Small eigenvalues of Witten Laplacian: old and new

L. Michel

Université de Bordeaux

La Thuile
February 10-16 2019

Plan

Semiclassical Witten Laplacian

Let $X = \mathbb{R}^d$ or a compact manifold and let $\phi : X \to \mathbb{R}$ be a smooth Morse function. Consider the semiclassical Witten Laplacian associated to ϕ:

$$\Delta_\phi = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$$

where $h \in]0,1]$ denotes the semiclassical parameter. Assume there exists $C > 0$ and a compact $K \subset \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d \setminus K$, one has

$$|\nabla\phi(x)| \geq \frac{1}{C}, \; |\mathrm{Hess}(\phi(x))| \leq C|\nabla\phi|^2, \text{ and } \phi(x) \geq C|x|.$$

Elementary properties

Under the preceding assumptions, one has the following properties on Δ_ϕ.

- Δ_ϕ is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$.
- $\Delta_\phi \geq 0$
- there exists $C_0, h_0 > 0$ such that for all $0 < h < h_0$

$$\sigma_{ess}(\Delta_\phi) \subset [C_0, \infty[$$

- 0 is an eigenvalue of Δ_ϕ associated to the eigenstate $e^{-\phi/h}$.

Goal:

Study the small eigenvalues of Δ_ϕ.

Brownian Dynamics

Consider a Brownian Particle x_t in a force field $-\nabla\phi(x)$ in a low temperature regime. Its movement is driven by the overdamped Langevin equation

$$\dot{x}_t = -2\nabla\phi(x_t) + \sqrt{2h}\dot{B}_t$$

where B_t is the brownian motion. At a macroscopic level, the probability $\rho(t,x)$ of presence of the particle in position x at time t satisfies the Kramers-Schmoluchovsky equation:

$$\partial_t\rho = h\,\mathrm{div}\circ(h\nabla + 2\nabla\phi)(\rho).$$

Change of unknown $\tilde{\rho} = e^{\phi/h}\rho$ yields

$$\partial_t\tilde{\rho} + \Delta_\phi\tilde{\rho} = 0$$

The behavior of $\tilde{\rho}$ when $t\to\infty$ is driven by the eigenvalues of Δ_ϕ. Eigenvalues which are exponentially close to 0 are associated with the so-called metastable states.

Analytic proof of Morse inequalities

Introduce the Hodge Laplacian on X:

$$\Delta = d^* \circ d + d \circ d^*$$

where $d : \Omega^p(X) \to \Omega^{p+1}(X)$ denotes the exterior derivative from p-forms into $p+1$ forms. The Betti numbers are defined by

$$b_p(X) := \dim(\mathrm{Ker}(d : \Omega^p \to \Omega^{p+1}) / \mathrm{Ran}(d : \Omega^{(p-1)} \to \Omega^p))$$

Hodge Theorem:

For all $p = 0, \ldots, d$, one has $b_p(X) = \dim \mathrm{Ker}\, \Delta^{(p)}$ with $\Delta^{(p)} = \Delta_{|\Omega^{(p)}(X)}$.

The Morse inequalities

Denote

- $\mathcal{U}$ the set of critical points of ϕ (since ϕ is a Morse function, then $\mathcal{U}$ is finite).
- $\mathcal{U}^{(p)}$ the set of critical points of ϕ of index p
- $n_p = \sharp\mathcal{U}^{(p)}$.

Hence $\mathcal{U}^{(0)}$ is the set of minima and $\mathcal{U}^{(1)}$ the set of saddle points of ϕ.

Theorem: Weak Morse Inequalities

For all $p = 0, \ldots, d$, one has $n_p(\phi) \geq b_p(X)$.

Witten Laplacian on forms

Witten's idea was to introduce the operator

$$\Delta_\phi = d_\phi^* \circ d_\phi + d_\phi \circ d_\phi^*$$

where $h > 0$ is a parameter and $d_\phi : \Omega^p(X) \to \Omega^{p+1}(X)$ denotes the twisted exterior derivative

$$d_\phi = e^{-\phi/h} \circ hd \circ e^{\phi/h} = hd + d\phi^\wedge.$$

Fact:

For any $p = 0, \ldots, d$, $\dim \operatorname{Ker} \Delta_\phi^{(p)} = b_p(X)$.

