_k t), & \text{for } 1 \leq k \leq n_-, \\ \cos(\omega_k t), & \text{for } n_- + 1 \leq k \leq n_- + n_+, \\ 1, & \text{for } k > n_- + n_+. \end{cases}$$

Recall that if $n_+ \geq 1$, then e^{-itH_0} has some singularities (see, e.g., [24]). This affects the above formula with phase factors we did not write (which can be incorporated in the definition of $(ig_k(2t))^{1/2}$), but not the computations we shall make in Section 6.1.

The group generated by H_0 is given by Mehler’s formula (2.10), and can be factored in an agreeable way, in the same fashion as $e^{it\Delta}$ (see for instance [26,22,16]). Recalling (2.10) and (2.11), we have:

$$e^{-itH_0} = \mathcal{M}_t \mathcal{D}_t \mathcal{F} \mathcal{M}_t e^{-i\frac{t^3}{12}|E|^2}, \tag{2.12}$$

where $E = (E_{n_- + n_+ + 1}, \dots, E_{n_- + n_+ + n_E})$,

$$\mathcal{M}_t = \mathcal{M}_t(x) = \exp \left(i \sum_{k=1}^n x_k^2 \frac{h_k(2t)}{2g_k(2t)} - i \frac{t}{2} \sum_{k=n_- + n_+ + 1}^{n_- + n_+ + n_E} E_k x_k \right),$$

$$(\mathcal{D}_t \varphi)(x) = \prod_{k=1}^n \left(\frac{1}{ig_k(2t)} \right)^{1/2} \varphi \left(\frac{x_1}{g_1(2t)}, \dots, \frac{x_n}{g_n(2t)} \right),$$

and

$$\mathcal{F} \varphi(\xi) = \hat{\varphi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \varphi(x) dx, \tag{2.13}$$

denotes the Fourier transform.

2.3. Absence of eigenvalues

We prove the absence of embedded eigenvalues under the unique continuation property. This result is very similar to [31, Theorem XIII.58]. We recall the notion of unique continuation property.

Definition 2.7. A Hamiltonian H has the *unique continuation property* if the following holds: Suppose that $Hu = 0$ for some $u \in L^2$, and that u vanishes outside a compact subset of \mathbb{R}^n ; then u is identically zero.

Theorem 2.8. Let $V_\alpha = V_\alpha^1 + V_\alpha^2$ be a real-valued potential satisfying (1.5) and (1.6). Assume that $-\Delta + (V_\alpha(x) - \langle x \rangle^\alpha)\mathbf{1}_{|x| < R}$ has the unique continuation property for all R large enough and

$$\langle x \rangle^{1-\alpha/2} \ln \langle x \rangle |V_\alpha^2(x)| \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \tag{2.14}$$

Then H_α has no eigenvalue.

Remark 2.9. The unique continuation property for Schrödinger operators is known in many situations but we recall only two cases. The works of M. Schechter and B. Simon [32] for $n = 1, 2$ and D. Jerison and C.E. Kenig [23] imply:

- If $V_\alpha^2 \in L^p(\mathbb{R}^n)$ with $p > 1$ for $n = 1, 2$, and $p \geq n/2$ for $n \geq 3$, then the unique continuation property holds for $-\Delta + (V_\alpha(x) - \langle x \rangle^\alpha)\mathbf{1}_{|x| < R}$.

We recall [31, Theorem XIII.57]:

- Assume there is a closed set S of measure zero so that $\mathbb{R}^n \setminus S$ is connected, and so that V_α^2 is bounded on any compact subset of $\mathbb{R}^n \setminus S$, then the unique continuation property holds for $-\Delta + (V_\alpha(x) - \langle x \rangle^\alpha)\mathbf{1}_{|x| < R}$.

Proof of Theorem 2.8. We follow the proof of [31, Theorem XIII.58]. Suppose that $u \in D(H_\alpha)$ is an eigenfunction for H_α with eigenvalue E . As in [31], we define a function w from $[0, \infty[$ to $L^2(\mathbb{S}^{n-1})$ by:

$$w(r, \omega) = r^{(n-1)/2} u(r\omega).$$

We have:

$$\int_0^{+\infty} \|w(r)\|_{L^2(\mathbb{S}^{n-1}, d\omega)}^2 dr = \|u\|_{L^2(\mathbb{R}^n)}^2 < +\infty. \tag{2.15}$$

Since $u \in D(H_{\alpha,0})$, we have $\langle x \rangle^{-\alpha} \Delta u \in L^2(\mathbb{R}^n)$ and we get, using the pseudo-differential calculus, that $(\partial_{x_j} \langle x \rangle^{-\alpha} \partial_{x_k} u)_{1 \leq j, k \leq n}$ and $\nabla \langle x \rangle^{-\alpha/2} u$ are in $L^2(\mathbb{R}^n)$. We have:

$$\int_0^{+\infty} ((\partial_r \langle r \rangle^{-\alpha/2} w, \partial_r \langle r \rangle^{-\alpha/2} w) - r^{-2} \langle r \rangle^{-\alpha} (w, Bw)) dr = \|\nabla \langle x \rangle^{-\alpha/2} u\|^2,$$

where B is the Laplace–Beltrami operator on $L^2(\mathbb{S}^{n-1})$ such that $-B \geq 0$, and

$$\mathfrak{d}_r f(r) = r^{(n-1)/2} \partial_r (r^{-(n-1)/2} f(r)).$$

Here (\cdot, \cdot) is the scalar product on $L^2(\mathbb{S}^{n-1})$. Using this formula, we get:

$$\int_0^{+\infty} \langle r \rangle^{-\alpha} \|w'\|^2 dr < +\infty; \quad \int_0^{+\infty} r^{-2} \langle r \rangle^{-\alpha} (w, -Bw) dr < +\infty, \tag{2.16}$$

and the quantities w' and (w, Bw) are defined almost everywhere on $]0, +\infty[$.

Now we define, for r large enough,

$$F(r) = r^{-\alpha} \|w'\|^2 + r^{-2-\alpha} (w, Bw) + (r^{-\alpha} \langle r \rangle^\alpha + Er^{-\alpha}) \|w\|^2.$$

From (2.15) and (2.16), $F(r)$ is integrable. On the other hand, we have:

$$\begin{aligned} (r \ln(r) F(r))' &= 2r^{1-\alpha} \ln(r) (w', w'') + r^{-\alpha} ((1 - \alpha) \ln(r) + 1) \|w'\|^2 \\ &\quad + 2r^{-1-\alpha} \ln(r) (w', Bw) - r^{-2-\alpha} ((1 + \alpha) \ln(r) - 1) (w, Bw) \\ &\quad + r^{-\alpha} \langle r \rangle^\alpha ((1 - \alpha) \ln(r) + \alpha \ln(r) r^2 \langle r \rangle^{-2} + 1) \|w\|^2 \\ &\quad + Er^{-\alpha} ((1 - \alpha) \ln(r) + 1) \|w\|^2 + 2r \ln(r) (r^{-\alpha} \langle r \rangle^\alpha + Er^{-\alpha}) (w', w). \end{aligned}$$

Since u is an eigenfunction of H_α , we have:

$$w'' = -r^{-2} Bw + \frac{1}{4} (n-1)(n-3) r^{-2} w - \langle r \rangle^\alpha w + V_\alpha w - Ew. \tag{2.17}$$

Then

$$\begin{aligned} (r \ln(r) F(r))' &= \ln(r) ((1 - \alpha) r^{-\alpha} \|w'\|^2 + \|w\|^2) + r^{-\alpha} \|w'\|^2 + \|w\|^2 \\ &\quad - r^{-2-\alpha} ((1 + \alpha) \ln(r) - 1) (w, Bw) \\ &\quad + \frac{1}{2} (n-1)(n-3) r^{-1-\alpha} \ln(r) (w', w) \\ &\quad + Er^{-\alpha} ((1 - \alpha) \ln(r) + 1) \|w\|^2 + \mathcal{O}(r^{-1} \ln(r)) \|w\|^2 \\ &\quad + 2r^{1-\alpha} \ln(r) (w', V_\alpha w). \end{aligned}$$

Using $-B \geq 0$ and

$$\begin{aligned} r^{-1-\alpha} \ln(r) (w', w) &= o(1) r^{-\alpha} \|w'\|^2 + o(1) \|w\|^2, \\ r^{1-\alpha} \ln(r) (w', V_\alpha w) &= o(1) r^{-\alpha/2} \|w'\| \|w\| = o(1) r^{-\alpha} \|w'\|^2 + o(1) \|w\|^2, \end{aligned}$$

we get:

$$(r \ln(r)F(r))' \geq \ln(r)((1 - \alpha)r^{-\alpha}\|w'\|^2 + \|w\|^2) + \frac{r^{-\alpha}}{2}\|w'\|^2 + \frac{1}{2}\|w\|^2, \quad (2.18)$$

for r large enough. Here $o(1)$ denotes a function which tends to 0 as r tends to $+\infty$. This computation is formal, but we can give, as in [31], a rigorous proof of the integral version of (2.18).

If $0 < \alpha < 1$, we get that $r \ln(r)F(r)$ is monotone increasing for $r \geq R_1$ large enough. Integrate (2.18) between R_1 and r : $F(r) \geq \frac{R_1 \ln R_1}{r \ln r} F(R_1)$. Since F is integrable but $(r \ln r)^{-1}$ is not, we have $F(r) \leq 0$ for $r > R_1$.

Now, we assume that $1 \leq \alpha \leq 2$. For $1 \ll a < b$, we have:

$$\begin{aligned} b \ln b F(b) - a \ln a F(a) &\geq \int_a^b \left(\ln(r)((1 - \alpha)r^{-\alpha}\|w'\|^2 + \|w\|^2) \right. \\ &\quad \left. + \frac{r^{-\alpha}}{2}\|w'\|^2 + \frac{1}{2}\|w\|^2 \right) dr. \end{aligned} \quad (2.19)$$

Integration by parts yields:

$$\begin{aligned} \int_a^b r^{-\alpha} \ln r (w'', w) dr &= r^{-\alpha} \ln r (w', w)|_a^b - \int_a^b r^{-\alpha} \ln r \|w'\|^2 dr \\ &\quad - \int_a^b (1 - \alpha \ln r) r^{-1-\alpha} (w', w) dr. \end{aligned}$$

The eigenfunction relation (2.17) yields:

$$\begin{aligned} \int_a^b r^{-\alpha} \ln r (w'', w) dr &= - \int_a^b r^{-2-\alpha} \ln r (w, Bw) dr + \int_a^b r^{-\alpha} \ln r (w, Vw) dr \\ &\quad + \int_a^b r^{-\alpha} \ln r \left(\frac{1}{4}(n-1)(n-3)r^{-2} - \langle r \rangle^\alpha - E \right) \|w\|^2 dr \\ &= - \int_a^b (\ln r + o(1)) \|w\|^2 dr - \int_a^b r^{-2-\alpha} \ln r (w, Bw) dr. \end{aligned}$$

We infer:

$$\int_a^b r^{-\alpha} \ln r \|w'\|^2 dr = \int_a^b \ln r \|w\|^2 dr + \int_a^b r^{-2-\alpha} \ln r (w, Bw) dr$$

$$+ [r^{-\alpha} \ln r (w', w)]_a^b + \int_a^b o(1)(r^{-\alpha} \|w'\|^2 + \|w\|^2) dr,$$

and (2.19) becomes,

$$b \ln b F(b) - a \ln a F(a) \geq \int_a^b ((2 - \alpha) \ln r \|w\|^2 + r^{-\alpha} \|w'\|^2/3 + \|w\|^2/3) dr$$

$$+ (1 - \alpha) [r^{-\alpha} \ln r (w', w)]_a^b$$

$$+ (1 - \alpha) \int_a^b r^{-2-\alpha} \ln r (w, Bw) dr$$

$$\geq \int_a^b ((2 - \alpha) \ln r \|w\|^2 + r^{-\alpha} \|w'\|^2/3 + \|w\|^2/3) dr$$

$$+ (1 - \alpha) [r^{-\alpha} \ln r (w', w)]_a^b, \tag{2.20}$$

since $(1 - \alpha)B \geq 0$. Let \tilde{F} be defined by:

$$\tilde{F}(r) = F(r) + (\alpha - 1)r^{-\alpha} (w', w), \tag{2.21}$$

which is integrable from (2.15) and (2.16). Inequality (2.20) implies that $r \ln r \tilde{F}(r)$ is monotone increasing, and reasoning as before, $\tilde{F}(r) \leq 0$ for $r > R_1$.

We now prove that $w(r) = 0$ for $r > R_2$ large enough. For $m \in \mathbb{N}$, let $w_m = r^m w$. It satisfies:

$$w_m'' = 2mr^{-1}w_m' - r^{-2}Bw_m$$

$$- \left(E + \langle r \rangle^\alpha - V + m(m+1)r^{-2} - \frac{1}{4}(n-1)(n-3)r^{-2} \right) w_m. \tag{2.22}$$

We also define:

$$G(r) = r^2 \|w_m'\|^2 + m(m+1) \|w_m\|^2 + (w_m, Bw_m)$$

$$+ (r^2 \langle r \rangle^\alpha + Er^2 - r) \|w_m\|^2, \tag{2.23}$$

and we have:

$$G'(r) = (4m + 2)r\|w'_m\|^2 + 2\left(\left(r^2V + \frac{1}{4}(n-1)(n-3) - r\right)w_m, w'_m\right) \\ + (2r\langle r \rangle^\alpha + \alpha r^3\langle r \rangle^{\alpha-2} + 2Er - 1)\|w_m\|^2.$$

Using (2.14) and the Cauchy–Schwarz inequality, we get:

$$G'(r) \geq (4m + 1)r\|w'_m\|^2 + r^{1+\alpha}\|w_m\|^2,$$

for $r > R_2 > R_1$ independent of m . Then $G(r)$ is monotone increasing on $]R_2, +\infty[$.

Suppose that $w(r_0) \neq 0$ for some $r_0 > R_2$. Since we have:

$$G(r) = r^{2m}(r^2\|w'\|^2 + mr^{-1}\|w\|^2 + m(m+1)\|w\|^2) \\ + (w, Bw) + (r^2\langle r \rangle^\alpha + Er^2 - r)\|w\|^2,$$

we get $G(r_0) > 0$ for $m \geq 1$ large enough, and now fixed. So, $G(r) > 0$ for all $r > r_0$. On the other hand, if $r > r_1 > r_0$, with r_1 large enough, we have $m(2m+1) - r < 0$. Since $\|w\|^2$ is integrable on $[r_1, +\infty[$, the function $\|w\|^2$ is not monotone increasing; there exists $r > r_1$ such that

$$(\|w\|^2)'(r) = 2(w', w)(r) \leq 0.$$

Then

$$G(r) = r^{2m}(r^2\|w'\|^2 + 2mr(w', w) + m(2m+1)\|w\|^2) \\ + (w, Bw) + (r^2\langle r \rangle^\alpha + Er^2 - r)\|w\|^2 \\ \leq r^{2m+2+\alpha}(r^{-\alpha}\|w'\|^2 + r^{-2-\alpha}(w, Bw) \\ + (r^{-\alpha}\langle r \rangle^\alpha + Er^{-\alpha})\|w\|^2) + 2mr^{-1-\alpha}(w', w).$$

Therefore, we get:

$$0 < G(r) \leq r^{2m+2+\alpha}F(r) \leq 0 \quad \text{if } 0 < \alpha < 1, \\ 0 < G(r) \leq r^{2m+2+\alpha}\tilde{F}(r) \leq 0 \quad \text{if } 1 \leq \alpha \leq 2,$$

which is impossible, so $w(r) = 0$ for $r > R_2$. Theorem 2.8 follows from unique continuation. \square

3. Mourre estimates

In the following, we use the Weyl calculus of L. Hörmander, for which we refer to [20, Section XVIII]. More precisely, we work with the simple metrics which are σ -temperate:

$$\begin{aligned}
 g_0 &= |dx|^2 + |d\xi|^2, \\
 g_1 &= \frac{|dx|^2}{1+x^2+\xi^2} + \frac{|d\xi|^2}{1+x^2+\xi^2}, \\
 g_1^\beta &= \frac{|dx|^2}{(1+x^2+\xi^2)^\beta} + \frac{|d\xi|^2}{(1+x^2+\xi^2)^\beta}, \\
 g_2 &= \frac{|dx|^2}{1+x^2} + \frac{|d\xi|^2}{1+\xi^2},
 \end{aligned}$$

for $\beta > 0$. We refer to [20] for the definition of the space of symbol $S(m, g)$ and we note $\Psi(m, g)$ the set of pseudo-differential operators whose symbol is in a space $S(m, g)$. We set $\Psi(g) = \bigcup_m \Psi(m, g)$.

The crucial point of the Mourre theory is the construction of the conjugate operator. This is a self-adjoint operator A_α such that $i[H_\alpha, A_\alpha]$ is positive on the energy level, and H_α -bounded. In our case, the generator of dilations $(xD + Dx)/2$ is not satisfactory for $H_{\alpha,0}$ since,

$$i[H_{\alpha,0}, (xD + Dx)/2] = -2\Delta + \alpha x^2 \langle x \rangle^{\alpha-2},$$

which is positive, but not $H_{\alpha,0}$ -bounded for $\alpha > 0$. So we must find another conjugate operator. We look for A_α as a pseudo-differential operator of symbol $a_\alpha(x, \xi)$. Consider the case of dimension one, and start with $\alpha = 2$. Formally, we want to solve:

$$\{\xi^2 - x^2, a_2(x, \xi)\} \equiv 2\xi \partial_x a_2 + 2x \partial_\xi a_2 = 1, \quad \text{on } \{(x, \xi); \xi^2 - x^2 = E\}. \quad (3.1)$$

We saw in the introduction that a solution to this equation is given by:

$$a_2(x, \xi) = \frac{1}{4} (\ln(\xi + x) - \ln(\xi - x)). \quad (3.2)$$

Now consider the case $0 < \alpha < 2$. Replacing $\langle x \rangle$ by $x > 0$, we try to solve:

$$\begin{aligned}
 \{\xi^2 - x^\alpha, a_\alpha(x, \xi)\} &\equiv 2\xi \partial_x a_\alpha + \alpha x^{\alpha-1} \partial_\xi a_\alpha = 2 - \alpha, \\
 &\text{on } \{(x, \xi); \xi^2 - x^\alpha = E\}.
 \end{aligned} \quad (3.3)$$

As we saw in the introduction, $a_\alpha(x, \xi) = \xi x^{1-\alpha}$ should do the job, up to an error which is compact on the energy level (because it decays like $x^{-\alpha}$). In this section, we make this heuristic approach rigorous. The main results (Mourre estimates) are proved in Section 3.5. Here, we can find again the short range condition: on the energy level, we have formally:

$$\langle \xi \rangle \approx \langle x \rangle^{\alpha/2} \quad \text{and then} \quad |a_\alpha(x, \xi)| \approx \begin{cases} \ln \langle x \rangle & \text{if } \alpha = 2, \\ \langle x \rangle^{1-\alpha/2} & \text{if } 0 < \alpha < 2. \end{cases}$$

By (3.1) and (3.3), we obtain that the position variable increases exactly like t along the evolution. Then, in Section 4.2, we will replace $a_\alpha(x, \xi)$ by $p_\alpha(x)$ and we will require that the potential decays as $p_\alpha(x)^{-1-\varepsilon}$.

3.1. General framework

We recall some results that we will use to prove regularity results on the groups generated by H_α . A full presentation of such issues can be found in the book of O. Amrein, A. Boutet de Monvel and V. Georgescu [1]. We start with the definition of $C^1(A)$.

Definition 3.1. Let A and H be self-adjoint operators on a Hilbert space \mathcal{H} . We say that H is of class $C^r(A)$ for $r > 0$, if there is $z \in \mathbb{C} \setminus \sigma(H)$ such that

$$\mathbb{R} \ni t \rightarrow e^{itA}(H - z)^{-1}e^{-itA},$$

is C^r for the strong topology of $\mathcal{L}(\mathcal{H})$.

We have the following useful characterization of the regularity $C^1(A)$.

Theorem 3.2 [1, Theorem 6.2.10]. *Let A and H be self-adjoint operators on a Hilbert space \mathcal{H} . Then H is of class $C^1(A)$ if and only if the following conditions are satisfied:*

(1) *There exists $c < \infty$ such that for all $u \in D(A) \cap D(H)$,*

$$|(Au, Hu) - (Hu, Au)| \leq c(\|Hu\|^2 + \|u\|^2).$$

(2) *For some $z \in \mathbb{C} \setminus \sigma(H)$, the set $\{u \in D(A); (H - z)^{-1}u \in D(A) \text{ and } (H - \bar{z})^{-1}u \in D(A)\}$ is a core for A .*

If H is of class $C^1(A)$, then the following holds:

- (1) *The space $(H - z)^{-1}D(A)$ is independent of $z \in \mathbb{C} \setminus \sigma(H)$, and contained in $D(A)$. It is a core for H , and a dense subspace of $D(A) \cap D(H)$ for the intersection topology (i.e., the topology associated to the norm $\|Hu\| + \|Au\| + \|u\|$).*
- (2) *The space $D(A) \cap D(H)$ is a core for H , and the form $[A, H]$ has a unique extension to a continuous sesquilinear form on $D(H)$ (equipped with the graph topology). If this extension is denoted by $[A, H]$, the following identity holds on \mathcal{H} (in the form sense):*

$$[A, (H - z)^{-1}] = -(H - z)^{-1}[A, H](H - z)^{-1},$$

for $z \in \mathbb{C} \setminus \sigma(H)$.

We also have the following theorem from [1, Theorem 6.3.4]:

Theorem 3.3. *Let A and H be self-adjoint operators in a Hilbert space \mathcal{H} . Assume that the unitary one-parameter group $\{\exp(iA\tau)\}_{\tau \in \mathbb{R}}$ leaves the domain $D(H)$ of H invariant. Then H is of class $C^1(A)$ if and only if $[H, A]$ is bounded from $D(H)$ to $D(H)^*$.*

A criterion for the above assumption to be satisfied is given by the following result:

Lemma 3.4 [13, Lemma 2]. *Let A and H be self-adjoint operators in a Hilbert space \mathcal{H} . Let $H \in C^1(A)$ and suppose that the commutator $[iH, A]$ can be extended to a bounded operator from $D(H)$ to \mathcal{H} . Then e^{itA} preserves $D(H)$.*

3.2. A technical result

It is not clear that the energy cut-offs $\chi(H_{\alpha,0})$, $\chi \in C_0^\infty(\mathbb{R})$, are pseudo-differential operators, since $H_{\alpha,0}$ is not elliptic, and $\chi(\xi^2 - \langle x \rangle^\alpha)$ is not a good symbol. The next proposition will allow us to use pseudo-differential calculus. Such techniques have been used by M. Dimassi and V. Petkov [11].

Proposition 3.5. *Let $0 < \alpha \leq 2$, $1/2 < \beta \leq 1$, $z \in \mathbb{C} \setminus \mathbb{R}$ and $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ near 0. Then*

$$(H_{\alpha,0} - z)^{-1} = (H_{\alpha,0} - z)^{-1} \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) + \mathcal{O}(1) \text{Op}(r),$$

with $r(x, \xi) \in S((\xi^2 + \langle x \rangle^\alpha)^{-\beta}, g_0)$.

