Sharp spectral gap for non-reversible metastable diffusions

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Dispersive Waves and Related Topics
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Plan

1. Introduction
   - General framework
   - The reversible case
   - The non-reversible case
2. The labelling procedure
   - Separating saddle points
   - The generic assumption
3. Sketch of proof
   - General strategy
   - Quasimodal estimates
   - Spectrum of non-symmetric matrices

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Brownian Dynamics

Given a vector field $U(x)$ on $\mathbb{R}^d$, consider the overdamped Langevin equation

$$dx_t = U(x_t) + \sqrt{2h}dB_t \tag{1}$$

where B_t is the Brownian motion , $h > 0$ is proportional to the temperature of the system. The generator of this process is

$$\mathscr{L} = \mathscr{L}_U := h^2\Delta + U(x)h\partial_x$$

Recall that

- for any bounded measurable function f, $u(t,x) = \mathbb{E}^x(f(x_t))$ solves

$$h\partial_t u = \mathscr{L}u$$

- the law $\mu(t,x)$ of the Markov process $(x_t)_{t\geq 0}$ is governed by the Fokker-Planck equation

$$h\partial_t \mu = \mathscr{L}^t \mu$$

Assumption 1

The vector field decomposes $U(x) = U_0(x) + h\nu(x)$ and there exists a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathscr{L}_U^t(e^{-2\phi/h}) = 0$$

Consequence

Denoting $b_0(x) = U_0 - 2\nabla\phi(x)$, one has $U = -2\nabla\phi + b_0 + h\nu$ with

$$\begin{cases} b_0 \cdot \nabla\phi = 0, \\ \operatorname{div}(\nu) = 0, \\ \operatorname{div}(b_0) = 2\nu \cdot \nabla\phi. \end{cases}$$

Particular case

A particular case is $\nu = 0$, div $b_0 = 0$ and $b_0 \perp \nabla\phi$ which can be obtained by taking $b_0(x) = J\nabla\phi(x)$ for any antisymmetric matrix J independent of x.

- Throughout, we denote $b_h = b_0 + h\nu$. We have

$$\mathscr{L}_U = h^2\Delta - 2\nabla\phi(x)h\partial x - b_h(x)h\partial_x$$

We sometime denote $\mathscr{L}_U = \mathscr{L}_{\phi,b_h}$

- Let $\Omega\psi = e^{-\phi/h}\psi$, then

$$\Omega\mathscr{L}_{\phi,b_h}\Omega^{-1} = -P_{\phi,b_h}$$

with

$$P_{\phi,b_h} = \Delta_\phi + b_h(x)\cdot\nabla_{\phi,h}$$

where

  - $\Delta_\phi = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$ is the Witten Laplacian associated to the function ϕ
  - $\nabla_{\phi,h} = e^{-\phi/h}\circ h\nabla\circ e^{\phi/h}$.

Spectral study of the Witten Laplacian

Assumption 2

There exists $C > 0$ and a compact $K \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d \setminus K$, one has

$$|\nabla\phi(x)| \geq \frac{1}{C}, \ |\mathrm{Hess}(\phi(x))| \leq C|\nabla\phi|^2, \ \text{and} \ \phi(x) \geq C|x|.$$

Under this assumptions, one has the following properties on Δ_ϕ.

- Δ_ϕ is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$.
- $\Delta_\phi \geq 0$
- there exists $C_0, h_0 > 0$ such that for all $0 < h < h_0$

$$\sigma_{ess}(\Delta_\phi) \subset [C_0, \infty[$$

- 0 is an eigenvalue of Δ_ϕ associated to the eigenstate $e^{-\phi/h}$.

Assumption 3

ϕ is a Morse function

We denote

- $\mathcal{U}$ the set of critical points of ϕ (since ϕ is a Morse function, then $\mathcal{U}$ is finite).
- $\mathcal{U}^{(p)}$ the set of critical points of ϕ of index p
- $n_p = \sharp\mathcal{U}^{(p)}$.

Theorem **(Witten 82, Helffer-Sjöstrand 84):**

There exists $\epsilon_0, h_0 > 0$ such that for all $0 < h < h_0$, one has

$$\sharp\sigma(\Delta_\phi) \cap [0, \epsilon_0 h] = n_0(\phi).$$

Moreover, these n_0 eigenvalues are $\mathcal{O}(e^{-c/h})$ for some $c > 0$.

Sharp asymptotics of small eigenvalues

Let us write $\lambda(\mathbf{m}, h)$, $\mathbf{m} \in \mathcal{U}^{(0)}$ the n_0 small eigenvalues of Δ_ϕ.