One has $\operatorname{Ker}(d_\phi) = e^{-\phi/h} \operatorname{Ker}(d)$ and $\operatorname{Ran}(d_\phi) = e^{-\phi/h} \operatorname{Ran}(d)$. Hence

$$b_p(X) = \dim \operatorname{Ker} d^{(p)} / \operatorname{Ran} d^{(p-1)} = \dim \operatorname{Ker} d_\phi^{(p)} / \operatorname{Ran} d_\phi^{(p-1)}$$
$$= \dim \operatorname{Ker} \Delta_\phi^{(p)}$$

Analytic proof of Morse inequalities

Theorem (Witten 82, Helffer-Sjöstrand 84):

There exists $\epsilon_0, h_0 > 0$ such that for all $0 < h < h_0$ and all $p = 0, \dots, d$, one has

$$\sharp \sigma(\Delta_\phi^{(p)}) \cap [0, \epsilon_0 h] = n_p(\phi).$$

Consequence:

$\dim \mathrm{Ker}\, \Delta_\phi^{(p)} \leq n_p(\phi)$.

Proof for $p = 0$:

- **Lower bound:** use the quasimodes

$$f_{\mathbf{m}}^{(0)} = c_{\mathbf{m}} h^{-\frac{d}{4}} \chi_{\mathbf{m}}(x) e^{(\phi(\mathbf{m}) - \phi(x))/h}.$$

Analytic proof of Morse inequalities

- **Upper bound:** On 0-forms, one has

$$\Delta_\phi = -h^2\Delta + |\nabla\phi|^2 - h\Delta\phi$$

  - away from critical points, Δ_ϕ is elliptic.
  - near critical points of index p, one has

$$\phi(x) \sim \frac{1}{2}((x')^2 - (x'')^2) \text{ with } x = (x', x'') \in \mathbb{R}^{d-p} \times \mathbb{R}^p$$

    and

$$\Delta_\phi \sim -h^2\Delta + |x|^2 - h(d-2p) := N$$

    Since

$$\sigma(-h^2\Delta + |x|^2) = \{h\sum_{i=1}^{d} n_i,\ n_i \in \mathbb{N}^*\}$$

    then $0 \in \sigma(N) \iff p = 0$.
  - This permit to find a n_0 dimensional vector space E_0 such that $\Delta_\phi \geq \epsilon h$ on $E_0^\perp$.

Some remarks about the small eigenvalues

- It is easy to see that the n_0 small eigenvalues of $\Delta_{\phi}^{(0)}$ are actually $\mathcal{O}(e^{-C/h})$ for some $C > 0$.
- One sees that C is related to the heights $\phi(\mathbf{s}) - \phi(\mathbf{m})$, $\mathbf{s} \in \mathcal{U}^{(1)}$, $\mathbf{m} \in \mathcal{U}^{(0)}$. Compute the constant C associated to each eigenvalue is not totally clear.
- First step is to identify which heights are relevant for this problem.
- First result in this direction are due to Bovier-Gayrard-Klein 04 (probabilistic approach) and Helffer-Klein-Nier 04 (PDE approach).
- The first step is the following labelling procedure.

The labelling procedure I

For any $\mathbf{s} \in \mathcal{U}^{(1)}$ and $r > 0$ small enough, the set

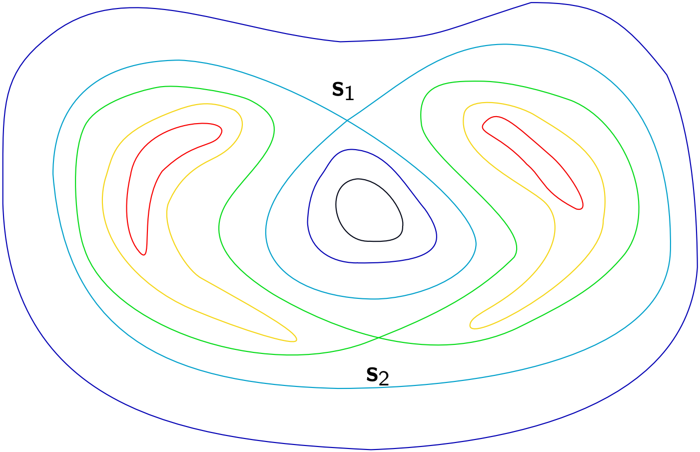
$$B(\mathbf{s}, r) \cap \{x \in X,\ \phi(x) < \phi(\mathbf{s})\}$$

has exactly two connected components $C_j(\mathbf{s}, r)$, $j = 1, 2$.

