Sharp spectral gap for non-reversible metastable diffusions

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Dispersive Waves and Related Topics
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Brownian Dynamics

Given a vector field $U(x)$ on $\mathbb{R}^d$, consider the overdamped Langevin equation

$$dx_t = U(x_t) + \sqrt{2h}dB_t$$

(1)

where $B_t$ is the Brownian motion, $h > 0$ is proportional to the temperature of the system. The generator of this process is

$$\mathcal{L} = \mathcal{L}_U := h^2 \Delta + U(x)h \partial_x$$

Recall that

- for any bounded measurable function $f$, $u(t, x) = \mathbb{E}^x(f(x_t))$ solves

$$h \partial_t u = \mathcal{L} u$$

- the law $\mu(t, x)$ of the Markov process $(x_t)_{t \geq 0}$ is governed by the Fokker-Planck equation

$$h \partial_t \mu = \mathcal{L}^t \mu$$
Assumption 1

The vector field decomposes $U(x) = U_0(x) + h\nu(x)$ and there exists a smooth function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$L^t_U(e^{-2\phi/h}) = 0$$

Consequence

Denoting $b_0(x) = U_0 - 2\nabla\phi(x)$, one has $U = -2\nabla\phi + b_0 + h\nu$ with

$$\begin{cases}
  b_0 \cdot \nabla\phi = 0, \\
  \text{div}(\nu) = 0, \\
  \text{div}(b_0) = 2\nu \cdot \nabla\phi.
\end{cases}$$

Particular case

A particular case is $\nu = 0$, $\text{div} b_0 = 0$ and $b_0 \perp \nabla\phi$ which can be obtained by taking $b_0(x) = J\nabla\phi(x)$ for any antisymmetric matrix $J$ independent of $x$. 
Throughout, we denote $b_h = b_0 + h\nu$. We have

$$L_U = h^2\Delta - 2\nabla\phi(x)h\partial x - b_h(x)h\partial x$$

We sometime denote $L_U = L_{\phi,b_h}$

Let $\Omega \psi = e^{-\phi/h}\psi$, then

$$\Omega L_{\phi,b_h} \Omega^{-1} = -P_{\phi,b_h}$$

with

$$P_{\phi,b_h} = \Delta_{\phi} + b_h(x) \cdot \nabla_{\phi,h}$$

where

- $\Delta_{\phi} = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$ is the Witten Laplacian associated to the function $\phi$
- $\nabla_{\phi,h} = e^{-\phi/h} \circ h\nabla \circ e^{\phi/h}$. 
Assumption 2

There exists $C > 0$ and a compact $K \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d \setminus K$, one has

$$|\nabla \phi(x)| \geq \frac{1}{C}, \ |\text{Hess}(\phi(x))| \leq C|\nabla \phi|^2, \text{ and } \phi(x) \geq C|x|.$$ 

Under this assumptions, one has the following properties on $\Delta \phi$.

- $\Delta \phi$ is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$.
- $\Delta \phi \geq 0$
- There exists $C_0, h_0 > 0$ such that for all $0 < h < h_0$

$$\sigma_{\text{ess}}(\Delta \phi) \subset [C_0, \infty[$$

- $0$ is an eigenvalue of $\Delta \phi$ associated to the eigenstate $e^{-\phi/h}$. 
Assumption 3

\( \phi \) is a Morse function

We denote

- \( \mathcal{U} \) the set of critical points of \( \phi \) (since \( \phi \) is a Morse function, then \( \mathcal{U} \) is finite).
- \( \mathcal{U}^{(p)} \) the set of critical points of \( \phi \) of index \( p \)
- \( n_p = \# \mathcal{U}^{(p)} \).

Theorem (Witten 82, Helffer-Sjöstrand 84):

There exists \( \epsilon_0, h_0 > 0 \) such that for all \( 0 < h < h_0 \), one has

\[
\# \sigma(\Delta \phi) \cap [0, \epsilon_0 h] = n_0(\phi).
\]

Moreover, these \( n_0 \) eigenvalues are \( \mathcal{O}(e^{-c/h}) \) for some \( c > 0 \).
Let us write $\lambda(m, h), m \in U^{(0)}$ the $n_0$ small eigenvalues of $\Delta \phi$.

