Scattering amplitude for the Schrödinger equation with strong magnetic field

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I Introduction

I.1 Framework

We consider the Schrödinger operator with constant magnetic field

$$\mathrm{H}(b) = \mathrm{H}_0(b) + b^\gamma V(x, y, z)$$

where

$$\mathrm{H}_0(b) = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2 - \Delta_z.$$

Here $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}^{n-2}$, $b > 0$ is a parameter and $\gamma \in [0, 1]$ is fixed.

In all this talk, we will assume that V satisfies the following hypothesis

Assumption 1

$$V(x, y, z) = V^\infty(z) + W(x, y, z)$$

with $V^\infty \in C_0^\infty(\mathbb{R}^{n-2})$, $W \in C_0^\infty(\mathbb{R}^n)$ *and* $V, V^\infty \geq 0$.

Consequence: *The scattering operator* $S(b) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *associate to the pair* $(\mathrm{H}_0(b), \mathrm{H}(b))$ *is well-defined, [Avron-Herbst-Simon 78']*.

Recall that the scattering operator admits the following definition. For $\psi_1 \in L^2(\mathbb{R}^n)$ there exists a unique function $\psi \in L^\infty(\mathbb{R}_t, L^2(\mathbb{R}^n))$ solution of

$$\begin{cases} i\partial_t \psi = H(b)\psi \\ \lim_{t\to-\infty} \|\psi(t,.) - e^{-itH_0(b)}\psi_1\|_{L^2(\mathbb{R}^n)} = 0, \end{cases}$$

where $e^{-itH_0(b)}\psi_1$ denotes the solution of the free Schrödinger equation ($V = 0$) with initial condition ψ_0 in $t = 0$. Moreover, there exists a unique $\psi_2 \in L^2(\mathbb{R}^n)$ such that

$$\lim_{t\to+\infty} \|\psi(t,.) - e^{-itH_0(b)}\psi_2\|_{L^2(\mathbb{R}^n)} = 0.$$

The **scattering operator** $S(b) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$\psi_2 = S(b)\psi_1.$$

I.2 Scattering matrix

Let us denote $\widehat{\mathrm{H}}_0(b) = \left(i\frac{\partial}{\partial x} + \frac{b}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} - \frac{b}{2}x\right)^2$ acting on $L^2(\mathbb{R}^2)$. Then, there exists U unitary on $L^2(\mathbb{R}^2)$ such that

$$U\widehat{\mathrm{H}}_0(b)U^* = bN_x \otimes I_y$$

where $N_x = -\frac{d^2}{dx^2} + x^2$ is the harmonic oscillator. Therefore

$$\sigma(\widehat{\mathrm{H}}_0(b)) = \sigma_{pp}(\widehat{\mathrm{H}}_0(b)) = b(2\mathbb{N}^* - 1).$$

These eigenvalues are called the Landau levels. We denote by

$$\Pi_q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

the projector onto $\ker(\widehat{\mathrm{H}}_0(b) - b(2q-1)),\ q \in \mathbb{N}^*$ and we define

$$\mathcal{F}_0 : L^2(\mathbb{R}^{n-2}) \to L^2(\mathbb{R}_+^*, L^2(S^{n-3}), dE),$$

by

$$\mathcal{F}_0\varphi(E) = E^{\frac{n-4}{4}}\hat{\varphi}(\sqrt{E}\,.\,)$$

where $\hat{\varphi}$ denotes the Fourier transfor of φ. Next, we define

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}_+^*, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$$

by

$$\mathcal{F}\varphi(E) = \sum_{1 \leq q < \frac{1+E/b}{2}} \Pi_q \otimes \mathcal{F}_0\varphi(E - b(2q-1)).$$

Then $\mathcal{F}$ is a unitary isomorphism and

- $\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*$ is the multiplication by E on $L^2(\mathbb{R}_+^*, L^2(\mathbb{R}^2 \times S^{n-3}), dE)$
- For all $t > 0$, $\mathcal{F}S(b)\mathcal{F}^*$ and $e^{it\mathcal{F}\,\mathrm{H}_0\,\mathcal{F}^*}$ commute.

Hence (cf. [Reed-Simon, T4]), there exists a function $E \mapsto S(E, b)$ in $L^\infty(\mathbb{R}_+^*, \mathcal{L}(L^2(\mathbb{R}^2 \times S^{n-3})))$ such that

$$\forall \varphi \in L^2(\mathbb{R}^n),\ S(b)\varphi = \mathcal{F}^* S(E, b)\mathcal{F}\varphi.$$

For $E > 0$, $S(E, b)$ is called the **scattering matrix** (it is a matrix only in the case $n = 3$).

II Main results

II.1 Representation formula

For $\alpha \in \mathbb{R}$ we use the L^2-weighted space $L^2_\alpha = L^2(\mathbb{R}^n, \langle z\rangle^\alpha dxdydz)$ where $\langle z\rangle = (1+|z|^2)^{1/2}$ and for $E > 0$, $\alpha > 1/2$ we define

$$\mathcal{F}(E) : L^2_\alpha \to L^2(\mathbb{R}^2 \times S^{n-3})$$

by $\mathcal{F}(E)\varphi = \mathcal{F}\varphi(E)$.