Definition (Hérau-Hitrik-Sjöstrand, 2011)

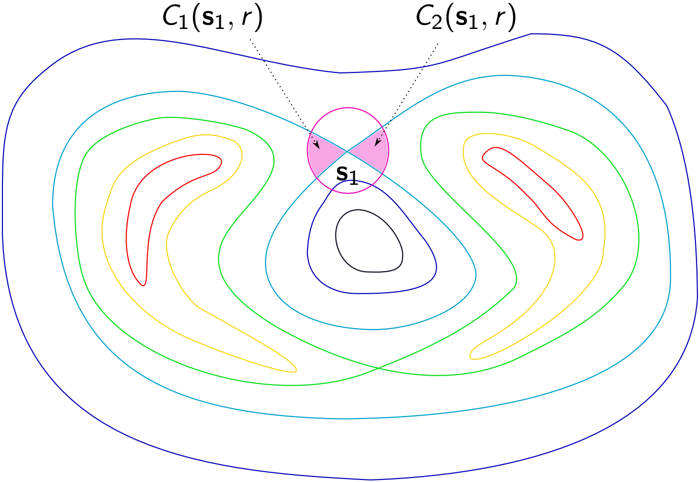
- $\mathbf{s} \in \mathcal{U}^{(1)}$ is a separating saddle point (ssp) iff $C_1(\mathbf{s}, r)$ and $C_2(\mathbf{s}, r)$ are contained in two different connected components of $\{x \in X,\ \phi(x) < \phi(\mathbf{s})\}$. We denote by $\mathcal{V}^{(1)}$ the set of ssp.
- $\sigma \in \mathbb{R}$ is a separating saddle value (ssv) if it is of the form $\sigma = \phi(\mathbf{s})$ with $\mathbf{s} \in \mathcal{V}^{('1)}$. We denote $\underline{\Sigma} = \phi(\mathcal{V}^{(1)}) = \{\sigma_2 > \sigma_3 > \ldots > \sigma_N\}$.

Example of SSP I

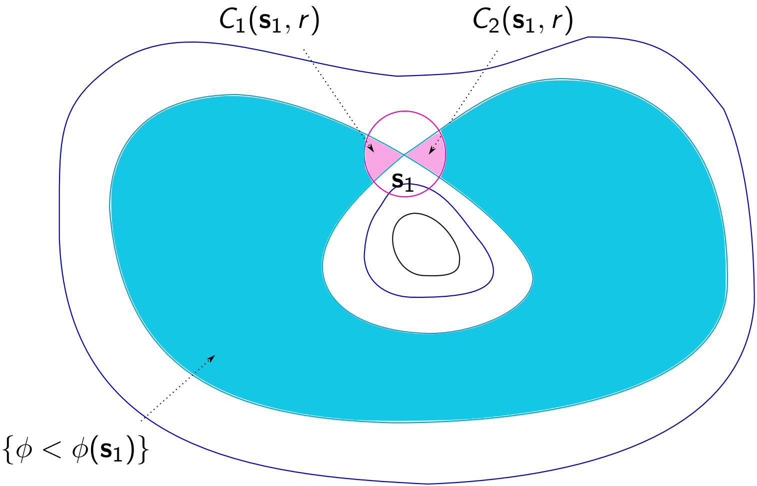


Level set of a potential with 2 minima, 2 saddle points and 1 maximum

Example of SSP II

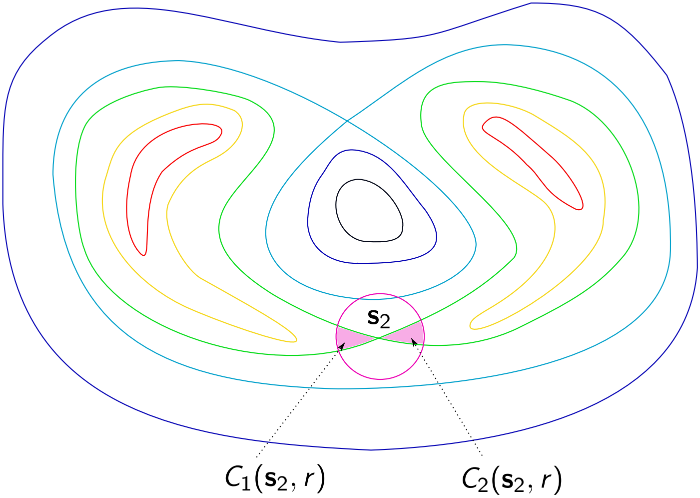


Example of SSP II

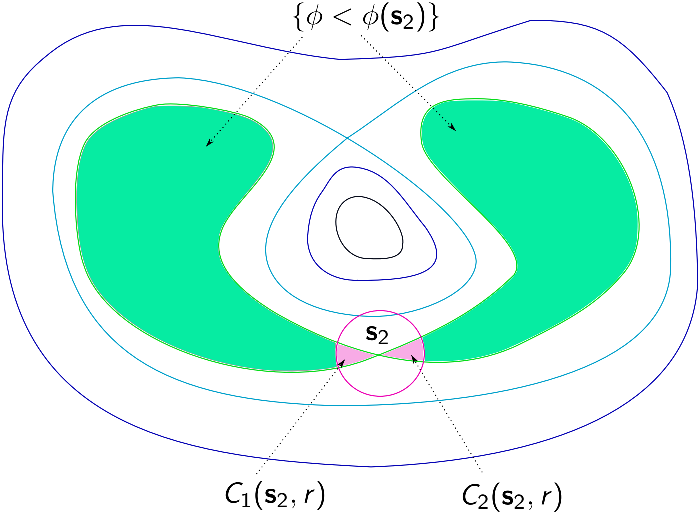


$\mathbf{s}_1$ is not separating

Example of SSP III



Example of SSP III



$\mathbf{s}_2$ is separating

The labelling procedure II

Add a fictive infinite saddle value $\sigma_1 = +\infty$ to $\underline{\Sigma}$ and let