Theorem (Bovier-Gayrard-Klein, Helffer-Klein-Nier 2004)

Under a generic assumption, there exist a injective map $\mathfrak{s} : \mathcal{U}^{(0)} \to \mathcal{U}^{(1)}$ such that the n_0 small eigenvalues of Δ_ϕ satisfy

$$\lambda(\mathbf{m}, h) = h\zeta(\mathbf{m}, h)e^{-2S(\mathbf{m})/h}$$

where $\zeta(\mathbf{m}, h) \sim \sum_{r=0}^{\infty} h^r \zeta_r(\mathbf{m})$ and

$$\zeta_0(\mathbf{m}) = \pi^{-1}|\mu(\mathbf{s}(\mathbf{m}))|\sqrt{\frac{|\det \phi''(\mathbf{m})|}{|\det \phi''(\mathbf{s}(\mathbf{m}))|}}$$

where $S(\mathbf{m}) = \phi(\mathfrak{s}(\mathbf{m})) - \phi(\mathbf{m})$ and $\mu(\mathbf{s})$ is the unique negative eigenvalue of ϕ'' in $\mathbf{s}$.

Let us go back to the general situation $U(x) = -2\nabla\phi(x) - b_h(x)$ with $b_h(x) = b_0(x) + h\nu(x)$ such that

$$\begin{cases} b_0 \cdot \nabla\phi = 0, \\ \mathrm{div}(\nu) = 0, \\ \mathrm{div}(b_0) = 2\nu \cdot \nabla\phi. \end{cases}$$

In that case

$$P_\phi = P_{\phi,b_0} = \Delta_\phi + b_h(x) \cdot \nabla_{\phi,h}$$

Assumption 4

$$\exists C > 0,\ \forall x \in \mathbb{R}^d,\ |b_0(x)| + |\nu(x)| \le C(1 + |\nabla\phi(x)|)$$

One aims to study the spectral properties of the unbounded operator P_{ϕ,b_h} on $L^2(dx)$.

- One has $P^*_{\phi,b_h} = P_{\phi,-b_h}$, hence our result hold also for P^*_{ϕ,b_h}.
- One has $\mathscr{L}_{\phi,b_h} = \Omega P_{\phi,b_h} \Omega^{-1}$ with

$$\Omega : L^2(dx) \to L^2(e^{-2\phi/h}dx)$$

isometry. Hence, spectral properties of P_{ϕ,b_h} yield

  - spectral properties of $\mathscr{L}_{\phi,b_h}$ on $L^2(e^{-2\phi/h}dx)$
  - spectral properties of $\mathscr{L}^t_{\phi,b_h}$ on $L^2(e^{2\phi/h}dx)$

Accretivity

Theorem (Le Peutrec-Michel)

1. The operator $P_\phi := P_{\phi,b_h}$ with domain $D(\Delta_\phi)$ is accretive.
2. It admits a unique maximal accretive extension P_ϕ with domain $D(P_\phi)$ and one has $D(\Delta_\phi) \subset D(P_\phi^*)$.
3. There exists $C, \Lambda_0 > 0$ such that $\sigma(P_\phi) \subset \Gamma_{\Lambda_0}$ where
$$\Gamma_{\Lambda_0} = \{z \in \mathbb{C},\ \mathrm{Re}(z) \geq 0,\ |\operatorname{Im} z| \leq \Lambda_0(\mathrm{Re}(z) + \sqrt{\mathrm{Re}(z)}\}$$
4. One has
$$\|(P_\phi - z)^{-1}\|_{L^2 \to L^2} \leq \frac{C}{\mathrm{Re}(z)}$$
for all $z \in \Gamma_{\Lambda_0}^c \cap \{\mathrm{Re}(z) \geq 0\}$.
5. There exists $c_1 > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$ the map $z \mapsto (P_\phi - z)^{-1}$ is meromorphic in $\{\mathrm{Re}(z) < c_1\}$ with finite rank residues.

First spectral localization

Theorem (Le Peutrec-Michel) continued

There exists $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $\sigma(P_\phi) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h\}$ is finite and

$$\sharp\sigma(P_\phi) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h\} \leq n_0$$

Moreover , one has

$$\sigma(P_\phi) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h\} \subset B(0, e^{-C/h})$$

for some $C > 0$. Eventually, for any $0 < \epsilon < \epsilon_0$, one has

$$(P_\phi - z)^{-1} = \mathcal{O}(h^{-1})$$

uniformly with respect to z such that $\epsilon h < |z| < \epsilon_0 h$.