**Theorem (Bovier-Gayrard-Klein, Helffer-Klein-Nier 2004)**

Under a generic assumption, there exist a injective map $\varsigma : U^{(0)} \to U^{(1)}$ such that the $n_0$ small eigenvalues of $\Delta \phi$ satisfy

$$\lambda(m, h) = h \zeta(m, h) e^{-2S(m)/h}$$

where $\zeta(m, h) \sim \sum_{r=0}^{\infty} h^r \zeta_r(m)$ and

$$\zeta_0(m) = \pi^{-1} |\mu(s(m))| \sqrt{\frac{|\det \phi''(m)|}{|\det \phi''(s(m))|}}$$

where $S(m) = \phi(s(m)) - \phi(m)$ and $\mu(s)$ is the unique negative eigenvalue of $\phi''$ in $s$. 
Let us go back to the general situation \( U(x) = -2 \nabla \phi(x) - b_h(x) \) with \( b_h(x) = b_0(x) + h \nu(x) \) such that

\[
\begin{aligned}
&b_0 \cdot \nabla \phi = 0, \\
&\text{div}(\nu) = 0, \\
&\text{div}(b_0) = 2 \nu \cdot \nabla \phi.
\end{aligned}
\]

In that case

\[
P_\phi = P_{\phi,b_0} = \Delta \phi + b_h(x) \cdot \nabla \phi, h
\]

**Assumption 4**

\[
\exists C > 0, \ \forall x \in \mathbb{R}^d, \ |b_0(x)| + |\nu(x)| \leq C(1 + |\nabla \phi(x)|)
\]
One aims to study the spectral properties of the unbounded operator $P_{\phi,b_h}$ on $L^2(dx)$.

- One has $P^*_{\phi,b_h} = P_{\phi,-b_h}$, hence our result hold also for $P^*_{\phi,b_h}$.
- One has $\mathcal{L}_{\phi,b_h} = \Omega P_{\phi,b_h} \Omega^{-1}$ with

$$\Omega : L^2(dx) \to L^2(e^{-2\phi/h}dx)$$

isometry. Hence, spectral properties of $P_{\phi,b_h}$ yield

- spectral properties of $\mathcal{L}_{\phi,b_h}$ on $L^2(e^{-2\phi/h}dx)$
- spectral properties of $\mathcal{L}_{\phi,b_h}^t$ on $L^2(e^{2\phi/h}dx)$
Accretivity

**Theorem (Le Peutrec-Michel)**

1. The operator $P_\phi := P_\phi, b_h$ with domain $D(\Delta_\phi)$ is accretive.
2. It admits a unique maximal accretive extension $P_\phi$ with domain $D(P_\phi)$ and one has $D(\Delta_\phi) \subset D(P_\phi^*)$.
3. There exists $C, \Lambda_0 > 0$ such that $\sigma(P_\phi) \subset \Gamma_{\Lambda_0}$ where

$$\Gamma_{\Lambda_0} = \{ z \in \mathbb{C}, \, \text{Re}(z) \geq 0, \, |\text{Im} \, z| \leq \Lambda_0(\text{Re}(z) + \sqrt{\text{Re}(z)}) \}$$

4. One has

$$\|(P_\phi - z)^{-1}\|_{L^2 \rightarrow L^2} \leq \frac{C}{\text{Re}(z)}$$

for all $z \in \Gamma_{\Lambda_0}^c \cap \{ \text{Re}(z) \geq 0 \}$.

5. There exists $c_1 > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$ the map $z \mapsto (P_\phi - z)^{-1}$ is meromorphic in $\{ \text{Re}(z) < c_1 \}$ with finite rank residues.
There exists $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in [0, h_0]$, 
$\sigma(P\phi) \cap \{\text{Re}(z) \leq \epsilon_0 h\}$ is finite and 

$$\|\sigma(P\phi) \cap \{\text{Re}(z) \leq \epsilon_0 h\} \| \leq n_0$$

Moreover, one has 

$$\sigma(P\phi) \cap \{\text{Re}(z) \leq \epsilon_0 h\} \subset B(0, e^{-C/h})$$

for some $C > 0$. Eventually, for any $0 < \epsilon < \epsilon_0$, one has 

$$(P\phi - z)^{-1} = O(h^{-1})$$

uniformly with respect to $z$ such that $\epsilon h < |z| < \epsilon_0 h$. 
Proof 1

- Use

\[
2 \text{Re} \langle P_\phi, b_h u, u \rangle = \langle (P_\phi, b_h + P_\phi, -b_h) u, u \rangle = \langle \Delta \phi u, u \rangle = \| \nabla \phi u \|^2 \geq 0
\]

and

\[
| \text{Im} \langle P_\phi, b_h u, u \rangle | \leq C(\| \nabla \phi u \|^2 + \| u \| \| \nabla \phi u \|)
\]

to prove accretivity and first spectral estimates.