Theorem 1 *Suppose that the potential V satisfies Assumption 1 and denote by $\sigma_{pp}(\mathrm{H})$ the point spectrum of $\mathrm{H}(b)$. Then, for all $E \in]b, +\infty[\setminus(b(2\mathbb{N}^* - 1) \cup \sigma_{pp}(\mathrm{H}))$, one has*

$$\begin{aligned} S(E,b) - Id = &- 2i\pi b^\gamma \mathcal{F}(E) V \mathcal{F}(E)^* \\ &+ 2i\pi b^{2\gamma} \mathcal{F}(E) V R(E+i0) V \mathcal{F}(E)^*, \end{aligned}$$

where

$$R(E+i0) = \lim_{\mu\to 0^+} (\mathrm{H}(b) - E - i\mu)^{-1}$$

exists in the space $\mathcal{L}(L^2_\alpha, L^2_{-\alpha})$ for $\alpha > 1/2$.

Corollary 1 *For* $E \in]b, +\infty[\backslash (b(2\mathbb{N}^* - 1) \cup \sigma_{pp}(\mathrm{H}))$, $T(E, b) := S(E, b) - Id$ *has a kernel*

$$(\omega, \omega') \in S^{n-3} \times S^{n-3} \to T(\omega, \omega', E, b) \in \mathcal{L}(L^2(\mathbb{R}^2))$$

which is smooth on $S^{n-3} \times S^{n-3}$. *The map* $(\omega, \omega') \mapsto T(\omega, \omega', E, b)$ *is called the* **scattering amplitude**.

Goal: Study $T(\omega, \omega', E, b)$ when $b \to \infty$. We work with *energies far from the Landau levels*; $E = \lambda b$ with $\lambda \notin 2\mathbb{N}^* - 1$.

Two different regimes according to γ:

- $\gamma \in [0, 1/2[$: High energy behavior.
- $\gamma \in [1/2, 1]$: Semiclassical behavior.

II.2 Very short bibliography

II.2.1 Representation formula for the Schrödinger equation (without magnetic field)

- S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

II.2.2 Scattering amplitude for the semiclassical Schrödinger equation in the non-trapping case

- B. R. Vainberg, Quasiclassical approximation in stationnary scattering problems, *Funct. Anal. Appl.*, 11, no 4, 247-257.
- D. Robert and H. Tamura, Asymptotic behavior of the scattering amplitudes in semiclassical and low energy limit, *Ann. Inst. Fourier*, 39 (1989), no. 1, 155-192.

II.2.3 Spectral Shift Function for Schrödinger equation with strong magnetic field

- V. Bruneau and M. Dimassi, Weak asymptotics of the spectral shift function in strong constant magnetic field, *Math. Nachr.*, to appear.
- V. Bruneau, A. Pushnitski and G. Raikov, Spectral shift function in strong magnetic fields, *Algebra i Analiz*, 16 (2004), no. 1, 207–238.

II.3 Asymptotics in the case $\gamma = 0$

Theorem 2 *Suppose that Assumption 1 is satisfied and let* $\lambda \in]2q_0 - 1, 2q_0 + 1[$ *for some* $q_0 \in \mathbb{N}^*$. *When* b *tends to infinity,* $\lambda b \notin \sigma_{pp}(H(b))$ *and*

$$T(\omega, \omega', \lambda b, b) = \frac{i b^{\frac{n-4}{2}}}{2(2\pi)^{n-1}} \sum_{q=1}^{q_0} \lambda_q^{\frac{n-4}{2}} \widehat{V}^z(x, y, , b^{1/2} \lambda_q^{1/2} (\omega - \omega')) \Pi_q + \mathcal{O}(b^{\frac{n-5}{2}})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, *where* $\lambda_q = \lambda - 2q + 1$.

From this theorem we deduce the following inverse scattering result.

Corollary 2 *Suppose that* V_1, V_2 *satisfy Assumption 1. Assume that the associate scattering operators* S_1 *and* S_2 *are equal. Then* $V_1 = V_2$.

II.4 Asymptotics in the case $\gamma = 1$

From now, we assume that $\lambda \in]2q_0 - 1, 2q_0 + 1[$, $q_0 \in \mathbb{N}^*$ and for $q \in \{1, \dots, q_0\}$ we set $\lambda_q = \lambda - 2q + 1$. For $(x, y) \in \mathbb{R}^2$, let us denote

$$p_{x,y}(z, z^*) = |z^*|^2 + V(x, y, z), \ \forall (z, z^*) \in T^*\mathbb{R}^{n-2},$$

$$H_{p_{x,y}} = \partial_{z^*} p_{x,y} \partial_z - \partial_z p_{x,y} \partial_{z^*}$$

the associated Hamiltonian vector field and $t \mapsto \exp(tH_{p_{x,y}})(z, z^*)$ the solution of the Hamiltonian system

$$\dot{Z} = 2Z^*, \ \dot{Z}^* = -\nabla_z V(x, y, Z) \tag{1}$$

with initial condition $(Z, Z^*)_{|t=0} = (z, z^*)$.

We introduce the following non-trapping condition.

Assumption 2 *For all $q = 1, \dots, q_0$ and all $(x, y) \in \mathbb{R}^2$,*

$$\lim_{|t| \to \infty} |\exp(tH_{p_{x,y}})(z, z^*)| = +\infty$$

for all $(z, z^*) \in T^*\mathbb{R}^{n-2}$ such that $|z^*|^2 + V(x, y, z) = \lambda_q$.