Proof I

- Use

$$\begin{aligned} 2\,\mathrm{Re}\langle P_{\phi,b_h}u,u\rangle &= \langle (P_{\phi,b_h}+P_{\phi,-b_h})u,u\rangle \\ &= \langle \Delta_\phi u,u\rangle = \|\nabla_\phi u\|^2 \geq 0 \end{aligned}$$

  and

$$|\,\mathrm{Im}\langle P_{\phi,b_h}u,u\rangle| \leq C(\|\nabla_\phi u\|^2 + \|u\|\|\nabla_\phi u\|)$$

  to prove accretivity and first spectral estimates.
- The spectral localization 'in the small" is obtained by mean of a Grushin problem associated to the eigenvectors $(e_k)_{k=1,\dots,n_0}$ associated to small eigenvalues of Δ_ϕ noticing
  - $\Delta_\phi \geq Ch$ on $\mathrm{Span}(e_1,\dots,e_{n_0})^\perp$
  - $b_h\cdot\nabla_\phi e_k = \mathcal{O}(h^{1/2}e^{-S/h})$.

A geometric Lemma

Lemma [Landim-Seo, 17]

Let $\mathbf{s} \in \mathcal{U}^{(1)}$ be saddle point of ϕ. Denote $B(\mathbf{s}) = db(\mathbf{s})$.

i) The (in general non symmetric) matrix $2\,\mathrm{Hess}\,\phi(\mathbf{s}) + B^*(\mathbf{s}) \in \mathcal{M}_d(\mathbb{R})$ admits precisely one eigenvalue with negative real part. This eigenvalue, denoted by $\mu(\mathbf{s})$, is real and has geometric multiplicity one.

ii) We denote by $\xi = \xi(\mathbf{s})$ one of the two (real) unitary eigenvectors of $2\,\mathrm{Hess}\,\phi(\mathbf{s}) + B^*(\mathbf{s})$ associated with $\mu(\mathbf{s})$. The real symmetric matrix

$$M_\phi \ := \ \mathrm{Hess}\,\phi(\mathbf{s}) + |\mu|\,\xi\,\xi^*$$

is then positive definite and its determinant satisfies:

$$\det M_\phi \ = \ -\det \mathrm{Hess}\,\phi(\mathbf{s})\,.$$

Sharp asymptotics of small spectral values

Theorem [Le Peutrec-Michel]

Suppose that the above Assumptions and that the generic assumption of Helffer-Klein-Nier hold true and let $\mathfrak{s} : \mathcal{U}^{(0)} \to \mathcal{U}^{(1)}$ denote the corresponding map. Then, there exists $c > 0$ such that the following holds for every $h > 0$ small enough:

$$\mathrm{Spec}\,(P_{\phi,b_h}) \cap \{\mathrm{Re}\, z < ch\} \ = \{\lambda(\mathbf{m}, h),\ \mathbf{m} \in \mathcal{U}^{(0)}\},$$

where $\lambda(\underline{\mathbf{m}}, h) = 0$ and for all $\mathbf{m} \neq \underline{\mathbf{m}}$

$$\lambda(\mathbf{m}, h) \ = \ h\,\frac{|\mu(\mathfrak{s}(\mathbf{m}))|}{2\pi}\,\frac{\det \mathrm{Hess}\,\phi(\mathbf{m})^{\frac12}}{|\det \mathrm{Hess}\,\phi(\mathfrak{s}(\mathbf{m}))|^{\frac12}}\,e^{-2\frac{\phi(\mathfrak{s}(\mathbf{m}))-\phi(\mathbf{m})}{h}}\,(1+\mathcal{O}(h^{\frac12}))\,.$$

Return to equilibrium

Corollary

Suppose that the above Assumptions and that the generic assumption of Helffer-Klein-Nier holds true. Suppose also that for all $\mathbf{m}, \mathbf{m}' \in \mathcal{U}^{(0)}$ one has

$$(\mathbf{m} \neq \mathbf{m}' \text{ and } S(\mathbf{m}) = S(\mathbf{m}')) \Longrightarrow \zeta(\mathbf{m}) \neq \zeta(\mathbf{m}')$$

Then, $\mathrm{Spec}\,(P_{\phi,b_h}) \cap \{\mathrm{Re}\, z < ch\}$ is made of n_0 real eigenvalues and there exists $C > 0$ and $h_0 > 0$ such that for all $0 < h < h_0$ and all $s > 0$, one has

$$\|e^{-s\mathscr{L}^t_{\phi,b_h}} - \Pi_0\|_{\mathscr{L}(L^2(e^{2\phi/h}dx))} \leq Ce^{-\lambda(h)s}$$

where $\lambda(h) = \min\{\lambda(\mathbf{m}, h), \mathbf{m} \neq \underline{\mathbf{m}}\}$ and Π_0 is the orthogonal projection onto $\mathbb{R}e^{-2\phi/h}$ in $L^2(e^{2\phi/h}dx)$.