- The spectral localization 'in the small' is obtained by mean of a Grushin problem associated to the eigenvectors \((e_k)_{k=1,...,n_0}\) associated to small eigenvalues of \(\Delta \phi\) noticing

  - \(\Delta \phi \geq Ch\) on \(\text{Span}(e_1, \ldots, e_{n_0})^\perp\)
  - \(b_h \cdot \nabla \phi e_k = O(h^{1/2}e^{-S/h})\).
A geometric Lemma

Lemma [Landim-Seo, 17]

Let \( s \in \mathcal{U}^{(1)} \) be saddle point of \( \phi \). Denote \( B(s) = db(s) \).

i) The (in general non symmetric) matrix
\[
2 \text{Hess} \phi(s) + B^*(s) \in \mathcal{M}_d(\mathbb{R})
\]
admits precisely one eigenvalue with negative real part. This eigenvalue, denoted by \( \mu(s) \), is real and has geometric multiplicity one.

ii) We denote by \( \xi = \xi(s) \) one of the two (real) unitary eigenvectors of \( 2 \text{Hess} \phi(s) + B^*(s) \) associated with \( \mu(s) \). The real symmetric matrix
\[
M_\phi := \text{Hess} \phi(s) + |\mu| \xi \xi^*
\]
is then positive definite and its determinant satisfies:
\[
det M_\phi = - \det \text{Hess} \phi(s).
\]
The non-reversible case

Sharp asymptotics of small spectral values

**Theorem [Le Peutrec-Michel]**

Suppose that the above Assumptions and that the generic assumption of Helffer-Klein-Nier hold true and let $s : \mathcal{U}^{(0)} \to \mathcal{U}^{(1)}$ denote the corresponding map. Then, there exists $c > 0$ such that the following holds for every $h > 0$ small enough:

$$\text{Spec } (P_{\phi, b_h}) \cap \{\text{Re } z < ch\} = \{\lambda(m, h), \ m \in \mathcal{U}^{(0)}\},$$

where $\lambda(m, h) = 0$ and for all $m \neq m$

$$\lambda(m, h) = h \frac{\left| \mu(s(m)) \right|}{2\pi} \frac{\det \text{Hess } \phi(m)^{1/2}}{\left| \det \text{Hess } \phi(s(m)) \right|^{1/2}} e^{-2 \frac{\phi(s(m)) - \phi(m)}{h}} \left(1 + O(h^{1/2})\right).$$
Corollary

Suppose that the above Assumptions and that the generic assumption of Helffer-Klein-Nier holds true. Suppose also that for all $m, m' \in U^{(0)}$ one has

$$(m \not= m' \text{ and } S(m) = S(m')) \implies \zeta(m) \not= \zeta(m')$$

Then, $\text{Spec}(P_{\phi, b_h}) \cap \{\text{Re } z < ch\}$ is made of $n_0$ real eigenvalues and there exists $C > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$ and all $s > 0$, one has

$$\|e^{-sL^t_{\phi, b_h}} - \Pi_0\|_{L(L^2(e^{2\phi/h}dx))} \leq Ce^{-\lambda(h)s}$$

where $\lambda(h) = \min\{\lambda(m, h), m \not= m\}$ and $\Pi_0$ is the orthogonal projection onto $\text{Re}e^{-2\phi/h}$ in $L^2(e^{2\phi/h}dx)$. 

Bibliography

- [Bouchet-Reygnier, 2016] obtained similar result for the exit time of a domain. Computation without proof.

- [Landim-Mariani-Seo, 2019] rigorous result for the exit time of a domain by capacity approach. Only for double well potential and particular form of drift $b = J\nabla \phi$ with $J$ antisymmetric.

- [Hérau-Hitrik-Sjöstrand, 2011] Results for the Kramers-Fokker-Planck equation. More difficult situation since it is hypoelliptic only. Uses supersymmetry and PT-symmetry in a crucial way.
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The labelling procedure

For any \( s \in U^{(1)} \) and \( r > 0 \) small enough, the set

\[
B(s, r) \cap \{ x \in X, \, \phi(x) < \phi(s) \}
\]

has exactly two connected components \( C_j(s, r), \, j = 1, 2. \)

**Definition (Hérau-Hitrik-Sjöstrand, 2011)**

- \( s \in U^{(1)} \) is a separating saddle point (ssp) iff \( C_1(s, r) \) and \( C_2(s, r) \) are contained in two different connected components of \( \{ x \in X, \, \phi(x) < \phi(s) \} \). We denote by \( V^{(1)} \) the set of ssp.