II.4.1 Asymptotics in dimension 3

In dimension $n = 3$, the structure of the classical scattered trajectories is rather simple and the preceding assumption is sufficient to state a theorem. Let us denote

$$S(E,b) = \begin{pmatrix} S_{11}(E,b) & S_{12}(E,b) \\ S_{21}(E,b) & S_{22}(E,b) \end{pmatrix}$$

the scattering matrix in dimension 3, with $S_{ij}(E,b) \in \mathcal{L}(L^2(\mathbb{R}^2))$. From Assumption 2, we deduce easily that there exists $q_1 \in \{1, \dots, q_0 + 1\}$ such that:

• for all $q \in \{1, \dots, q_1 - 1\}$ and all $(x,y) \in \mathbb{R}^2$, $\lambda_q > \sup_{z \in \mathbb{R}} V(x,y,z)$

• for all $q \in \{q_1, \dots, q_0\}$ and all $(x,y) \in \mathbb{R}^2$, the equation in z $V(x,y,z) = \lambda_q$ has exactly two solutions $\alpha_q(x,y) < \beta_q(x,y)$ and these solutions are non-critical points of $z \mapsto V(x,y,z)$.

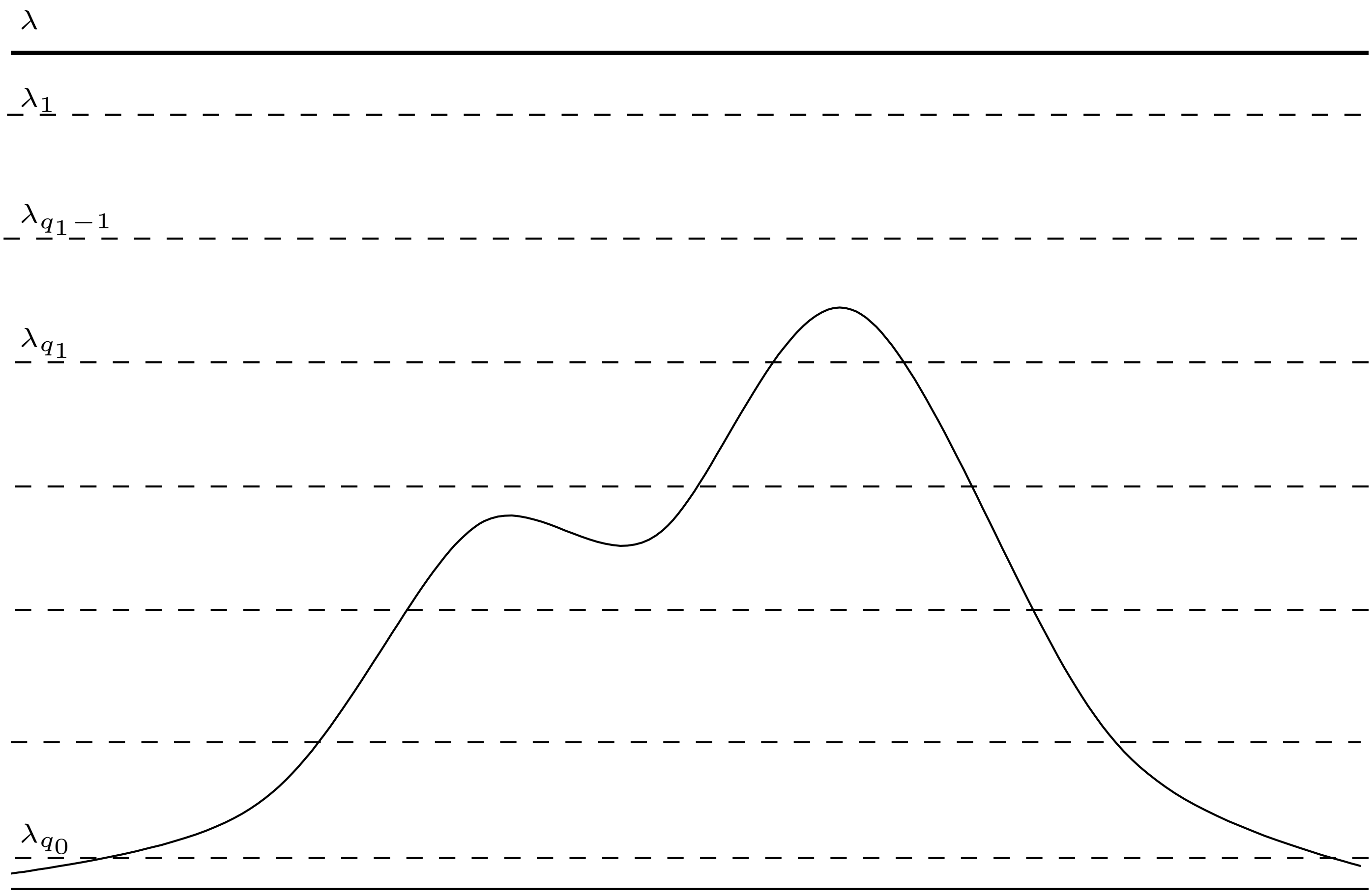

Figure 1: A potential satisfying Assumption 2.