Bibliography

- **[Bouchet-Reygnier, 2016]** obtained similar result for the exit time of a domain. Computation without proof.
- **[Landim-Mariani-Seo,2019]** rigorous result for the exit time of a domain by capacity approach. Only for double well potential and particular form of drift $b = J\nabla\phi$ with J antisymmetric.
- **[Hérau-Hitrik-Sjöstrand, 2011]** Results for the Kramers-Fokker-Planck equation. More difficult situation since it is hypoelliptic only. Uses supersymmetry and PT-symmetry in a crucial way.

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The labelling procedure I

For any $\mathbf{s} \in \mathcal{U}^{(1)}$ and $r > 0$ small enough, the set

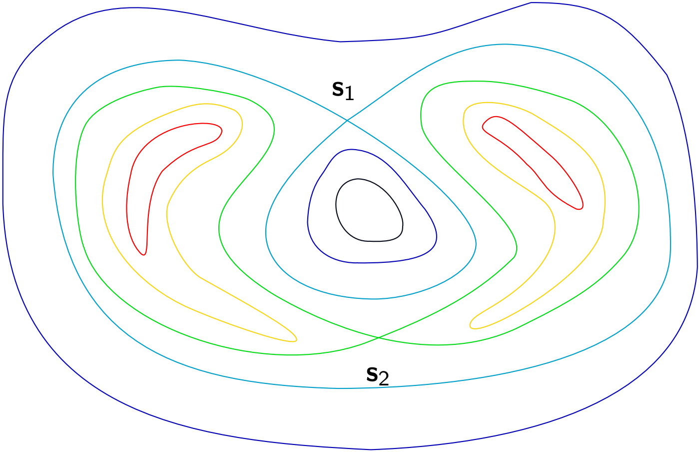
$$B(\mathbf{s}, r) \cap \{x \in X,\ \phi(x) < \phi(\mathbf{s})\}$$

has exactly two connected components $C_j(\mathbf{s}, r)$, $j = 1, 2$.

Definition (Hérau-Hitrik-Sjöstrand, 2011)

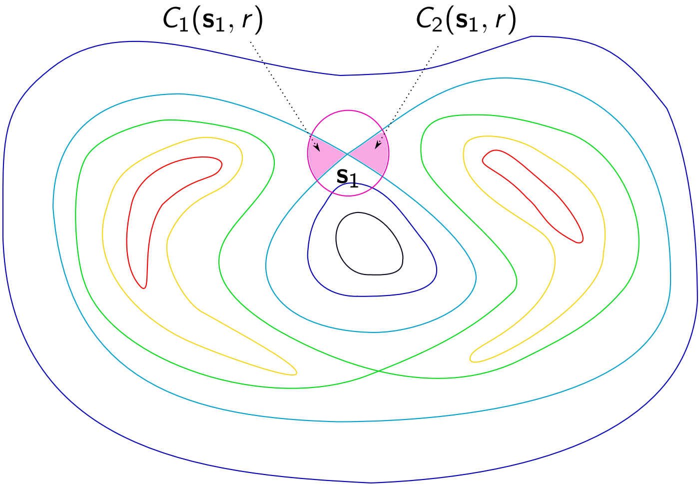
- $\mathbf{s} \in \mathcal{U}^{(1)}$ is a separating saddle point (ssp) iff $C_1(\mathbf{s}, r)$ and $C_2(\mathbf{s}, r)$ are contained in two different connected components of $\{x \in X,\ \phi(x) < \phi(\mathbf{s})\}$. We denote by $\mathcal{V}^{(1)}$ the set of ssp.
- $\sigma \in \mathbb{R}$ is a separating saddle value (ssv) if it is of the form $\sigma = \phi(\mathbf{s})$ with $\mathbf{s} \in \mathcal{V}^{(`1)}$. We denote $\underline{\Sigma} = \phi(\mathcal{V}^{(1)}) = \{\sigma_2 > \sigma_3 > \ldots > \sigma_N\}$.