$$\Sigma = \{\sigma_1\} \cup \underline{\Sigma} = \{\sigma_1 > \sigma_2 > \ldots > \sigma_N\}$$

- To $\sigma_1 = +\infty$ associate the unique connected component $E_{1,1} = X$ of $\{\phi < \sigma_1\}$. In $E_{1,1}$, pick up $m_{1,1}$ one (non necessarily unique) minimum of $\phi_{|E_{1,1}}$.
- The set $\{\phi < \sigma_2\}$ has finitely many connected components. One of them contains $m_{1,1}$. The others are denoted $E_{2,1}, \ldots, E_{2,N_2}$. In each of these CC, one choses one absolute minimum $m_{2,j}$ of $\phi_{|E_{2,j}}$.
- The set $\{\phi < \sigma_k\}$ has finitely many CC. One denotes by $E_{k,1}, \ldots, E_{k,N_k}$ those of these CC which do not contain any $m_{i,j}$, $i < k$. In each $E_{k,j}$ one choses one absolute minimum $m_{k,j}$ of $\phi_{|E_{k,j}}$.

The labelling procedure III

Denote $\underline{\mathbf{m}} = \mathbf{m}_{1,1}$ the absolute minimum of ϕ that was chosen at the first step of the labelling procedure, and let

$$\underline{\mathcal{U}}^{(0)} = \mathcal{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}.$$

Let $\mathcal{O}(X)$ denote the connected open subsets of X. Using the preceding labelling one constructs the following applications:

- $\boldsymbol{\sigma} : \mathcal{U}^{(0)} \to \Sigma$, defined by $\boldsymbol{\sigma}(\mathbf{m}_{i,j}) = \sigma_i$.
- $E : \mathcal{U}^{(0)} \to \mathcal{O}(X)$, defined by $E(\mathbf{m}_{i,j}) = E_{i,j}$.
- $S = \boldsymbol{\sigma} - \phi$

The Generic case I

The following hypothesis introduced by Hérau-Hitrik-Sjöstrand (2011) is a generalization of Helffer-Klein-Nier assumption (2004).

Generic Assumption (GA):

For all $\mathbf{m} \in \mathcal{U}^{(0)}$, the following hold true:

i) $\phi_{|E(\mathbf{m})}$ has a unique point of minimum

ii) if $\overline{E(\mathbf{m})} \cap \mathcal{V}^{(1)} \neq \emptyset$, there exists a unique $\mathbf{s} \in \mathcal{V}^{(1)}$ such that $\phi(\mathbf{s}) = \sup \phi(\overline{E(\mathbf{m})} \cap \mathcal{V}^{(1)})$.

Under this assumption, there exists a bijection

$$\mathbf{s} : \mathcal{U}^{(0)} \to \mathcal{V}^{(1)} \cup \{\infty\}$$

such that $S(\mathbf{m}) = \phi(\mathbf{s}(\mathbf{m})) - \phi(\mathbf{m})$ with the convention $\phi(\infty) = \infty$.

The Generic case II

Let us write $\lambda(\mathbf{m}, h)$, $\mathbf{m} \in \mathcal{U}^{(0)}$ the n_0 small eigenvalues of Δ_ϕ.

Theorem (Helffer-Klein-Nier 2004, Hérau-Hitrik-Sjöstrand 2011)

Suppose the the Generic Assumption is satisfied. Then the n_0 small eigenvalues of Δ_ϕ satisfy

$$\lambda(\mathbf{m}, h) = h\zeta(\mathbf{m}, h)e^{-2S(\mathbf{m})/h}$$

where $\zeta(\mathbf{m}, h) \sim \sum_{r=0}^{\infty} h^r \zeta_r(\mathbf{m})$ and

$$\zeta_0(\mathbf{m}) = \pi^{-1}|\mu(\mathbf{s}(\mathbf{m}))|\sqrt{\frac{|\det \phi''(\mathbf{m})|}{|\det \phi''(\mathbf{s}(\mathbf{m}))|}}$$

where $\mu(\mathbf{s})$ is the unique negative eigenvalue of ϕ'' in $\mathbf{s}$.