Proof. For $\gamma \geq 0$, let $g_3^{\alpha,\gamma}$ be the σ -temperate metric:

$$g_3^{\alpha,\gamma} = |dx|^2 + \frac{|d\xi|^2}{(\xi^2 + \langle x \rangle^\alpha)^\gamma}.$$

Let $B = \text{Op}(b)$, with $b = (\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta \in S((\xi^2 + \langle x \rangle^\alpha)^{1-\beta}, g_3^{\alpha,1})$. It satisfies

$$\partial_{x,\xi}^\delta b \in S((\xi^2 + \langle x \rangle^\alpha)^{1-\beta-\min(1,|\delta|/2)}, g_3^{\alpha,1}), \quad \forall \delta \in \mathbb{N}^{2n}. \tag{3.4}$$

We have:

$$(H_{\alpha,0} - z)^{-1}(1 - \psi(B)) = (H_{\alpha,0} - z)^{-1} B B^{-1}(1 - \psi(B)). \tag{3.5}$$

Theorem 18.5.4 of [20] on the composition of pseudo-differential operators in $\Psi(g_3^{\alpha,1})$, and (3.4) imply that

$$(H_{\alpha,0} - z)^{-1} B = (H_{\alpha,0} - z)^{-1} (H_{\alpha,0} \text{Op}((\xi^2 + \langle x \rangle^\alpha)^{-\beta}) + \text{Op}(r)),$$

where $r \in S((\xi^2 + \langle x \rangle^\alpha)^{-\beta}, g_3^{\alpha,1})$. So we have:

$$(H_{\alpha,0} - z)^{-1} B = \mathcal{O}(1) \text{Op}(r), \tag{3.6}$$

for some other $r \in S((\xi^2 + \langle x \rangle^\alpha)^{-\beta}, g_3^{\alpha,1})$.

Let $\varphi(y) = (1 - \psi(y))/y$, and $\tilde{\varphi}$ be an almost analytic extension of φ (see [19,9] and [10, Appendix C.2]). This is a $C^\infty(\mathbb{C})$ function which coincides with φ on \mathbb{R} , whose support is contained in a region like $|\operatorname{Im} z| < C|\operatorname{Re} z|$, and which satisfies:

$$|\partial_{\bar{z}} \tilde{\varphi}(z)| \leq C_k |z|^{-2-k} |\operatorname{Im} z|^k, \quad \forall k \in \mathbb{N}.$$

Using the Helffer–Sjöstrand formula (see [17] or [9]), we can write:

$$\operatorname{Op}(r) B^{-1} (1 - \psi(B)) = \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\varphi}(z) \operatorname{Op}(r) (B - z)^{-1} L(dz). \tag{3.7}$$

For $\operatorname{Im} z \neq 0$ and $1/2 < \beta \leq 1$, we have $(b(x, \xi) - z)^{-1} \in S(1, g_3^{\alpha, 2\beta-1})$ and

$$\partial_{x, \xi}^\delta (b(x, \xi) - z)^{-1} \in S((\xi^2 + \langle x \rangle^\alpha)^{-(\beta-1/2)\min(2, |\delta|)}, g_3^{\alpha, 2\beta-1}), \quad \forall \delta \in \mathbb{N}^{2n}. \tag{3.8}$$

Using [20, Theorem 18.5.5] on the composition of pseudo-differential operators in $\Psi(g_3^{\alpha, 1})$, and $\Psi(g_3^{\alpha, 2\beta-1})$, (3.4) and (3.8), we have, for $\operatorname{Im} z \neq 0$,

$$(B - z) \operatorname{Op}((b(x, \xi) - z)^{-1}) = 1 + \operatorname{Op}(d(z)),$$

where $d(z) \in S((\xi^2 + \langle x \rangle^\alpha)^{1-3\beta}, g_3^{\alpha, 2\beta-1})$, and each semi-norm of $d(z)$ in this space is bounded by some power of $1 + |\operatorname{Im} z|^{-1}$. On the other hand, we have:

$$\operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{1-3\beta}) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{3\beta-1}) = 1 + \mathcal{O}(\lambda^{-1}),$$

and then, for λ large enough,

$$\begin{aligned} \operatorname{Op}(d(z)) &= \operatorname{Op}(d(z)) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{3\beta-1}) \mathcal{O}(1) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{3\beta-1})^{-1} \\ &= \operatorname{Op}(\tilde{d}(z)) \mathcal{O}(1) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{1-3\beta}), \end{aligned}$$

with $\tilde{d}(z) \in S(1, g_3^{\alpha, 2\beta-1})$ and each semi-norm is bounded by some power of $1 + |\operatorname{Im} z|^{-1}$. The continuity in $L^2(\mathbb{R}^n)$ of pseudo-differential operators yields:

$$(B - z) \operatorname{Op}((b(x, \xi) - z)^{-1}) = 1 + \mathcal{O}(1 + |\operatorname{Im} z|^{-M}) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{1-3\beta}),$$

for some $M > 0$. We infer:

$$\begin{aligned} (B - z)^{-1} &= \operatorname{Op}((b(x, \xi) - z)^{-1}) \\ &\quad + \mathcal{O}(1 + |\operatorname{Im} z|^{-M-1}) \operatorname{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{1-3\beta}). \end{aligned} \tag{3.9}$$

Then, using the pseudo-differential calculus, (3.7) becomes:

$$\begin{aligned} \text{Op}(r)B^{-1}(1 - \psi(B)) &= \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\varphi}(z) \text{Op}(r) \text{Op}((b(x, \xi) - z)^{-1})L(dz) \\ &\quad + \mathcal{O}(1) \text{Op}((\xi^2 + \langle x \rangle^\alpha + i\lambda)^{1-3\beta}) \\ &= \mathcal{O}(1) \text{Op}(r), \end{aligned} \tag{3.10}$$

for some other $r \in S((\xi^2 + \langle x \rangle^\alpha)^{-\beta}, g_3^{\alpha, 2\beta-1})$. From (3.5), (3.6) and (3.10), we have:

$$(H_{\alpha,0} - z)^{-1} = (H_{\alpha,0} - z)^{-1}\psi(B) + \mathcal{O}(1) \text{Op}(r). \tag{3.11}$$

Using the Helffer–Sjöstrand formula:

$$\begin{aligned} \psi(B) &= \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\psi}(z)(B - z)^{-1}L(dz), \\ \text{Op}(\psi(b(x, \xi))) &= \frac{1}{\pi} \int \partial_{\bar{z}} \tilde{\psi}(z) \text{Op}((b(x, \xi) - z)^{-1})L(dz), \end{aligned}$$

and (3.16), we obtain:

$$\psi(B) = \text{Op}(\psi(b(x, \xi))) + \mathcal{O}(1) \text{Op}(r), \tag{3.12}$$

with $r \in S((\xi^2 + \langle x \rangle^\alpha)^{-\beta}, g_3^{\alpha, 2\beta-1})$. The proposition follows from (3.11) and (3.12). \square

3.3. Conjugate operator

Following the discussion of the beginning of Section 3, we choose for the conjugate operator if $\alpha = 2$,

$$A_2 = \text{Op}(a_2(x, \xi)), \quad \text{with } a_2 = (\ln \langle \xi + x \rangle - \ln \langle \xi - x \rangle). \tag{3.13}$$

One can see that $a_2(x, \xi) \in S(\langle \ln(x) \rangle, g_0)$. Indeed, we have, for $|\xi| < 2|x|$,

$$\ln \langle \xi + x \rangle - \ln \langle \xi - x \rangle \leq \ln \langle 3x \rangle \leq \langle \ln(x) \rangle + C, \tag{3.14}$$

with $C > 0$. On the other hand, we get for $|\xi| \geq 2|x|$,

$$\ln \langle \xi + x \rangle - \ln \langle \xi - x \rangle = \frac{1}{2} \ln \left(\frac{1 + (\xi + x)^2}{1 + (\xi - x)^2} \right) \leq \frac{1}{2} \ln \left(\frac{1 + 9\xi^2/4}{1 + \xi^2/4} \right) \leq C, \tag{3.15}$$

with $C > 0$. For computational reasons, it is better to have a another writing for A_2 .

Lemma 3.6. *We have:*

$$A_2 = U(\ln \langle \sqrt{2}x \rangle - \ln \langle \sqrt{2}D \rangle)U^*.$$

Proof. Using the exact composition of pseudo-differential operators (Theorem 18.5.4 of [20]), we have:

$$\begin{aligned} \text{Op}(\ln\langle\xi + x\rangle) &= \text{Op}((\xi + x)^2 + 1) \text{Op}(\ln\langle\xi + x\rangle((\xi + x)^2 + 1)^{-1}), \\ \text{Op}((\xi + x)^2 + z)^{-1} &= ((D + x)^2 + z)^{-1}, \end{aligned}$$

for $\text{Im } z \neq 0$. Let $\gamma \subset \mathbb{C}$ be a contour enclosed $[0, +\infty[$ in the region where $\ln(z + 1)(z + 1)^{-1}$ is holomorphic and coinciding with $\text{Re } z = |\text{Im } z|$ for z large enough. Using the Cauchy formula and (2.5), we get:

$$\begin{aligned} \text{Op}(\ln\langle\xi + x\rangle) &= \frac{1}{2i\pi} ((D + x)^2 + 1) \int_{\gamma} \ln(z + 1)(z + 1)^{-1} \text{Op}(((\xi + x)^2 - z)^{-1}) dz \\ &= \frac{1}{2i\pi} ((D + x)^2 + 1) \int_{\gamma} \ln(z + 1)(z + 1)^{-1} ((D + x)^2 - z)^{-1} dz \\ &= U \frac{1}{2i\pi} (2x^2 + 1) \int_{\gamma} \ln(z + 1)(z + 1)^{-1} (2x^2 - z)^{-1} dz U^* \\ &= U \ln\langle\sqrt{2}x\rangle U^*, \end{aligned} \tag{3.16}$$

and the lemma follows. \square

In the case $0 < \alpha < 2$, we choose for the conjugate operator $A_{\alpha} = \text{Op}(a_{\alpha}(x, \xi))$, where

$$\begin{aligned} a_{\alpha}(x, \xi) &= x \cdot \xi \langle x \rangle^{-\alpha} \psi \left(\frac{\xi^2 - \langle x \rangle^{\alpha}}{\xi^2 + \langle x \rangle^{\alpha}} \right) \\ &\in S(\langle x \rangle^{1-\alpha/2}, g_1^{\alpha/2}) \cap S(\langle \xi \rangle \langle x \rangle^{1-\alpha}, g_2), \end{aligned} \tag{3.17}$$

with $\psi \in C_0^{\infty}([-1/2, 1/2])$, and $\psi = 1$ near 0. Notice that on $\text{supp } a_{\alpha}$, $|\xi|$ is like $\langle x \rangle^{\alpha/2}$.

3.4. Regularity results

The aim of this section is to prove some regularity results for H_{α} . First, we give a common core for the operators H_{α} and A_{α} . Using the results of the previous section, we get:

Lemma 3.7. *Let $0 < \alpha \leq 2$. The operator A_{α} is essentially self-adjoint on $D(N_2)$.*

Proof. As in the proof of Lemma 2.2, we use Theorem 2.1. We distinguish the cases $\alpha = 2$ and $0 < \alpha < 2$. First, we suppose that $0 < \alpha < 2$; for $u \in D(N)$, we have:

$$\|A_{\alpha}u\| \lesssim \|\langle x \rangle^{1-\alpha/2}u\| \lesssim \|N_2u\|,$$

and the composition rules for an operator in $\Psi(g_1)$ by an operator in $\Psi(g_1^{\alpha/2})$ yield

$$[A_\alpha, N_2] \in \Psi(\langle x, \xi \rangle^{1-\alpha/2} \langle x \rangle^{1-\alpha/2}, g_1^{\alpha/2}),$$

which implies (2.2), from [20, Theorem 18.6.3]. The lemma follows for $0 < \alpha < 2$. When $\alpha = 2$,

$$\|A_2 u\| \lesssim \|(\ln \langle x \rangle) u\| \lesssim \|N_2 u\|,$$

which proves (2.1). Moreover, since $a_2(x, \xi) \in S(\langle \ln \langle x \rangle \rangle, g_0)$ and $N_2 \in \Psi(\langle x, \xi \rangle^2, g_1)$, we get $[N_2, A_2] \in \Psi(\langle \ln \langle x \rangle \rangle \langle x, \xi \rangle, g_0)$ and then

$$|([N_2, A_2]u, u)| \lesssim \|N_2^{1/2} u\|^2,$$

which yields (2.2) and the lemma. \square

3.4.1. Regularity for $H_{2,0}$

Lemma 3.8. For $z \in \mathbb{C} \setminus \mathbb{R}$, $(H_{2,0} - z)^{-1}$ maps $D(N_2)$ into itself.

Proof. We use the notations of Section 2.2. For $u \in D(N_2)$, we have:

$$\begin{aligned} \|x^2 e^{-itH_{2,0}} u\| &= \|x^2 \mathcal{M}_t \mathcal{D}_t \mathcal{F} \mathcal{M}_t u\| = \|x^2 \mathcal{D}_t \mathcal{F} \mathcal{M}_t u\| \\ &= \|(\sinh 2t)^2 x^2 \mathcal{F} \mathcal{M}_t u\| = \|-(\sinh 2t)^2 \Delta \mathcal{M}_t u\| \\ &= \|\mathcal{M}_t (-(\sinh 2t)^2 \Delta + (\cosh 2t)^2 x^2 - \tanh 2t (xD + Dx)) u\| \\ &\lesssim e^{4|t|} \|N_2 u\|. \end{aligned}$$

We also have:

$$\|-\Delta e^{-itH_{2,0}} u\| = \|(H_{2,0} + x^2) e^{-itH_{2,0}} u\| \lesssim \|H_{2,0} u\| + e^{4|t|} \|N_2 u\| \lesssim e^{4|t|} \|N_2 u\|.$$

So, for $\text{Im } z > 4$, we get:

$$\|N_2 (H_{2,0} - z)^{-1} u\| = \left\| iN_2 \int_0^{+\infty} e^{itz} e^{-itH_{2,0}} u \, dt \right\| \lesssim \int_0^{+\infty} e^{-t \text{Im } z} e^{4t} \, dt \|N_2 u\| \lesssim \|N_2 u\|,$$

which shows that $(H_{2,0} - z)^{-1}$ maps $D(N_2)$ into itself for $\text{Im } z > 4$. Then the lemma follows from [1, Lemma 6.2.1]. \square

Lemma 3.9. $H_{2,0}$ is in $C^1(A_2)$ and $[H_{2,0}, A_2]$ is bounded on $L^2(\mathbb{R}^n)$.

Proof. From Theorem 3.2, it is enough to estimate $[H_{2,0}, A_2]$. Recall that from (2.7), $\tilde{H}_{2,0} = D_x x + x D_x - 1$. Using (2.6), Lemma 3.6 and $\tilde{H}_{2,0} + 1 = -\mathcal{F}(\tilde{H}_{2,0} + 1)\mathcal{F}^*$, we have:

$$\begin{aligned} [H_{2,0}, A_2] &= U[\tilde{H}_{2,0}, \ln\langle\sqrt{2}x\rangle - \ln\langle\sqrt{2}D\rangle]U^* \\ &= U[\tilde{H}_{2,0}, \ln\langle\sqrt{2}x\rangle]U^* + U\mathcal{F}[\tilde{H}_{2,0}, \ln\langle\sqrt{2}x\rangle]\mathcal{F}^*U^* \\ &= -iU\frac{4x^2}{\langle\sqrt{2}x\rangle^2}U^* - iU\mathcal{F}\frac{4x^2}{\langle\sqrt{2}x\rangle^2}\mathcal{F}^*U^* \\ &= -iU\left(\frac{4x^2}{\langle\sqrt{2}x\rangle^2} + \frac{4D^2}{\langle\sqrt{2}D\rangle^2}\right)U^*, \end{aligned} \tag{3.18}$$

which is bounded on $L^2(\mathbb{R}^n)$. \square

For the asymptotic completeness, we need more regularity. We begin with:

Lemma 3.10. $H_{2,0}$ is in $C^2(A_2)$ and $[[H_{2,0}, A_2], A_2]$ is bounded on $L^2(\mathbb{R}^n)$.

Proof. Since we know that $H_{2,0}$ is in $C^1(A_2)$, it is enough to prove that $[[H_{2,0}, A_2], A_2]$ is bounded. From (3.18), we can write:

$$[[H_{2,0}, A_2], A_2] = -iU\left[\frac{4x^2}{\langle\sqrt{2}x\rangle^2} + \frac{4D^2}{\langle\sqrt{2}D\rangle^2}, \ln\langle\sqrt{2}x\rangle - \ln\langle\sqrt{2}D\rangle\right]U^*.$$

The symbols

$$\begin{aligned} f(x, \xi) &= \frac{2x^2}{\langle x \rangle^2} + \frac{2\xi^2}{\langle \xi \rangle^2}, \\ g(x, \xi) &= \ln\langle x \rangle - \ln\langle \xi \rangle, \end{aligned}$$

satisfy $f \in S(1, g_2)$ and $g \in S(\ln\langle x \rangle + \ln\langle \xi \rangle, g_2)$. Then, from Theorems 18.5.4 and 18.6.3 of [20], we have $[\text{Op}(f), \text{Op}(g)] = \mathcal{O}(1)$ which completes the proof. \square

3.4.2. Regularity for $H_{\alpha,0}$ for $0 < \alpha < 2$

Lemma 3.11. The operator $[H_{\alpha,0}, A_\alpha]$ is in $\Psi(1, g_2)$, and its symbol is supported inside the support of $a_\alpha(x, \xi)$, modulo $S(\langle x, \xi \rangle^{-\infty}, g_2)$.

Proof. Since $a_\alpha(x, \xi) \in S(\langle x \rangle^{1-\alpha}\langle \xi \rangle, g_2)$ and $\xi^2 - \langle x \rangle^\alpha \in S(\langle \xi \rangle^2 + \langle x \rangle^\alpha, g_2)$, we get $[A_\alpha, H_{\alpha,0}] \in \Psi(1 + \langle x \rangle^{-\alpha}\langle \xi \rangle^2, g_2)$, and each term in the development of its symbol is supported inside the support of $a_\alpha(x, \xi)$. Since $\langle \xi \rangle$ is like $\langle x \rangle^{\alpha/2}$ on the support of $a_\alpha(x, \xi)$, we get the lemma. \square

Lemma 3.12. $H_{\alpha,0}$ is in $C^1(A_\alpha)$, and $[H_{\alpha,0}, A_\alpha]$ is bounded on $L^2(\mathbb{R}^n)$.

Proof. As for Lemma 3.11, we have:

$$[A_\alpha, N_\alpha] \in \Psi(1, g_2), \tag{3.19}$$

and its symbol is supported inside the support of $a_\alpha(x, \xi)$, modulo a term in $S(\langle x, \xi \rangle^{-\infty}, g_2)$. Then $[A_\alpha, N_\alpha]$ is bounded on $L^2(\mathbb{R}^n)$. On the other hand, from the pseudo-differential calculus, one can show that $(N_\alpha + i)^{-1}$ maps $D(N_2)$ into itself. Then, from Theorem 3.2, N_α is $C^1(A_\alpha)$.

Since $[A_\alpha, N_\alpha]$ is bounded, we get from the proof of [13, Lemma 2], that e^{itA_α} preserves $D(N_\alpha)$ and that

$$N_\alpha e^{itA_\alpha} = e^{itA_\alpha} N_\alpha + i \int_0^t e^{i(t-s)A_\alpha} [N_\alpha, A_\alpha] e^{isA_\alpha} ds, \tag{3.20}$$

on $D(N_\alpha)$. From (3.19), we infer that $[[A_\alpha, N_\alpha], N_\alpha] \in \Psi(\langle x \rangle^{-1} \langle \xi \rangle + \langle x \rangle^{\alpha-1} \langle \xi \rangle^{-1}, g_2)$ and that each term in the development of its symbol is supported inside the support of $a_\alpha(x, \xi)$. Then $[[A_\alpha, N_\alpha], N_\alpha] \in \Psi(\langle x \rangle^{\alpha/2-1}, g_2) \subset \Psi(1, g_2)$ because $0 < \alpha < 2$. By induction, we obtain that

$$[\dots [[A_\alpha, N_\alpha], N_\alpha], \dots N_\alpha] \in \Psi(1, g_2). \tag{3.21}$$

Using (3.20) and (3.21), we obtain that e^{itA_α} preserves $D(N_\alpha^k)$ for all $k \in \mathbb{N}$ and that

$$\|N_\alpha^k e^{itA_\alpha} u\| \lesssim \|N_\alpha^k u\| + \|u\|, \tag{3.22}$$

for all $u \in D(N_\alpha^k)$. Since $\alpha \neq 0$, there is $k \in \mathbb{N}$ such that e^{itA_α} maps continuously $D(N_\alpha^k)$ into $D(N_\alpha^2)$. Then

$$t \mapsto e^{i(t-s)A_\alpha} H_{\alpha,0} e^{isA_\alpha} u,$$

is well-defined and C^1 for $u \in D(N_\alpha^k)$. It follows that

$$H_{\alpha,0} e^{itA_\alpha} = e^{itA_\alpha} H_{\alpha,0} + i \int_0^t e^{i(t-s)A_\alpha} [H_{\alpha,0}, A_\alpha] e^{isA_\alpha} ds, \tag{3.23}$$

on $D(N_\alpha^k)$. Using Lemma 3.11, $[H_{\alpha,0}, A_\alpha]$ can be extended as a bounded operator on $L^2(\mathbb{R}^n)$. On the other hand $H_{\alpha,0}$ satisfies Nelson’s theorem 2.1 with N_α^k as reference operator. Then (3.23) can be extended on $D(H_{\alpha,0})$ and e^{itA_α} preserves $D(H_{\alpha,0})$.

Since $[H_{\alpha,0}, A_\alpha]$ is bounded on $L^2(\mathbb{R}^n)$, Theorem 3.3 shows that $H_{\alpha,0}$ is in $C^1(A_\alpha)$. \square

Lemma 3.13. $H_{\alpha,0}$ is in $C^2(A_\alpha)$ and $[[H_{\alpha,0}, A_\alpha], A_\alpha]$ is bounded on $L^2(\mathbb{R}^n)$.

Proof. As in Lemma 3.10, it is enough to estimate $[[H_{\alpha,0}, A_\alpha], A_\alpha]$. Since $[H_{\alpha,0}, A_\alpha] \in \Psi(1, g_2)$ and $A_\alpha \in \Psi(\langle x \rangle^{1-\alpha} \langle \xi \rangle, g_2)$, we get $[[H_{\alpha,0}, A_\alpha], A_\alpha] \in \Psi(\langle x \rangle^{-\alpha}, g_2)$ which implies the lemma. \square

3.4.3. Regularity for H_α

Proposition 3.14. Assume that V_α satisfies the assumptions of Theorem 1.1. Then H_α is of class $C^{1+\delta}(A_\alpha)$ for some $\delta > 0$. Moreover $[H_\alpha, A_\alpha]$ is bounded from $D(H_\alpha)$ to $L^2(\mathbb{R}^n)$.

Proof. We use an interpolation argument as in [14, Proposition 3.7.5]. We begin by proving that H_α is in $C^1(A_\alpha)$ if V_α^2 satisfies (1.6) with $\varepsilon \geq 0$. Since $H_{\alpha,0}$ is $C^1(A_\alpha)$ and $[H_\alpha, A_\alpha]$ is bounded from $D(H_{\alpha,0})$ to $L^2(\mathbb{R}^n)$, e^{itA_α} preserves $D(H_{\alpha,0}) = D(H_\alpha)$, from Lemma 3.4. Then, from Theorem 3.3, it is enough to show that $[H_\alpha, A_\alpha]$ is bounded from $D(H_\alpha)$ to $L^2(\mathbb{R}^n)$.