Example of SSP I

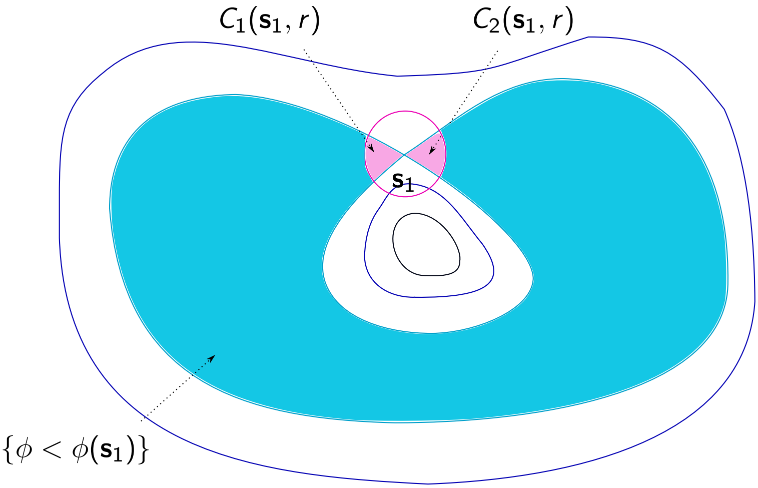


Level set of a potential with 2 minima, 2 saddle points and 1 maximum

Example of SSP II

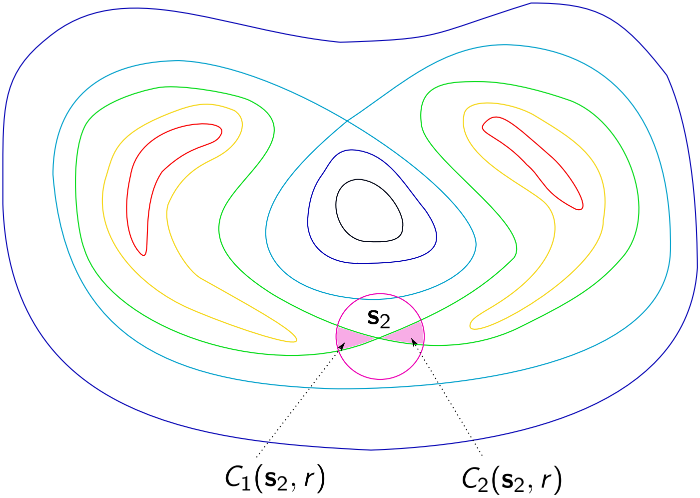


Example of SSP II

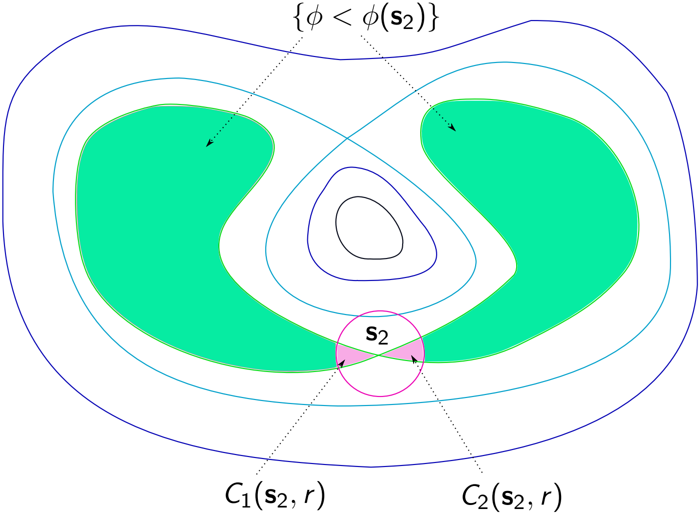


$\mathbf{s}_1$ is not separating

Example of SSP III



Example of SSP III



$\mathbf{s}_2$ is separating

The labelling procedure II

Add a fictive infinite saddle value $\sigma_1 = +\infty$ to $\underline{\Sigma}$ and let

$$\Sigma = \{\sigma_1\} \cup \underline{\Sigma} = \{\sigma_1 > \sigma_2 > \ldots > \sigma_N\}$$

- To $\sigma_1 = +\infty$ associate the unique connected component $E_{1,1} = X$ of $\{\phi < \sigma_1\}$. In $E_{1,1}$, pick up $m_{1,1}$ one (non necessarily unique) minimum of $\phi_{|E_{1,1}}$.
- The set $\{\phi < \sigma_2\}$ has finitely many connected components. One of them contains $m_{1,1}$. The others are denoted $E_{2,1}, \ldots, E_{2,N_2}$. In each of these CC, one choses one absolute minimum $m_{2,j}$ of $\phi_{|E_{2,j}}$.
- The set $\{\phi < \sigma_k\}$ has finitely many CC. One denotes by $E_{k,1}, \ldots, E_{k,N_k}$ those of these CC which do not contain any $m_{i,j}$, $i < k$. In each $E_{k,j}$ one choses one absolute minimum $m_{k,j}$ of $\phi_{|E_{k,j}}$.

The labelling procedure III

Denote $\underline{\mathbf{m}} = \mathbf{m}_{1,1}$ the absolute minimum of ϕ that was chosen at the first step of the labelling procedure, and let

$$\underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}.$$

Using the preceding labelling one constructs the following applications:

- $\boldsymbol{\sigma} : \mathcal{U}^{(0)} \to \Sigma$, defined by $\boldsymbol{\sigma}(\mathbf{m}_{i,j}) = \sigma_i$.
- $E(\mathbf{m})$ is the connected component of $\{\phi < \boldsymbol{\sigma}(\mathbf{m})\}$ that contains $\mathbf{m}$.
- $S(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}) - \phi(\mathbf{m})$

The Generic Assumption

The following hypothesis introduced by Hérau-Hitrik-Sjöstrand (2011) is a generalization of Helffer-Klein-Nier assumption (2004).