A simple example

Suppose that the following hypothesis are verified:

- The set of minimal values is reduced to one point:
  $\exists c_0,\ \forall \mathbf{m} \in \mathcal{U}^{(0)},\ \phi(\mathbf{m}) = c_0$
- The set of saddle values is reduced to one point:
  $\exists c_1,\ \forall \mathbf{m} \in \mathcal{U}^{(1)},\ \phi(\mathbf{m}) = c_1$

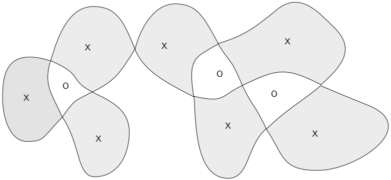


Figure 1: The sublevel set $\{\varphi < \sigma\}$ (dashed region) associated to a potential φ satisfying the assumptions. The x's represent local minima, the o's, local maxima.

Theorem

The n_0 small eigenvalues of Δ_ϕ satisfy $\lambda_1 = 0$ and for all $k = 2, \ldots n_0$,

$$\lambda_k(h) = h\zeta_k(h)e^{-2S/h}$$

where $S = c_1 - c_0$ and

$$\zeta_k(h) \sim \sum_{r=0}^{\infty} h^r \zeta_{k,r}$$

and $\zeta_{k,0}$ are the non zero eigenvalues of the weighted graph $\mathcal{G}$ defined by

- The vertices of the graph are the minima $\mathbf{m} \in \widehat{\mathcal{U}}^{(0)}$.
- The edges between two vertices $\mathbf{m}$, $\mathbf{m}'$ are the saddle points $\mathbf{s} \in \mathcal{U}^{(1)}$ such that $\mathbf{s} \in \overline{E}(\mathbf{m}) \cap \overline{E}(\mathbf{m}')$.
- The weights explicitly depend on the values of ϕ'' on $\mathcal{U}^{(0)}$ and $\mathcal{U}^{(1)}$.

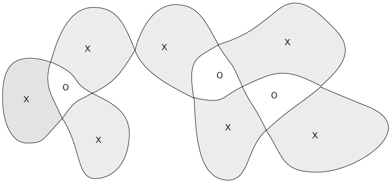


Figure 2: The sublevel set $\{\varphi < \sigma\}$ (dashed region) associated to a potential φ satisfying the assumptions. The x's represent local minima, the o's, local maxima.

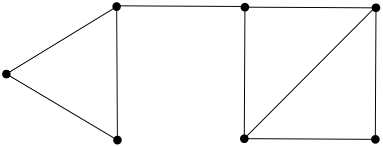


Figure 3: The graph associated to the potential represented in Figure **??**

Gathering interacting minima

For any $\mathbf{m} \in \underline{\mathcal{U}}^{(0)}$, let

- $G(\mathbf{m})$ denotes the connected component of $\{\phi \leq \sigma(\mathbf{m})\}$ that contains $\mathbf{m}$.

Fact:

For any $\mathbf{m} \neq \underline{\mathbf{m}}$, there exists a unique $\hat{\mathbf{m}} = \hat{\mathbf{m}}(\mathbf{m}) \in G(\mathbf{m}) \cap \mathcal{U}^{(0)}$ such that $\sigma(\hat{\mathbf{m}}) > \sigma(\mathbf{m})$

We denote by $\widehat{E}(\mathbf{m})$ the connected component of $\{\phi < \sigma(\mathbf{m})\}$ that contains $\hat{\mathbf{m}}(\mathbf{m})$. This defines two applications

$$\hat{\mathbf{m}} : \underline{\mathcal{U}}^{(0)} \to \mathcal{U}^{(0)} \text{ and } \widehat{E} : \underline{\mathcal{U}}^{(0)} \to \mathcal{O}(X)$$

Two different types of minima

Observe that by definition, we have

$$\forall \mathbf{m} \in \underline{\mathcal{U}}^{(0)},\ \phi(\hat{\mathbf{m}}(\mathbf{m})) \leq \phi(\mathbf{m}).$$

The fact that the above inequality is large or strict plays an important role in our analysis.

Definition

Let $\mathbf{m} \in \underline{\mathcal{U}}^{(0)}$. We say that $\mathbf{m}$ is of type I if $\phi(\hat{\mathbf{m}}(\mathbf{m})) < \phi(\mathbf{m})$. If $\phi(\hat{\mathbf{m}}(\mathbf{m})) = \phi(\mathbf{m})$, we say that $\mathbf{m}$ is of type II. We will denote

$$\underline{\mathcal{U}}^{(0),I} = \{\mathbf{m} \in \underline{\mathcal{U}}^{(0)},\ \mathbf{m} \text{ is of type I}\}$$

$$\underline{\mathcal{U}}^{(0),II} = \{\mathbf{m} \in \underline{\mathcal{U}}^{(0)},\ \mathbf{m} \text{ is of type II}\}$$

We have clearly the following disjoint union $\underline{\mathcal{U}}^{(0)} = \underline{\mathcal{U}}^{(0),I} \cup \underline{\mathcal{U}}^{(0),II}$.

