Small eigenvalues of Witten Laplacian: old and new

L. Michel

Université de Bordeaux

La Thuile
February 10-16 2019
Plan

1. Introduction
   - General framework
   - Motivations

2. Sharp Asymptotics of the small eigenvalues
   - The labelling procedure
   - Results in a generic case

3. The degenerate Case
   - A simple example
   - Gathering interacting minima
   - Asymptotics without assumption on the Morse function

4. Sketch of proofs
   - General strategy
   - Proof by example
1 Introduction
   • General framework
   • Motivations

2 Sharp Asymptotics of the small eigenvalues
   • The labelling procedure
   • Results in a generic case

3 The degenerate Case
   • A simple example
   • Gathering interacting minima
   • Asymptotics without assumption on the Morse function

4 Sketch of proofs
   • General strategy
   • Proof by example
Let \( X = \mathbb{R}^d \) or a compact manifold and let \( \phi : X \to \mathbb{R} \) be a smooth Morse function. Consider the semiclassical Witten Laplacian associated to \( \phi \):

\[
\Delta_\phi = -h^2 \Delta + |\nabla \phi|^2 - h \Delta \phi
\]

where \( h \in ]0, 1] \) denotes the semiclassical parameter. Assume there exists \( C > 0 \) and a compact \( K \subset \mathbb{R}^d \) such that for all \( x \in \mathbb{R}^d \setminus K \), one has

\[
|\nabla \phi(x)| \geq \frac{1}{C}, \quad |\text{Hess}(\phi(x))| \leq C|\nabla \phi|^2, \quad \text{and} \quad \phi(x) \geq C|x|.
\]
Under the preceding assumptions, one has the following properties on $\Delta \phi$.

- $\Delta \phi$ is essentially self-adjoint on $C^\infty_c(X)$.
- $\Delta \phi \geq 0$
- there exists $C_0, h_0 > 0$ such that for all $0 < h < h_0$

$$\sigma_{\text{ess}}(\Delta \phi) \subset [C_0, \infty[$$

- $0$ is an eigenvalue of $\Delta \phi$ associated to the eigenstate $e^{-\phi/h}$.

**Goal:**

Study the small eigenvalues of $\Delta \phi$. 
Brownian Dynamics

Consider a Brownian Particle $x_t$ in a force field $-\nabla \phi(x)$ in a low temperature regime. Its movement is driven by the overdamped Langevin equation

$$\dot{x}_t = -2\nabla \phi(x_t) + \sqrt{2h}\dot{B}_t$$

where $B_t$ is the brownian motion. At a macroscopic level, the probability $\rho(t, x)$ of presence of the particle in position $x$ at time $t$ satisfies the Kramers-Schmoluchovsky equation:

$$\partial_t \rho = h \text{div} \circ (h\nabla + 2\nabla \phi)(\rho).$$

Change of unknown $\tilde{\rho} = e^{\phi/h} \rho$ yields

$$\partial_t \tilde{\rho} + \Delta \phi \tilde{\rho} = 0$$

The behavior of $\tilde{\rho}$ when $t \to \infty$ is driven by the eigenvalues of $\Delta \phi$. Eigenvalues which are exponentially close to 0 are associated with the so-called metastable states.
Introduce the Hodge Laplacian on $X$:

$$\Delta = d^* \circ d + d \circ d^*$$

where $d : \Omega^p(X) \to \Omega^{p+1}(X)$ denotes the exterior derivative from $p$-forms into $p+1$ forms. The Betti numbers are defined by

$$b_p(X) := \dim(\ker(d : \Omega^p \to \Omega^{p+1})) / \dim(\text{Ran}(d : \Omega^{p-1} \to \Omega^p))$$

**Hodge Theorem:**

For all $p = 0, \ldots, d$, one has $b_p(X) = \dim \ker \Delta^{(p)}$ with

$$\Delta^{(p)} = \Delta|_{\Omega^{(p)}(X)}.$$
Denote

- $\mathcal{U}$ the set of critical points of $\phi$ (since $\phi$ is a Morse function, then $\mathcal{U}$ is finite).
- $\mathcal{U}^{(p)}$ the set of critical points of $\phi$ of index $p$
- $n_p = \# \mathcal{U}^{(p)}$.