Class of symbols:

Let $m : \mathbb{R}^d \to [0, +\infty[$ be an order function, that is:

$$\exists C, N > 0, \forall x, y \in \mathbb{R}^d,\ m(x) \leq C\langle x - y\rangle^N m(y).$$

For $\delta \in [0, 1]$, we say that a function $a(x, h) \in C^\infty(\mathbb{R}^d \times]0, 1])$ belongs to the class $S^\delta(\mathbb{R}^d, m, h)$ if

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0,\ |\partial_x^\alpha a(x, h)| \leq C_\alpha h^{-\delta|\alpha|} m(x).$$

For a in a suitable class of symbol, we will denote by $a^w(x, hD_x)$ the standard Weyl-quantization of a.

Theorem 3 *Suppose that $n = 3$ and that Assumption 1 and 2 are fulfilled, then we have the following asymptotics.*

Diagonal coefficients

$$S_{11}(\lambda b, b) = \sum_{q=1}^{q_1-1} s_{d,q}^w(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\Pi_q + \sum_{q=q_0+1}^{+\infty} \Pi_q + \mathcal{O}(b^{-\infty})$$

in $\mathcal{L}(L^2(\mathbb{R}^2)$, with $s_{d,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$ and

$$s_{d,q}(\lambda, y, \xi) = \exp(ib^{1/2}\int_{-\infty}^{+\infty} \sqrt{\lambda_q - V(\xi, y, z)} - \sqrt{\lambda_q} dz) + \mathcal{O}(b^{-1/2}).$$

Off-diagonal coefficients

$$S_{21}(\lambda b, b) = \sum_{q=q_1}^{q_0} s_{a,q}^w(\lambda, y/2 - b^{-1}D_x, x/2 - b^{-1}D_y)\Pi_q + \mathcal{O}(b^{-\infty})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, with $s_{a,q} \in S^{1/2}(\mathbb{R}^2, b^{-1})$ and

$$s_{a,q}(\lambda, y, \xi) =$$
$$i \exp(2ib^{1/2}(\sqrt{\lambda_q}\alpha_q(\xi, y) + \int_{-\infty}^{\alpha_q(\xi,y)} \sqrt{\lambda_q - V(\xi, y, z)} - \sqrt{\lambda_q} dz))$$
$$+ \mathcal{O}(b^{-1/2})$$

II.4.2 Asymptotics in dimension $n \geq 4$

From now, we fix a couple of directions $(\omega, \omega') \in S^{n-3} \times S^{n-3}$. As V is compactly supported in the variable z, out of a compact set the solutions of (1) are straight lines and it is easy to see that for all $(x, y) \in \mathbb{R}^2$, $q = 1, \ldots, q_0$ and $\tilde{z} \in \omega^\perp$, there exists a unique solution $(Z_{q,\infty}, Z^*_{q,\infty})(t, x, y, \tilde{z}, \omega)$ of (1) such that for $-t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\omega t + \tilde{z}.$$

Under Assumption 2, we can precise the behavior of these particles when t goes to $+\infty$. There exists $\theta_{q,\infty}(x, y, \tilde{z}, \omega) \in S^{n-3}$ and $r_{q,\infty}(x, y, \tilde{z}, \omega) \in \mathbb{R}^{n-2}$ such that for $t > 0$ large enough

$$Z_{q,\infty}(t, x, y, \tilde{z}, \omega) = 2\sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega)t + r_{q,\infty}(x, y, \tilde{z}, \omega)$$
$$Z^*_{q,\infty}(t, x, y, \tilde{z}, \omega) = \sqrt{\lambda_q}\theta_{q,\infty}(x, y, \tilde{z}, \omega).$$

For $\tilde{z} \in \omega^\perp \simeq \mathbb{R}^{n-3}$, we define the angular densities by

$$\widehat{\sigma}_q(x, y, \tilde{z}) = |\det(\theta_{q,\infty}, \partial_{\tilde{z}_1}\theta_{q,\infty}, \ldots, \partial_{\tilde{z}_{n-3}}\theta_{q,\infty})|$$

Assumption 3 *We suppose that for all $q \in \{1, \ldots, q_0\}$, $(x, y) \in \mathbb{R}^2$ and all $\tilde{z} \in \omega^\perp$ with $\theta_{q,\infty}(x, y, \tilde{z}) = \omega'$, we have $\widehat{\sigma}_q(x, y, \tilde{z}) \neq 0$.*

Consequence: It follows from this assumption and implicit function theorem that for all $q \in \{1, \ldots, q_0\}$ and all $(x, y) \in \mathbb{R}^2$, the equation

$$\theta_{q,\infty}(x, y, \tilde{z}, \omega) = \omega'$$

has a finite number of solutions $\tilde{z}_{q,1}(x, y), \ldots, \tilde{z}_{q,l_q}(x, y)$ smooth with respect to (x, y). Moreover, the number l_q does not depend on (x, y).