- \( \sigma \in \mathbb{R} \) is a separating saddle value (ssv) if it is of the form \( \sigma = \phi(s) \) with \( s \in V^{(1)} \). We denote \( \Sigma = \phi(V^{(1)}) = \{ \sigma_2 > \sigma_3 > \ldots > \sigma_N \} \).
Example of SSP I

Level set of a potential with 2 minima, 2 saddle points and 1 maximum
Example of SSP II

\[ C_1(s_1, r) \]

\[ C_2(s_1, r) \]
Example of SSP II

\[ C_1(s_1, r) \]
\[ C_2(s_1, r) \]

\( s_1 \) is not separating

\( \{ \phi < \phi(s_1) \} \)
Example of SSP III
Example of SSP III

\[ \{ \phi < \phi(s_2) \} \]

\[ C_1(s_2, r) \]

\[ C_2(s_2, r) \]

\( s_2 \) is separating
The labelling procedure II

Add a fictive infinite saddle value $\sigma_1 = +\infty$ to $\Sigma$ and let

$$\Sigma = \{\sigma_1\} \cup \Sigma = \{\sigma_1 > \sigma_2 > \ldots > \sigma_N\}$$

- To $\sigma_1 = +\infty$ associate the unique connected component $E_{1,1} = X$ of $\{\phi < \sigma_1\}$. In $E_{1,1}$, pick up $m_{1,1}$ one (non necessarily unique) minimum of $\phi|_{E_{1,1}}$.

- The set $\{\phi < \sigma_2\}$ has finitely many connected components. One of them contains $m_{1,1}$. The others are denoted $E_{2,1}, \ldots, E_{2,N_2}$. In each of these CC, one choses one absolute minimum $m_{2,j}$ of $\phi|_{E_{2,j}}$.

- The set $\{\phi < \sigma_k\}$ has finitely many CC. One denotes by $E_{k,1}, \ldots, E_{k,N_k}$ those of these CC which do not contain any $m_{i,j}$, $i < k$. In each $E_{k,j}$ one choses one absolute minimum $m_{k,j}$ of $\phi|_{E_{k,j}}$. 
Denote $\underline{m} = m_{1,1}$ the absolute minimum of $\phi$ that was chosen at the first step of the labelling procedure, and let

$$\underline{U}^{(0)} = U^{(0)} \setminus \{m\}.$$  

Using the preceding labelling one constructs the following applications:

- $\sigma : \underline{U}^{(0)} \to \Sigma$, defined by $\sigma(m_{i,j}) = \sigma_i$.
- $E(m)$ is the connected component of $\{\phi < \sigma(m)\}$ that contains $m$.
- $S(m) = \sigma(m) - \phi(m)$
The Generic Assumption

The following hypothesis introduced by Hérau-Hitrik-Sjöstrand (2011) is a generalization of Helffer-Klein-Nier assumption (2004).

**Generic Assumption (GA):**

For all \( m \in \mathcal{U}^{(0)} \), the following hold true:

i) \( \phi|_{E(m)} \) has a unique point of minimum

ii) if \( E \) is any connected component of \( \{ \phi < \sigma(m) \} \) and \( \overline{E} \cap \mathcal{V}^{(1)} \neq \emptyset \), there exists a unique \( s \in \mathcal{V}^{(1)} \) such that \( \phi(s) = \sup \phi(\overline{E} \cap \mathcal{V}^{(1)}) \).

Under this assumption, there exists a bijection

\[
\varsigma : \mathcal{U}^{(0)} \rightarrow \mathcal{V}^{(1)} \cup \{ \infty \}
\]

such that \( S(m) = \phi(\varsigma(m)) - \phi(m) \) with the convention \( \phi(\infty) = \infty \).
The simplified Generic Assumption

Simplified Generic Assumption:
The map
\[(m, s) \in U^{(0)} \times V^{(1)} \mapsto \phi(s) - \phi(m)\]
is injective.

Consequence
For any \(m \in U^{(0)}\), there exists a unique \(s \in \partial E(m) \cap V^{(1)}\) and the map
\[s : U^{(0)} \rightarrow V^{(1)} \cup \{\infty\}\]
\[m \mapsto s\]
is injective. Moreover, one has \(S(m) = \phi(s(m)) - \phi(m)\) with the convention \(\phi(\infty) = \infty\).
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Let
\[
\Pi_h = \frac{1}{2i\pi} \int_{|z|=\epsilon h} (P_{\phi} - z)^{-1} dz
\]
and \( E_h = \text{Ran} \Pi_h \). Then \( \dim E_h = n_0 \) and \( P_{\phi} : E_h \to E_h \).