Hence $\mathcal{U}^{(0)}$ is the set of minima and $\mathcal{U}^{(1)}$ the set of saddle points of $\phi$.

**Theorem: Weak Morse Inequalities**

For all $p = 0, \ldots, d$, one has $n_p(\phi) \geq b_p(X)$. 
Witten Laplacian on forms

Witten’s idea was to introduce the operator

$$\Delta_{\phi} = d_{\phi}^* \circ d_{\phi} + d_{\phi} \circ d_{\phi}^*$$

where $h > 0$ is a parameter and $d_{\phi} : \Omega^p(X) \to \Omega^{p+1}(X)$ denotes the twisted exterior derivative

$$d_{\phi} = e^{-\phi/h} \circ hd \circ e^{\phi/h} = hd + d_{\phi}^\wedge.$$  

Fact:

For any $p = 0, \ldots, d$, $\dim \ker \Delta_{\phi}^{(p)} = b_p(X)$. 

One has $\ker(d_{\phi}) = e^{-\phi/h} \ker(d)$ and $\operatorname{ran}(d_{\phi}) = e^{-\phi/h} \operatorname{ran}(d)$. 

Hence

$$b_p(X) = \dim \ker d^{(p)}/\operatorname{ran} d^{(p-1)} = \dim \ker d_{\phi}^{(p)}/\operatorname{ran} d_{\phi}^{(p-1)}$$

$$= \dim \ker \Delta_{\phi}^{(p)}$$
Theorem (Witten 82, Helffer-Sjöstrand 84):

There exists \( \epsilon_0, h_0 > 0 \) such that for all \( 0 < h < h_0 \) and all \( p = 0, \ldots, d \), one has

\[
\# \sigma(\Delta_\phi^{(p)}) \cap [0, \epsilon_0 h] = n_p(\phi).
\]

Consequence:

\[
\dim \text{Ker} \Delta_\phi^{(p)} \leq n_p(\phi).
\]

Proof for \( p = 0 \):

- **Lower bound:** use the quasimodes

\[
f_m^{(0)} = c_m h^{-\frac{d}{4}} \chi_m(x) e^{(\phi(m) - \phi(x))/h}.
\]
Analytic proof of Morse inequalities

**Upper bound:** On 0-forms, one has

\[ \Delta \phi = -h^2 \Delta + |\nabla \phi|^2 - h \Delta \phi \]

- away from critical points, \( \Delta \phi \) is elliptic.
- near critical points of index \( p \), one has

\[ \phi(x) \sim \frac{1}{2}((x')^2 - (x'')^2) \] with \( x = (x', x'') \in \mathbb{R}^{d-p} \times \mathbb{R}^p \)

and

\[ \Delta \phi \sim -h^2 \Delta + |x|^2 - h(d - 2p) := N \]

Since

\[ \sigma(-h^2 \Delta + |x|^2) = \{ h \sum_{i=1}^d n_i, n_i \in \mathbb{N}^* \} \]

then \( 0 \in \sigma(N) \iff p = 0. \)

- This permit to find a \( n_0 \) dimensional vector space \( E_0 \) such that \( \Delta \phi \geq \epsilon h \) on \( E_0 \).
1 Introduction
   • General framework
   • Motivations

2 Sharp Asymptotics of the small eigenvalues
   • The labelling procedure
   • Results in a generic case

3 The degenerate Case
   • A simple example
   • Gathering interacting minima
   • Asymptotics without assumption on the Morse function

4 Sketch of proofs
   • General strategy
   • Proof by example
Some remarks about the small eigenvalues

- It is easy to see that the $n_0$ small eigenvalues of $\Delta^{(0)}_\phi$ are actually $O(e^{-C/h})$ for some $C > 0$.
- One sees that $C$ is related to the heights $\phi(s) - \phi(m)$, $s \in U^{(1)}$, $m \in U^{(0)}$. Compute the constant $C$ associated to each eigenvalue is not totally clear.
- First step is to identify which heights are relevant for this problem.
- First result in this direction are due to Bovier-Gayrard-Klein 04 (probabilistic approach) and Helffer-Klein-Nier 04 (PDE approach).
- The first step is the following labelling procedure.
The labelling procedure