Theorem 4 *Suppose that $n \geq 4$ and that Assumptions 1, 2 and 3 are satisfied. Then $\lambda b \notin \sigma_{pp}(\mathrm{H}(b))$ and*

$$T(\omega, \omega', \lambda b, b) =$$

$$b^{\frac{n-3}{4}} \sum_{q=1}^{q_0} \lambda_q^{\frac{n-3}{4}} T_q^w(\omega, \omega', y/2 - b^{-1}D_x, x/2 + b^{-1}D_y)\Pi_q + \mathcal{O}(b^{\frac{n-5}{4}})$$

in $\mathcal{L}(L^2(\mathbb{R}^2))$, where

$$T_q(\omega, \omega', y, \xi) = c(n) \sum_{l=1}^{l_q} \widehat{\sigma}_q(\xi, y, \tilde{z}_{q,l}(\xi, y))^{-1/2} e^{ib^{1/2}\mathbf{S_{q,l}}(y,\xi) - i\mu_{q,l}\pi/2}$$

is a symbol of class $S^{1/2}(\mathbb{R}^2, b^{-1})$. Here

$$\begin{aligned}\mathbf{S_{q,l}}(y, \xi) = \int_{-\infty}^{+\infty} (|Z^*_{q,\infty}(t, \xi, y, \tilde{z}_{q,l}(\xi, y), \omega)|^2 \\ - V(\xi, y, Z_{q,\infty}(t, \xi, y, \tilde{z}_{q,l}(\xi, y), \omega)) - \lambda_q)dt \\ - r_{q,\infty}(\xi, y, \tilde{z}_{q,l}(\xi, y), \omega),\end{aligned}$$

$\mu_{q,l}$ is the Maslov index of $(Z_{q,\infty}, Z^*_{q,\infty})(t, \xi, y, \tilde{z}_{q,l}(\xi, y), \omega)$ on the Lagrangian manifold

$$\{(z, z^*) \in T^*\mathbb{R}^{n-2} \mid \\ z = Z_{q,\infty}(t, \xi, y, \tilde{z}, \omega), z^* = Z^*_{q,\infty}(t, \xi, y, \tilde{z}, \omega)), \tilde{z} \in \omega^\perp,\ t \in \mathbb{R}\}$$

and $\mu_{q,l}$ is independent on (y, ξ).

III Sketch of the proof

III.1 Case $\gamma = 0$

Proposition 1 *Suppose that Assumption 1 is satisfied and let* $\lambda \in]1, +\infty[\setminus (2\mathbb{N}^* - 1)$. *Then*

$$\| \langle z \rangle^{-\alpha} R(\lambda b + i0) \langle z \rangle^{-\alpha} \|_{L^2, L^2} = \mathcal{O}(b^{-1/2}).$$

Proof: in the case where $V = V^\infty(z)$. Then

$$R(\lambda b + i0) = (-\Delta_z + V(z) + b(\sum_q (2q-1)\Pi_q - \lambda) - i0)^{-1}$$
$$= \sum_q (-\Delta_z + V(z) - b\lambda_q - i0)^{-1} \, \Pi_q$$

To conclude, we use the well-known high energy estimate for the Schrödinger equation when $E \rightarrow +\infty$

$$\| \langle z \rangle^{-\alpha} (-\Delta_z + V(z) - E - i0)^{-1} \langle z \rangle^{-\alpha} \|_{L^2, L^2} = \mathcal{O}(E^{-1/2})$$

for all $\alpha > 1/2$. □

Theorem 2 is proved by combining Theorem 1 and Proposition 1.

III.2 Case $\gamma = 1$

Starting point: For $q = 1, \ldots, q_0$, denote $\lambda_q = \lambda - 2q + 1$ and let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}_z^{n-2})$ such that $V \prec \chi_1 \prec \chi_2$. Then

$$T(\omega, \omega', \lambda b, b) = \sum_{p,q=1}^{q_0} \Pi_p f_{p,q}(\omega, \omega', \lambda, b) \Pi_q$$

with

$$f_{p,q}(\omega, \omega', \lambda, b) = \int_{\mathbb{R}^{n-2}} e^{-ib^{1/2}\sqrt{\lambda_p}\langle z, \omega' \rangle} [\Delta_z, \chi_2] \\ R(\lambda b + i0)[\Delta_z, \chi_1] e^{ib^{1/2}\sqrt{\lambda_q}\langle z, \omega \rangle} dz$$

Ouline of the proof:

1. Effective Hamiltonian
2. Non-trapping Resolvent estimate
3. Microlocal Resolvent estimate and Egorov Lemma
4. Approximation of the evolution
5. Stationnary phase method

III.2.1 Effective Hamiltonian

Projection on the Landau levels

Lemma 1 *([Dimassi-Raikov 01'], [Dimassi 01'],...) There exists a symplectic change of coordinates involving a unitary operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that*

$$UH(b)U^* = h^{-2}\widetilde{P}(h)$$

with $h = b^{-1/2}$ and

$$\widetilde{P}(h) = -h^2\Delta_z + N_x + V^w(h^2D_y + hD_x, y - hx, z),$$

where $N_x = -\frac{d^2}{dx^2} + x^2$.

Remark 1 *Let us notice that the pseudo-differential operators we are dealing with have two scales: h for the variable z and h^2 for the variable y.*

For $q = 1, \dots, q_0$, let us set

$$\widetilde{\Pi}_q = U\Pi_q U^* : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Denoting $\Phi_q \in L^2(\mathbb{R}_x)$ the eigenfunction of N_x associated to the eigenvalue $2q - 1$ we have

$$\widetilde{\Pi}_q\varphi = \langle\varphi, \Phi_q\rangle_{L^2(\mathbb{R}_x)}\Phi_q.$$

Let us denote $\widetilde{\Pi} = \sum_{q=1}^{q_0}\widetilde{\Pi}_q$ and identify $\mathrm{Ran}\,\widetilde{\Pi}$ with $L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}$. Then we have to analyze the operator

$$\widetilde{\Pi}(\widetilde{P}(h) - \lambda - i\mu)^{-1}\widetilde{\Pi} : L^2(\mathbb{R}^{n-1}_{y,z})^{q_0} \to L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}.$$