An equivalence relation on $\mathcal{U}^{(0)}$

For $\sigma \in \Sigma$, let Ω_σ be defined by

$$\Omega_\sigma = \{E(\mathbf{m}), \mathbf{m} \in \boldsymbol{\sigma}^{-1}(\sigma)\} \bigcup \{\widehat{E}(\mathbf{m}), \mathbf{m} \in \boldsymbol{\sigma}^{-1}(\sigma) \cap \underline{\mathcal{U}}^{(0),II}\}$$

Definition:

We define an equivalence relation $\mathcal{R}$ on $\mathcal{U}^{(0)}$ by $\mathbf{m}\mathcal{R}\mathbf{m}'$ if and only if the two following properties hold true

- $\boldsymbol{\sigma}(\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{m}') = \sigma$
- $\mathbf{m}$ and $\mathbf{m}'$ belong to the same connected component of $\cup_{\omega \in \Omega_\sigma} \overline{\omega}$.

We denote by $\mathrm{Cl}(\mathbf{m})$ the equivalence class of any $\mathbf{m} \in \mathcal{U}^{(0)}$ and by $(\mathcal{U}_\alpha^{(0)})_{\alpha \in \mathcal{A}} = \mathcal{U}^{(0)}/\mathcal{R}$.

Main Theorem

For any $\alpha \in \mathcal{A}$, denote $\mathcal{S}_\alpha = S(\mathcal{U}_\alpha^{(0)})$ and $p(\alpha) = \sharp \mathcal{S}_\alpha$ and

$$\mathcal{S}_\alpha = \{S_{\nu_1^\alpha}, \dots, S_{\nu_{p(\alpha)}^\alpha}\}$$

for some integers $\nu_1^\alpha < \nu_2^\alpha < \dots < \nu_{p(\alpha)}^\alpha$.

Theorem

There exist $c > 0$ and some symmetric positive definite matrices $\mathcal{M}^\alpha$, $\alpha \in \mathcal{A}$ such that counted with multiplicity, on has $\sigma(\Delta_\phi) \cap [0, \epsilon_0 h] = \bigcup_{\alpha \in \mathcal{A}} \sigma(\mathcal{M}^\alpha)(1 + \mathcal{O}(e^{-c/h}))$ with

$$\sigma(\mathcal{M}^\alpha) = \bigcup_{j=1}^{p(\alpha)} h e^{-2h^{-1} S_{\nu_j^\alpha}} \sigma(M^{\alpha,j})$$

for some symmetric positive definite matrices $M^{\alpha,j}$ having a classical expansion with explicit invertible leading term

Comments

- The way to construct the matrices $\mathcal{M}_\alpha$ depends on
  - the number of equivalence class of $\mathcal{R}$
  - the number $p(\alpha)$ of values taken by φ on each equivalence class $\mathcal{U}_\alpha^{(0)}$.
- If there is only one equivalence class $\mathcal{U}_{\alpha_0}^{(0)}$ and if $p(\alpha_0) = 1$ then we are in the case where $\mathcal{M}_{\alpha_0}$ is a graph Laplacian.
- If $p(\alpha_0) \geq 2$, the situation is more complicated.

Finite dimensional reduction

The general strategy of Helffer-Klein-Nier is the following:

- Introduce
  - $F^{(0)}$ = eigenspace associated to the n_0 low lying eigenvalues on 0-forms
  - $\Pi^{(0)}$ = projector on $F^{(0)}$.
  - M = restriction of Δ_ϕ to $F^{(0)}$.

  We have to compute the eigenvalues of M.
- We compute suitable BKW approximated eigenfunctions $f_{\mathbf{m}}^{(0)}$ indexed by $\mathbf{m} \in \mathcal{U}^{(0)}$, and show that

$$\Pi^{(0)} f_{\mathbf{m}}^{(0)} = f_{\mathbf{m}}^{(0)} + error$$

  and compute the matrix of M in the base $\Pi^{(0)} f_{\mathbf{m}}^{(0)}$.
- Doing that leads to error terms which are too big.
- In order to overcome this difficulty, they use the supersymmetric structure.