The labelling procedure I

For any \( s \in U^{(1)} \) and \( r > 0 \) small enough, the set

\[
B(s, r) \cap \{ x \in X, \phi(x) < \phi(s) \}
\]

has exactly two connected components \( C_j(s, r), j = 1, 2. \)

Definition (Hérau-Hitrik-Sjöstrand, 2011)

- \( s \in U^{(1)} \) is a separating saddle point (ssp) iff \( C_1(s, r) \) and \( C_2(s, r) \) are contained in two different connected components of \( \{ x \in X, \phi(x) < \phi(s) \} \). We denote by \( V^{(1)} \) the set of ssp.
- \( \sigma \in \mathbb{R} \) is a separating saddle value (ssv) if it is of the form \( \sigma = \phi(s) \) with \( s \in V^{(1)} \). We denote \( \Sigma = \phi(V^{(1)}) = \{ \sigma_2 > \sigma_3 > \ldots > \sigma_N \} \).
Example of SSP 1

Level set of a potential with 2 minima, 2 saddle points and 1 maximum
The labelling procedure

Example of SSP II

\[ C_1(s_1, r) \]

\[ C_2(s_1, r) \]
The labelling procedure

Example of SSP II

\[ C_1(s_1, r) \]
\[ C_2(s_1, r) \]

\[ \{ \phi < \phi(s_1) \} \]

\[ s_1 \text{ is not separating} \]
Example of SSP III

\[ s_2 \]

\[ C_1(s_2, r) \quad C_2(s_2, r) \]
The labelling procedure

Example of SSP III

\[ \{ \phi < \phi(s_2) \} \]

\[ C_1(s_2, r) \]

\[ C_2(s_2, r) \]

s_2 is separating
The labelling procedure II

Add a fictive infinite saddle value $\sigma_1 = +\infty$ to $\Sigma$ and let

$$\Sigma = \{\sigma_1\} \cup \Sigma = \{\sigma_1 > \sigma_2 > \ldots > \sigma_N\}$$

- To $\sigma_1 = +\infty$ associate the unique connected component $E_{1,1} = X$ of $\{\phi < \sigma_1\}$. In $E_{1,1}$, pick up $m_{1,1}$ one (not necessarily unique) minimum of $\phi|_{E_{1,1}}$.

- The set $\{\phi < \sigma_2\}$ has finitely many connected components. One of them contains $m_{1,1}$. The others are denoted $E_{2,1}, \ldots, E_{2,N_2}$. In each of these CC, one chooses one absolute minimum $m_{2,j}$ of $\phi|_{E_{2,j}}$.

- The set $\{\phi < \sigma_k\}$ has finitely many CC. One denotes by $E_{k,1}, \ldots, E_{k,N_k}$ those of these CC which do not contain any $m_{i,j}$, $i < k$. In each $E_{k,j}$ one chooses one absolute minimum $m_{k,j}$ of $\phi|_{E_{k,j}}$. 
Denote $\underline{m} = m_{1,1}$ the absolute minimum of $\phi$ that was chosen at the first step of the labelling procedure, and let

$$\mathcal{U}^{(0)} = \mathcal{U}^{(0)} \setminus \{\underline{m}\}.$$

Let $\mathcal{O}(X)$ denote the connected open subsets of $X$. Using the preceding labelling one constructs the following applications:

- $\sigma : \mathcal{U}^{(0)} \to \Sigma$, defined by $\sigma(m_{i,j}) = \sigma_i$.
- $E : \mathcal{U}^{(0)} \to \mathcal{O}(X)$, defined by $E(m_{i,j}) = E_{i,j}$.
- $S = \sigma - \phi$
The following hypothesis introduced by Hérau-Hitrik-Sjöstrand (2011) is a generalization of Helffer-Klein-Nier assumption (2004).