Proposition 2 *For all $\mu > 0$,*

$$\widetilde{\Pi}(\widetilde{P}(h) - \lambda - i\mu)^{-1}\widetilde{\Pi} = E(h, \lambda + i\mu)^{-1}$$

where $E(h, \lambda + i\mu)$ has the following properties:

i) There exists a sequence of matrix valued symbols $(E_j(y, \xi, z, z^*, \lambda + i\mu))_{j\in\mathbb{N}}$ such that $E_0 \in S^0(\mathbb{R}^{2n-2}, \langle z^* \rangle^2, \mathcal{L}(\mathbb{R}^{q_0}))$, $E_j \in S^0(\mathbb{R}^{2n-2}, 1, \mathcal{L}(\mathbb{R}^{q_0})), \forall j \geq 1$ and for all $N \in \mathbb{N}^*$,

$$E(h, \lambda + i\mu) = \sum_{j=0}^{N} h^j E_j^w(y, h^2 D_y, z, hD_z, \lambda + i\mu) + \mathcal{O}(h^N).$$

ii)

$$E_0(y, \xi, z, z^*, \lambda + i\mu) = \mathrm{diag}((|z^*|^2 + V(\xi, y, z) - \lambda_q - i\mu)_{q=1,\ldots,q_0}),$$

iii) E_1 is off-diagonal and for all $j \geq 1$ the seminorms of E_j are bounded uniformly with respect to $\mu > 0$.

Proof: Let us denote $\widehat{\Pi} = 1 - \widetilde{\Pi}$ and $\widehat{P} = \widehat{\Pi} P \widehat{\Pi}$. Solving a suitable Grushin problem, we get

$$\widetilde{\Pi}(\widetilde{P}(h) - \lambda - i\mu)^{-1}\widetilde{\Pi} = E(h, \lambda + i\mu)^{-1}$$

with

$$\begin{aligned} E(h, \lambda + i\mu) = \widetilde{\Pi}(\widetilde{P}(h) - \lambda - i\mu)\widetilde{\Pi} \\ - \widetilde{\Pi}V^w(\ldots)\widehat{\Pi}(\widehat{P}(h) - \lambda - i\mu)^{-1}\widehat{\Pi}V^w(\ldots)\widetilde{\Pi} \\ = E_D(h) + E_A(h) \end{aligned}$$

We have

$$E_D(h) = (a_{pq}(y, z, h^2 D_y, hD_z))_{p,q \in \{1,\ldots,q_0\}}$$

with

$$a_{pq}(y, z, \xi, z^*) = \langle (|z^*|^2 + V^w(\xi + hD_x, y - hx, z) - \lambda - i\mu)\phi_p, \phi_q \rangle.$$

By Taylor expansion, we get

$$\begin{aligned} a_{pq}(y, z, \xi, z^*) = \delta_{pq}(|z^*|^2 + V(\xi, y, z) + h^2 p_2 + h^3 p_3 + \ldots) \\ + (1 - \delta_{pq})hb_{pq}, \end{aligned}$$

with $b_{pq} \in S^0(\mathbb{R}^{2n-2})$. Therefore, $E_D(h)$ has the required properties.

To analyze the term $E_A(h)$ it suffices to remark that

- $\widehat{\Pi}V^w(\ldots)\widehat{\Pi} = \mathcal{O}(h)$
- As $V \geq 0$, $\tilde{P}(h) - \lambda$ is elliptic and we can construct a parametrix.

□

Diagonalization of the Hamiltonian

Proposition 3 *For all $N_0 \in \mathbb{N}^*$, there exists a unitary transformation U_{N_0} on $L^2(\mathbb{R}^{n-1}_{y,z})$ such that*

$$U_{N_0} E(h, \lambda + i\mu) U^*_{N_0} = \mathcal{P}_{N_0}(\lambda + i\mu) + \mathcal{O}(h^{N_0})$$

with

$$\mathcal{P}_{N_0}(\lambda+i\mu) = \mathrm{diag}((p^w_q(y, z, h^2 D_y, hD_z, N_0) - \lambda_q - i\mu),\ q = 1, \ldots, q_0)$$

and $p_q(., N_0) \in S^0(\mathbb{R}^{2n-2}, \langle z^* \rangle^2)$. Moreover,

$$p_q(y, z, \xi, z^*, N_0) = \sum_{m=0}^{N_0} h^m p_{q,m}(y, z, \xi, z^*)$$

with $p_{q,0} = |z^*|^2 + V(\xi, y, z)$, $p_{q,m} \in S^0(T^*\mathbb{R}^{n-1})$ for $m \geq 1$.

Proof: Look for U_1 under the form

$$U_1 = \exp(hu^w_1(y, z, h^2 D_y, hD_z)).$$

Then

$$\begin{aligned} U_1 E(h, \lambda + i\mu) U^*_1 &= E^w_0(y, z, h^2 D_y, hD_z) \\ &\quad + h(E^w_1(\ldots) + u^w_1(\ldots)E^w_0(\ldots) + E^w_0(\ldots)u^w_1(\ldots)^*) + \ldots \end{aligned}$$

We remark that E_0 has the required form and we can chose u_1 so that the term in h in the preceding expansion vanishes. We conclude by induction. □

III.2.2 Resolvent estimate

Proposition 4 *Supspose that Assumptions 1 and 2 are satisfied, then λb is not an eigenvalue of $H(b)$ and*

$$\|\langle z\rangle^{-\alpha}\mathcal{P}_{N_0}(\lambda+i0)^{-1}\langle z\rangle^{-\alpha}\|_{L^2(\mathbb{R}^{n-1}_{y,z})^{q_0},L^2(\mathbb{R}^{n-1}_{y,z})^{q_0}}=\mathcal{O}(h^{-1})$$

for all $\alpha>1/2$.