**Goal**

Compute the spectrum of the restriction of \( P_{\phi} \) to \( E_h \). This is a problem in finite dimension.

The general strategy is the following:

1) Construct suitable approximated eigenfunctions \( f_m, m \in \mathcal{U}(0) \) of the operator \( P_{\phi} \)

2) Project these eigenfunctions on \( E_h \), \( e_m = \Pi_h f_m \) and estimate the difference \( e_m - f_m \).

3) Compute the matrix \( M_{\phi} \) of \( P_{\phi} \) in the base \( (e_m, m \in \mathcal{U}(0)) \)

4) Compute the spectrum of \( M_{\phi} \)
Some comments

• The better the quasimodes are, the smaller $\|e_m - f_m\|$ is.

Indeed:

$$e_m - f_m = \prod_h f_m - f_m = \frac{1}{2i\pi} \int_{|z|=\epsilon h} ((P_\phi - z)^{-1} - z^{-1}) f_m dz$$

$$= \frac{-1}{2i\pi} \int_{|z|=\epsilon h} (P_\phi - z)^{-1} z^{-1} P_\phi f_m dz$$

• Standard quasimodes $\tilde{f}_m = \chi_m e^{-(\phi - \phi(m))/h}$ with $\chi_m$ cut-off function in $E(m)$ yield

$$P_\phi \tilde{f}_m = O(e^{-(S(m) - \epsilon)/h})$$

• We need to construct accurate quasimodes

• The operator $P_\phi$ is non-self-adjoint, hence the matrix $M_\phi$ is not symmetric. We have to be careful of Jordan’s block.
Construction of quasimodes

- Let $s = s(m)$ the saddle point associated to $m$ and let $\xi(s)$ be given by the geometric Lemma:
  - $\xi(s)$ is a unitary eigenvector associated to the unique negative eigenvalue $\mu(s)$ of $2\text{Hess}(\phi)(s) + db(s)$.
  - the matrix $\text{Hess} \phi(s) + |\mu(s)|\xi\xi^*$ is positive definite.

- Define the quasimode

\[
  f_m(x) = c_h^{-1} \kappa_h((x - s) \cdot \xi(s)) \chi_m(x) e^{-(\phi(x) - \phi(m))/h}
\]

where $c_h$ is a $L^2$-normalization constant and $\kappa_h : \mathbb{R} \to \mathbb{R}$ is a cut-off function such that

- $\kappa_h(t) = \begin{cases} 
  0 & \text{if } t < -1 \\
  1 & \text{if } t > 1 
\end{cases}$

- $\frac{d\kappa_h}{dt}(t) = h^{-1/2} e^{-|\mu(s)|t^2/2h}$ if $t \in [-\frac{1}{2}, \frac{1}{2}]$.

- $\text{supp}(\kappa_h(.)\partial_x \chi) \subset \{\phi > \phi(s) + \epsilon\}$. 
The cutoff function $\chi_m$

\[ \{ \phi < \phi(s) \} \]

\[ \text{supp}(\chi_m) \]
The cutoff function $\kappa_h$

The diagram shows the support of $\chi_m$ and the sets $\{\phi = \phi(s)\}$ with $\kappa_h = 0$ and $\kappa_h = 1$. The expression $\xi(s)$ indicates a specific function or variable in the context.
Lemma

Suppose that the generic assumption is satisfied and let $m \neq m'$ in $U(0)$. Then, either $\text{supp}(f_m) \cap \text{supp}(f_{m'}) = \emptyset$ or $f_m = 1$ on $\text{supp}(f_{m'})$ or $f_{m'} = 1$ on $\text{supp}(f_m)$. In particular, there exists $c > 0$ such that

$$\langle f_m, f_{m'} \rangle = \delta_{mm'} + O(e^{-c/h}).$$

Lemma

For all $m \in U(0)$, one has

$$\langle P_\phi f_m, f_m \rangle = h \frac{|\mu(s)|}{2\pi} \frac{D_m}{D_s} e^{-2S(m)/h} (1 + O(h))$$

where $s = s(m)$, $S(m) = \phi(s) - \phi(m)$, $D_x = |\det \text{Hess}(\phi)(x)|^{1/2}$. 
Quasimodal estimates II