Generic Assumption (GA):

For all $\mathbf{m} \in \mathcal{U}^{(0)}$, the following hold true:

i) $\phi_{|E(\mathbf{m})}$ has a unique point of minimum

ii) if E is any connected component of $\{\phi < \sigma(\mathbf{m})\}$ and $\overline{E} \cap \mathcal{V}^{(1)} \neq \emptyset$, there exists a unique $\mathbf{s} \in \mathcal{V}^{(1)}$ such that $\phi(\mathbf{s}) = \sup \phi(\overline{E} \cap \mathcal{V}^{(1)})$.

Under this assumption, there exists a bijection

$$\mathfrak{s} : \mathcal{U}^{(0)} \to \mathcal{V}^{(1)} \cup \{\infty\}$$

such that $S(\mathbf{m}) = \phi(\mathfrak{s}(\mathbf{m})) - \phi(\mathbf{m})$ with the convention $\phi(\infty) = \infty$.

The simplified Generic Assumption

Simplified Generic Assumption:

The map

$$(\mathbf{m}, \mathbf{s}) \in \mathcal{U}^{(0)} \times \mathcal{V}^{(1)} \mapsto \phi(\mathbf{s}) - \phi(\mathbf{m})$$

is injective.

Consequence

For any $\mathbf{m} \in \mathcal{U}^{(0)}$, there exists a unique $\mathbf{s} \in \partial E(\mathbf{m}) \cap \mathcal{V}^{(1)}$ and the map

$$\mathfrak{s} : \mathcal{U}^{(0)} \to \mathcal{V}^{(1)} \cup \{\infty\}$$
$$\mathbf{m} \mapsto \mathbf{s}$$

is injective. Moreover, one has $S(\mathbf{m}) = \phi(\mathfrak{s}(\mathbf{m})) - \phi(\mathbf{m})$ with the convention $\phi(\infty) = \infty$.

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Let

$$\Pi_h = \frac{1}{2i\pi}\int_{|z|=\epsilon h}(P_\phi - z)^{-1}dz$$

and $E_h = \mathrm{Ran}\,\Pi_h$. Then $\dim E_h = n_0$ and $P_\phi : E_h \to E_h$.

Goal

Compute the spectrum of the restriction of P_ϕ to E_h. This is a problem in finite dimension.

The general strategy is the following:

1) Construct suitable approximated eigenfunctions $f_{\mathbf{m}}$, $\mathbf{m} \in \mathcal{U}^{(0)}$ of the operator P_ϕ
2) Project these eigenfunctions on E_h, $e_{\mathbf{m}} = \Pi_h f_{\mathbf{m}}$ and estimate the difference $e_{\mathbf{m}} - f_{\mathbf{m}}$.
3) Compute the matrix M_ϕ of P_ϕ in the base $(e_{\mathbf{m}}, \mathbf{m} \in \mathcal{U}^{(0)})$
4) Compute the spectrum of M_ϕ

Some comments

- The better the quasimodes are, the smaller $\|e_{\mathbf{m}} - f_{\mathbf{m}}\|$ is. Indeed:

$$e_{\mathbf{m}} - f_{\mathbf{m}} = \Pi_h f_m - f_m = \frac{1}{2i\pi}\int_{|z|=\epsilon h}((P_\phi - z)^{-1} - z^{-1}) f_{\mathbf{m}} dz$$
$$= \frac{-1}{2i\pi}\int_{|z|=\epsilon h}(P_\phi - z)^{-1} z^{-1} P_\phi f_{\mathbf{m}} dz$$

- Standard quasimodes $\tilde{f}_{\mathbf{m}} = \chi_{\mathbf{m}} e^{-(\phi - \phi(\mathbf{m}))/h}$ with $\chi_{\mathbf{m}}$ cut-off function in $E(\mathbf{m})$ yield

$$P_\phi \tilde{f}_{\mathbf{m}} = \mathcal{O}(e^{-(S(\mathbf{m})-\epsilon)/h})$$

- We need to construct accurate quasimodes
- The operator P_ϕ is non-self-adjoint, hence the matrix M_ϕ is not symmetric. We have to be careful of Jordan's block.