Proof: Apply Mourre theory and search a conjugate operator for $\mathcal{P}_{N_0}(\lambda)$. It suffices to build a conjugate operator for each $p^w_{q,0}(y,z,h^2D_y,hD_z)$ at the energy λ_q.

Assumption 2 $\iff$ $\forall(y,\xi)\in\mathbb{R}^2,\ \lambda_q$ is non-trapping for the symbol $p_{\xi,y}(z,z^*)=|z^*|^2+V(\xi,y,z)$

$\Longrightarrow$ For all (y,ξ) one can find an escape function $(z,z^*)\mapsto G(y,\xi,z,z^*)$

More precisely, using [Gerard-Martinez 88'], for all $(y,\xi)\in\mathbb{R}^2$ one can find a bounded function $(z,z^*)\mapsto G(y,\xi,z,z^*)$ such that:

$$\forall(z,z^*)\in T^*\mathbb{R}^{n-2},\ H_{p_{\xi,y}}G(y,\xi,z,z^*)\geq 0$$

$$\forall(z,z^*)\in p^{-1}_{\xi,y}([\lambda_q-\epsilon,\lambda_q+\epsilon]),\ H_{p_{\xi,y}}G(y,\xi,z,z^*)\geq 1.$$

Therefore, for $\chi \in C_0^\infty(\mathbb{R})$ localizing near $\{\lambda_q\}$ and all (y, ξ)

$$i\chi(p_{\xi,y}^w(z, hD_z))[p_{\xi,y}^w(z, hD_z), G^w(y, \xi, z, hD_z)]\chi(p_{\xi,y}^w(z, hD_z)) \\ \geq h\chi^2(p_{\xi,y}(z, hD_z))$$

From the symbolic calculus (in the variable y) with operator-valued symbols, we have

$$[p_{q,0}^w(y, z, h^2 D_y, hD_z), G^w(y, h^2 D_y, z, hD_z)] \\ = [p_{\xi,y}^w(z, hD_z), G^w(y, \xi, z, hD_z)](y, h^2 D_y) + \mathcal{O}(h^2)$$

Combining the two last equations, we obtain the Mourre estimate.

□

III.2.3 Egorov Lemma

For $t > 0$, let $\phi_t : T^*\mathbb{R}^{n-1} \to T^*\mathbb{R}^{n-1}$ be defined by

$$\phi_t(y, z, \xi, z^*) = \exp(tH_{p_{\xi,y}})(z, z^*).$$

Lemma 2 *Let $\omega_1, \omega_2 \in S^0(T^*\mathbb{R}^{n-1}_{y,z})$ such that $\operatorname{supp}\omega_2 \cap \phi_t(\operatorname{supp}(\omega_1)) = \emptyset$ and ω_1 is compactly supported, then*

$$\|\omega_2^w(y, z, h^2 D_y, hD_z) e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda)} \omega_1^w(y, z, h^2 D_y, hD_z)\| = \mathcal{O}(h^\infty),$$

for all $\alpha > 1/2$.

Proof: Use the fact that the dynamics is governed by the variable z, whereas the variable y produces terms of order $\mathcal{O}(h^2)$. □

Consequences: Using this lemma, microlocal resolvent estimates and the following formula

$$\mathcal{P}_{N_0}(\lambda + i0)^{-1} = ih^{-1} \int_0^T e^{-ih^{-1}t\mathcal{P}_{N_0}(\lambda+i0)} dt + \mathcal{P}_{N_0}(\lambda)^{-1} e^{-ih^{-1}T\mathcal{P}_{N_0}(\lambda+i0)},$$

we obtain

$$f_{p,q}(\omega, \omega', \lambda, h) = \int_0^{T_0} \int_{\mathbb{R}^{n-2}} e^{-ih^{-1}\sqrt{\lambda_p}\langle z,\omega'\rangle} [\Delta_z, \chi_2] e^{ih^{-1}t(\lambda_q - P_q(h))} [\Delta_z, \chi_1] e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} dz dt + \mathcal{O}(h),$$

with $P_q(h) = p_q^w(y, z, h^2 D_y, hD_z)$.