Since we know from Lemma 3.9 and Lemma 3.12 that $[H_{\alpha,0}, A_\alpha]$ is bounded from $D(H_\alpha)$ to $L^2(\mathbb{R}^n)$, it is enough to show that

$$\begin{aligned} [V_\alpha, A_\alpha](H_{\alpha,0} + i)^{-1} &\text{ is compact (respectively, continuous)} \\ &\text{ on } L^2(\mathbb{R}^n) \text{ if } \varepsilon > 0 \text{ (respectively, } \varepsilon = 0), \end{aligned} \tag{3.24}$$

where ε is the constant in (1.6). We can write:

$$\begin{aligned} [V_\alpha, A_\alpha](H_{\alpha,0} + i)^{-1} &= V_\alpha^1 A_\alpha (H_{\alpha,0} + i)^{-1} - A_\alpha V_\alpha^1 (H_{\alpha,0} + i)^{-1} \\ &\quad + [V_\alpha^2, A_\alpha](H_{\alpha,0} + i)^{-1}. \end{aligned} \tag{3.25}$$

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 near the support of V_α^1 . Since $A_\alpha \in \Psi(\langle p_\alpha(x) \rangle, g_0)$, $A_\alpha \chi$ and χA_α are bounded. So

$$A_\alpha V_\alpha^1 (H_{\alpha,0} + i)^{-1} = A_\alpha \chi V_\alpha^1 (H_{\alpha,0} + i)^{-1} = \mathcal{O}(1) V_\alpha^1 (H_{\alpha,0} + i)^{-1}, \tag{3.26}$$

which is compact because V_α^1 is $H_{\alpha,0}$ -compact. Since χA_α is bounded, we can write:

$$\begin{aligned} \chi A_\alpha (H_{\alpha,0} + i)^{-1} &= (H_{\alpha,0} + i)^{-1} \chi A_\alpha + (H_{\alpha,0} + i)^{-1} [H_{\alpha,0}, \chi A_\alpha] (H_{\alpha,0} + i)^{-1} \\ &= (H_{\alpha,0} + i)^{-1} \mathcal{O}(1) + (H_{\alpha,0} + i)^{-1} [H_{\alpha,0}, \chi] A_\alpha (H_{\alpha,0} + i)^{-1} \\ &\quad + (H_{\alpha,0} + i)^{-1} \chi [H_{\alpha,0}, A_\alpha] (H_{\alpha,0} + i)^{-1}. \end{aligned} \tag{3.27}$$

We have:

$$\text{Op}(c(x, \xi)) := [H_{\alpha,0}, \chi] A_\alpha \in \Psi(\langle \xi \rangle \langle x \rangle^{-\infty}, g_0).$$

Then Proposition 3.5 and the pseudo-differential calculus imply:

$$[H_{\alpha,0}, \chi] A_\alpha (H_{\alpha,0} + i)^{-1} = \left(\text{Op} \left(c(x, \xi) \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) + R \right) \mathcal{O}(1),$$

with $R \in \Psi(\langle x \rangle^{-\infty}, g_0)$. Since we also have $c(x, \xi)\psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)) \in S(\langle x \rangle^{-\infty}, g_0)$, we get:

$$[H_{\alpha,0}, \chi]A_\alpha(H_{\alpha,0} + i)^{-1} = \mathcal{O}(1). \tag{3.28}$$

As $[H_{\alpha,0}, A_\alpha](H_{\alpha,0} - z)^{-1}$ is bounded, (3.27) becomes:

$$\chi A_\alpha(H_{\alpha,0} + i)^{-1} = (H_{\alpha,0} + i)^{-1}\mathcal{O}(1),$$

and then

$$V_\alpha^1 A_\alpha(H_{\alpha,0} + i)^{-1} = V_\alpha^1 (H_{\alpha,0} + i)^{-1}\mathcal{O}(1), \tag{3.29}$$

which is compact. Since $A_\alpha \in \Psi(\langle p_\alpha(x) \rangle, g_0)$ and V_α^2 satisfies (1.6), we have:

$$\begin{aligned} [V_\alpha^2, A_\alpha](H_{\alpha,0} + i)^{-1} &= (V_\alpha^2 A_\alpha - A_\alpha V_\alpha^2)(H_{\alpha,0} + i)^{-1} \\ &= \mathcal{O}(1)\langle p_\alpha(x) \rangle^{-\varepsilon} (H_{\alpha,0} + i)^{-1}, \end{aligned} \tag{3.30}$$

which is compact (respectively, bounded) if $\varepsilon > 0$ (respectively, $\varepsilon = 0$) from Lemma 2.3. So we have (3.24), H_α is of class $C^1(A_\alpha)$ and $[H_\alpha, A_\alpha]$ is bounded from $D(H_\alpha)$ to $L^2(\mathbb{R}^n)$.

To have H_α in $C^{1+\delta}(A_\alpha)$, it remains to show that

$$T(V_\alpha^2) := [(H_\alpha + i)^{-1}, A_\alpha] = (H_\alpha + i)^{-1}[H_\alpha, A_\alpha](H_\alpha + i)^{-1},$$

is of class $C^\delta(A_\alpha)$. We use an interpolation argument as in [14]. For $\rho > 0$, we set:

$$S_\alpha^\rho = \{W \in L^\infty(\mathbb{R}^n; \mathbb{R}); |W(x)| \lesssim \langle p_\alpha(x) \rangle^{-\rho} \text{ a.e. } x \in \mathbb{R}^n\}.$$

Then S_α^ρ is a Banach space, equipped with the norm $\|W\|_{\rho,\alpha} = \|\langle p_\alpha(x) \rangle^\rho W(x)\|_{L^\infty(\mathbb{R}^n)}$. We already have proved that $T(\cdot)$ maps S_α^1 into $C^0(A_\alpha)$. From (3.30), we get:

$$\|T(W) - T(\tilde{W})\| \lesssim \|W - \tilde{W}\|_{1,\alpha},$$

and then T is continuous. We now show that $T(V_\alpha^2)$ is of class $C^1(A_\alpha)$ for $V_\alpha^2 \in S_\alpha^2$. Using $H_\alpha \in C^1(A_\alpha)$, $[H_\alpha, A_\alpha](H_\alpha + i)^{-1} = \mathcal{O}(1)$ and Lemma 6.2.9 of [1], it is enough to show that $[[H_\alpha, A_\alpha], A_\alpha](H_\alpha + i)^{-1}$ is bounded. From Lemmas 3.10 and 3.13, it is enough to show that $[[V_\alpha, A_\alpha], A_\alpha](H_\alpha + i)^{-1}$ is bounded. As for (3.26) and (3.29), we have:

$$\begin{aligned} [[V_\alpha^1, A_\alpha], A_\alpha](H_\alpha + i)^{-1} &= (V_\alpha^1 A_\alpha^2 - 2A_\alpha V_\alpha^1 A_\alpha + A_\alpha^2 V_\alpha^1)(H_\alpha + i)^{-1} \\ &= V_\alpha^1 (H_\alpha + i)^{-1}\mathcal{O}(1) - 2\mathcal{O}(1)V_\alpha^1 (H_\alpha + i)^{-1}\mathcal{O}(1) \\ &\quad + \mathcal{O}(1)V_\alpha^1 (H_\alpha + i)^{-1}, \end{aligned}$$

which is bounded. On the other hand,

$$[[V_\alpha^2, A_\alpha], A_\alpha](H_\alpha + i)^{-1} = (V_\alpha^2 A_\alpha^2 - 2A_\alpha V_\alpha^2 A_\alpha + A_\alpha^2 V_\alpha^2)(H_\alpha + i)^{-1},$$

is bounded since $V_\alpha^2 \in S_\alpha^2$ and $A_\alpha \in \Psi(\langle p_\alpha(x) \rangle, g_0)$. Then $T(V_\alpha^2) \in C^1(A_\alpha)$ for $V_\alpha^2 \in S_\alpha^2$.

Moreover, we know that for $0 < \varepsilon' < 1$, $C^{\varepsilon'}(A_\alpha)$ is a real interpolation space between $C^0(A_\alpha)$ and $C^1(A_\alpha)$. Using the notation of [1], [1, Eq. (5.2.22)] implies:

$$C^{\varepsilon'}(A_\alpha) = (C^0(A_\alpha), C^1(A_\alpha))_{1-\varepsilon', \infty}.$$

On the other hand, mimicking the proof of Lemma A.3 of [14] with $\chi_R = \chi(p_\alpha(x)/R)$, we prove that for $\rho \in]1, 2[$,

$$S_\alpha^\rho \subset (S_\alpha^1, S_\alpha^2)_{\rho-1, \infty}.$$

By interpolation (see, e.g., [1, Theorem 2.6.1]), there is $\delta > 0$ such that H_α is of class $C^{1+\delta}(A_\alpha)$ for $V_\alpha \in S_\alpha^{1+\varepsilon}$ with $\varepsilon > 0$. \square

3.5. Mourre estimates

First, we prove a Mourre estimate for $H_{2,0}$.

Lemma 3.15. *Let $\eta > 0$ and $\chi \in C_0^\infty(\mathbb{R})$. There exists K compact on $L^2(\mathbb{R}^n)$ such that*

$$\chi(H_{2,0})[iH_{2,0}, A_2]\chi(H_{2,0}) \geq (2 - \eta)\chi^2(H_{2,0}) + \chi(H_{2,0})K\chi(H_{2,0}). \quad (3.31)$$

Proof. Using (3.18), we have:

$$\chi(H_{2,0})[iH_{2,0}, A]\chi(H_{2,0}) = 2\chi(H_{2,0})U\left(\frac{2x^2}{\langle \sqrt{2}x \rangle^2} + \frac{2D^2}{\langle \sqrt{2}D \rangle^2}\right)U^*\chi(H_{2,0}). \quad (3.32)$$

For $x^2 + \xi^2 > C$ with $C \gg 1$, the symbol

$$f(x, \xi) = \frac{2x^2}{\langle \sqrt{2}x \rangle^2} + \frac{2D^2}{\langle \sqrt{2}D \rangle^2} \in S(1, g_2)$$

satisfies $f \geq 1 - \eta/2$. Then Gårding inequality (Theorem 18.6.7 of [20]) yields

$$\text{Op}(f) \geq (1 - \eta/2) - \tilde{C} \text{Op}(\chi(x, \xi)) - R,$$

with $\tilde{C} > 0$, $\chi \in C_0^\infty(\mathbb{R}^{2n})$ and $R \in \Psi(\langle x \rangle^{-1} \langle \xi \rangle^{-1}, g_2)$. Then

$$\chi(H_{2,0})[iH_{2,0}, A]\chi(H_{2,0}) \geq (2 - \eta)\chi(H_{2,0})^2 + \chi(H_{2,0})K\chi(H_{2,0}),$$

where K is compact. \square

We have also a Mourre estimate for $H_{\alpha,0}$ with $0 < \alpha < 2$.

Lemma 3.16. *Let $\eta > 0$. If the support of ψ in (3.17) is close enough to 0 and $\chi \in C_0^\infty(\mathbb{R})$, there exists a compact operator K on $L^2(\mathbb{R}^n)$ such that*

$$\chi(H_{\alpha,0})[iH_{\alpha,0}, A_\alpha]\chi(H_{\alpha,0}) \geq (2 - \alpha - \eta)\chi^2(H_{\alpha,0}) + \chi(H_{\alpha,0})K\chi(H_{\alpha,0}). \quad (3.33)$$

Proof. Since $a_\alpha(x, \xi) \in S(\langle x \rangle^{1-\alpha} \langle \xi \rangle, g_2)$, $\xi^2 - \langle x \rangle^\alpha \in S(\xi^2 + \langle x \rangle^\alpha, g_2)$ and $\langle \xi \rangle$ is like $\langle x \rangle^{\alpha/2}$ on the support of a_α , we have:

$$[iH_{\alpha,0}, A_\alpha] = \text{Op}(b_1) + \text{Op}(b_2) + K_1,$$

where

$$b_1(x, \xi) = (2\xi^2 \langle x \rangle^{-\alpha} - 2\alpha \langle x \rangle \xi^2 \langle x \rangle^{-\alpha-2} + \alpha x^2 \langle x \rangle^{-2}) \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \in S(1, g_2),$$

$b_2(x, \xi) \in S(1, g_2)$ with support inside the support of $\psi'((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha))$ and $K_1 \in \Psi(\langle x \rangle^{-1} \langle \xi \rangle^{-1}, g_2)$. If the support of ψ is close enough to 0, we have:

$$b_1(x, \xi) \geq (2 - \alpha - \eta) \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right),$$

for (x, ξ) large enough, and the Gårding inequality implies:

$$\text{Op}(b_1) \geq (2 - \alpha - \eta) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) + K_2,$$

with $K_2 \in \Psi(\langle x \rangle^{-1} \langle \xi \rangle^{-1}, g_2)$. Therefore

$$\begin{aligned} \chi(H_{\alpha,0})[iH_{\alpha,0}, A_\alpha]\chi(H_{\alpha,0}) &\geq (2 - \alpha - \eta)\chi(H_{\alpha,0}) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) \chi(H_{\alpha,0}) \\ &\quad + \chi(H_{\alpha,0})(\text{Op}(b_2) + K_1 + K_2)\chi(H_{\alpha,0}). \end{aligned} \quad (3.34)$$

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ be equal to 1 near the support of χ . Using Proposition (3.5), we get:

$$\begin{aligned} &\text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) \chi(H_{\alpha,0}) \\ &= \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) (H_{\alpha,0} + i)^{-1} (H_{\alpha,0} + i) \tilde{\chi}(H_{\alpha,0}) \chi(H_{\alpha,0}) \\ &= \chi(H_{\alpha,0}) - \mathcal{O}(1) \text{Op}(r)(H_{\alpha,0} + i) \tilde{\chi}(H_{\alpha,0}) \chi(H_{\alpha,0}). \end{aligned}$$

Then we have:

$$\chi(H_{\alpha,0}) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) \chi(H_{\alpha,0}) = \chi^2(H_{\alpha,0}) + \chi(H_{\alpha,0})K_3\chi(H_{\alpha,0}),$$

where K_3 is compact. Let $\tilde{\psi} \in C_0^\infty([-1/2, 1/2])$ such that $\tilde{\psi} = 1$ near 0 and $\psi = 1$ on the support of $\tilde{\psi}$. Using Proposition 3.5 with $\tilde{\psi}$, we get:

$$\begin{aligned} \text{Op}(b_2)(H_{\alpha,0} + i)^{-1} &= \text{Op}\left(b_2(x, \xi)\tilde{\psi}\left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha}\right)\right)\mathcal{O}(1) + \text{Op}(s)\mathcal{O}(1) \\ &= 0 + \text{Op}(s)\mathcal{O}(1), \end{aligned}$$

with $s(x, \xi) \in S(\langle x \rangle^{-1}\langle \xi \rangle^{-1}, g_0)$. Here we have used the fact that $b_2 = 0$ on the support of $\tilde{\psi}((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha))$. Then

$$\chi(H_{\alpha,0})\text{Op}(b_2)\chi(H_{\alpha,0}) = \chi(H_{\alpha,0})K_4\chi(H_{\alpha,0}),$$

with K_4 compact. Then (3.34) becomes:

$$\begin{aligned} \chi(H_{\alpha,0})[iH_{\alpha,0}, A_\alpha]\chi(H_{\alpha,0}) &\geq (2 - \alpha - \eta)\chi^2(H_{\alpha,0}) \\ &\quad + \chi(H_{\alpha,0})(K_1 + K_2 + (2 - \alpha - \eta)K_3 + K_4)\chi(H_{\alpha,0}), \end{aligned}$$

which implies the lemma. \square

Finally, we obtain a Mourre estimate for H_α for $0 < \alpha \leq 2$.

Proposition 3.17. *Let $\eta > 0$ and $0 < \alpha \leq 2$. If the support of ψ in (3.17) is close enough to 0 and $\chi \in C_0^\infty(\mathbb{R})$, there exists a compact operator K on $L^2(\mathbb{R}^n)$ such that*

$$\chi(H_\alpha)[iH_\alpha, A_\alpha]\chi(H_\alpha) \geq (\sigma_\alpha - \eta)\chi^2(H_\alpha) + \chi(H_\alpha)K\chi(H_\alpha). \quad (3.35)$$

Proof. Let $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\chi} = 1$ near the support of χ . $\tilde{\chi}(H_\alpha) - \tilde{\chi}(H_{\alpha,0})$ is compact because V_α is $H_{\alpha,0}$ -compact. Since $[H_\alpha, A_\alpha]$ is H_α -bounded from Proposition 3.14,

$$\begin{aligned} \chi(H_\alpha)[iH_\alpha, A_\alpha]\chi(H_\alpha) &= \chi(H_\alpha)\tilde{\chi}(H_{\alpha,0})\left([iH_{\alpha,0}, A_\alpha] + i[V_\alpha, A_\alpha]\right)\tilde{\chi}(H_{\alpha,0})\chi(H_\alpha) \\ &\quad + \chi(H_\alpha)K\chi(H_\alpha), \end{aligned}$$

with K compact. From (3.24), Lemmas 3.15 and 3.16, we have:

$$\chi(H_\alpha)[iH_\alpha, A_\alpha]\chi(H_\alpha) \geq (\sigma_\alpha - \eta)\chi(H_\alpha)\tilde{\chi}^2(H_{\alpha,0})\chi(H_\alpha) + \chi(H_\alpha)K\chi(H_\alpha),$$

with another K compact. Therefore,

$$\chi(H_\alpha)[iH_\alpha, A_\alpha]\chi(H_\alpha) \geq (\sigma_\alpha - \eta)\chi^2(H_\alpha) + \chi(H_\alpha)K\chi(H_\alpha),$$

which implies the lemma. \square

4. Asymptotic completeness

4.1. Limiting absorption principle

From Proposition 3.14, H_α is of class $C^{1+\delta}(A_\alpha)$, and $[H_\alpha, A_\alpha]$ maps $D(H_\alpha)$ into $L^2(\mathbb{R}^n)$. Using Proposition 3.17, Theorems 1.1 and 4.13 of [6] yield:

Theorem 4.1 (Limiting absorption principle). *Let $0 < \alpha \leq 2$. The singular continuous spectrum of H_α is empty, and its point spectrum is locally finite. For $\Lambda \subset \mathbb{R} \setminus \sigma_{pp}(H)$ and $\nu > 1/2$, we have, for some $\eta > 0$,*

$$\sup_{z \in \Lambda + i[-\eta, \eta]} \|\langle A_\alpha \rangle^{-\nu} (H_\alpha - z)^{-1} \langle A_\alpha \rangle^{-\nu}\| < \infty. \tag{4.1}$$

As a corollary, we have:

Proposition 4.2. *Let $0 < \alpha \leq 2$ and $\eta > 0$. Assume that the support of ψ in (3.17) is close enough to 0. For $\lambda \in \mathbb{R} \setminus \sigma_{pp}(H_\alpha)$, there is a real open interval Λ containing λ such that*

$$\mathbf{1}_\Lambda(H_\alpha)[iH_\alpha, A_\alpha]\mathbf{1}_\Lambda(H_\alpha) \geq (\sigma_\alpha - \eta)\mathbf{1}_\Lambda(H_\alpha). \tag{4.2}$$

Proof. Since the spectrum of H_α is absolutely continuous near λ , we have:

$$s\text{-}\lim_{\delta \rightarrow 0} \mathbf{1}_{[\lambda-\delta, \lambda+\delta]}(H_\alpha) = 0.$$

Using Proposition 3.17, we infer that $K\mathbf{1}_{[\lambda-\delta, \lambda+\delta]}(H_{2,0})$ goes to 0 in norm when $\delta \rightarrow 0$, since K is compact. So we can find Λ such that

$$\mathbf{1}_\Lambda(H_\alpha)[iH_\alpha, A_\alpha]\mathbf{1}_\Lambda(H_\alpha) \geq (\sigma_\alpha - 2\eta)\mathbf{1}_\Lambda(H_\alpha).$$

Since $\eta > 0$ is arbitrary, this yields the proposition. \square

4.2. Minimal velocity estimate

The following proposition is a simplified version of Proposition A.1 of [15], due to ideas of I.M. Sigal and A. Soffer.

Proposition 4.3 [15]. *Let H and A be two self-adjoint operators on a separable Hilbert space \mathcal{H} . We suppose that*

- (i) H is in $C^{1+\delta}(A)$ for some $\delta > 0$.
- (ii) There exists an interval Λ such that: $\mathbf{1}_\Lambda(H)[H, iA]\mathbf{1}_\Lambda(H) \geq c\mathbf{1}_\Lambda(H)$, with $c > 0$.

Then for any $g \in C_0^\infty(]-\infty, c[)$ and any $f \in C_0^\infty(A)$, we have:

$$\int_1^{+\infty} \left\| g\left(\frac{A}{t}\right) e^{-itH} f(H)u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2, \tag{4.3}$$

for $u \in \mathcal{H}$, and

$$s\text{-}\lim_{t \rightarrow +\infty} g\left(\frac{A}{t}\right) e^{-itH} f(H) = 0. \tag{4.4}$$

From Propositions 3.14, 4.2 and 4.3, we get:

Proposition 4.4. *For any $g \in C_0^\infty(]-\infty, \sigma_\alpha[)$ and any $f \in C_0^\infty(\mathbb{R})$, we have:*

$$\int_1^{+\infty} \left\| g\left(\frac{A_\alpha}{t}\right) e^{-itH_\alpha} f(H_\alpha)u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2, \tag{4.5}$$

for $u \in L^2(\mathbb{R}^n)$, and

$$s\text{-}\lim_{t \rightarrow +\infty} g\left(\frac{A_\alpha}{t}\right) e^{-itH_\alpha} f(H_\alpha) = 0. \tag{4.6}$$

We want to replace A_α by $p_\alpha(x)$ in Proposition 4.4. For that, we use a slight modification of Lemma A.3 of C. Gérard and F. Nier [15].

Lemma 4.5 [15]. *Let \mathcal{A} and \mathcal{B} be two self-adjoint operators on a separable Hilbert space \mathcal{H} such that, for each $\mu > 0$, we have:*

$$D(\mathcal{B}) \subset D(\mathcal{A}) \quad \text{and} \quad 1 \leq \mathcal{B}, \tag{4.7}$$

$$\mathcal{A} \leq (1 + \mu)\mathcal{B} + C_\mu, \tag{4.8}$$

with $C_\mu \geq 0$, and

$$[\mathcal{A}, \mathcal{B}]\mathcal{B}^{-1} \in \mathcal{L}(\mathcal{H}). \tag{4.9}$$

Then for each $\lambda \in \mathbb{R}$, let $\varphi \in C^\infty(\mathbb{R})$ with $\text{supp}(\varphi) \subset]-\infty, \lambda[$, $\varphi = 1$ near $-\infty$ and $\psi \in C^\infty(\mathbb{R})$ with $\text{supp}(\psi) \subset]\lambda, +\infty[$, $\psi = 1$ near $+\infty$. We have:

$$\|\varphi(\mathcal{B}/t)\psi(\mathcal{A}/t)\| = \mathcal{O}(t^{-1}) \quad \text{as } t \rightarrow +\infty. \tag{4.10}$$

Proof. We follow the proof of [15, Lemma A.3]. Let $\varphi_1 \in C^\infty(\mathbb{R})$ with $\text{supp}(\varphi_1) \subset]-\infty, \lambda[$ and $\varphi_1 = 1$ near the support of φ . We have by (4.8),

$$\varphi_1(\mathcal{B}/t)\mathcal{A}\varphi_1(\mathcal{B}/t) \leq (1 + \mu)\varphi_1(\mathcal{B}/t)\mathcal{B}\varphi_1(\mathcal{B}/t) + \mathcal{O}(1) \leq (1 + \mu)\lambda t + \mathcal{O}(1).$$

So, if μ is small enough and t large enough, we get:

$$\psi\left(\frac{\varphi_1(\mathcal{B}/t)\mathcal{A}\varphi_1(\mathcal{B}/t)}{t}\right) = 0,$$

and it remains to show that

$$\phi(\mathcal{B}/t)\left(\psi(\mathcal{A}/t) - \psi\left(\frac{\varphi_1(\mathcal{B}/t)\mathcal{A}\varphi_1(\mathcal{B}/t)}{t}\right)\right) = \mathcal{O}(t^{-1}). \tag{4.11}$$

The proof of (4.11) is the same as in [15]. \square

To apply the above result, we distinguish the cases $\alpha = 2$ and $0 < \alpha < 2$.