**Generic Assumption (GA):**

For all $m \in U^{(0)}$, the following hold true:

1. $\phi|_{E(m)}$ has a unique point of minimum
2. if $\overline{E(m)} \cap V^{(1)} \neq \emptyset$, there exists a unique $s \in V^{(1)}$ such that $\phi(s) = \sup \phi(\overline{E(m)} \cap V^{(1)})$.

Under this assumption, there exists a bijection

$$s : U^{(0)} \rightarrow V^{(1)} \cup \{\infty\}$$

such that $S(m) = \phi(s(m)) - \phi(m)$ with the convention $\phi(\infty) = \infty$. 
The Generic case II

Let us write $\lambda(m, h), m \in U^{(0)}$ the $n_0$ small eigenvalues of $\Delta \phi$.


Suppose the the Generic Assumption is satisfied. Then the $n_0$ small eigenvalues of $\Delta \phi$ satisfy

$$\lambda(m, h) = h \zeta(m, h) e^{-2S(m)/h}$$

where $\zeta(m, h) \sim \sum_{r=0}^{\infty} h^r \zeta_r(m)$ and

$$\zeta_0(m) = \pi^{-1} |\mu(s(m))| \sqrt{\frac{|\det \phi''(m)|}{|\det \phi''(s(m))|}}$$

where $\mu(s)$ is the unique negative eigenvalue of $\phi''$ in $s$. 
1 Introduction
   - General framework
   - Motivations

2 Sharp Asymptotics of the small eigenvalues
   - The labelling procedure
   - Results in a generic case

3 The degenerate Case
   - A simple example
   - Gathering interacting minima
   - Asymptotics without assumption on the Morse function

4 Sketch of proofs
   - General strategy
   - Proof by example
A simple example

Suppose that the following hypothesis are verified:

- The set of minimal values is reduced to one point:
  \[ \exists c_0, \forall m \in U^{(0)}, \phi(m) = c_0 \]

- The set of saddle values is reduced to one point:
  \[ \exists c_1, \forall m \in U^{(1)}, \phi(m) = c_1 \]

**Figure 1:** The sublevel set \( \{ \varphi < \sigma \} \) (dashed region) associated to a potential \( \varphi \) satisfying the assumptions. The x’s represent local minima, the o’s, local maxima.
Theorem

The $n_0$ small eigenvalues of $\Delta \phi$ satisfy $\lambda_1 = 0$ and for all $k = 2, \ldots, n_0$, 

$$\lambda_k(h) = h \zeta_k(h)e^{-2S/h}$$

where $S = c_1 - c_0$ and 

$$\zeta_k(h) \sim \sum_{r=0}^{\infty} h^r \zeta_{k,r}$$

and $\zeta_{k,0}$ are the non zero eigenvalues of the weighted graph $G$ defined by

- The vertices of the graph are the minima $m \in \hat{U}(0)$.
- The edges between two vertices $m, m'$ are the saddle points $s \in U(1)$ such that $s \in \overline{E}(m) \cap \overline{E}(m')$.
- The weights explicitly depend on the values of $\phi''$ on $U(0)$ and $U(1)$. 

The degenerate Case

Sketch of proofs
Figure 2: The sublevel set $\{ \varphi < \sigma \}$ (dashed region) associated to a potential $\varphi$ satisfying the assumptions. The x’s represent local minima, the o’s, local maxima.

Figure 3: The graph associated to the potential represented in Figure ??
For any $m \in \mathcal{U}^{(0)}$, let

- $G(m)$ denotes the connected component of $\{\phi \leq \sigma(m)\}$ that contains $m$.

Fact:

For any $m \neq \hat{m}$, there exists a unique $\hat{m} = \hat{m}(m) \in G(m) \cap \mathcal{U}^{(0)}$ such that $\sigma(\hat{m}) > \sigma(m)$

We denote by $\hat{E}(m)$ the connected component of $\{\phi < \sigma(m)\}$ that contains $\hat{m}(m)$. This defines two applications

$$\hat{m} : \mathcal{U}^{(0)} \to \mathcal{U}^{(0)} \text{ and } \hat{E} : \mathcal{U}^{(0)} \to \mathcal{O}(X)$$
Two different types of minima

Observe that by definition, we have

\[ \forall m \in U^{(0)}, \, \phi(\hat{m}(m)) \leq \phi(m). \]

The fact that the above inequality is large or strict plays an important role in our analysis.