Using Supersymmetry

The fondamental remarks are the following:

- $\Delta_\phi^{(p+1)} d_\phi^{(p)} = d_\phi^{(p)} \Delta_\phi^{(p)}$ and $d_\phi^{(p),*} \Delta_\phi^{(p+1)} = \Delta_\phi^{(p)} d_\phi^{(p),*}$
- Denote $F^{(1)}$ the eigenspace associated to low lying eigenvalues on 1 forms, then $d_\phi^{(0)}(F^{(0)}) \subset F^{(1)}$ and $d_\phi^{(0),*}(F^{(1)}) \subset F^{(0)}$. Hence

$$M = L^* L$$

where L is the matrix of $d_\phi^{(0)} : F^{(0)} \to F^{(1)}$.

- The matrix L is well approximated by

$$L \simeq \mathcal{L} := (\langle d_\phi^{(0)} f_{\mathbf{m}}^{(0)}, f_{\mathbf{s}}^{(1)} \rangle)_{,\mathbf{s} \in \mathcal{U}^{(1)} \mathbf{m} \in \mathcal{U}^{(0)}}$$

where $f_{\mathbf{s}}^{(1)}$ are BKW approximated eigenfunctions on 1-form.

Form of the quasimodes

- On 0-forms, one takes $f_{\mathbf{m}}^{(0)} = \sum_{\mathbf{m}' \in \mathrm{Cl}(\mathbf{m})} \theta_{\mathbf{m}}(\mathbf{m}') g_{\mathbf{m}'}^{(0)}$ with

$$g_{\mathbf{m}'}^{(0)} = h^{-\frac{d}{4}} \chi_{\mathbf{m}'}(x) e^{-(\phi(x) - \phi(\mathbf{m}'))/h}$$

  and $\chi_{\mathbf{m}'} \simeq 1\!\!1_{E(\mathbf{m}')}$.

- On 1-forms, one takes

$$f_{\mathbf{s}}^{(1)} = h^{-\frac{d}{4}} \chi_{\mathbf{s}}(x) b_s(x,h) e^{-(\phi_+(x) - \phi(\mathbf{s}))/h}$$

  with $\chi_{\mathbf{s}}$ cut-off function near $\mathbf{s}$, ϕ_+ phase function solving the eikonal equation

$$|\nabla \phi_+|^2 = |\nabla \phi|^2$$

  and $b_{\mathbf{s}}(x,h)$ a 1 form obtained by solving some transport equations.

Computation of the singular values

- One has $\mathcal{L} = \widehat{\mathcal{L}}\mathcal{T}$ with $\mathcal{T} = (\theta_{\mathbf{m}}(\mathbf{m}'))_{\mathbf{m},\mathbf{m}'}$ an orthogonal matrix and

$$\widehat{\mathcal{L}} = (\langle d_\phi^{(0)} g_{\mathbf{m}}^{(0)}, f_{\mathbf{s}}^{(1)} \rangle)_{\mathbf{s}\in\mathcal{U}^{(1)},\mathbf{m}\in\mathcal{U}^{(0)}}$$

- The computation of the coefficients of $\widehat{\mathcal{L}}$ is performed by Laplace Method.
- Under the Generic Assumption (GA), the matrix $\widehat{\mathcal{L}}$ is diagonal. Its singular values are then given by its diagonal coefficients.
- In the general case, $\widehat{\mathcal{L}}$ is only block-diagonal

$$\widehat{\mathcal{L}} = diag(\widehat{\mathcal{L}}^{\alpha}, \alpha \in \mathcal{A})$$

where each block $\mathcal{L}^{\alpha}$ corresponds to an equiv. class of $\mathcal{R}$.
- If $p(\alpha) = 1$, each block has a typical size $e^{-S_\alpha/h}$.
- If $p(\alpha) \geq 2$, we can perform a Schur type argument on each block.

Computations in dimension when $p(\alpha) = 1$

Suppose ϕ is given by

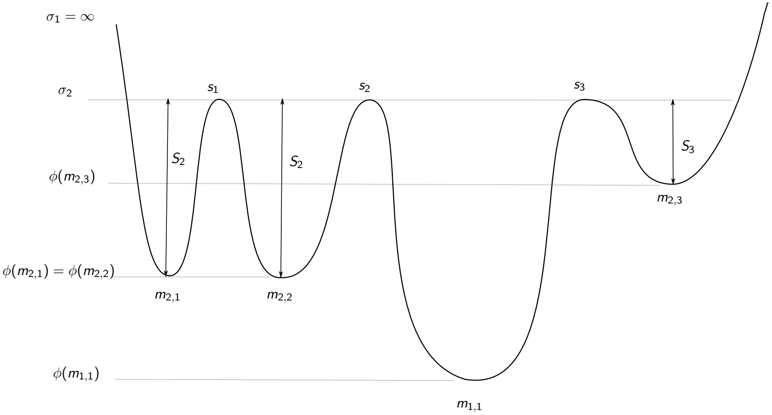


Figure 4: A potential satisfying H(1)