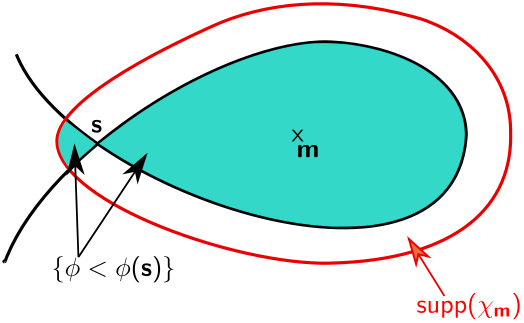
Construction of quasimodes

- Let $\mathbf{s} = \mathfrak{s}(\mathbf{m})$ the saddle point associated to $\mathbf{m}$ and let $\xi(\mathbf{s})$ be given by the geometric Lemma:
  - $\xi(\mathbf{s})$ is a unitary eigenvector associated to the unique negative eigenvalue $\mu(\mathbf{s})$ of $2\,\mathrm{Hess}(\phi)(\mathbf{s}) + db(\mathbf{s})$.
  - the matrix $\mathrm{Hess}\,\phi(\mathbf{s}) + |\mu(s)|\xi\xi^*$ is positive definite.
- Define the quasimode

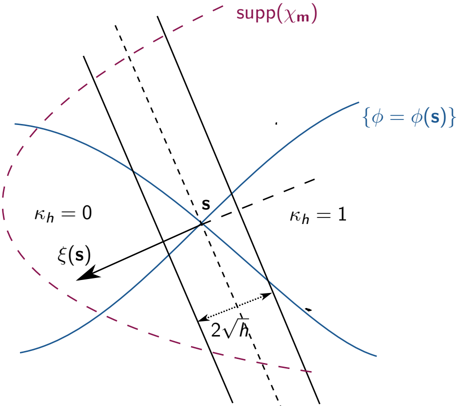
$$f_{\mathbf{m}}(x) = c_h^{-1}\kappa_h((x-\mathbf{s})\cdot\xi(\mathbf{s}))\chi_{\mathbf{m}}(x)e^{-(\phi(x)-\phi(\mathbf{m}))/h}$$

where c_h is a L^2-normalization constant and $\kappa_h : \mathbb{R} \to \mathbb{R}$ is a cut-off function such that
  - $\kappa_h(t) = \begin{cases} 0 \text{ if } t < -1 \\ 1 \text{ if } t > 1 \end{cases}$
  - $\frac{d\kappa_h}{dt}(t) = h^{-\frac{1}{2}}e^{-|\mu(\mathbf{s})|t^2/2h}$ si $t \in [-\frac{1}{2}, \frac{1}{2}]$.
  - $\mathrm{supp}(\kappa_h(.)\partial_x\chi) \subset \{\phi > \phi(\mathbf{s}) + \epsilon\}$.

The cutoff function $\chi_{\mathbf{m}}$



The cutoff function κ_h



Quasimodal estimates I

Lemma

Suppose that the generic assumption is satisfied and let $\mathbf{m} \neq \mathbf{m}'$ in $\mathcal{U}^{(0)}$. Then, either $\mathrm{supp}(f_m) \cap \mathrm{supp}(f_{\mathbf{m}'}) = \emptyset$ or $f_{\mathbf{m}} = 1$ on $\mathrm{supp}(f_{\mathbf{m}'})$ or $f_{\mathbf{m}'} = 1$ on $\mathrm{supp}(f_{\mathbf{m}})$. In particular, there exists $c > 0$ such that

$$\langle f_{\mathbf{m}}, f_{\mathbf{m}'} \rangle = \delta_{\mathbf{m}\mathbf{m}'} + \mathcal{O}(e^{-c/h}).$$

Lemma

For all $\mathbf{m} \in \mathcal{U}^{(0)}$, one has

$$\langle P_\phi f_{\mathbf{m}}, f_{\mathbf{m}} \rangle = h \frac{|\mu(\mathbf{s})|}{2\pi} \frac{D_{\mathbf{m}}}{D_{\mathbf{s}}} e^{-2S(\mathbf{m})/h}(1 + \mathcal{O}(h))$$

where $\mathbf{s} = \mathfrak{s}(\mathbf{m})$, $S(\mathbf{m}) = \phi(\mathbf{s}) - \phi(\mathbf{m})$, $D_x = |\det \mathrm{Hess}(\phi)(x)|^{\frac{1}{2}}$.