Lemma 4.6. *The pairs $(\mathcal{A}, \mathcal{B}) = (A_2, \langle \ln(x) \rangle)$ and $(\mathcal{A}, \mathcal{B}) = (-A_2, \langle \ln(x) \rangle)$ satisfy the assumptions of Lemma 4.5, provided that the support of ψ in (3.13) is small enough according to μ .*

Proof. We prove the lemma only for $(\mathcal{A}, \mathcal{B}) = (A_2, \langle \ln(x) \rangle)$; the proof is the same in the other case. Since $\mathcal{A} \in \Psi(\langle \ln(x) \rangle, g_0)$, \mathcal{A} is well-defined and symmetric on $D(\mathcal{B}) = \{u \in L^2(\mathbb{R}^n); \langle \ln(x) \rangle u \in L^2(\mathbb{R}^n)\}$. So the assumption (4.7) of Lemma 4.5 is true from Theorem 2.1. Moreover $\mathcal{B} \in \Psi(\langle \ln(x) \rangle, g_1)$ implies $[\mathcal{A}, \mathcal{B}]\mathcal{B}^{-1} \in \Psi(\langle \ln(x) \rangle \langle x \rangle^{-1}, g_0)$, from [20, Theorem 18.5.5]. Then the assumption (4.9) is also true.

Let $f, g \in C_0^\infty([0, 1]; [0, 1])$ be equal to 1 near 0. For $\delta, M > 0$, we can write:

$$\mathcal{A} = \text{Op}(s_1) + \text{Op}(s_2) + \text{Op}(s_3), \tag{4.12}$$

with

$$\begin{aligned} s_1 &= \left(\ln\langle \xi + x \rangle f\left(\frac{\langle \xi + x \rangle}{\langle x \rangle^\delta}\right) - \ln\langle \xi - x \rangle f\left(\frac{\langle \xi - x \rangle}{\langle x \rangle^\delta}\right) \right) g(\langle x \rangle/M), \\ s_2 &= \left(\ln\langle \xi + x \rangle f\left(\frac{\langle \xi + x \rangle}{\langle x \rangle^\delta}\right) - \ln\langle \xi - x \rangle f\left(\frac{\langle \xi - x \rangle}{\langle x \rangle^\delta}\right) \right) (1 - g)(\langle x \rangle/M), \\ s_3 &= \left(\ln\langle \xi + x \rangle (1 - f)\left(\frac{\langle \xi + x \rangle}{\langle x \rangle^\delta}\right) - \ln\langle \xi - x \rangle (1 - f)\left(\frac{\langle \xi - x \rangle}{\langle x \rangle^\delta}\right) \right). \end{aligned}$$

Then

$$|(\text{Op}(s_1)u, u)| \leq C\|u\|^2, \tag{4.13}$$

because $s_1 \in S(\langle x \rangle^{-\infty}, g_0)$. On the other hand, since

$$s_2 \in S(\langle \ln \langle x \rangle \rangle, g_0) \quad \text{and} \quad \langle \ln \langle x \rangle \rangle^{-1/2} \in S\left(\langle \ln \langle x \rangle \rangle^{-1/2}, \frac{|dx|^2}{\langle x \rangle^2} + \frac{|d\xi|^2}{\langle x \rangle^2}\right),$$

we get:

$$\langle \ln \langle x \rangle \rangle^{-1/2} \text{Op}(s_2) \langle \ln \langle x \rangle \rangle^{-1/2} = \text{Op}(\langle \ln \langle x \rangle \rangle^{-1} s_2(x, \xi)) + R, \tag{4.14}$$

with $R \in \Psi(\langle x \rangle^{-1}, g_0)$. Since $\text{supp } f \subset [0, 1]$, we get:

$$|\langle \ln \langle x \rangle \rangle^{-1} s_2(x, \xi)| \leq 2\delta.$$

We also have:

$$|\partial_x^\alpha \partial_\xi^\beta \langle \ln \langle x \rangle \rangle^{-1} s_2(x, \xi)| \leq C_{\alpha, \beta, \delta} \ln(M)^{-1},$$

where $C_{\alpha, \beta, \delta}$ depends on δ . Fix δ small enough, and then, M large enough. Theorem 18.6.3 of [20] yields:

$$\|\text{Op}(\langle \ln \langle x \rangle \rangle^{-1} s_2(x, \xi))\| < \eta/2.$$

Then (4.14) implies

$$(\langle \ln \langle x \rangle \rangle^{-1/2} \text{Op}(s_2) \langle \ln \langle x \rangle \rangle^{-1/2} u, u) \leq \eta/2 \|u\|^2 + (Ru, u),$$

and since $R \in \Psi(\langle x \rangle^{-1}, g_0)$,

$$(\text{Op}(s_2)u, u) \leq \eta/2 (\mathcal{B}u, u) + C \|u\|^2. \tag{4.15}$$

So it remains to study $s_3(x, \xi)$. Using (3.14) and (3.15), we get:

$$\begin{aligned} s_3(x, \xi) &\leq \ln \langle \xi + x \rangle - \ln \langle \xi - x \rangle + \ln \langle \xi - x \rangle f\left(\frac{\langle \xi - x \rangle}{\langle x \rangle^\delta}\right) \\ &\leq (1 + \delta) \langle \ln \langle x \rangle \rangle + C. \end{aligned} \tag{4.16}$$

We also have:

$$s_3(x, \xi) \in S\left(\langle \ln \langle x \rangle \rangle, \frac{|dx|^2}{\langle x \rangle^{2\delta}} + \frac{|d\xi|^2}{\langle x \rangle^{2\delta}}\right).$$

If we assume $\delta < \eta/2$, Gårding inequality implies:

$$(\text{Op}(s_3)u, u) \leq (1 + \eta/2)(\mathcal{B}u, u) + C \|u\|^2 + (Ru, u),$$

with $R \in \Psi(\langle \ln \langle x \rangle \rangle \langle x \rangle^{-2\delta}, g_0)$. So,

$$(\text{Op}(s_3)u, u) \leq (1 + \eta/2)(\mathcal{B}u, u) + C\|u\|^2. \tag{4.17}$$

Combining with (4.13), (4.15) and (4.17), we get:

$$(\mathcal{A}u, u) \leq (1 + \eta)(\mathcal{B}u, u) + C\|u\|^2,$$

which is (4.8). \square

Lemma 4.7. *Let $0 < \alpha < 2$. The pairs of operators $(\mathcal{A}, \mathcal{B}) = (A_\alpha, \langle x \rangle^{1-\alpha/2})$ and $(\mathcal{A}, \mathcal{B}) = (-A_\alpha, \langle x \rangle^{1-\alpha/2})$ satisfy the assumptions of Lemma 4.5, provided that the support of ψ in (3.17) is small enough according to μ .*

Proof. We prove the lemma only for $(\mathcal{A}, \mathcal{B}) = (A_\alpha, \langle x \rangle^{1-\alpha/2})$; the proof is the same in the other case. Since $\mathcal{A} \in \Psi(\langle x \rangle^{1-\alpha/2}, g_1^{\alpha/2})$, \mathcal{A} is well-defined and symmetric on $D(\mathcal{B}) = \{u \in L^2(\mathbb{R}^n); \langle x \rangle^{1-\alpha/2}u \in L^2(\mathbb{R}^n)\}$. So the hypothesis (4.7) of Lemma 4.5 is true from Theorem 2.1. Moreover, $\mathcal{B} \in \Psi(\langle x \rangle^{1-\alpha/2}, g_1)$ implies $[\mathcal{A}, \mathcal{B}]\mathcal{B}^{-1} \in \Psi(\langle x \rangle^{-\alpha}, g_1^{\alpha/2})$ from Theorem 18.5.5 of [20]. Then the assumption (4.9) is also true.

On the support of $a_\alpha(x, \xi)$, with (x, ξ) large enough, we have $|\xi| = \langle x \rangle^{\alpha/2}(1 + o(1))$, where $o(1)$ stands for an arbitrary small function as $\text{supp } \psi \rightarrow \{0\}$. Then,

$$a_\alpha(x, \xi) \leq (1 + \eta/2)\langle x \rangle^{1-\alpha/2} + C,$$

with $C > 0$. The Gårding inequality in $\Psi(g_1^{\alpha/2})$ implies that

$$(\mathcal{A}u, u) \leq (1 + \eta/2)(\mathcal{B}u, u) + C\|u\|^2 + (\text{Op}(r)u, u), \tag{4.18}$$

with $r \in S(\langle x \rangle^{1-3\alpha/2}, g_1^{\alpha/2})$. We have:

$$\begin{aligned} |(\text{Op}(r)u, u)| &= |(\langle x \rangle^{-1/2+\alpha/4} \text{Op}(r)\langle x \rangle^{-1/2+5\alpha/4} \langle x \rangle^{-\alpha} \langle x \rangle^{1/2-\alpha/4}u, \langle x \rangle^{1/2-\alpha/4}u)| \\ &\lesssim \|\langle x \rangle^{-\alpha} \langle x \rangle^{1/2-\alpha/4}u\| \|\langle x \rangle^{1/2-\alpha/4}u\| \\ &\leq (\eta/4\|\langle x \rangle^{1/2-\alpha/4}u\| + C\|u\|)\|\langle x \rangle^{1/2-\alpha/4}u\| \\ &\leq \eta/2\|\langle x \rangle^{1/2-\alpha/4}u\|^2 + C\|u\|^2. \end{aligned}$$

So, (4.18) becomes

$$(\mathcal{A}u, u) \leq (1 + \eta)(\mathcal{B}u, u) + C\|u\|^2, \tag{4.19}$$

which proves (4.8) and the lemma. \square

From Proposition 4.4, Lemmas 4.5, 4.6 and 4.7, we obtain:

Proposition 4.8 (Minimal velocity estimate). *For any $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi \cap \sigma_{pp}(H_\alpha) = \emptyset$, $0 < \theta < \sigma_\alpha$, and $u \in L^2(\mathbb{R}^n)$ we have:*

$$\int_1^\infty \left\| \mathbf{1}_{[0,\theta]} \left(\frac{p_\alpha(x)}{t} \right) e^{itH_\alpha} \chi(H_\alpha) u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2,$$

$$\text{s-} \lim_{t \rightarrow +\infty} \mathbf{1}_{[0,\theta]} \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \mathbf{1}^c(H_\alpha) = 0.$$

4.3. Proof of Theorem 1.1

As mentioned in the introduction, we prove Theorem 1.1, with $H_{\alpha,0}$ (respectively, H_α) replaced by $H_{\alpha,0}$ (respectively, H_α), since this substitution will turn out to yield a short range perturbation. We prove (1.8); it will be clear from the proof that (1.7) follows the same way. By a density argument and using that $\sigma_{pp}(H_{\alpha,0}) = \emptyset$, and $\sigma_{pp}(H_\alpha)$ has no accumulating point, it is enough to show the existence of

$$\text{s-} \lim_{t \rightarrow +\infty} e^{itH_{\alpha,0}} e^{-itH_\alpha} \chi^2(H_\alpha),$$

with $\text{supp } \chi \cap \sigma_{pp}(H_\alpha) = \emptyset$. We have:

$$e^{itH_{\alpha,0}} e^{-itH_\alpha} \chi^2(H_\alpha) = \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} e^{-itH_\alpha} \chi(H_\alpha) + e^{itH_{\alpha,0}} (\chi(H_{\alpha,0}) - \chi(H_\alpha)) e^{-itH_\alpha} \chi(H_\alpha). \tag{4.20}$$

As the spectrum of H_α is absolutely continuous on $\text{supp}(\chi)$, $e^{-itH_\alpha} \chi(H_\alpha) \rightarrow 0$ weakly. Since $\chi(H_{\alpha,0}) - \chi(H_\alpha)$ is compact, the second term in (4.20) converges strongly to 0.

Let $g_1 \in C_0^\infty]-\infty, \sigma_\alpha[$ such that $g_1 = 1$ near 0. From Proposition 4.8, we deduce that

$$\text{s-} \lim_{t \rightarrow +\infty} \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} g_1 \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \chi(H_\alpha) = 0. \tag{4.21}$$

Now, let us consider:

$$G(t) = \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} (1 - g_1) \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \chi(H_\alpha) u.$$

The function $G(t)$ is differentiable and

$$G'(t) = \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} \left[g_1 \left(\frac{p_\alpha(x)}{t} \right), iH_{\alpha,0} \right] e^{-itH_\alpha} \chi(H_\alpha) u + \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} \frac{p_\alpha(x)}{t^2} g_1' \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \chi(H_\alpha) u + \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} (1 - g_1) \left(\frac{p_\alpha(x)}{t} \right) V_\alpha(x) e^{-itH_\alpha} \chi(H_\alpha) u. \tag{4.22}$$

The first term of (4.22) is equal to

$$I(t) = \frac{1}{t} \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} \left(\text{Op}(f) g'_1 \left(\frac{p_\alpha(x)}{t} \right) - i \frac{|\nabla_x p_\alpha(x)|^2}{t} g''_1 \left(\frac{p_\alpha(x)}{t} \right) \right) e^{-itH_\alpha} \chi(H_\alpha) u,$$

with

$$f(x, \xi) = -2 \nabla_x p_\alpha(x) \xi \in S(\langle x \rangle^{-\alpha/2} \langle \xi \rangle, g_2).$$

Using Proposition 3.5 and the fact that $|\nabla_x p_\alpha(x)|$ is bounded, we get:

$$\begin{aligned} I(t) &= \frac{1}{t} \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) \text{Op}(f) g'_1 \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \chi(H_\alpha) u \\ &\quad + \mathcal{O}(t^{-1}) \left\| \text{Op}(r) \text{Op}(f) g'_1 \left(\frac{p_\alpha(x)}{t} \right) \right\| + \mathcal{O}(t^{-2}), \end{aligned}$$

with $r \in S(\langle \xi \rangle^{-2}, g_0)$. Using the pseudo-differential calculus and the fact that $\langle x \rangle$ is like $t^{1/(1-\alpha/2)}$ (respectively, e^t) if $0 < \alpha < 2$ (respectively, $\alpha = 2$) on the support of $g'_1(p_\alpha(x)/t)$, we get:

$$\left\| \text{Op}(r) \text{Op}(f) g'_1 \left(\frac{p_\alpha(x)}{t} \right) \right\| = \begin{cases} \mathcal{O}(t^{-\frac{\alpha/2}{1-\alpha/2}}) & \text{for } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\alpha t/2}) & \text{for } \alpha = 2. \end{cases}$$

On the other hand, the pseudo-differential calculus in $\Psi(g_1)$ implies:

$$\text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{\xi^2 + \langle x \rangle^\alpha} \right) \right) \text{Op}(f) = \text{Op}(m),$$

with $m(x, \xi) \in S(1, g_2)$. Let $g_2 \in C_0^\infty(]-\infty, \sigma_\alpha[)$ such that $g_2 = 0$ near 0 and $g_2 = 1$ near the support of g'_1 . Using the pseudo-differential calculus in $\Psi(|dx|^2 \langle x \rangle^{-\alpha} + |d\xi|^2)$, we get:

$$\begin{aligned} \text{Op}(m) g'_1 \left(\frac{p_\alpha(x)}{t} \right) &= \text{Op} \left(m(x, \xi) g'_1 \left(\frac{p_\alpha(x)}{t} \right) \right) g_2 \left(\frac{p_\alpha(x)}{t} \right) + \mathcal{O}(1) \langle x \rangle^{-\alpha/2} g_2 \left(\frac{p_\alpha(x)}{t} \right) \\ &= g_2 \left(\frac{p_\alpha(x)}{t} \right) \mathcal{O}(1) g_2 \left(\frac{p_\alpha(x)}{t} \right) + \mathcal{O}(t^{-\delta}), \end{aligned}$$

with $\delta > 0$. Then

$$I(t) = \frac{1}{t} \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} g_2 \left(\frac{p_\alpha(x)}{t} \right) \mathcal{O}(1) g_2 \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha} \chi(H_\alpha) u + \mathcal{O}(t^{-1-\delta}).$$

Proposition 4.8 and a duality argument imply that $I(t)$ is integrable.

The second term in (4.22) can be written:

$$\begin{aligned} & \chi(H_{\alpha,0})e^{itH_{\alpha,0}}\frac{p_{\alpha}(x)}{t^2}g_1'\left(\frac{p_{\alpha}(x)}{t}\right)e^{-itH_{\alpha}}\chi(H_{\alpha})u \\ &= \frac{1}{t}\chi(H_{\alpha,0})e^{itH_{\alpha,0}}g_2\left(\frac{p_{\alpha}(x)}{t}\right)\mathcal{O}(1)g_2\left(\frac{p_{\alpha}(x)}{t}\right)e^{-itH_{\alpha}}\chi(H_{\alpha})u. \end{aligned} \quad (4.23)$$

Like for $I(t)$, we get that (4.23) is integrable.

Finally, using assumptions (1.5) and (1.6), we get, for $t \gg 1$,

$$\left| (1 - g_1)\left(\frac{p_{\alpha}(x)}{t}\right)V_{\alpha}(x) \right| = \left| (1 - g_1)\left(\frac{p_{\alpha}(x)}{t}\right)V_{\alpha}^2(x) \right| = \mathcal{O}(t^{-1-\varepsilon}),$$

which is integrable. This implies that the third term in (4.22), and then $G'(t)$, is integrable. So $G(t)$ has a limit when $t \rightarrow +\infty$ and the theorem follows from (4.20) and (4.21).

5. Asymptotic velocity

In this section, we construct the asymptotic velocity and describe its spectrum. In (1.4), we defined the position variable so that it increases like t along the evolution. We define the local velocity as

$$V_{\alpha} := [iH_{\alpha}, p_{\alpha}(x)].$$

(We use typewriter style letters to avoid any confusion with previous notations.) We denote $N = N_2$ the harmonic oscillator. The observable V_{α} is defined as a quadratic form on $D(N)$. By a direct calculation and an application of Theorem 2.1, we obtain that V_{α} is (well defined as an operator and) essentially self-adjoint with domain $D(N)$; we note again V_{α} the self-adjoint extension. Thanks to Theorem 1.1, it is sufficient to construct the asymptotic velocity and to describe its spectrum in the free case. Nevertheless, the local velocity does not commute with the free evolution, in particular the asymptotic velocity is different from the local velocity even in the free case. First, we establish some propagation estimates in the free case and the general case will follow. For an observable $\Theta(t)$, we denote $\mathbf{D}\Theta(t)$ its Heisenberg derivative with respect to $H_{\alpha,0}$, i.e.,

$$\mathbf{D}\Theta(t) := \frac{d}{dt}\Theta(t) + [iH_{\alpha,0}, \Theta(t)].$$

The main result we prove in this section is Theorem 1.2. It can be proved for all $0 < \alpha \leq 2$ in the same way. Like for the asymptotic completeness, we give some generalizations of this result in the case $\alpha = 2$, see Section 6.3.

5.1. Local velocity

This section is devoted to the study of the local velocity and the local acceleration. A direct calculation yields $2\nabla p_\alpha(x) = \sigma_\alpha \langle x \rangle^{-1-\alpha/2} x$, and:

$$\nabla_\alpha = \frac{\sigma_\alpha}{2} \left(\frac{x}{\langle x \rangle^{1+\alpha/2}} D + hc \right), \tag{5.1}$$

where hc stands for the adjoint of the first term. We set:

$$v_\alpha(x, \xi) = \sigma_\alpha \frac{x \cdot \xi}{\langle x \rangle^{1+\alpha/2}}. \tag{5.2}$$

Lemma 5.1. *The operator $(\nabla_\alpha, D(N))$ is well defined as an operator, and essentially self-adjoint. We have $\nabla_\alpha \in \Psi(\langle \xi \rangle \langle x \rangle^{-\alpha/2}, g_2)$, and the symbol of ∇_α is v_α .*

Proof. We clearly have: $\|\nabla_\alpha u\| \lesssim \|Nu\|$, for $u \in D(N)$. We also have:

$$\{\xi^2 + x^2, v_\alpha(x, \xi)\} = \sigma_\alpha \left(\frac{2\xi^2}{\langle x \rangle^{1+\alpha/2}} - (2 + \alpha) \frac{(x \cdot \xi)^2}{\langle x \rangle^{3+\alpha/2}} - 2 \frac{x^2}{\langle x \rangle^{1+\alpha/2}} \right),$$

and the lemma follows from Theorem 2.1. \square

Define the acceleration:

$$\begin{aligned} A_\alpha &:= [iH_{\alpha,0}, \nabla_\alpha]; \\ a_\alpha(x, \xi) &:= \sigma_\alpha \left(\frac{2\xi^2}{\langle x \rangle^{1+\alpha/2}} - (2 + \alpha) \frac{(x \cdot \xi)^2}{\langle x \rangle^{3+\alpha/2}} + \alpha \frac{x^2}{\langle x \rangle^{3-\alpha/2}} \right). \end{aligned} \tag{5.3}$$

The operator A_α is a pseudo-differential operator, with principal symbol

$$a_\alpha(x, \xi) \in S(\langle \xi \rangle^2 \langle x \rangle^{-1+\alpha/2}, g_2).$$

We will often use the decomposition $a_\alpha(x, \xi) = a_\alpha^1(x, \xi) + a_\alpha^2(x, \xi)$, with $a_\alpha^2(x, \xi) = \sigma_\alpha \alpha \frac{x^2}{\langle x \rangle^{3-\alpha/2}} \in S(\langle x \rangle^{-1+\alpha/2}, g_2)$, and $a_\alpha^1(x, \xi) \in S(\langle \xi \rangle^2 \langle x \rangle^{-1-\alpha/2}, g_2)$.

Lemma 5.2. *The operators $\nabla_\alpha(i + H_{\alpha,0})^{-1}$ and $A_\alpha(i + H_{\alpha,0})^{-1}$, defined on $D(N)$, can be extended to bounded operators.*

Proof. We prove a slightly more general result. Let $c \in S(\langle \xi \rangle^m \langle x \rangle^{-k}, g_2)$, with $\alpha m/2 - k \leq 0, 0 \leq m \leq 2$. We prove:

The operator $\text{Op}(c)(i + H_{\alpha,0})^{-1}$, defined on $D(N)$, can be extended to a bounded operator.

The lemma then follows using the decomposition $a_\alpha = a_\alpha^1 + a_\alpha^2$. Recall from Proposition 3.5 that for all $1 \geq \beta > 1/2$,

$$(i + H_{\alpha,0})^{-1} = \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right)(i + H_{\alpha,0})^{-1} + R_\beta, \tag{5.4}$$

with $N_\alpha^\beta R_\beta$ bounded, and $\psi \in C_0^\infty(\mathbb{R})$, $\psi = 1$ in a neighborhood of zero. Since $\text{Op}(c)N_\alpha^{-1}$ is bounded, it is sufficient to prove:

$$\text{Op}(c)\text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) \text{ is bounded.} \tag{5.5}$$

This is a pseudo-differential operator, with principal symbol

$$\left|c(x, \xi)\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right| \lesssim \langle x \rangle^{\alpha m/2-k} \psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right) \lesssim 1,$$

where we have used $\langle \xi \rangle \lesssim \langle x \rangle^{\alpha/2}$ on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/((\xi^2 + \langle x \rangle^\alpha)^\beta))$. This yields (5.5), and the lemma. \square

Lemma 5.3. *Let $f, \chi \in C_0^\infty(\mathbb{R})$. Then, as $t \rightarrow \infty$:*

(i)
$$\left[\chi(H_\alpha), f\left(\frac{p_\alpha(x)}{t}\right)\right] = \mathcal{O}(t^{-1}).$$

(ii) *If f is constant in a neighborhood of 0, then there exists $\varepsilon > 0$ such that*

$$\left[f\left(\frac{p_\alpha(x)}{t}\right), \mathbb{V}_\alpha\right] = \begin{cases} \mathcal{O}(t^{-(2+\alpha)/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon t}) & \text{if } \alpha = 2. \end{cases}$$

(iii) *If f is constant in a neighborhood of 0, then there exists $\varepsilon > 0$ such that*

$$\chi(H_{\alpha,0})\left[f\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha\right]\chi(H_{\alpha,0}) = \begin{cases} \mathcal{O}(t^{-4/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon t}) & \text{if } \alpha = 2. \end{cases}$$

Proof. (i) Using Helffer–Sjöstrand formula, it is sufficient to show:

$$(z - H_\alpha)^{-1} \frac{1}{2t} \left(\mathbb{V}_\alpha f'\left(\frac{p_\alpha(x)}{t}\right) + hc\right) (z - H_\alpha)^{-1} = \mathcal{O}(t^{-1}), \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

The above relation follows from Lemma 5.2, and the fact that $D(H_\alpha) = D(H_{\alpha,0})$.