• One has

\[ \langle P_\phi f_m, f_m \rangle = \langle \Delta_\phi f_m, f_m \rangle = \langle d_\phi^* d_\phi f_m, f_m \rangle = \| d_\phi f_m \|^2 \]

with

\[ f_m(x) = c_h^{-1} \chi_m(x) \kappa_h((x - s)\xi) e^{-(\phi(x) - \phi(m))/h} \]

Hence

\[ d_\phi f_m = h c_h^{-1} \chi_m(x) \kappa'_h((x - s)\xi) \xi e^{-(\phi(x) - \phi(m))/h} + O(e^{-(S(m) + \epsilon)/h}) \]

• near \( s \), one has

\[ \kappa'_h((x - s)\xi) e^{-(\phi(x) - \phi(m))/h} = e^{\langle (\frac{1}{2} \text{Hess} \phi(s) + |\mu|\xi\xi^*)(x-s), (x-s) \rangle + O((x-s)^2)} \]

and \( \frac{1}{2} \text{Hess} \phi(s) + |\mu|\xi\xi^* \) positive definite.

• Apply Laplace method to complete the computation.
Quasimodal estimates III

Lemma

For all $m \in \mathcal{U}(0)$, one has

$$\|P_{\phi}f_m\|^2 = \langle P_{\phi}f_m, f_m \rangle \mathcal{O}(h^2)$$

$$\|P_{\phi}^*f_m\|^2 = \langle P_{\phi}f_m, f_m \rangle \mathcal{O}(h)$$

Proof. One has

$$P_{\phi}(f_m) = (-h^2 \Delta + h(2\nabla \phi + b_h \nabla)) (\kappa_h) c_h^{-1} \chi(x) e^{-(\phi(x) - \phi(s))/h}$$

$$+ \mathcal{O}(e^{-(S(m) + \epsilon)/h})$$

and for $x$ close to $s$, one has

$$(-h^2 \Delta + h(2\nabla \phi + b_h \nabla))(\kappa_h) = h((2\nabla \phi + b) \xi + |\mu| \xi(x - s) + \mathcal{O}(h)) e^{-}$$

$$= h\mathcal{O}((x - s)^2 + h)e^{-|\mu|((x-s)^2/2h)}$$
Spectrum of non-symmetric matrices

Graded structure of the interaction matrix

We enumerate the minima $\mathcal{U}^{(0)} = \{\mathbf{m}_1, \ldots, \mathbf{m}_{n_0}\}$ in such way that the sequence $(S(\mathbf{m}_j))_{j}$ is non-decreasing. We denote

- $\tilde{\lambda}_j = \langle P_\phi f_{m_j}, f_{m_j} \rangle$
- $(e_{m_j})_{1 \leq j \leq n_0}$ the basis of $E_h$ obtained from $\Pi_h f_{m_j}$ by Graam-Schmidt procedure.

Proposition

For all $j, k = 1, \ldots, n_0$, one has

$$\langle P_\phi e_j, e_k \rangle = \delta_{jk} \tilde{\lambda}_j + \mathcal{O}(\sqrt{h\tilde{\lambda}_j \tilde{\lambda}_k})$$

- Let $\mathcal{M}_\phi = (\langle P_\phi e_j, e_k \rangle)_{j,k}$ be the matrix of $P_\phi$ in the basis $(e_j)$.
- Let $\Omega = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d})$, then

$$\mathcal{M}_\phi = \Omega(\text{Id} + \mathcal{O}(\sqrt{h}))\Omega$$
Suppose that $\mathcal{M}_\phi = \Omega(\text{Id} + O(h))\Omega$ with $\Omega$ as above. We can compute the spectrum of $\mathcal{M}_\phi$ by Schur complement method.

Computation for 2x2 matrices. Suppose

$$\mathcal{M}_\phi = \begin{pmatrix} \tilde{\lambda}_1 & B_h \\ B_h^* & \tilde{\lambda}_2 \end{pmatrix}$$

with $B_h = O(\sqrt{h\tilde{\lambda}_1\tilde{\lambda}_2})$. The spectral values of $\mathcal{M}_\phi$ are the poles of

$$z \mapsto (\tilde{\lambda}_1 - z - B_h^*(\tilde{\lambda}_2 - z)^{-1}B_h)^{-1}$$

and

$$z \mapsto (\tilde{\lambda}_2 - z - B_h^*(\tilde{\lambda}_1 - z)^{-1}B_h)^{-1}$$