**Definition**

Let \( m \in U^{(0)} \). We say that \( m \) is of type I if \( \phi(\hat{m}(m)) < \phi(m) \). If \( \phi(\hat{m}(m)) = \phi(m) \), we say that \( m \) is of type II. We will denote

\[ U^{(0)},I = \{ m \in U^{(0)}, \, m \text{ is of type I} \} \]

\[ U^{(0)},II = \{ m \in U^{(0)}, \, m \text{ is of type II} \} \]

We have clearly the following disjoint union \( U^{(0)} = U^{(0)},I \cup U^{(0)},II \).
Gathering interacting minima

An equivalence relation on $\mathcal{U}^{(0)}$

For $\sigma \in \Sigma$, let $\Omega_{\sigma}$ be defined by

$$\Omega_{\sigma} = \{E(m), m \in \sigma^{-1}(\sigma)\} \bigcup \{\hat{E}(m), m \in \sigma^{-1}(\sigma) \cap \mathcal{U}^{(0);II}\}$$

**Definition:**

We define an equivalence relation $\mathcal{R}$ on $\mathcal{U}^{(0)}$ by $m \mathcal{R} m'$ if and only if the two following properties hold true

- $\sigma(m) = \sigma(m') = \sigma$
- $m$ and $m'$ belong to the same connected component of $\bigcup_{\omega \in \Omega_{\sigma}} \overline{\omega}$.

We denote by $\text{Cl}(m)$ the equivalence class of any $m \in \mathcal{U}^{(0)}$ and by $(\mathcal{U}^{(0)}_\alpha)_{\alpha \in \mathcal{A}} = \mathcal{U}^{(0)}/\mathcal{R}$.
**Main Theorem**

For any $\alpha \in \mathcal{A}$, denote $S\alpha = S(U^{(0)}_\alpha)$ and $p(\alpha) = \#S\alpha$ and

$$S\alpha = \{S_{\nu^\alpha_1}, \ldots, S_{\nu^\alpha_{p(\alpha)}}\}$$

for some integers $\nu^\alpha_1 < \nu^\alpha_2 < \ldots < \nu^\alpha_{p(\alpha)}$.

**Theorem**

There exist $c > 0$ and some symmetric positive definite matrices $M^\alpha$, $\alpha \in \mathcal{A}$ such that counted with multiplicity, one has

$$\sigma(\Delta_\phi) \cap [0, \epsilon_0 h] = \bigcup_{\alpha \in \mathcal{A}} \sigma(M^\alpha)(1 + O(e^{-c/h}))$$

with

$$\sigma(M^\alpha) = \bigcup_{j=1}^{p(\alpha)} he^{-2h^{-1}S_{\nu^\alpha_j}} \sigma(M^{\alpha,j})$$

for some symmetric positive definite matrices $M^{\alpha,j}$ having a classical expansion with explicit invertible leading term.
The way to construct the matrices $M_\alpha$ depends on
- the number of equivalence class of $R$
- the number $p(\alpha)$ of values taken by $\varphi$ on each equivalence class $U^{(0)}_\alpha$.

If there is only one equivalence class $U^{(0)}_{\alpha_0}$ and if $p(\alpha_0) = 1$ then we are in the case where $M_{\alpha_0}$ is a graph Laplacian.
If $p(\alpha_0) \geq 2$, the situation is more complicated.
1 Introduction
   - General framework
   - Motivations

2 Sharp Asymptotics of the small eigenvalues
   - The labelling procedure
   - Results in a generic case

3 The degenerate Case
   - A simple example
   - Gathering interacting minima
   - Asymptotics without assumption on the Morse function

4 Sketch of proofs
   - General strategy
   - Proof by example
Finite dimensional reduction

The general strategy of Helffer-Klein-Nier is the following:

- Introduce
  - $F^{(0)} = \text{eigenspace associated to the } n_0 \text{ low lying eigenvalues on } 0\text{-forms}$
  - $\Pi^{(0)} = \text{projector on } F^{(0)}$
  - $M = \text{restriction of } \Delta_{\phi} \text{ to } F^{(0)}$

We have to compute the eigenvalues of $M$.