(ii) We have:

$$\left[i\mathbb{V}_\alpha, f\left(\frac{p_\alpha(x)}{t}\right)\right] = \frac{\sigma_\alpha^2}{2t} f'\left(\frac{p_\alpha(x)}{t}\right) \frac{x^2}{\langle x \rangle^{2+\alpha}} = \begin{cases} \mathcal{O}(t^{-(2+\alpha)/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon t}) & \text{if } \alpha = 2, \end{cases}$$

for some $\varepsilon > 0$.

(iii) First notice that

$$\left[\text{i}f\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha \right] = \frac{1}{t} f' \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c),$$

with $c \in S(\langle \xi \rangle \langle x \rangle^{-1-\alpha}, g_2)$. We now use Proposition 3.5:

$$\chi(H_{\alpha,0}) = \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) \chi(H_{\alpha,0}) + R_\beta, \quad \forall 1 \geq \beta > 1/2,$$

with $N_\alpha^\beta R_\beta$ bounded, and $\psi \in C_0^\infty(\mathbb{R})$, $\psi = 1$ in a small neighborhood of zero. Let $0 < \alpha < 2$. We have:

$$\text{Op}(c) N_\alpha^{-\beta} \in S(\langle x \rangle^{-1-\alpha-\alpha\beta/2}, g_0),$$

and thus

$$\frac{1}{t} f' \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) R_\beta = \mathcal{O}(t^{-(4+\alpha\beta+\alpha)/(2-\alpha)}).$$

Furthermore

$$\text{Op}(c) \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) \in S(\langle x \rangle^{-1-\alpha/2}, g_2)$$

and thus

$$\frac{1}{t} f' \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) = \mathcal{O}(t^{-4/(2-\alpha)}).$$

For $\alpha = 2$ all these terms are in $\mathcal{O}(e^{-\varepsilon t})$ for some $\varepsilon > 0$. This completes the proof of (iii). \square

We now set $g_{\gamma,\eta} = |dx|^2/\langle x \rangle^{2\gamma} + |d\xi|^2/\langle \xi \rangle^{2\eta}$.

Lemma 5.4. *Let $J \in C_b^\infty(\mathbb{R})$, $J = 0$ on $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, and $a \in S(\langle \xi \rangle^{\eta m}, g_{\gamma,\eta})$ for some real m . Then for all δ with $\varepsilon > \delta > 0$, there exists $\tilde{J} \in C_b^\infty(\mathbb{R})$, $\tilde{J} = 0$ on $[-\delta, \delta]$, with $\tilde{J}J = J$, such that:*

$$\text{Op}(a)J\left(\frac{p_\alpha(x)}{t}\right) = \tilde{J}\left(\frac{p_\alpha(x)}{t}\right) \text{Op}(a)J\left(\frac{p_\alpha(x)}{t}\right) + \begin{cases} \mathcal{O}(t^{-\infty}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\infty t}) & \text{if } \alpha = 2. \end{cases}$$

Proof. Let $\varepsilon > \delta' > \delta$, $\tilde{J} \in C_b^\infty(\mathbb{R})$, with $\text{supp } \tilde{J} \subset \mathbb{R} \setminus [-\delta, \delta]$, and $\tilde{J} = 1$ on $\mathbb{R} \setminus [-\delta', \delta']$. Let $\hat{J} := 1 - \tilde{J}$. We have to estimate:

$$R(t, x) := \hat{J}\left(\frac{p_\alpha(x)}{t}\right) \left(\text{Op}(a)J\left(\frac{p_\alpha(\cdot)}{t}\right) \Phi \right)(x).$$

By definition,

$$R(t, x) = \iint e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) \hat{J}\left(\frac{p_\alpha(x)}{t}\right) J\left(\frac{p_\alpha(y)}{t}\right) \Phi(y) \, dy \, d\xi.$$

Introduce the operator ${}^tL_\xi = (x - y) \cdot \partial_\xi / (i|x - y|^2)$. We have for any $k \in \mathbb{N}$:

$$R(t, x) = \iint e^{i(x-y)\cdot\xi} L_\xi^k a\left(\frac{x+y}{2}, \xi\right) \hat{J}\left(\frac{p_\alpha(x)}{t}\right) J\left(\frac{p_\alpha(y)}{t}\right) \Phi(y) \, dy \, d\xi.$$

We treat the case $0 < \alpha < 2$, the other case is analogous. Notice that

$$\left. \begin{aligned} y \in \text{supp } J(p_\alpha(\cdot)/t) &\Rightarrow |y| \geq \varepsilon^{2/(2-\alpha)} t^{2/(2-\alpha)} - 1 \\ x \in \text{supp } \hat{J}(p_\alpha(\cdot)/t) &\Rightarrow |x| \leq \delta'^{2/(2-\alpha)} t^{2/(2-\alpha)} \end{aligned} \right\} \\ \Rightarrow |x - y| \geq (\varepsilon^{2/(2-\alpha)} - \delta'^{2/(2-\alpha)}) t^{2/(2-\alpha)} - 1.$$

We infer:

$$\begin{aligned} |R(t, x)| &\lesssim \left| \hat{J}\left(\frac{p_\alpha(x)}{t}\right) \right| \iint |x - y|^{-k} \langle \xi \rangle^{\eta m - k} |\Phi(y)| \, dy \, d\xi \\ &\lesssim \left| \hat{J}\left(\frac{p_\alpha(x)}{t}\right) \right| t^{-k/(2-\alpha)} \|\Phi\|_{L^2}, \end{aligned}$$

for any k , and t sufficiently large. Thus,

$$\left\| \hat{J}\left(\frac{p_\alpha(x)}{t}\right) \left(\text{Op}(a) J\left(\frac{p_\alpha(\cdot)}{t}\right) \Phi \right)(x) \right\| \lesssim t^{-k/(2(2-\alpha))} \|\Phi\|_{L^2},$$

for any k , and t sufficiently large. \square

Lemma 5.5. Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R}_+)$, $J \in C_b^\infty(\mathbb{R}; \mathbb{R}_+)$, with $J = 0$ in a neighborhood of zero. Then we have for some $\varepsilon > 0$:

(i) Denote $\Theta(t) = \sigma_\alpha \chi(H_{\alpha,0}) J^2(p_\alpha(x)/t) \chi(H_{\alpha,0})$.

Then:

$$-\Theta(t) + \mathcal{O}(t^{-\varepsilon}) \leq \chi(H_{\alpha,0}) J\left(\frac{p_\alpha(x)}{t}\right) \mathbb{V}_\alpha J\left(\frac{p_\alpha(x)}{t}\right) \chi(H_{\alpha,0}) \leq \Theta(t) + \mathcal{O}(t^{-\varepsilon}).$$

(ii) $\chi(H_{\alpha,0}) J\left(\frac{p_\alpha(x)}{t}\right) \mathbb{A}_\alpha J\left(\frac{p_\alpha(x)}{t}\right) \chi(H_{\alpha,0}) \geq \mathcal{O}(t^{-1-\varepsilon})$.

(iii) $\chi(H_{\alpha,0}) J\left(\frac{p_\alpha(x)}{t}\right) \mathbb{A}_\alpha (\sigma_\alpha - \mathbb{V}_\alpha) J\left(\frac{p_\alpha(x)}{t}\right) \chi(H_{\alpha,0}) \geq \mathcal{O}(t^{-1-\varepsilon})$.

Proof. We start with an inequality which we will use in the following. Let $\varepsilon > 0$ such that $J = 0$ on $[-\varepsilon', \varepsilon']$, for some $\varepsilon' > \varepsilon$. For $1 \leq j \leq d$, let $c_j \in S(\langle \xi \rangle^{m_j} \langle x \rangle^{-k_j}, g_2)$, with $0 \leq m_j \leq 3$, and let $l := \min_j \{k_j - \alpha m_j / 2\} \geq 0$. We suppose that for $\psi \in C_0^\infty(\mathbb{R})$, $\psi = 1$ in a small neighborhood of 0, we have:

$$c(x, \xi) = \sum_{j=1}^d c_j(x, \xi) \geq -C \langle x \rangle^{-q(\beta)}, \quad \text{on } \text{supp } \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right), \quad (5.6)$$

with $q(\beta) > 0$. For $1 \geq \beta > 3/4$, let $\gamma := \min\{1, q(\beta), \alpha\beta + l\}$. We prove that:

$$\begin{aligned} & \chi(H_{\alpha,0}) J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ & \geq \begin{cases} \mathcal{O}(t^{-2\gamma/(2-\alpha)}), & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon\gamma t}), & \text{if } \alpha = 2. \end{cases} \end{aligned} \quad (5.7)$$

Before proving (5.7), we show that it implies the lemma. First, on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$, we have:

$$|\xi^2 - \langle x \rangle^\alpha| \lesssim \langle x \rangle^{\alpha\beta}. \quad (5.8)$$

We start with proving (i). We have on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$:

$$|\mathbf{v}_\alpha(x, \xi)| \leq \sigma_\alpha \frac{|x| \langle x \rangle^{\alpha/2} + C \langle x \rangle^{\alpha\beta/2}}{\langle x \rangle^{1+\alpha/2}} \leq \sigma_\alpha + C \langle x \rangle^{(\beta-1)\alpha/2}. \quad (5.9)$$

We have $q(\beta) = (1 - \beta)\alpha/2 > 0$ and $\alpha\beta + l = \alpha\beta > 0$, for all $1 > \beta > 3/4$. This yields (i). In order to prove (ii), we decompose $\mathbf{a}_\alpha = \mathbf{a}_\alpha^1 + \mathbf{a}_\alpha^2$. We then use that on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$:

$$\mathbf{a}_\alpha(x, \xi) \geq 2\sigma_\alpha \xi^2 \frac{\langle x \rangle^2 - x^2}{\langle x \rangle^{3+\alpha/2}} + \alpha\sigma_\alpha x^2 \frac{\langle x \rangle^\alpha - \xi^2}{\langle x \rangle^{3+\alpha/2}} \geq -C \langle x \rangle^{\alpha\beta-1-\alpha/2}. \quad (5.10)$$

Now observe that $2/(2 - \alpha) > 1$, and:

$$\left(1 + \alpha \left(\frac{1}{2} - \beta \right) \right) \frac{2}{2 - \alpha} > 1 \quad \text{and} \quad \left(\alpha\beta + 1 - \frac{\alpha}{2} \right) \frac{2}{2 - \alpha} > 1, \quad \text{for } \frac{3}{4} < \beta < 1.$$

This yields (ii). Let us prove (iii). We have:

$$\mathbf{A}_\alpha(\sigma_\alpha - \mathbf{V}_\alpha) = \text{Op}(\mathbf{a}_\alpha(\sigma_\alpha - \mathbf{v}_\alpha)) + \text{Op}(r_1 + r_2),$$

with

$$r_1 \in S(\langle x \rangle^{-2}, g_2), \quad r_2 \in S(\langle \xi \rangle^2 \langle x \rangle^{-2-\alpha}, g_2).$$

In particular we have, on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$:

$$r_j \geq -C \langle x \rangle^{-2} \quad (j = 1, 2).$$

We apply (5.7) to $r_1 + r_2$, and find $l = 2 = q(\beta)$. Therefore $2\gamma/(2 - \alpha) > 1$.

Let us consider $\text{Op}(a_\alpha(\sigma_\alpha - \nu_\alpha))$. We have:

$$a_\alpha = \frac{1}{p_\alpha(x)} \frac{2 + \alpha}{\sigma_\alpha} (\sigma_\alpha^2 - \nu_\alpha^2) + r_3; \quad r_3 = 2\sigma_\alpha \frac{\xi^2 - \langle x \rangle^\alpha}{\langle x \rangle^{1+\alpha/2}} - \alpha\sigma_\alpha \frac{1}{\langle x \rangle^{3-\alpha/2}} = r_3^1 + r_3^2,$$

$$r_3^1 \in S(\langle \xi \rangle^2 \langle x \rangle^{-1-\alpha/2}, g_2), \quad r_3^2 \in S(\langle x \rangle^{-1+\alpha/2}, g_2).$$

We find on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$:

$$r_3 \geq -C \langle x \rangle^{-1-\alpha/2+\alpha\beta}.$$

Using that $|\nu_\alpha| \lesssim 1$ on $\text{supp } \psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta)$, we find:

$$e_3 := r_3(\sigma_\alpha - \nu_\alpha) \geq -C \langle x \rangle^{-1-\alpha/2+\alpha\beta} \quad \text{on } \text{supp } \psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right).$$

We decompose $e_3 = e_3^1 + e_3^2 + e_3^3 + e_3^4$ with

$$e_3^1 \in S(\langle \xi \rangle^2 \langle x \rangle^{-1-\alpha/2}, g_2); \quad e_3^2 \in S(\langle x \rangle^{-1+\alpha/2}, g_2),$$

$$e_3^3 \in S(\langle \xi \rangle^3 \langle x \rangle^{-1-\alpha}, g_2); \quad e_3^4 \in S(\langle \xi \rangle \langle x \rangle^{-1}, g_2).$$

We apply (5.7), and find $l = 1 - \alpha/2$, $q(\beta) = 1 + \alpha/2 - \alpha\beta$. In particular, $2\gamma/(2 - \alpha) > 1$ if $\beta < 1$. It remains to consider:

$$b_\alpha := \frac{2 + \alpha}{\sigma_\alpha} \frac{1}{p_\alpha} (\sigma_\alpha - \nu_\alpha)^2 (\sigma_\alpha + \nu_\alpha).$$

We use (5.9) and find:

$$b_\alpha \geq -C \langle x \rangle^{-1+\alpha\beta/2} \quad \text{on } \text{supp } \psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right).$$

Notice that $b_\alpha = \sum_j b_\alpha^j$, with $b_\alpha^j \in S(\langle x \rangle^{-1+\alpha/2} \langle \xi \rangle^{m_j} \langle x \rangle^{-\alpha m_j/2}, g_2)$, $0 \leq m_j \leq 3$. We have $l = 1 - \alpha/2$ and $q(\beta) = 1 - \alpha\beta/2$. In particular, $2\gamma/(2 - \alpha) > 1$ if $\beta < 1$. This yields (iii).

It remains to show (5.7). Recall from Proposition 3.5 that

$$\chi(H_{\alpha,0}) = \chi(H_{\alpha,0}) \text{Op}\left(\psi\left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta}\right)\right) + R_\beta,$$

with $N_\alpha^\beta R_\beta$ bounded. Let

$$g_4 = \frac{|dx|^2}{\langle x \rangle^{2(\alpha\beta+1-\alpha)}} + \frac{|d\xi|^2}{\langle x \rangle^{2\alpha(\beta-1/2)}}.$$

Notice that $\psi((\xi^2 - \langle x \rangle^\alpha)/(\xi^2 + \langle x \rangle^\alpha)^\beta) \in S(1, g_4) \cap S(1, g_{\alpha\beta+1-\alpha, 2\beta-1})$. We have:

$$\begin{aligned} & \chi(H_{\alpha,0}) J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ &= \chi(H_{\alpha,0}) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) \right) J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) \\ & \quad \times J \left(\frac{p_\alpha(x)}{t} \right) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) \right) \chi(H_{\alpha,0}) \\ & \quad + R_\beta J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) J \left(\frac{p_\alpha(x)}{t} \right) \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) \right) \chi(H_{\alpha,0}) \\ & \quad + R_\beta J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) J \left(\frac{p_\alpha(x)}{t} \right) R_\beta \\ & =: F_1 + F_2 + F_3. \end{aligned}$$

Let us start with estimating F_2 . Using Lemma 5.4, we find $\tilde{J} \in C_b^\infty(\mathbb{R})$, $\tilde{J}J = J$, $\tilde{J} = 0$ on $[-\varepsilon, \varepsilon]$, such that:

$$\begin{aligned} F_2 &= R_\beta N_\alpha^\beta \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) N_\alpha^{-\beta} J \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(c) J \left(\frac{p_\alpha(x)}{t} \right) \\ & \quad \times \text{Op} \left(\psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) \right) \chi(H_{\alpha,0}) \\ & \quad + \mathcal{O}(t^{-\infty}) \\ &= R_\beta N_\alpha^\beta \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \text{Op} \left(\frac{c(x, \xi)}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) J^2 \left(\frac{p_\alpha(x)}{t} \right) \right) \\ & \quad \times \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ & \quad + R_\beta N_\alpha^\beta \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(e_2^t) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) + \mathcal{O}(t^{-\infty}). \end{aligned}$$

We estimate:

$$\begin{aligned}
 |f_2(x, \xi)| &= \left| \frac{c(x, \xi)}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) J^2 \left(\frac{p_\alpha(x)}{t} \right) \right| \\
 &\lesssim \sum_{j=1}^d \langle x \rangle^{-\alpha\beta} \langle \xi \rangle^{m_j} \langle x \rangle^{-k_j} \psi \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) \\
 &\lesssim \sum_{j=1}^d \langle x \rangle^{-\alpha\beta} \langle x \rangle^{\alpha m_j / 2 - k_j} \lesssim \langle x \rangle^{-\alpha\beta - l}, \quad \text{uniformly in } t.
 \end{aligned}$$

Thus,

$$F_2 = \begin{cases} \mathcal{O}(t^{-2(\alpha\beta+l)/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon(\alpha\beta+l)t}) & \text{if } \alpha = 2. \end{cases}$$

The estimate for F_3 is analogous; we have to estimate:

$$|f_3(x, \xi)| \lesssim \sum_{j=1}^d \frac{1}{(\xi^2 + \langle x \rangle^\alpha)^{2\beta}} \langle \xi \rangle^{m_j} \langle x \rangle^{-k_j} \lesssim \langle x \rangle^{-2\beta\alpha - l}, \quad \text{if } 1 \geq \beta > 3/4.$$

Using the same arguments as in the estimates for F_2, F_3 , we find:

$$\begin{aligned}
 F_1 &= \chi(H_{\alpha,0}) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \text{Op} \left(c(x, \xi) \psi^2 \left(\frac{\xi^2 - \langle x \rangle^\alpha}{(\xi^2 + \langle x \rangle^\alpha)^\beta} \right) J^2 \left(\frac{p_\alpha(x)}{t} \right) \right) \\
 &\quad \times \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\
 &\quad + \chi(H_{\alpha,0}) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(e_1^t) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) + \mathcal{O}(t^{-\infty}).
 \end{aligned}$$

Using [20, Theorem 18.5.5] on the composition of pseudo-differential operators in $\Psi(g_{\alpha\beta+1-\alpha, 2\beta-1})$ and $\Psi(g_2)$, as well as the fact that $\langle \xi \rangle \lesssim \langle x \rangle^{\alpha/2}$ on $\text{supp } \psi$, we find:

$$e_1^t \in S(\langle x \rangle^{\alpha/2 - \alpha\beta - 1}, 1/2(g_2 + g_4)) \subset S(\langle x \rangle^{-1}, 1/2(g_2 + g_4)),$$

because $\beta > 1/2$. Using (5.6) and the Gårding inequality in $\Psi(g_4)$ we get:

$$F_1 \geq \chi(H_{\alpha,0}) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \text{Op}(\hat{e}_1^t) \tilde{J} \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) + \mathcal{O}(t^{-2/(2-\alpha)}),$$

with $\hat{e}_1^t \in S(\langle x \rangle^{-\min\{q(\beta), 2\alpha\beta+1-3\alpha/2\}}, g_4) \subset S(\langle x \rangle^{-\min\{q(\beta), 1\}}, g_4)$ ($\beta > 3/4$), uniformly in $t \geq 1$. Therefore:

$$F_1 \geq \begin{cases} \mathcal{O}(t^{-2\min\{q(\beta), 1\}/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\min\{q(\beta), 1\}\varepsilon t}) & \text{if } \alpha = 2. \end{cases}$$

The estimates for F_1, F_2 and F_3 yield (5.7). \square

5.2. Propagation estimates

Lemma 5.6. *Let H be a self-adjoint operator on a Hilbert space \mathcal{H} , and let \mathbf{D}_H be the associated Heisenberg derivative. Let $\Theta(t)$ be a uniformly bounded observable. We suppose:*

$$\mathbf{D}_H \Theta(t) \geq f(t) \in L^1(\mathbb{R}^+, dt). \tag{5.11}$$

Then the limit

$$\lim_{t \rightarrow \infty} (\Theta(t) e^{-itH} \Phi | e^{-itH} \Phi)$$

exists for all $\Phi \in \mathcal{H}$.

Proof. We set $\Phi_t = e^{-itH} \Phi$. By (5.11), we have:

$$\frac{d}{dt} ((\Theta(t)\Phi_t | \Phi_t) - F(t)) \geq 0, \quad \text{with } F(t) = \int_1^t (f(s)\Phi | \Phi) ds.$$

Thus $(\Theta(t)\Phi_t | \Phi_t) - F(t)$ is increasing and bounded. Therefore, the limit

$$\lim_{t \rightarrow \infty} ((\Theta(t)\Phi_t | \Phi_t) - F(t))$$

exists. Since $\lim_{t \rightarrow \infty} F(t) = \int_1^\infty (f(s)\Phi | \Phi) ds$ exists, this gives the lemma. \square

Proposition 4.8 yields a minimal velocity estimate. There is also a maximal velocity estimate:

Proposition 5.7 (Maximal velocity estimate). *Let $\chi \in C_0^\infty(\mathbb{R})$, $\sigma_\alpha < \theta_2 < \theta_3 < \infty$. Then*

(i)
$$\int_1^\infty \left\| \mathbf{1}_{[\theta_2, \theta_3]} \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_{\alpha,0}} \chi(H_{\alpha,0}) \Phi \right\|^2 \frac{dt}{t} \lesssim \|\Phi\|^2.$$

(ii) *Let $F \in C^\infty(\mathbb{R})$, with $F' \in C_0^\infty(\mathbb{R})$ and $\text{supp } F \subset]\theta_2, \infty[$. Then*

$$\text{s-} \lim_{t \rightarrow \infty} F \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_{\alpha,0}} = 0.$$

Proof. (i) Let $f \in C_0^\infty(\mathbb{R})$, with $f = 1$ on $[\theta_2, \theta_3]$, and $\text{supp } f \subset]\sigma_\alpha, \infty[$. Let

$$F(s) := \int_{-\infty}^s f^2(s) ds; \quad \Theta(t) := \chi(H_{\alpha,0}) F \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}).$$

We compute:

$$\begin{aligned}
 -\mathbf{D}\Theta(t) &= \chi(H_{\alpha,0})f^2\left(\frac{p_\alpha(x)}{t}\right)\frac{p_\alpha(x)}{t^2}\chi(H_{\alpha,0}) \\
 &\quad - \frac{1}{t}\chi(H_{\alpha,0})f\left(\frac{p_\alpha(x)}{t}\right)\nabla_\alpha f\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\
 &\geq \frac{\mu}{t}\chi(H_{\alpha,0})f^2\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-1-\varepsilon}),
 \end{aligned}$$

for some $\mu, \varepsilon > 0$. We have used Lemma 5.5. This proves (i), using [10, Lemma B.4.1].