The case $p(\alpha) = 1$ continued

The interaction matrix is

$$\widehat{\mathcal{L}} = \begin{array}{c} \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{m}_{1,1} & \mathbf{m}_{2,1} & \mathbf{m}_{2,2} & \mathbf{m}_{2,3} \end{array} \\ \begin{pmatrix} 0 & e^{-S_2/h} & -e^{-S_2/h} & 0 \\ 0 & 0 & e^{-S_2/h} & 0 \\ 0 & 0 & 0 & e^{-S_3/h} \end{pmatrix} \end{array}$$

This block structure implies:

$$SV(\widehat{\mathcal{L}}) = \{0\} \cup e^{-S_2/h} SV\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cup \{e^{-S_3/h}\}$$

Computations in dimension one when $p(\alpha) = 2$

Suppose that ϕ is given by

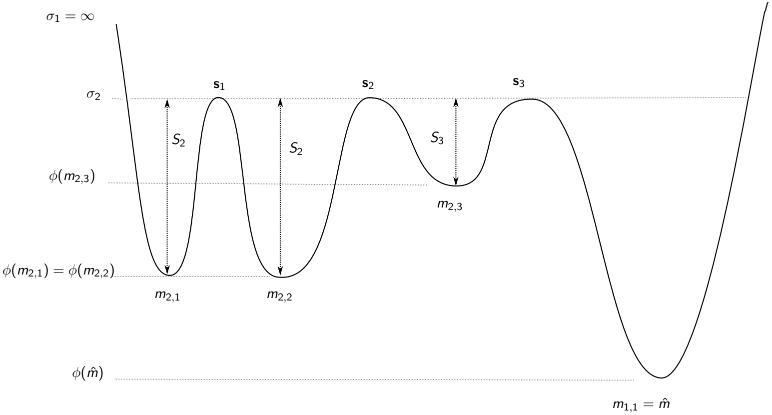


Figure 5: A potential with $p(\alpha) = 2$

The interaction matrix is

$$\widehat{\mathcal{L}} = e^{-S_3/h} \begin{array}{c} \\ \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{array} \begin{array}{c} \begin{matrix} \mathbf{m}_{1,1} & \mathbf{m}_{2,1} & \mathbf{m}_{2,2} & \mathbf{m}_{2,3} \end{matrix} \\ \begin{pmatrix} 0 & e^{-(S_2-S_3)/h} & -e^{-(S_2-S_3)/h} & 0 \\ 0 & 0 & e^{-(S_2-S_3)/h} & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

This is not a block-diagonal matrix, but $\mathcal{M} := \widehat{\mathcal{L}}^* \widehat{\mathcal{L}}$ has the form:

$$\mathcal{M} = e^{-2S_3/h} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^2 A & \epsilon B^* \\ 0 & \epsilon B & J \end{pmatrix}$$

with $\epsilon = e^{-(S_2-S_3)/h} << 1$. So we can use Schur complement method to compute its eigenvalues. We get

$$\sigma(\mathcal{M}) = e^{-2S_3/h}(\sigma(J) + \mathcal{O}(e^{-2(S_2-S_3)/h})) \\ \cup\, e^{-2S_2/h}(\sigma(A - B^* J^{-1} B) + \mathcal{O}(e^{-2(S_2-S_3)/h})) \cup \{0\}$$

The general case

- Block-diagonalize the interaction matrix: $\widehat{\mathcal{L}} = diag(\widehat{\mathcal{L}}^{\alpha}, \alpha \in \mathcal{A})$ where each block $\mathcal{L}^{\alpha}$ corresponds to an equiv. class of $\mathcal{R}$.
- Observe that $\mathcal{M}_{\alpha} := (\mathcal{L}^{\alpha})^{*}\mathcal{L}^{\alpha}$ has a nice structure: $\mathcal{M}_{\alpha} = \Omega_{\alpha}\tilde{\mathcal{M}}_{\alpha}\Omega_{\alpha}$ with $\mathcal{M}_{\alpha}$ "independent" of h and

$$\Omega^{\alpha} = \begin{pmatrix} I_{r_p} & 0 & \dots & \dots & 0 \\ 0 & \tau_2 I_{r_{p-1}} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \tau_2\tau_3\dots\tau_p I_{r_1} \end{pmatrix}$$

where $\tau_j = e^{(S_{\nu_{p-(j-2)}} - S_{\nu_{p-(j-1)}})/h}$ for any $j = 2, \dots, p$ and $p = p(\alpha)$ is the number of values taken by ϕ on $\mathcal{U}_{\alpha}^{(0)}$.
- Apply Schur complement's method and induction on $p(\alpha)$.