- We compute suitable BKW approximated eigenfunctions $f^{(0)}_m$ indexed by $m \in U^{(0)}$, and show that

$$\Pi^{(0)} f^{(0)}_m = f^{(0)}_m + \text{error}$$

and compute the matrix of $M$ in the base $\Pi^{(0)} f^{(0)}_m$.

- Doing that leads to error terms which are too big.
- In order to overcome this difficulty, they use the supersymmetric structure.
Using Supersymmetry

The fundamental remarks are the following:

- $\Delta^{(p+1)} d^{(p)} = d^{(p)} \Delta^{(p)}$ and $d^{(p),*} \Delta^{(p+1)} = \Delta^{(p)} d^{(p),*}$

- Denote $F^{(1)}$ the eigenspace associated to low lying eigenvalues on 1 forms, then $d^{(0)}(F^{(0)}) \subset F^{(1)}$ and $d^{(0),*}(F^{(1)}) \subset F^{(0)}$. Hence

$$M = L^* L$$

where $L$ is the matrix of $d^{(0)} : F^{(0)} \to F^{(1)}$.

- The matrix $L$ is well approximated by

$$L \simeq \mathcal{L} := \langle \langle d^{(0)} f^{(0)}_m, f^{(1)}_s \rangle \rangle_{s \in U^{(1)} m \in U^{(0)}}$$

where $f^{(1)}_s$ are BKW approximated eigenfunctions on 1-form.
Form of the quasimodes

- **On 0-forms**, one takes
  \[ f_m^{(0)} = \sum_{m' \in \text{Cl}(m)} \theta_m(m') g_{m'}^{(0)} \]

  \[ g_{m'}^{(0)} = h^{-\frac{d}{4}} \chi_{m'}(x) e^{-\left(\phi(x) - \phi(m')\right)/h} \]

  and \( \chi_{m'} \simeq 1_{E(m')} \).

- **On 1-forms**, one takes
  \[ f_s^{(1)} = h^{-\frac{d}{4}} \chi_s(x) b_s(x, h) e^{-\left(\phi_+(x) - \phi(s)\right)/h} \]

  with \( \chi_s \) cut-off function near \( s \), \( \phi_+ \) phase function solving the eikonal equation
  \[ |\nabla \phi_+|^2 = |\nabla \phi|^2 \]

  and \( b_s(x, h) \) a 1 form obtained by solving some transport equations.
Computation of the singular values

- One has $\mathcal{L} = \hat{\mathcal{L}} \mathcal{T}$ with $\mathcal{T} = (\theta_m(m'))_{m,m'}$ an orthogonal matrix and
  
  $$\hat{\mathcal{L}} = (\langle d^{(0)}_\phi g^{(0)}_m, f^{(1)}_s \rangle)_{s \in \mathcal{U}^{(1)}, m \in \mathcal{U}^{(0)}}$$

- The computation of the coefficients of $\hat{\mathcal{L}}$ is performed by Laplace Method.

- **Under the Generic Assumption (GA),** the matrix $\hat{\mathcal{L}}$ is diagonal. Its singular values are then given by its diagonal coefficients.

- In the general case, $\hat{\mathcal{L}}$ is only block-diagonal
  
  $$\hat{\mathcal{L}} = \text{diag}(\hat{\mathcal{L}}^\alpha, \alpha \in \mathcal{A})$$

  where each block $\mathcal{L}^\alpha$ corresponds to an equiv. class of $\mathcal{R}$.

- If $p(\alpha) = 1$, each block has a typical size $e^{-S_\alpha/h}$.