(ii) Let the function F satisfy the conditions in (ii). Clearly we can assume that $F \geq 0$, and $F(s) = 1$ for $s \geq R_0$. Let $\tilde{f} \in C_0^\infty(\mathbb{R})$ be such that $\tilde{f} = 1$ on $\text{supp } F'$, and $\text{supp } \tilde{f} \subset]\theta_2, \infty[$. Then,

$$\mathbf{D}\Theta(t) = \frac{1}{t}\chi(H_{\alpha,0})\tilde{f}\left(\frac{p_\alpha(x)}{t}\right)B(t)\tilde{f}\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-1-\varepsilon}), \tag{5.12}$$

with $B(t)$ uniformly bounded in t . To see that (5.12) is true, we introduce $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ with $\tilde{\chi}\chi = \chi$. Using Lemma 5.3 to estimate the commutator $[\tilde{\chi}(H_{\alpha,0}), \tilde{f}(\frac{p_\alpha(x)}{t})]$, and arguments similar to the arguments used in the proof of Lemma 5.5, we check that $B(t)$ is uniformly bounded in $t \geq 1$. From (i), there exists

$$\text{s-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}}\Theta(t)e^{-itH_{\alpha,0}}. \tag{5.13}$$

If, in addition, F is compactly supported, then by (i) we have:

$$\int_1^\infty (\Theta(t)e^{-itH_{\alpha,0}}\Phi | e^{-itH_{\alpha,0}}\Phi) \frac{dt}{t} \lesssim \|\Phi\|^2.$$

Thus if F satisfies the conditions in (ii), and is compactly supported, then the limit (5.13) is zero. Now take $F_1 \in C^\infty(\mathbb{R})$, $f \in C_0^\infty(\mathbb{R})$ such that $\text{supp } F_1 \subset]\theta_2, \infty[$ with $F_1 = 1$ in a neighborhood of ∞ , and $F_1' = f^2$. Set

$$\Theta_R(t) := \chi(H_{\alpha,0})F_1\left(\frac{p_\alpha(x)}{Rt}\right)\chi(H_{\alpha,0}).$$

From the previous discussion, we know that, for $R > 0$, the limit $\text{s-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}}\Theta_R(t)e^{-itH_{\alpha,0}}$ exists. Repeating the computations of the proof of (i), and keeping track of R , we obtain:

$$\begin{aligned}
 -\mathbf{D}\Theta_R(t) &= \frac{1}{t}\chi(H_{\alpha,0})f^2\left(\frac{p_\alpha(x)}{Rt}\right)\frac{p_\alpha(x)}{Rt}\chi(H_{\alpha,0}) \\
 &\quad - \frac{1}{t}\chi(H_{\alpha,0})f\left(\frac{p_\alpha(x)}{Rt}\right)\frac{\nabla_\alpha}{R}f\left(\frac{p_\alpha(x)}{Rt}\right)\chi(H_{\alpha,0}) \\
 &\geq \frac{1}{t}(\theta_2 - \sigma_\alpha/R)\chi(H_{\alpha,0})f^2\left(\frac{p_\alpha(x)}{Rt}\right)\chi(H_{\alpha,0}) + \mathcal{O}((tR)^{-1-\varepsilon}).
 \end{aligned}$$

Hence for $R \geq 1$, $-\mathbf{D}\Theta_R(t) \geq \mathcal{O}((tR)^{-1-\varepsilon})$. Therefore, for $t_0 \geq 1$, we have:

$$\begin{aligned} \text{s-} \lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} \Theta_R(t) e^{-itH_{\alpha,0}} &= e^{it_0H_{\alpha,0}} \Theta_R(t_0) e^{-it_0H_{\alpha,0}} + \int_{t_0}^{\infty} e^{isH_{\alpha,0}} \mathbf{D}\Theta_R(s) e^{-isH_{\alpha,0}} ds \\ &\leq e^{it_0H_{\alpha,0}} \Theta_R(t_0) e^{-it_0H_{\alpha,0}} + \mathcal{O}(t_0^{-\varepsilon} R^{-1-\varepsilon}). \end{aligned}$$

For a fixed t_0 , the terms on the right hand side go strongly to 0 as $R \rightarrow \infty$, hence:

$$\text{s-} \lim_{R \rightarrow \infty} \left(\text{s-} \lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} \Theta_R(t) e^{-itH_{\alpha,0}} \right) = 0. \tag{5.14}$$

We remark now that, for $R \geq 1$, the function $F_1(p_{\alpha}(x)) - F_1(p_{\alpha}(x)/R)$ has a compact support included in $[\theta_2, \infty[$. So,

$$\text{s-} \lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} (\Theta_1(t) - \Theta_R(t)) e^{-itH_{\alpha,0}} = 0. \tag{5.15}$$

Letting R go to infinity in (5.15) and using (5.14), we obtain:

$$\text{s-} \lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} \Theta_1(t) e^{-itH_{\alpha,0}} = 0.$$

This completes the proof of (ii). \square

The next estimate is a weak propagation estimate:

Proposition 5.8. *Let $\chi \in C_0^\infty(\mathbb{R})$, $0 < \theta_1 < \theta_2$. Then:*

- (i)
$$\int_1^\infty \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{p_{\alpha}(x)}{t} \right) \left(\frac{p_{\alpha}(x)}{t} - \mathbb{V}_{\alpha} \right) e^{-itH_{\alpha,0}} \chi(H_{\alpha,0}) \Phi \right\|^2 \frac{dt}{t} \lesssim \|\Phi\|^2.$$
- (ii)
$$\text{s-} \lim_{t \rightarrow \infty} \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{p_{\alpha}(x)}{t} \right) \left(\frac{p_{\alpha}(x)}{t} - \mathbb{V}_{\alpha} \right) e^{-itH_{\alpha,0}} = 0.$$

Proof. Let $R \in C^\infty(\mathbb{R})$, with $R'' \geq 0$, $R' = 0$ on $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, and $R(x) = x^2/2$ for $|x| \geq \theta_1$. We take $\theta_3 > \max\{\theta_2, \sigma_{\alpha}\}$. Let $J \in C_0^\infty(\mathbb{R})$, with $J = 1$ on $[0, \theta_3]$. We set:

$$\begin{aligned} M(t) &:= \frac{1}{2} \left(\mathbb{V}_{\alpha} - \frac{p_{\alpha}(x)}{t} \right) R' \left(\frac{p_{\alpha}(x)}{t} \right) + \frac{1}{2} R' \left(\frac{p_{\alpha}(x)}{t} \right) \left(\mathbb{V}_{\alpha} - \frac{p_{\alpha}(x)}{t} \right) + R \left(\frac{p_{\alpha}(x)}{t} \right), \\ \Theta(t) &:= \chi(H_{\alpha,0}) J \left(\frac{p_{\alpha}(x)}{t} \right) M(t) J \left(\frac{p_{\alpha}(x)}{t} \right) \chi(H_{\alpha,0}). \end{aligned}$$

The observable $\Theta(t)$ is bounded uniformly in t (see Lemmas 5.4 and 5.5). We have:

$$\begin{aligned} \mathbf{D}\Theta(t) &= -\chi(H_{\alpha,0})J'\left(\frac{p_\alpha(x)}{t}\right)\frac{p_\alpha(x)}{t^2}M(t)J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad + \frac{1}{2}\chi(H_{\alpha,0})\left(J'\left(\frac{p_\alpha(x)}{t}\right)\frac{\mathbb{V}_\alpha}{t} + hc\right)M(t)J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + hc \\ &\quad + \chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\mathbf{D}M(t)J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &=: R_1 + R_2 + R_3. \end{aligned}$$

The first two terms are of the form:

$$\frac{1}{t}\chi(H_{\alpha,0})\tilde{j}\left(\frac{p_\alpha(x)}{t}\right)B(t)\tilde{j}\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-1-\varepsilon}),$$

with $\tilde{j} \in C_0^\infty(\mathbb{R})$, $\text{supp } \tilde{j} \subset]\sigma_\alpha, \infty[$ (see Lemmas 5.3 and 5.4). They are integrable, from Proposition 5.7. For the last term, we have:

$$\begin{aligned} R_3 &= \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\mathbb{A}_\alpha R'\left(\frac{p_\alpha(x)}{t}\right) + hc\right\}J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad - \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\frac{\mathbb{V}_\alpha}{t}R'\left(\frac{p_\alpha(x)}{t}\right) + hc\right\}J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad + \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)R''\left(\frac{p_\alpha(x)}{t}\right)\frac{\mathbb{V}_\alpha}{t} + hc\right\} \\ &\quad \times J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad + \chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\frac{p_\alpha(x)}{t^2}R'\left(\frac{p_\alpha(x)}{t}\right)J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad - \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)R''\left(\frac{p_\alpha(x)}{t}\right)\frac{p_\alpha(x)}{t^2} + hc\right\} \\ &\quad \times J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad + \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\frac{\mathbb{V}_\alpha}{t}R'\left(\frac{p_\alpha(x)}{t}\right) + hc\right\}J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ &\quad - \chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\frac{p_\alpha(x)}{t^2}R'\left(\frac{p_\alpha(x)}{t}\right)J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-2}) \\ &= \frac{1}{2t}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)R''\left(\frac{p_\alpha(x)}{t}\right)\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right) \end{aligned}$$

$$\begin{aligned} & \times J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) \\ & + \frac{1}{2}\chi(H_{\alpha,0})J\left(\frac{p_\alpha(x)}{t}\right)\left\{\mathbb{A}_\alpha R'\left(\frac{p_\alpha(x)}{t}\right) + hc\right\}J\left(\frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-2}). \end{aligned}$$

Consider the second term. Let $\tilde{J} \in C_0^\infty(\mathbb{R}; \mathbb{R}^+)$, with $\tilde{J}J = J$ and $R'\tilde{J} = \hat{J}^2$. We have:

$$\begin{aligned} \frac{1}{2}\left(\mathbb{A}_\alpha \hat{J}^2\left(\frac{p_\alpha(x)}{t}\right) + \hat{J}^2\left(\frac{p_\alpha(x)}{t}\right)\mathbb{A}_\alpha\right) &= \hat{J}\left(\frac{p_\alpha(x)}{t}\right)\mathbb{A}_\alpha \hat{J}\left(\frac{p_\alpha(x)}{t}\right) \\ &+ \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha\right]\right]. \end{aligned}$$

We estimate the double commutator. From Lemma 5.4, there exists \hat{J}_1 , with $\hat{J}_1 = 0$ in a neighborhood of zero and $\hat{J}_1 J = \hat{J}$, such that:

$$\begin{aligned} & \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha\right]\right] \\ &= \hat{J}_1\left(\frac{p_\alpha(x)}{t}\right)\left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha\right]\right]\hat{J}_1\left(\frac{p_\alpha(x)}{t}\right) + \mathcal{O}(t^{-\infty}). \end{aligned}$$

Using $[\hat{J}(p_\alpha(x)/t), [\hat{J}(p_\alpha(x)/t), \mathbb{A}_\alpha]] \in \Psi(\langle x \rangle^{-3-\alpha/2}, g_2)$ uniformly in t we get:

$$\left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \left[\hat{J}\left(\frac{p_\alpha(x)}{t}\right), \mathbb{A}_\alpha\right]\right] = \begin{cases} \mathcal{O}(t^{2(-3-\alpha/2)/(2-\alpha)}) & \text{if } 0 < \alpha < 2, \\ \mathcal{O}(e^{-\varepsilon t}) & \text{if } \alpha = 2, \end{cases}$$

for some $\varepsilon > 0$. Putting all together and using Lemmas 5.3, 5.5 we obtain:

$$R_3 \geq \frac{1}{2t}\chi(H_{\alpha,0})\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)\mathbf{1}_{[\theta_1, \theta_2]}\left(\frac{p_\alpha(x)}{t}\right)\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)\chi(H_{\alpha,0}) + \mathcal{O}(t^{-1-\varepsilon}).$$

This yields the desired estimate, thanks to [10, Lemma B.4.1].

(ii) We can suppose $\sigma_\alpha \in [\theta_1, \theta_2]$. In the other case, (ii) follows from Propositions 4.8, 5.7, and Lemma 5.2. Let us first observe that

$$\begin{aligned} & s\text{-}\lim_{t \rightarrow \infty} \mathbf{1}_{[\theta_1, \theta_2]}\left(\frac{p_\alpha(x)}{t}\right)\left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t}\right)e^{-itH_{\alpha,0}} \\ &= s\text{-}\lim_{t \rightarrow \infty} \mathbf{1}_{[\theta_1, \theta_2]}\left(\frac{p_\alpha(x)}{t}\right)\left(\mathbb{V}_\alpha - \sigma_\alpha\right)e^{-itH_{\alpha,0}}, \end{aligned}$$

that is:

$$s\text{-}\lim_{t \rightarrow \infty} \mathbf{1}_{[\theta_1, \theta_2]}\left(\frac{p_\alpha(x)}{t}\right)\left(\frac{p_\alpha(x)}{t} - \sigma_\alpha\right)e^{-itH_{\alpha,0}} = 0. \tag{5.16}$$

Indeed, let $\varepsilon > 0$. Then for $\Phi \in \mathcal{H}$, we have:

$$\begin{aligned} & \left\| \mathbf{1}_{[\theta_1, \theta_2]} \left(\frac{p_\alpha}{t} \right) \left(\frac{p_\alpha}{t} - \sigma_\alpha \right) e^{-itH_{\alpha,0}} \Phi \right\| \\ & \leq \varepsilon + \left\| \mathbf{1}_{[\theta_1, \theta_2] \setminus [\sigma_\alpha - \varepsilon, \sigma_\alpha + \varepsilon]} \left(\frac{p_\alpha}{t} \right) \left(\frac{p_\alpha}{t} - \sigma_\alpha \right) e^{-itH_{\alpha,0}} \Phi \right\| \\ & \leq 2\varepsilon, \end{aligned}$$

for t sufficiently large, using Propositions 4.8 and 5.7. This yields (5.16). Let $J \in C_0^\infty(\mathbb{R}^*)$, with $J \geq 0$ and $J(x) = 1$ in a neighborhood of $[\theta_1, \theta_2]$. Set

$$\Theta(t) := \chi(H_{\alpha,0})(\mathbb{V}_\alpha - \sigma_\alpha) J^2 \left(\frac{p_\alpha(x)}{t} \right) (\mathbb{V}_\alpha - \sigma_\alpha) \chi(H_{\alpha,0}).$$

We compute:

$$\begin{aligned} -\mathbf{D}\Theta(t) &= \chi(H_{\alpha,0}) \mathbb{A}_\alpha J^2 \left(\frac{p_\alpha(x)}{t} \right) (\sigma_\alpha - \mathbb{V}_\alpha) \chi(H_{\alpha,0}) \\ &\quad - \frac{1}{t} \chi(H_{\alpha,0}) (\mathbb{V}_\alpha - \sigma_\alpha) (J^2)' \left(\frac{p_\alpha(x)}{t} \right) \left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t} \right) (\mathbb{V}_\alpha - \sigma_\alpha) \chi(H_{\alpha,0}) + hc. \end{aligned}$$

The second term is integrable along the evolution, by Propositions 4.8, 5.7 and Lemma 5.2 (the derivative of J^2 is zero in a neighborhood of σ_α). Using Lemmas 5.3 and 5.5, we find:

$$\begin{aligned} & \chi(H_{\alpha,0}) \mathbb{A}_\alpha J^2 \left(\frac{p_\alpha(x)}{t} \right) (\mathbb{V}_\alpha - \sigma_\alpha) \chi(H_{\alpha,0}) \\ &= \chi(H_{\alpha,0}) J \left(\frac{p_\alpha(x)}{t} \right) \mathbb{A}_\alpha (\mathbb{V}_\alpha - \sigma_\alpha) J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) + \mathcal{O}(t^{-1-\varepsilon}) \\ &\geq \mathcal{O}(t^{-1-\varepsilon}), \end{aligned}$$

for some $\varepsilon > 0$. Set $\Phi_t = e^{-itH_{\alpha,0}} \Phi$. By Lemma 5.6, the limit $\lim_{t \rightarrow \infty} (\Theta(t) \Phi_t | \Phi_t)$ exists. Let

$$\tilde{\Theta}(t) = \chi(H_{\alpha,0}) \left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t} \right) J^2 \left(\frac{p_\alpha(x)}{t} \right) \left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}).$$

We have:

$$\begin{aligned} \tilde{\Theta}(t) - \Theta(t) &= \chi(H_{\alpha,0}) \left(\left(\sigma_\alpha - \frac{p_\alpha(x)}{t} \right) J^2 \left(\frac{p_\alpha(x)}{t} \right) \left(\mathbb{V}_\alpha - \frac{p_\alpha(x)}{t} \right) \right. \\ &\quad \left. + (\mathbb{V}_\alpha - \sigma_\alpha) J^2 \left(\frac{p_\alpha(x)}{t} \right) \left(\sigma_\alpha - \frac{p_\alpha(x)}{t} \right) \right) \chi(H_{\alpha,0}). \end{aligned}$$

Using (5.16), we obtain:

$$\lim_{t \rightarrow \infty} (\Theta(t)\Phi_t | \Phi_t) = \lim_{t \rightarrow \infty} (\tilde{\Theta}(t)\Phi_t | \Phi_t).$$

But by (i), we have $\int_1^\infty (\Phi_t | \tilde{\Theta}(t)\Phi_t) \frac{dt}{t} \lesssim \|\Phi\|^2$. Hence the limit is zero. \square

5.3. Asymptotic velocity

We now have all the technical tools to prove Theorem 1.2.

Proposition 5.9. *Let $J \in C_\infty(\mathbb{R})$. Then there exists*

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} J \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_\alpha}.$$

Moreover, if $J(0) = 1$, then

$$s\text{-}\lim_{R \rightarrow \infty} \left(s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} J \left(\frac{p_\alpha(x)}{Rt} \right) e^{-itH_\alpha} \right) = \mathbf{1}.$$

If we define:

$$P_\alpha^+ = s\text{-}C_\infty\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} \frac{p_\alpha(x)}{t} e^{-itH_\alpha},$$

then P_α^+ is a self-adjoint operator, which commutes with H_α .

Proof. We prove the proposition in two steps:

First step. We assume $V_\alpha \equiv 0$: $H_\alpha = H_{\alpha,0}$.

By density, we may assume that $J \in C_0^\infty(\mathbb{R})$, and that J is constant in a neighborhood of 0 and in a neighborhood of σ_α . It also suffices to prove the existence of

$$\begin{aligned} & s\text{-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} J \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_{\alpha,0}} \chi^2(H_{\alpha,0}) \\ &= s\text{-}\lim_{t \rightarrow \infty} \chi(H_{\alpha,0}) e^{itH_{\alpha,0}} J \left(\frac{p_\alpha(x)}{t} \right) e^{-itH_{\alpha,0}} \chi(H_{\alpha,0}), \end{aligned}$$

for any $\chi \in C_0^\infty(\mathbb{R})$, using Lemma 5.3. Let $\Theta(t) := \chi(H_{\alpha,0}) J(p_\alpha(x)/t) \chi(H_{\alpha,0})$. We compute:

$$\begin{aligned} \mathbf{D}\Theta(t) &= \chi(H_{\alpha,0}) \frac{1}{2} \left(J' \left(\frac{p_\alpha(x)}{t} \right) \frac{V_\alpha}{t} + hc \right) \chi(H_{\alpha,0}) \\ &\quad - \chi(H_{\alpha,0}) J' \left(\frac{p_\alpha(x)}{t} \right) \frac{p_\alpha(x)}{t^2} \chi(H_{\alpha,0}). \end{aligned}$$

This is integrable along the evolution by Lemmas 5.2, 5.3 and Propositions 4.8, 5.7.

Second step. General case.

Let $P_{\alpha,0}^+$ be the asymptotic velocity associated with $H_{\alpha,0}$. Using Theorem 1.1, we obtain the existence of P_α^+ by the formula:

$$J(P_\alpha^+) = \Omega^+ J(P_{\alpha,0}^+) (\Omega^+)^* + J(0) \mathbf{1}^{pp}(H_\alpha). \tag{5.17}$$

The fact that H_α commutes with P_α^+ follows from Lemma 5.3. \square

Proposition 5.10. *We have*

$$\sigma(P_\alpha^+) = \begin{cases} \{0, \sigma_\alpha\} & \text{if } \sigma_{pp}(H_\alpha) \neq \emptyset, \\ \{\sigma_\alpha\} & \text{if } \sigma_{pp}(H_\alpha) = \emptyset. \end{cases}$$

Proof. We first observe that $\sigma(P_{\alpha,0}^+) \subset \{\sigma_\alpha\}$, by Propositions 4.8 and 5.7. But the spectrum of $P_{\alpha,0}^+$ cannot be empty, thus $\sigma(P_{\alpha,0}^+) = \{\sigma_\alpha\}$. If $\sigma_{pp}(H_\alpha) = \emptyset$, then by (5.17), $\sigma(P_\alpha^+) = \sigma(P_{\alpha,0}^+) = \{\sigma_\alpha\}$. We suppose in the following that² $\sigma_{pp}(H_\alpha) \neq \emptyset$. By (5.17), we have $\sigma(P_\alpha^+) \subset \{0, \sigma_\alpha\}$. Let $J(0) \neq 0$, and $\Phi \neq 0$ be an eigenfunction of H_α . Then $J(P_\alpha^+)\Phi = J(0)\Phi \neq 0$, thus $0 \in \sigma(P_\alpha^+)$. Let now $J(\sigma_\alpha) \neq 0$, $J(P_{\alpha,0}^+)\Phi \neq 0$, and $\psi = \Omega^+\Phi$. Since $\text{Im } \Omega^+ \subset \mathbf{1}^c(H_\alpha)$, we obtain by (5.17): $J(P_\alpha^+)\psi = \Omega^+ J(P_{\alpha,0}^+)\Phi \neq 0$, in particular $\sigma_\alpha \in \sigma(P_\alpha^+)$. \square

Proposition 5.11. *For $0 < \alpha \leq 2$, we have: $\mathbf{1}_{\{0\}}(P_\alpha^+) = \mathbf{1}^{pp}(H_\alpha)$.*

Proof. Take (an approximation of) $J = \mathbf{1}_{\{0\}}$ in (5.17), and use $\mathbf{1}_{\{0\}}(P_{\alpha,0}^+) = 0$. \square

Proposition 5.12. *Let $g \in C_\infty(\mathbb{R})$. Then*

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} g(V_\alpha) e^{-itH_\alpha} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+) = g(P_\alpha^+) \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+).$$

Proof. We first treat the case $H_\alpha = H_{\alpha,0}$. It is enough to assume $g \in C_0^\infty(\mathbb{R})$ and to prove that

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} \left(g(V_\alpha) - g\left(\frac{p_\alpha(x)}{t}\right) \right) J\left(\frac{p_\alpha(x)}{t}\right) \chi(H_{\alpha,0}) e^{-itH_{\alpha,0}} = 0,$$

for any $J \in C_0^\infty(\mathbb{R}^*)$ and $\chi \in C_0^\infty(\mathbb{R})$. By the Helffer–Sjöstrand formula, it is sufficient to show that for all $z \in \mathbb{C} \setminus (\sigma(V_\alpha) \cup \mathbb{R}^+)$:

$$s\text{-}\lim_{t \rightarrow \infty} (z - V_\alpha)^{-1} \left(V_\alpha - \frac{p_\alpha(x)}{t} \right) \left(z - \frac{p_\alpha(x)}{t} \right)^{-1} J\left(\frac{p_\alpha(x)}{t}\right) \chi(H_{\alpha,0}) e^{-itH_{\alpha,0}} = 0.$$

² From Proposition 2.8 and Remark 2.9, this does not seem to be the generic case.