Quasimodal estimates II

- One has

$$\langle P_\phi f_{\mathbf{m}}, f_{\mathbf{m}}\rangle = \langle \Delta_\phi f_{\mathbf{m}}, f_{\mathbf{m}}\rangle = \langle d_\phi^* d_\phi f_{\mathbf{m}}, f_{\mathbf{m}}\rangle = \|d_\phi f_{\mathbf{m}}\|^2$$

  with

$$f_{\mathbf{m}}(x) = c_h^{-1}\chi_{\mathbf{m}}(x)\kappa_h((x-\mathbf{s})\xi)e^{-(\phi(x)-\phi(\mathbf{m}))/h}$$

  Hence

$$d_\phi f_{\mathbf{m}} = hc_h^{-1}\chi_{\mathbf{m}}(x)\kappa'_h((x-\mathbf{s})\xi)\xi e^{-(\phi(x)-\phi(\mathbf{m}))/h} + \mathcal{O}(e^{-(S(\mathbf{m})+\epsilon)/h})$$

- near $\mathbf{s}$, one has

$$\kappa'_h((x-\mathbf{s})\xi)e^{-(\phi(x)-\phi(\mathbf{m}))/h} = e^{-\langle(\frac{1}{2}\operatorname{Hess}\phi(\mathbf{s})+|\mu|\xi\xi^*)(x-\mathbf{s}),(x-s)\rangle+\mathcal{O}((x-\mathbf{s}}$$

  and $\frac{1}{2}\operatorname{Hess}\phi(\mathbf{s}) + |\mu|\xi\xi^*$ positive definite.
- Apply Laplace method to complete the computation.

Quasimodal estimates III

Lemma

For all $\mathbf{m} \in \mathcal{U}^{(0)}$, one has

$$\|P_\phi f_{\mathbf{m}}\|^2 = \langle P_\phi f_{\mathbf{m}}, f_{\mathbf{m}}\rangle \mathcal{O}(h^2)$$

$$\|P_\phi^* f_{\mathbf{m}}\|^2 = \langle P_\phi f_{\mathbf{m}}, f_{\mathbf{m}}\rangle \mathcal{O}(h)$$

Proof. One has

$$P_\phi(f_{\mathbf{m}}) = (-h^2\Delta + h(2\nabla\phi + b_h)\nabla)(\kappa_h) c_h^{-1}\chi(x) e^{-(\phi(x)-\phi(\mathbf{s}))/h} + \mathcal{O}(e^{-(S(\mathbf{m})+\epsilon)/h}).$$

and for x close to $\mathbf{s}$, one has

$$(-h^2\Delta + h(2\nabla\phi + b_h\nabla))(\kappa_h) = h((2\nabla\phi + b)\xi + |\mu|\xi(x-\mathbf{s}) + \mathcal{O}(h))e^{-}$$
$$= h\mathcal{O}((x-\mathbf{s})^2 + h)e^{-\frac{|\mu|(\xi(x-\mathbf{s}))^2}{2h}}$$

Graded structure of the interaction matrix

We enumerate the minima $\mathcal{U}^{(0)} = \{\mathbf{m}_1, \dots, \mathbf{m}_{n_0}\}$ in such way that the sequence $(S(\mathbf{m}_j))_j$ is non-decreasing. We denote

- $\tilde{\lambda}_j = \langle P_\phi f_{\mathbf{m}_j}, f_{\mathbf{m}_j} \rangle$
- $(e_{m_j})_{1 \leq j \leq n_0}$ the basis of E_h obtained from $\Pi_h f_{m_j}$ by Graam-Schmidt procedure.

Proposition

For all $j, k = 1, \dots, n_0$, one has

$$\langle P_\phi e_j, e_k \rangle = \delta_{jk} \tilde{\lambda}_j + \mathcal{O}(\sqrt{h \tilde{\lambda}_j \tilde{\lambda}_k})$$

- Let $\mathcal{M}_\phi = (\langle P_\phi e_j, e_k \rangle)_{j,k}$ be the matrix of P_ϕ in the basis (e_j).
- Let $\Omega = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})$, then

$$\mathcal{M}_\phi = \Omega(\mathrm{Id} + \mathcal{O}(\sqrt{h}))\Omega$$

Schur complement method for graded matrices

- Suppose that $\mathcal{M}_\phi = \Omega(\mathrm{Id} + \mathcal{O}(h))\Omega$ with Ω as above. We can compute the spectrum of $\mathcal{M}_\phi$ by Schur complement method.
- Computation for 2x2 matrices. Suppose

$$\mathcal{M}_\phi = \begin{pmatrix} \tilde{\lambda}_1 & B_h \\ B_h^* & \tilde{\lambda}_2 \end{pmatrix}$$

with $B_h = \mathcal{O}(\sqrt{h\tilde{\lambda}_1\tilde{\lambda}_2})$.

- The spectral values of $\mathcal{M}_\phi$ are the poles of

$$z \mapsto (\tilde{\lambda}_1 - z - B_h^*(\tilde{\lambda}_2 - z)^{-1}B_h)^{-1}$$

and

$$z \mapsto (\tilde{\lambda}_2 - z - B_h^*(\tilde{\lambda}_1 - z)^{-1}B_h)^{-1}$$