- If $p(\alpha) \geq 2$, we can perform a Schur type argument on each block.
Computations in dimension when $p(\alpha) = 1$

Suppose $\phi$ is given by

\[ \phi(m_{2,3}) = \phi(m_{2,1}) = \phi(m_{2,2}) \]

Figure 4: A potential satisfying H(1)
The case $p(\alpha) = 1$ continued

The interaction matrix is

$$
\hat{\mathcal{L}} = \begin{pmatrix}
    s_1 & m_{1,1} & m_{2,1} & m_{2,2} & m_{2,3} \\
    0 & e^{-S_2/h} & -e^{-S_2/h} & 0 \\
    0 & 0 & e^{-S_2/h} & 0 \\
    0 & 0 & 0 & e^{-S_3/h}
\end{pmatrix}
$$

This block structure implies:

$$
SV(\hat{\mathcal{L}}) = \{0\} \cup e^{-S_2/h} SV \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cup \{e^{-S_3/h}\}
$$
Computations in dimension one when $p(\alpha) = 2$

Suppose that $\phi$ is given by

$\sigma_1 = \infty$

$\sigma_2$

$\phi(m_{2,3})$

$\phi(m_{2,1}) = \phi(m_{2,2})$

$m_{2,1}$

$m_{2,2}$

$m_{2,3}$

$s_1$

$s_2$

$s_3$

$S_2$

$S_3$

$S_3$

$S_2$

$m_{1,1} = \hat{m}$

**Figure 5:** A potential with $p(\alpha) = 2$
The interaction matrix is

\[
\begin{pmatrix}
    m_{1,1} & m_{2,1} & m_{2,2} & m_{2,3} \\
    s_1 & 0 & e^{-(S_2-S_3)/h} & -e^{-(S_2-S_3)/h} & 0 \\
    s_2 & 0 & 0 & e^{-(S_2-S_3)/h} & -1 \\
    s_3 & 0 & 0 & 0 & 1 
\end{pmatrix}
\]

\(\hat{\mathcal{L}} = e^{-S_3/h}\)

This is not a block-diagonal matrix, but \(\mathcal{M} := \hat{\mathcal{L}}^* \hat{\mathcal{L}}\) has the form:

\[
\begin{pmatrix}
    0 & 0 & 0 \\
    0 & e^2A & \epsilon B^* \\
    0 & \epsilon B & J
\end{pmatrix}
\]

with \(\epsilon = e^{-(S_2-S_3)/h} \ll 1\). So we can use Schur complement method to compute its eigenvalues. We get

\[
\sigma(\mathcal{M}) = e^{-2S_3/h}(\sigma(J) + O(e^{-2(S_2-S_3)/h})) \\
\cup e^{-2S_2/h}(\sigma(A - B^* J^{-1} B) + O(e^{-2(S_2-S_3)/h})) \cup \{0\}
\]
The general case

- Block-diagonalize the interaction matrix:
  \[ \hat{\mathcal{L}} = \text{diag}(\hat{\mathcal{L}}^\alpha, \alpha \in A) \]
  where each block \( \mathcal{L}^\alpha \) corresponds to an equiv. class of \( \mathcal{R} \).

- Observe that \( \mathcal{M}_\alpha := (\mathcal{L}^\alpha)^* \mathcal{L}^\alpha \) has a nice structure:
  \[ \mathcal{M}_\alpha = \Omega_\alpha \tilde{\mathcal{M}}_\alpha \Omega_\alpha \]
  with \( \mathcal{M}_\alpha \) "independent" of \( h \) and

  \[ \Omega_\alpha = \begin{pmatrix}
  l_{rp} & 0 & \ldots & \ldots & 0 \\
  0 & \tau_2 l_{rp-1} & 0 & \ldots & 0 \\
  \vdots & 0 & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & 0 \\
  0 & \ldots & \ldots & 0 & \tau_2 \tau_3 \ldots \tau_p l_{r1}
\end{pmatrix} \]

  where \( \tau_j = e^{(S_{\nu_{p-(j-2)}} - S_{\nu_{p-(j-1)}})/h} \) for any \( j = 2, \ldots, p \) and

  \[ p = p(\alpha) \]
  is the number of values taken by \( \phi \) on \( \mathcal{U}^{(0)}_\alpha \).

- Apply Schur complement’s method and induction on \( p(\alpha) \).