We have:

$$\begin{aligned} & (z - V_\alpha)^{-1} \left(V_\alpha - \frac{p_\alpha(x)}{t} \right) \left(z - \frac{p_\alpha(x)}{t} \right)^{-1} J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ &= (z - V_\alpha)^{-1} \left(z - \frac{p_\alpha(x)}{t} \right)^{-1} \left(V_\alpha - \frac{p_\alpha(x)}{t} \right) J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ & \quad + (z - V_\alpha)^{-1} \left(z - \frac{p_\alpha(x)}{t} \right)^{-1} \frac{[p_\alpha, V_\alpha]}{t} \left(z - \frac{p_\alpha(x)}{t} \right)^{-1} J \left(\frac{p_\alpha(x)}{t} \right) \chi(H_{\alpha,0}) \\ &=: R_1 + R_2. \end{aligned}$$

A direct computation shows that the commutator $[V_\alpha, p_\alpha]$ is bounded. Thus $s\text{-}\lim_{t \rightarrow \infty} R_2 \times e^{-itH_{\alpha,0}} = 0$. We have $s\text{-}\lim_{t \rightarrow \infty} R_1 e^{-itH_{\alpha,0}} = 0$ by Proposition 5.8.

For the general case, we notice that

$$\begin{aligned} g(P_\alpha^+) \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+) &= g(P_\alpha^+) \mathbf{1}^c(H_\alpha) = \Omega^+ g(P_{\alpha,0}^+) (\Omega^+)^* \\ &= \Omega^+ s\text{-}\lim_{t \rightarrow \infty} e^{itH_{\alpha,0}} g(V_\alpha) e^{-itH_{\alpha,0}} (\Omega^+)^* \\ &= s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} g(V_\alpha) e^{-itH_\alpha} \mathbf{1}^c(H_\alpha) \\ &= s\text{-}\lim_{t \rightarrow \infty} e^{itH_\alpha} g(V_\alpha) e^{-itH_\alpha} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_\alpha^+), \end{aligned}$$

which implies the proposition. \square

Propositions 5.9–5.12 correspond to the three points in Theorem 1.2.

6. Generalizations in the case $\alpha = 2$

In this section, we generalize our results, in the special case where the reference Hamiltonian is exactly the Laplace operator plus a second order polynomial. Resuming the notations of Section 2.2, we may assume, as in (2.9) that

$$H_0 = -\Delta - \sum_{k=1}^{n_-} \omega_k^2 x_k^2 + \sum_{k=n_-+1}^{n_-+n_+} \omega_k^2 x_k^2 + \sum_{k=n_-+n_++1}^{n_-+n_++n_E} E_k x_k =: -\Delta + U(x), \quad (6.1)$$

on $L^2(\mathbb{R}^n)$, with $n_- + n_+ + n_E \leq n$, $\omega_k > 0$ and $E_k \neq 0$. By convention $\sum_{j=a}^b = 0$ if $b < a$. We study the scattering theory for the Hamiltonians $(H_0, H = H_0 + V(x))$. This setting includes the presence of a Stark potential ($n_E \neq 0$). Notice that, if $U(x)$ is a general second order polynomial with real coefficients, the operator

$$-\Delta + U(x),$$

can always be written as (6.1), modulo a constant term, after a change of orthonormal basis and origin (which leaves the Laplace operator invariant). With that approach, one could even demand $n_E \leq 1$. The reason why we do not reduce to that case is that the pointwise decay estimates required for the potential (see (6.4) below) are not invariant with respect to such reductions.

In Section 6.1, we prove the existence of wave operators under rather weak assumptions on the perturbative potential V . In the case $n_- = n_+ = 1$ for instance, the repulsive effects due to $-x_1^2$ overwhelm the other effects: confinement due to $+x_2^2$ and drift due to the Stark potential.

In Section 6.2, we give the asymptotic completeness if H_0 has no Stark effect and no Schrödinger part (this means that $n_+ + n_- = n$). The hypothesis on V are similar to (1.5)–(1.6).

In Section 6.3, we construct the asymptotic velocity under the previous hypothesis. As the free Hamiltonian H_0 is a sum of commuting self-adjoint “one-dimensional” operators, the existence of asymptotic velocities in each space direction is a corollary of Theorem 1.2 applied to the one-dimensional case. For the Hamiltonian H , the asymptotic velocity of Theorem 1.2 exists also and is equal to P_ℓ^+ , where $\omega_\ell = \max_{1 \leq j \leq n_-} \omega_j$, and P_ℓ^+ is the asymptotic velocity in the direction x_ℓ .

6.1. Existence of wave operators

In this section, we prove the existence of wave operators for perturbations of H_0 by Cook’s method. We consider the perturbation $H = H_0 + V$, where $V(x)$ is a real-valued potential which can be decomposed as

$$V(x) = V_1(x) + V_2(x) + W(x), \tag{6.2}$$

where

$$V_j \in L^{p_j}(\mathbb{R}^n; \mathbb{R}), \quad \text{for } j = 1, 2, \quad \text{with } \begin{cases} 2 \leq p_j < \infty & \text{if } n \leq 3, \\ 2 < p_j < \infty & \text{if } n = 4, \\ n/2 \leq p_j < \infty & \text{if } n \geq 5, \end{cases} \tag{6.3}$$

and W is a sum of terms in $L^\infty(\mathbb{R}^n)$ satisfying a.e.

$$|W(x)| \lesssim \left(\prod_{j=1}^{n_-} \langle \ln \langle x \rangle \rangle^{-\beta_j} \right) \left(\prod_{j=n_-+n_++1}^{n_-+n_++n_E} \langle x_j \rangle^{-\beta_j/2} \right) \left(\prod_{j=n_-+n_++n_E+1}^n \langle x_j \rangle^{-\beta_j} \right), \tag{6.4}$$

with $\beta_j \geq 0$ and $\sum \beta_j > 1$. Notice that the V_j ’s do not contain pointwise information.

Theorem 6.1. (i) *Suppose that the quadratic part of U has at most one (simple) positive eigenvalue ($n_+ \leq 1$), and at least one negative eigenvalue ($n_- \geq 1$). Let V satisfying the previous assumptions. Then $H = H_0 + V$ admits a unique self-adjoint extension, and the following strong limits exist in $L^2(\mathbb{R}^n)$,*

$$s\text{-}\lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}.$$

(ii) *If the quadratic part of U has at least one negative eigenvalue ($n_- \geq 1$) and V satisfies the previous assumptions with V_1 and V_2 compactly supported, then the same conclusions hold.*

(iii) *If W satisfies (6.4) and $V_1 = V_2 \equiv 0$, then the same conclusions hold.*

The self-adjointness property follows from Faris–Lavine theorem ([30], see also Section 2). Before the proof of this result, a few remarks are in order.

Remark 6.2. This result shows that wave operators exist even for very slowly decaying potentials. Potentials decaying even more slowly could be included (involving $\ln(\ln|x|)$ for instance); like for Theorem 1.1, we do not seek so general results, and rather focus on the method. Notice that in the first case, singular potentials, like

$$V(x) = \frac{1}{|x|^\delta} = \frac{1}{|x|^\delta} \mathbf{1}_{|x| \leq 1} + \frac{1}{|x|^\delta} \mathbf{1}_{|x| > 1} + 0,$$

are allowed, provided that $\delta < \min(2, n/2)$. This includes the case of Coulomb potentials in space dimension $n \geq 3$.

Remark 6.3. The dynamics associated to H_0 is known explicitly (see (2.10)), and cannot be compared to that of $-\Delta$.

Remark 6.4. If the quadratic part of U has more than one positive eigenvalue, results similar to the first point of the theorem can be proved, provided that U has at least one negative eigenvalue. This will be clear from the proof below, as well as the reasons why we did not wish to state too general a result.

Resuming the notations of Section 2.2, the dilation operator \mathcal{D}_t is crucial. As mentioned in Section 2.2, a formula similar to (2.12) is available for $e^{it\Delta}$. The factor \mathcal{M}_t in that case is different, but still of modulus one, while \mathcal{D}_t corresponds to dilations of size $2t$ instead of $g(2t)$. This is closely related to the properties of the classical trajectories. The operator \mathcal{D}_t enables us to prove the existence of wave operators with Cook’s method.

It is of course sufficient to study the case $t \rightarrow +\infty$, the case $t \rightarrow -\infty$ being similar. A density argument shows that Theorem 6.1 follows from:

Lemma 6.5. *Under the assumptions of Theorem 6.1, for any $\varphi = \varphi_1 \otimes \dots \otimes \varphi_n$, with $\varphi_j \in \mathcal{S}(\mathbb{R})$, there exists a unique $\phi \in L^2(\mathbb{R}^n)$ such that*

$$\|e^{itH} e^{-itH_0} \varphi - \phi\|_{L^2} \xrightarrow{t \rightarrow +\infty} 0.$$

Proof. Following Cook’s method, we compute:

$$\frac{d}{dt} e^{itH} e^{-itH_0} \varphi = i e^{itH} V e^{-itH_0} \varphi.$$

Taking the L^2 -norm,

$$\left\| \frac{d}{dt} e^{itH} e^{-itH_0} \varphi \right\|_{L^2} = \| V e^{-itH_0} \varphi \|_{L^2}.$$

Under the assumptions of Theorem 6.1, it is sufficient to prove that the maps

$$t \mapsto \| W e^{-itH_0} \varphi \|_{L^2} \quad \text{and} \quad t \mapsto \| V_j e^{-itH_0} \varphi \|_{L^2}, \quad j = 1, 2,$$

are integrable on $[1, +\infty[$. Let $t \geq 1$. Since the operator \mathcal{D}_t is unitary on L^2 , we have, from (2.12) and Hölder’s inequality,

$$\begin{aligned} \| V_j e^{-itH_0} \varphi \|_{L^2} &= \| V_j \mathcal{D}_t \mathcal{F} \mathcal{M}_t \varphi \|_{L^2} \\ &= \| \mathcal{D}_t V_j (\cdot g_1(2t), \dots, \cdot g_n(2t)) \mathcal{F} \mathcal{M}_t \varphi \|_{L^2} \\ &= \| V_j (\cdot g_1(2t), \dots, \cdot g_n(2t)) \mathcal{F} \mathcal{M}_t \varphi \|_{L^2} \\ &\leq \| V_j (\cdot g_1(2t), \dots, \cdot g_n(2t)) \|_{L^{p_j}} \| \mathcal{F} \mathcal{M}_t \varphi \|_{L^{q_j}} \\ &\lesssim \left(\frac{1}{g_{n_-+1}(2t)} \prod_{k=1}^{n_-} \frac{\omega_k}{\sinh(2\omega_k t)} \right)^{1/p_j} \| V_j \|_{L^{p_j}} \| \mathcal{M}_t \varphi \|_{L^{q_j}}, \end{aligned} \tag{6.5}$$

where $1/p_j + 1/q_j = 1/2$ and the last estimate stems from Hausdorff–Young inequality (q'_j denotes the Hölder conjugate exponent of q_j).

In the first case of Theorem 6.1, the function g_{n_-+1} is a sinus if the quadratic part of U has one positive eigenvalue ($n_+ = 1$), and is linear otherwise.

The exponential decay of $1/g_1(2t)$ is enough to ensure the integrability of the right-hand side of (6.5). The worst possible situation with our assumptions is $n_+ = 1$, where (6.5) yields, since $n_- \geq 1$,

$$\| V_j e^{-itH_0} \varphi \|_{L^2} \lesssim \left(\frac{1}{\sin(2\omega_{n_-+1} t) \sinh(2\omega_1 t)} \right)^{1/p_j} \| V_j \|_{L^{p_j}} \| \varphi \|_{L^{q'_j}}.$$

From assumption (6.3), $p_j \geq 2$, and the map $t \mapsto 1/(\sin(2\omega_{n_-+1} t) \sinh(2\omega_1 t))^{1/p_j}$ is integrable on $[1, +\infty[$. Since $L^{q'_j} \subset L^1 \cap L^2$, the lemma is proved for the V_j ’s parts, in the first case of Theorem 6.1.

In the second case of Theorem 6.1, we assume in addition that the V_j ’s are compactly supported: $\text{supp } V_j \subset \{|x| \leq R\} =: B$. In that case, they are H_0 -bounded (with bound zero); the assumptions on p_j are such that V_j is Δ -bounded, and the polynomial U is bounded on the support of V_j . To take advantage of this, we write

$$H_0 = -\partial_1^2 - \omega_1^2 x_1^2 + \tilde{H}_0,$$

where \tilde{H}_0 takes the last $(n - 1)$ variables into account, and use a factorization like (2.12) in the first variable only. Mimicking the above computations with x replaced by x_1 yields:

$$\|V_j e^{-itH_0} \varphi\|_{L^2} = \|V_j e^{-itH_0} \varphi\|_{L^2(B)} = \|V_j(\cdot, g_1(2t), \dots, \cdot) \mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi\|_{L^2(B_t)},$$

where \mathcal{F}_1 stands for the Fourier transform with respect to the first variable, and

$$\mathcal{M}_t^1 = \exp\left(ix_1^2 \frac{h_1(2t)}{2g_1(2t)}\right).$$

Notice that these two operators commute with \tilde{H}_0 . We also denoted $B_t = \{x_1^2 g_1(2t)^2 + x_2^2 + \dots + x_n^2 \leq R^2\}$. From Hölder’s inequality, this last term is estimated by:

$$\|V_j(\cdot, g_1(2t), \dots, \cdot)\|_{L^{p_j}(B_t)} \|\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi\|_{L^{q_j}(B_t)},$$

where $1/2 = 1/p_j + 1/q_j$. The first term is equal to $(g_1(2t))^{-1/p_j} \|V_j\|_{L^{p_j}}$; since we assume $n_- \geq 1$, g_1 has an exponential growth, and this term is integrable. It suffices to show that the second term is bounded. From our assumption on p_j , $H^2(\mathbb{R}^n) \subset L^{q_j}(\mathbb{R}^n)$, and

$$\begin{aligned} \|\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi\|_{L^{q_j}(B_t)} &\lesssim \|\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi\|_{L^2} + \|\Delta(\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi)\|_{L^2(B_t)} \\ &\lesssim \|\varphi\|_{L^2} + \|(\partial_1^2 - \tilde{H}_0)(\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi)\|_{L^2(B_t)}, \end{aligned}$$

since U is bounded on B_t , uniformly with respect to $t \geq 1$ ($B_t \subset B$ for $t \geq 1$). Finally, we have:

$$\|(\partial_1^2 - \tilde{H}_0)(\mathcal{F}_1 \mathcal{M}_t^1 e^{-it\tilde{H}_0} \varphi)\|_{L^2(\mathbb{R}^n)} = \|(x_1^2 + \tilde{H}_0)\varphi\|_{L^2(\mathbb{R}^n)}.$$

Therefore, the lemma is proved for the V_j ’s parts, in the second case of Theorem 6.1.

For the last component of V (the function W), we make no assumption on n_- or n_+ . This is where we use the tensor product structure for φ . The idea is to proceed as above, except for the components $(x_{n_-+n_++1}, \dots, x_n)$, for which we proceed “as usual”. We denote:

$$H_0^j = \begin{cases} -\Delta_{x_j} - \omega_j^2 x_j^2 & \text{for } 1 \leq j \leq n_-, \\ -\Delta_{x_j} + \omega_j^2 x_j^2 & \text{for } n_- < j \leq n_- + n_+, \\ -\Delta_{x_j} + E_j x_j & \text{for } n_- + n_+ < j \leq n_- + n_+ + n_E, \\ -\Delta_{x_j} & \text{for } n_- + n_+ + n_E < j \leq n, \end{cases} \tag{6.6}$$

and we have $e^{-itH_0} \varphi = e^{-itH_0^1} \varphi_1 \otimes \dots \otimes e^{-itH_0^n} \varphi_n$.

Since $e^{-itH_0^j}$ is unitary, we have, for $j = n_- + 1, \dots, n_- + n_+$:

$$\begin{aligned} \|We^{-itH_0}\varphi\|_{L^2} &\lesssim \prod_{j=1}^{n_-} \|\langle \ln(x_j) \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j\|_{L^2} \prod_{j=n_-+n_++1}^{n_-+n_++n_E} \|\langle x_j \rangle^{-\beta_j/2} e^{-itH_0^j} \varphi_j\|_{L^2} \\ &\times \prod_{j=n_-+n_++n_E+1}^n \|\langle x_j \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j\|_{L^2}, \end{aligned} \tag{6.7}$$

and we estimate each term in the product.

For $1 \leq j \leq n_-$, we have, from (2.12) in the one-dimensional case,

$$\begin{aligned} \|\langle \ln(x_j) \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j\|_{L^2} &= \|\langle \ln(x_j) \rangle^{-\beta_j} \mathcal{M}_t^j \mathcal{D}_t^j \mathcal{F}_j \mathcal{M}_t^j \varphi_j\|_{L^2} \\ &= \|\langle \ln(x_j g_j(2t)) \rangle^{-\beta_j} \mathcal{F}_j \mathcal{M}_t^j \varphi_j\|_{L^2}, \end{aligned}$$

where \mathcal{M}_t^j , \mathcal{D}_t^j and \mathcal{F}_j are given by (2.12) in dimension 1. Denoting $\tilde{\varphi}_j = \mathcal{F}_j \mathcal{M}_t^j \varphi_j$ and replacing $g_j(2t)$ with $e^{2\omega_j t}$, we get:

$$\begin{aligned} \|\langle \ln(x_j g_j(2t)) \rangle^{-\beta_j} \tilde{\varphi}_j\|_{L^2}^2 &= \int_{|x_j| > e^{-\omega_j t}} |\langle \ln(x_j g_j(2t)) \rangle^{-\beta_j} \tilde{\varphi}_t^j|^2 dx_j \\ &\quad + \int_{|x_j| \leq e^{-\omega_j t}} |\langle \ln(x_j g_j(2t)) \rangle^{-\beta_j} \tilde{\varphi}_t^j|^2 dx_j \\ &\lesssim \frac{1}{t^{2\beta_j}} \|\tilde{\varphi}_t^j\|_{L^2}^2 + \int \mathbf{1}_{|x_j| \leq e^{-\omega_j t}} |\tilde{\varphi}_t^j|^2 dx_j \\ &\lesssim \frac{1}{t^{2\beta_j}} \|\varphi_j\|_{L^2}^2 + e^{-\omega_j t/2} \|\tilde{\varphi}_t^j\|_{L^4}^2, \end{aligned}$$

where we used Cauchy–Schwarz inequality. From Hausdorff–Young inequality, we have:

$$\|\langle \ln(x_j) \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j\|_{L^2}^2 \lesssim \frac{1}{t^{2\beta_j}} \|\varphi_j\|_{L^2}^2 + e^{-\omega_j t/2} \|\varphi_j\|_{L^{4/3}}^2. \tag{6.8}$$

For the case $n_- + n_+ < j \leq n_- + n_+ + n_E$, we simply recall the approach of [2]. From Avron–Herbst formula (see, e.g., [2,8]),

$$(e^{-itH_0^j} \varphi_j)(x_j) = e^{-i(tE_j x_j + \frac{t^3}{3} E_j^2)} (e^{it\Delta_{x_j}} \varphi_j)(x_j + t^2 E_j).$$

Using Avron–Herbst formula, the term we have to estimate reads

$$\|\langle x_j \rangle^{-\beta_j/2} e^{-itH_0^j} \varphi_j\|_{L^2} = \|\langle x_j - t^2 E_j \rangle^{-\beta_j/2} e^{it\Delta_{x_j}} \varphi_j\|_{L^2}.$$

By a density argument, we may assume $\varphi_j \in \mathcal{F}_j(C_0^\infty(\mathbb{R}))$ ($\text{supp } \mathcal{F}_j(\varphi_j) \subset \{|\xi| \leq R\}$ for some positive R). For $|x_j| < 3Rt$, the drift caused by Stark effect accelerates the particle:

$$\| \langle x_j - t^2 E_j \rangle^{-\beta_j/2} e^{it\Delta_{x_j}} \varphi_j \|_{L^2(|x_j| < 3Rt)} \lesssim |t^2|^{-\beta_j/2} \|\varphi_j\| \lesssim t^{-\beta_j}. \tag{6.9}$$

For $|x_j| > 3Rt$, a nonstationary phase argument and $\text{supp } \mathcal{F}_j(\varphi_j) \subset \{|\xi| \leq R\}$ show that

$$\begin{aligned} \| e^{it\Delta_{x_j}} \varphi_j \|_{L^2(|x_j| > 3Rt)} &= \left\| \frac{1}{\sqrt{2\pi}} \int e^{-it\xi^2 + ix_j\xi} \mathcal{F}_j(\varphi_j)(\xi) \, d\xi \right\|_{L^2(|x_j| > 3Rt)} \\ &= \mathcal{O}(t^{-\infty}). \end{aligned} \tag{6.10}$$

These two estimates yield:

$$\| \langle x_j \rangle^{-\beta_j/2} e^{-itH_0^j} \varphi_j \|_{L^2} \lesssim t^{-\beta_j}. \tag{6.11}$$

We now study the term with $n_- + n_+ + n_E < j \leq n$. By a density argument, we can assume that φ_j satisfies:

$$\varphi_j(x_j) = \iint e^{i(x_j-y)\xi} \chi(\xi) \psi(y) \, dy \, d\xi,$$

where $\psi \in C_0^\infty(\mathbb{R})$ ($\text{supp } \psi \subset \{|y| < R\}$), $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ and $\chi = 0$ for $|\xi| \leq c$. We get:

$$\langle x_j \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j = \langle x_j \rangle^{-\beta_j} \iint e^{-it\xi^2 + i(x_j-y)\xi} \chi(\xi) \psi(y) \, dy \, d\xi.$$

Obviously, we have:

$$\| \langle x_j \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j \|_{L^2(|x| > ct)} \lesssim t^{-\beta_j} \|\varphi_j\|_{L^2}. \tag{6.12}$$

For $|x| < ct$, recall that on the support of $\psi(y)\chi(\xi)$, $|\xi| > c$. Differentiating the phase, and noting that

$$|2t\xi + x - y| \geq |2t\xi + x| - R \geq 2t|\xi| - |x| - R \geq ct - R,$$

a nonstationary phase argument shows that, for large t ,

$$\left\| \langle x_j \rangle^{-\beta_j} \iint e^{-it\xi^2 + i(x_j-y)\xi} \chi(\xi) \psi(y) \, dy \, d\xi \right\|_{L^2(|x| < ct)} = \mathcal{O}(t^{-\infty}). \tag{6.13}$$

Combining (6.12) and (6.13), we get, for $n_- + n_+ + n_E < j \leq n$,

$$\| \langle x_j \rangle^{-\beta_j} e^{-itH_0^j} \varphi_j \|_{L^2} \lesssim t^{-\beta_j}. \tag{6.14}$$

Gathering (6.7), (6.8) and (6.14) together yields:

$$\|W e^{-itH_0} \varphi_j\|_{L^2} \lesssim t^{-\sum \beta_j}.$$

Since we assumed $\sum \beta_j > 1$, the lemma follows. \square

If the quadratic part of U has more than one positive eigenvalue, the problem may become intricate for the estimates related to the V_j 's. For instance, if the quadratic part of U has one positive eigenvalue, whose order is $n_+ \geq 2$, then (6.5) becomes:

$$\|V_j e^{-itH_0} \varphi_+\|_{L^2} \lesssim \left(\frac{1}{|\sin(2\omega_{n_+}t)|^{n_+} \sinh(2\omega_1 t)} \right)^{1/p_j} \|V_j\|_{L^{p_j}} \|\varphi_+\|_{L^{q_j'}}.$$

and if $n_+ = 2$ and $n = 3$ (see the Assumption 6.3), one has to adapt the assumption on the power p_j for this map to be integrable near $+\infty$: the value $p_j = 2$ is not allowed, because for that value, the above map is not even locally integrable.

Reasoning the same way, we notice that if the quadratic part of U has several distinct positive eigenvalues, then arithmetic properties of these eigenvalues will have to be taken into account. We leave out the discussion at this stage.

6.2. Asymptotic completeness

In this part, we assume that $n_- + n_+ = n$. Then

$$H_0 = -\Delta - \sum_{k=1}^{n_-} \omega_k^2 x_k^2 + \sum_{k=n_-+1}^n \omega_k^2 x_k^2.$$

Like for Theorem 1.1, we assume that V is a real-valued function with

$$V(x) = V_1(x) + V_2(x), \tag{6.15}$$

where

$$V_1 \text{ is a compactly supported measurable function, and } \Delta\text{-compact}, \tag{6.16}$$

and $V_2 \in L^\infty(\mathbb{R}^n; \mathbb{R})$ satisfies the short range condition:

$$|V_2(x)| \lesssim \langle \ln|x_-| \rangle^{-1-\varepsilon}, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{6.17}$$

for some $\varepsilon > 0$, and $x = (x_-, x_+) \in \mathbb{R}^{n_-} \times \mathbb{R}^{n_+}$. Notice that there is propagation only in the x_- direction. It is therefore reasonable to impose decay only in that direction. Denote:

$$H_0^- = -\Delta_{x_-} - (\omega_- x_-)^2; \quad H_0^+ = -\Delta_{x_+} + (\omega_+ x_+)^2,$$

with $\omega_- = \text{diag}(\omega_1, \dots, \omega_{n_-})$, $\omega_+ = \text{diag}(\omega_{n_-+1}, \dots, \omega_n)$. Let $N_{\pm} = -\Delta_{x_{\pm}} + x_{\pm}^2$. As in Section 2.1, we can show that the operators $(H_0^{\pm}, D(N_{\pm}))$ and $(H, D(N))$ are essentially self-adjoint ($N = N_2$ is the harmonic oscillator on \mathbb{R}^n). The result of this section is:

Theorem 6.6. *Assume that V satisfies (6.15)–(6.17). Then there exist*

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH} e^{-itH_0}, \tag{6.18}$$

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_0} e^{-itH} \mathbf{1}^c(H). \tag{6.19}$$

If we denote (6.18) by Ω^+ , then (6.19) equals $(\Omega^+)^*$ and we have:

$$(\Omega^+)^* \Omega^+ = \mathbf{1}, \quad \Omega^+ (\Omega^+)^* = \mathbf{1}^c(H).$$

Remark 6.7. From the following discussion, it is clear that the conditions $\omega_j > 0$ for all j (respectively, $n_- + n_+ = n$) are crucial for the proof. If $\omega_j = 0$ for one j , then Eqs. (6.21), (6.22) below fail to be true. For the same reason, we cannot include linear terms like $E \cdot x$.

Proof. Since the proof is very similar to the case $n_- = n$ and $\omega_j = 1$ for all j , we will be very concise. The following points have to be addressed:

- (1) Definition of the conjugate operator A ,
- (2) The regularity $H_0 \in C^2(A)$,
- (3) The Mourre estimate for H_0 ,
- (4) The regularity $H \in C^{1+\delta}(A)$ and the Mourre estimate for H ,
- (5) Replacement of the conjugate operator A by $\langle \ln \langle x_{n_-} \rangle \rangle$,
- (6) Proof of the asymptotic completeness.

(1) As in (3.13), we choose for the conjugate operator $A = \text{Op}(a(x, \xi))$, with

$$a(x, \xi) = \ln \langle \xi_- + \omega_- x_- \rangle - \ln \langle \xi_- - \omega_- x_- \rangle.$$

One can show as in Lemma 3.7 that $(A, D(N))$ is essentially self-adjoint. As in Proposition 3.5, we can prove that, for $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ near 0,

$$(H + i)^{-1} = (H + i)^{-1} \text{Op} \left(\psi \left(\frac{\xi_-^2 - (\omega_- x_-)^2 + (\omega_+ x_+)^2}{(\xi_-^2 + x_-^2 + 1)^\beta} \right) \right) + \mathcal{O}(1) \text{Op}(r), \tag{6.20}$$

with $r \in S(\langle x, \xi \rangle^{-\beta}, g_0)$ for $1/2 < \beta \leq 1$. If the support of ψ is small enough, we have:

$$\xi_-^2 + \xi_+^2 + x_+^2 \lesssim x_-^2 + 1, \tag{6.21}$$

on the support of $\psi((\xi_-^2 - (\omega_- x_-)^2 + (\omega_+ x_+)^2)/(\xi_-^2 + x_-^2 + 1))$.

(2) First, note that $[H_0, A]$ and $[[H_0, A], A]$ are bounded on $L^2(\mathbb{R}^n)$, by the same arguments as in Lemmas 3.9 and 3.10. Notice that $N = N_- + N_+$ and $D(N) = D(N_-) \cap D(N_+)$. Clearly, $D(N_+) = D(H_0^+)$. Since $[H_0^+, H_0] = 0$, we have:

$$(z - H_0)^{-1} : D(N_+) \rightarrow D(N_+).$$

To prove $(z - H_0)^{-1} : D(N_-) \rightarrow D(N_-)$, we can proceed as in the proof of Lemma 3.8; we use

$$(z - H_0)^{-1} = \int_0^\infty e^{-itH_0^+} e^{-itH_0^-} e^{itz} dt,$$

and $[N_-, H_0^+] = 0$.

(3) As in Lemma 3.15, one can show that, for $\chi \in C_0^\infty(\mathbb{R})$,

$$\chi(H_0)[iH_0, A]\chi(H_0) = \chi(H_0) \sum_{j=1}^{n_-} 2\omega_j \left(\frac{(D_j + \omega_j x_j)^2}{\langle D_j + \omega_j x_j \rangle^2} + \frac{(D_j - \omega_j x_j)^2}{\langle D_j - \omega_j x_j \rangle^2} \right) \chi(H_0).$$

Using Gårding inequality, we get, for any $\mu > 0$,

$$\chi(H_0)[iH_0, A]\chi(H_0) \geq (2\hat{\omega} - \mu)\chi^2(H_0) + \chi(H_0)R\chi(H_0),$$

where $\hat{\omega} = \min_{j \in \{1, \dots, n_-\}} \omega_j$, and R is a pseudo-differential operator whose symbol is decreasing in (x_-, ξ_-) . Using (6.20) and (6.21), this decay becomes a decay in (x, ξ) on the energy level, and then

$$\chi(H_0)[iH_0, A]\chi(H_0) \geq (2\hat{\omega} - \mu)\chi^2(H_0) + \chi(H_0)K\chi(H_0),$$

with K compact. If the support of χ is sufficiently small, we therefore obtain:

$$\chi(H_0)[iH_0, A]\chi(H_0) \geq (2\hat{\omega} - \mu)\chi^2(H_0).$$

(4) Using (6.21), one can show that VA is compact from $D(H_0)$ to $L^2(\mathbb{R}^n)$, since the decay of V_2 in x_- (6.17) yields decay in all the variables. Thus, the Mourre estimate and the regularity $C^{1+\delta}(A)$ for H can be obtained as in Section 3.

(5) We apply the same arguments as in Section 4.2. We have to use that

$$\langle \xi_- - x_- \rangle \lesssim \langle x_- \rangle; \quad \langle \xi_- + x_- \rangle \lesssim \langle x_- \rangle, \tag{6.22}$$

on the energy levels, and (6.21). Then we show that the assumptions of Lemma 4.5 are fulfilled.

(6) The proof of the asymptotic completeness is exactly as in Section 4.3, using the minimal velocity estimate and the fact that $\chi(H) - \chi(H_0)$ is compact for $\chi \in C_0^\infty(\mathbb{R})$. \square

6.3. Asymptotic velocity

We assume that the hypothesis of Theorem 6.6 are satisfied. Let $\mathcal{H}^k = L^2(\mathbb{R})$ for $k \in \{1, \dots, n\}$. Clearly $\mathcal{H} = \bigotimes_{k=1}^n \mathcal{H}^k$. We write, as in (6.6),

$$H_0 = \sum_{j=1}^n H_{0,\omega_j}^j, \quad \text{with } H_{0,\omega_j}^j = -\Delta_{x_j} \pm \omega_j^2 x_j^2. \tag{6.23}$$

Obviously

$$[H_{0,\omega_j}^j, H_{0,\omega_k}^k] = 0. \tag{6.24}$$

If we use this separation of variables and apply Theorem 1.2 to the one-dimensional case, we obtain asymptotic velocities in each space direction. Let $V_j = [H, \ln\langle x_j \rangle]$. Like for V we can show that $(V_j, D(N))$ is well defined as an operator, and essentially self-adjoint. We denote again V_j its self-adjoint extension.

Theorem 6.8 (Asymptotic velocities). *There exists a vector $\vec{P}^+ = (P_1^+, \dots, P_n^+)$ of bounded self-adjoint commuting operators P_j^+ which commute with H , such that*

$$(i) \quad \vec{P}^+ = s\text{-}C_{\infty^-} \lim_{t \rightarrow \infty} e^{itH} \left(\frac{\ln\langle x_1 \rangle}{t}, \dots, \frac{\ln\langle x_n \rangle}{t} \right) e^{-itH}.$$

(ii) *The operator P_j^+ satisfies*

$$P_j^+ = \begin{cases} 2\omega_j \mathbf{1}^c(H) & \text{for } j \in \{1, \dots, n_-\}, \\ 0 & \text{for } j \in \{n_- + 1, \dots, n\}. \end{cases}$$

(iii) *For any $J \in C_{\infty}(\mathbb{R})$, $J(P_j^+) \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_j^+) = s\text{-}\lim_{t \rightarrow \infty} e^{itH} J(v_j) e^{-itH} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(P_j^+)$.*

Proof. We denote $H_{\omega} = H$, and $H_{0,\omega} = H_0$. We prove Theorem 6.8 in two steps.

First step. We assume $V \equiv 0$, that is $H_{\omega} = H_{0,\omega}$.

(i) We first treat the case $n_- = n = 1$. If $\omega = (-1)$, then the claim follows from Theorem 1.2. For $\omega = (-\omega_1^2)$, set:

$$(Dv)(x) = \frac{1}{\omega_1^{1/4}} v \left(\frac{x}{\sqrt{\omega_1}} \right).$$

Then $D : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is unitary, and we have:

$$H_{0,\omega} = \omega_1 D^* H_{0,(-1)} D. \tag{6.25}$$

Therefore:

$$\begin{aligned} s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,\omega}} J\left(\frac{\ln\langle x \rangle}{t}\right) e^{-itH_{0,\omega}} &= D^* s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,(-1)}} J\left(\frac{\ln\langle x/\sqrt{\omega_1} \rangle \omega_1}{t}\right) e^{-itH_{0,(-1)}} D \\ &= D^* s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,(-1)}} J\left(\frac{\ln\langle x \rangle \omega_1}{t}\right) e^{-itH_{0,(-1)}} D, \end{aligned} \tag{6.26}$$

because

$$J\left(\frac{\ln\langle x/\sqrt{\omega_1} \rangle \omega_1}{t}\right) - J\left(\frac{\ln\langle x \rangle \omega_1}{t}\right) = \mathcal{O}(t^{-1}).$$

Thus the result for general ω follows from the result for $\omega = (-1)$. Let now $n_- = 0$ and $n_+ = 1$. Then

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,\omega}} J\left(\frac{\ln\langle x \rangle}{t}\right) e^{-itH_{0,\omega}} = J(0), \tag{6.27}$$

because $L^2(\mathbb{R})$ possesses a basis of eigenfunctions of $H_{0,\omega}$. The general case follows from the one-dimensional cases using (6.23), (6.24). We denote \vec{P}_0^+ the vector associated to $H_{0,\omega}$.

(ii) First note that

$$J(P_{0,j}^+) = s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,\omega_j}^j} J\left(\frac{\ln\langle x_j \rangle}{t}\right) e^{-itH_{0,\omega_j}^j}.$$

The result on the spectrum follows from (6.26), (6.27), and from Theorem 1.2(ii).

(iii) By (ii), $P_{0,j}^+$ depends only on ω_j , and we note it P_{0,ω_j}^+ . We have $P_{0,\omega_j}^+ = \omega_j P_{0,1}^+$. If $j \in \{n_- + 1, \dots, n\}$, both operators are zero. For $j \in \{1, \dots, n_-\}$, we have:

$$\begin{aligned} J(P_{0,\omega_j}^+) &= J(\omega_j P_{0,1}^+) = D^* J(\omega_j P_{0,1}^+) D = D^* s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,1}^j} J(\omega_j \mathbb{V}_j) e^{-itH_{0,1}^j} D \\ &= s\text{-}\lim_{t \rightarrow \infty} e^{itH_{0,\omega_j}^j} D^* J(\omega_j \mathbb{V}_j) D e^{-itH_{0,\omega_j}^j} \end{aligned}$$

Here we have used that $P_{0,1}^+ = 2$, and thus $D P_{0,1}^+ D^* = P_{0,1}^+$, as well as (6.25). To prove (iv), it is thus sufficient to show that:

$$(D^* J(\omega_j \mathbb{V}_j) D - J(\mathbb{V}_j)) \chi(H_{0,\omega_j}^j) \text{ is compact on } L^2(\mathbb{R}),$$

$$\text{respectively, } (J(\omega_j \mathbb{V}_j) - D J(\mathbb{V}_j) D^*) \chi(H_{0,1}^j) \text{ is compact on } L^2(\mathbb{R}),$$

for any $\chi \in C_0^\infty(\mathbb{R})$. All operators have to be understood as operators acting on $L^2(\mathbb{R})$. Let $\tilde{\mathbb{V}}_j = D \mathbb{V}_j D^*$. The operator $\tilde{\mathbb{V}}_j$ is a pseudo-differential operator, with symbol $\tilde{v}_j(x_j, \xi_j) = x_j \xi_j / \langle x_j / \sqrt{\omega_j} \rangle^2$. Recall from Proposition 3.5 that

$$\chi(H_{0,1}^j) = \text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right) \chi(H_{0,1}^j) + R_j,$$

with $N_j R_j$ bounded ($N_j = -\Delta_{x_j} + \langle x_j \rangle^2$), and $\psi \in C_0^\infty(\mathbb{R})$ with $\psi = 1$ in a neighborhood of zero. Clearly, $(J(\omega_j V_j) - J(\tilde{V}_j))R_j$ is compact, and it remains to show that

$$(J(\omega_j V_j) - J(\tilde{V}_j)) \text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right) \text{ is compact.}$$

By the Helffer–Sjöstrand formula, it is sufficient to show that for any $z \in \mathbb{C} \setminus (\sigma(V_j) \cup \sigma(\tilde{V}_j))$,

$$\begin{aligned} & (z - \tilde{V}_j)^{-1}(\tilde{V}_j - \omega_j V_j)(z - \omega_j V_j)^{-1} \text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right) \\ &= (z - \tilde{V}_j)^{-1}(\tilde{V}_j - \omega_j V_j) \text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right)(z - \omega_j V_j)^{-1} \\ & \quad + (z - \tilde{V}_j)^{-1}(\tilde{V}_j - \omega_j V_j)(z - \omega_j V_j)^{-1} \omega_j \\ & \quad \times \left[\text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right), V_j \right] (z - \omega_j V_j)^{-1}, \end{aligned} \tag{6.28}$$

is compact. We have:

$$\begin{aligned} \left| (\tilde{V}_j(x_j, \xi_j) - \omega_j V_j(x_j, \xi_j)) \psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right) \right| &= \left| x_j \xi_j \left(\frac{1 - \omega_j}{\langle x_j \rangle^2 \langle x_j / \sqrt{\omega_j} \rangle^2} \right) \psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + x_j^2}\right) \right| \\ &\lesssim \left| \frac{1 - \omega_j}{\langle x_j / \sqrt{\omega_j} \rangle^2} \psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right) \right|, \end{aligned}$$

and each derivative of this symbol satisfies the same estimate. Thus the first term in (6.28) is compact, by the pseudo-differential calculus. Next we compute:

$$\left[iV_j, \text{Op}\left(\psi\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right)\right) \right] = \text{Op}(c_1) + \text{Op}(c_2),$$

with

$$c_1 = \psi'\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right) \left\{ V_j, \frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2} \right\} = \psi'\left(\frac{\xi_j^2 - x_j^2}{\xi_j^2 + \langle x_j \rangle^2}\right) \tilde{c}_1.$$

We have $\tilde{c}_1 \in S(\langle x \rangle^{-2}, g_2)$, $c_1 \in S(\langle x \rangle^{-1} \langle \xi \rangle^{-1}, g_2)$ and $c_2 \in S(\langle x \rangle^{-3} \langle \xi \rangle^{-1}, g_2)$. Thus $\text{Op}(c_1)$ and $\text{Op}(c_2)$ are compact by the pseudo-differential calculus.

Second step. General case.

Let $J \in C_\infty(\mathbb{R}^n)$. We have:

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_\omega} J\left(\frac{\ln\langle x_1 \rangle}{t}, \dots, \frac{\ln\langle x_n \rangle}{t}\right) e^{-itH_\omega} = \Omega^+ J(\vec{P}_0^+) (\Omega^+)^* + J(0) \mathbf{1}^{PP}(H_\omega).$$

The existence of \vec{P}^+ follows from the existence of \vec{P}_0^+ . Specializing $\tilde{J}(x_1, \dots, x_n) = J(x_j)$ we obtain furthermore:

$$J(P_j^+) = \Omega^+ J(P_{0,j}^+) (\Omega^+)^* + J(0) \mathbf{1}^{PP}(H_\omega).$$

Then (ii), (iii) follow from this formula and the results on $P_{0,j}^+$, as in the proof of Theorem 1.2. \square

One can ask whether the construction of Theorem 1.2 works also in the more general case, and what is the possible link between the vector \vec{P}^+ and P^+ . The answer is given by the following theorem:

Theorem 6.9. *Let $\omega_\ell = \max_{1 \leq j \leq n_-} \omega_j$. There exists,*

$$P^+ = s\text{-}C_\infty\text{-}\lim_{t \rightarrow \infty} e^{itH} \frac{\ln\langle x \rangle}{t} e^{-itH},$$

and we have $P^+ = P_\ell^+$.

Proof. *First step.* We assume $V \equiv 0$, that is $H = H_0$.

We already know that $P_\ell^+ = 2\omega_\ell$. Thus we only have to show that for $J \in C_0^\infty(\mathbb{R})$,

$$s\text{-}\lim_{t \rightarrow \infty} e^{itH_0} J\left(\frac{\ln\langle x \rangle}{t}\right) e^{-itH_0} = J(2\omega_\ell). \tag{6.29}$$

We can suppose $\ell = n_-$ and $\omega_1 \leq \omega_2 \leq \dots \leq \omega_k < \omega_{k+1} = \dots = \omega_{n_-}$. Let $\varepsilon > 0$. For $j \in \{1, \dots, k+1\}$, we choose $\tilde{J}_j \in C_0^\infty([2\omega_j - \varepsilon, 2\omega_j + \varepsilon])$, with $\tilde{J}_j = 1$ near $2\omega_j$. For $j \in \{n_- + 1, \dots, n\}$, we choose $\tilde{J}_j \in C_0^\infty([-\varepsilon, \varepsilon])$ with $\tilde{J}_j = 1$ near 0. Then by Propositions 4.8, 5.7 and the separability of the variables, we have:

$$\begin{aligned} e^{itH_0} J\left(\frac{\ln\langle x \rangle}{t}\right) e^{-itH_0} &= e^{itH_0} J\left(\frac{\ln\langle x \rangle}{t}\right) \tilde{J}_{k+1}\left(\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t}\right) \\ &\quad \times \prod_{\substack{j \in \{1, \dots, n\} \\ j \notin \{k+1, \dots, n_-\}}} \tilde{J}_j\left(\frac{\ln\langle x_j \rangle}{t}\right) e^{-itH_0} + R(t), \end{aligned}$$

with $s\text{-}\lim R(t) = 0$. It is clearly sufficient to show

$$\begin{aligned} s\text{-}\lim_{t \rightarrow \infty} e^{itH_0} \left(J\left(\frac{\ln\langle x \rangle}{t}\right) - J\left(\frac{\ln\langle x_{k+1}, \dots, x_n \rangle}{t}\right) \right) \\ \times \tilde{J}_{k+1}\left(\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t}\right) \prod_{\substack{j \in \{1, \dots, n\} \\ j \notin \{k+1, \dots, n_-\}}} \tilde{J}_j\left(\frac{\ln\langle x_j \rangle}{t}\right) e^{-itH_0} = 0. \end{aligned} \tag{6.30}$$

We have:

$$\begin{aligned} \left| J\left(\frac{\ln\langle x \rangle}{t}\right) - J\left(\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t}\right) \right| &\lesssim \frac{\ln\langle x \rangle}{t} - \frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t} \\ &\lesssim \frac{1}{t} \ln\left(1 + \sum_{\substack{j \in \{1, \dots, n\} \\ j \notin \{k+1, \dots, n_-\}}} \frac{x_j^2}{\langle x_{k+1}, \dots, x_{n_-} \rangle^2}\right). \end{aligned} \tag{6.31}$$

For $j \leq k$, we have on $\text{supp } \tilde{J}_j$

$$\frac{\ln\langle x_j \rangle^2}{t} \leq 4\omega_j + 2\varepsilon \implies x_j^2 \leq e^{(4\omega_j + 2\varepsilon)t} - 1. \tag{6.32}$$

We have, on $\text{supp } \tilde{J}_{k+1}$,

$$\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle^2}{t} \geq 4\omega_j - 2\varepsilon \implies \langle x_{k+1}, \dots, x_{n_-} \rangle^2 \geq e^{(4\omega_n - 2\varepsilon)t}. \tag{6.33}$$

For $j \geq n_- + 1$ we have, on $\text{supp } \tilde{J}_j$,

$$x_j^2 \leq e^{2\varepsilon t} - 1. \tag{6.34}$$

Gathering (6.31)–(6.34) together, we obtain:

$$\begin{aligned} &\left| \left(J\left(\frac{\ln\langle x \rangle}{t}\right) - J\left(\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t}\right) \right) \tilde{J}_{k+1}\left(\frac{\ln\langle x_{k+1}, \dots, x_{n_-} \rangle}{t}\right) \right. \\ &\quad \left. \times \prod_{\substack{j \in \{1, \dots, n\} \\ j \notin \{k+1, \dots, n_-\}}} \tilde{J}_j\left(\frac{\ln\langle x_j \rangle}{t}\right) \right| \lesssim f(t), \end{aligned}$$

where $f(t)$ is defined by

$$f(t) = \frac{1}{t} \ln\left(1 + \sum_{j=1}^k \frac{e^{(4\omega_j + 2\varepsilon)t} - 1}{e^{(4\omega_n - 2\varepsilon)t}} + \sum_{j=n_-+1}^n \frac{e^{2\varepsilon t} - 1}{e^{(4\omega_n - 2\varepsilon)t}}\right).$$

If ε is small enough, $\lim_{t \rightarrow \infty} f(t) = 0$. This yields (6.30).

Second step. General case. We have:

$$\begin{aligned} s\text{-}\lim_{t \rightarrow \infty} e^{itH} J\left(\frac{\ln\langle x \rangle}{t}\right) e^{-itH} &= \Omega^+ J(P_0^+) (\Omega^+)^* + J(0) \mathbf{1}^{pp}(H) \\ &= \Omega^+ J(P_{0,\ell}^+) (\Omega^+)^* + J(0) \mathbf{1}^{pp}(H) \\ &= \Omega^+ J(2\omega_\ell) (\Omega^+)^* + J(0) \mathbf{1}^{pp}(H) \\ &= J(2\omega_\ell) \mathbf{1}^c(H) + J(0) \mathbf{1}^{pp}(H). \end{aligned}$$

Thus P^+ exists, and

$$J(P^+) = J(2\omega_\ell)\mathbf{1}^c(H) + J(0)\mathbf{1}^{pp}(H). \quad (6.35)$$

Then $P^+ = 2\omega_\ell\mathbf{1}^c(H) = P_\ell^+$, which proves the theorem. \square

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