III.2.4 Approximation of the evolution

Our aim is to approximate the operator $\Omega(t) : L^2(\mathbb{R}_y) \to L^2(\mathbb{R}^2_{y,z})$ defined by

$$(\Omega(t)\varphi)(y,z) = e^{-ih^{-1}tP_q(h)}[h^2\Delta_z, \chi_1]e^{ih^{-1}\sqrt{\lambda_q}\langle z, \rangle\omega}\varphi(y)$$

for all $\varphi \in L^2(\mathbb{R}_y)$. We look for $\Omega(t)\varphi$ under the form $\tau^w(t,y,z,h^2D_y,h)\varphi$. We have to solve

$$\begin{cases} ih\partial_t\tau^w(t,y,z,h^2D_y,h) - P_q(h)\tau^w(t,y,z,h^2D_y,h) = 0 \\ \tau(t=0,y,z,h^2D_y,h)^w\varphi = [h^2\Delta_z,\chi_1]e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle}\varphi(y) \end{cases}$$

From the symbolic calculus, it follows that τ must be solution of

$$\begin{cases} (ih\partial_t - L^w(y,z,\xi,hD_y,hD_z,hD_\xi,h))\tau(t,y,z,\xi,h) = 0 \\ \tau(0,y,z,\xi,h) = [h^2\Delta_z,\chi_1]e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} \end{cases}$$

where formally,

$$L(y,z,\xi,y^*,z^*,\xi^*) = \sum_{\alpha,\beta,m\in\mathbb{N}} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!}\partial_y^\alpha\partial_\xi^\beta p_{q,m}(y,z,\xi,z^*)(\xi^*)^\alpha(y^*)^\beta.$$

In fact, for $N \in \mathbb{N}^*$ to be chosen large enough, we look for τ_N solution of

$$\begin{cases} (ih\partial_t - L_N^w(y,z,\xi,hD_y,hD_z,hD_\xi,h))\tau_N(t,y,z,\xi,h) = \mathcal{O}(h^N) \\ \quad \tau_N(0,y,z,\xi,h) = [\Delta_z,\chi_1]e^{ih^{-1}\sqrt{\lambda_q}\langle z,\omega\rangle} \end{cases} \tag{2}$$

with

$$L_N(y,z,\xi,y^*,z^*,\xi^*) = \sum_{|\alpha+\beta+m|\leq N} \frac{h^{\alpha+\beta+m}(-1)^\alpha}{2^{\alpha+\beta}\alpha!\beta!}\partial_y^\alpha\partial_\xi^\beta p_{q,m}(y,z,\xi,z^*)(\xi^*)^\alpha(y^*)^\beta.$$

Remark that the principal symbol of $L_N(y,z,\xi,y^*,z^*,\xi^*)$ is given by

$$l_0(y,z,\xi,y^*,z^*,\xi^*) = |z^*|^2 + V(\xi,y,z),$$

so that the corresponding Hamiltonian system is

$$\begin{cases} \dot{Z} = 2Z^*, \quad \dot{Z}^* = -\nabla_z V(\Xi,Y,Z) \\ \dot{Y} = 0, \quad \dot{Y}^* = -\nabla_y V(\Xi,Y,Z) \\ \dot{\Xi} = 0, \quad \dot{\Xi}^* = -\nabla_x V(\Xi,Y,Z) \end{cases}$$

In particular, Y and Ξ are constant, say $(Y,\Xi) = (y,\xi)$ so that the solution of the two first lines with initial condition (z,z^*) is given by

$$(Z,Z^*)(t,y,z,\xi,z^*) = \exp(tH_{p_{\xi,y}})(z,z^*)$$

From Assumption 3, we deduce that for $T > 0$ large enough and (y,ξ,z) in a suitable compact set, the point $(Y,Z,\Xi)(t,y,z,\xi,\sqrt{\lambda_q}\omega)$ is non-focal in the Maslov sense:

$$D_q(t,y,,\xi) := \det\frac{\partial(Y,Z,\Xi)}{\partial(y,z,\xi)}(y,z,\xi,\sqrt{\lambda_q}\omega) \neq 0.$$

From Maslov's work, we deduce

Proposition 5 *There exists some functions* $\tau_{N,j} \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_{y,z,\xi})$, $j \in \mathbb{N}$ *such that*

$$\tau_N(y,z,\xi,h) = e^{ih^{-1}S_q(t,y,\tilde{z},\xi) - i\mu_q\pi/2} \quad |D_q(t,y,\tilde{z},\xi)|^{-1/2} \sum_{j=1}^{N} h^j \tau_{N,j}(t,y,\tilde{z},\xi)$$

solves (2). Here, $z = Z(t,\xi,y,\tilde{z},\sqrt{\lambda_q}\omega)$, S_q *is the action along the trajectory joining* $\tilde{z}$ *and* z

$$S_q(t,y,\tilde{z},\xi) = \int_0^t (|Z^*(s,\xi,y,\tilde{z},\sqrt{\lambda_q}\omega)|^2 - V(\xi,y,Z(s,\xi,y,\tilde{z},\sqrt{\lambda_q}\omega))ds + \sqrt{\lambda_q}\langle \tilde{z},\omega\rangle$$

and μ_q *is the path index of this trajectory. Moreover,* $\tau_{N,0}(t,y,\tilde{z},\xi) = c_l(y,\tilde{z},\xi)$ *and* μ_q *is independent on* (y,ξ).

Remark 2 *The symbol* τ_N *is in the class* $S^{1/2}(\mathbb{R}^n, h^2)$. *In particular there is a symbolic calculus for the product of* $\tau_N(y,z,h^2D_y)$ *with pseudo whose symbol is in* S^0 *(cf. [Dimassi-Sjostrand]). This permits to justify our approximation.*

III.2.5 Stationnary phase method

The end of the proof follows [Robert-Tamura,89'].

- We replace $\Omega(t)$ by τ_N in the representation formula.
- In the integral giving the scattering amplitude do the changes of variable

$$(t, \tilde{z}) \in \mathbb{R} \times \omega^{\perp} \mapsto z = Z_{q,\infty}(t, x, y, \tilde{z}, \omega) \in \mathbb{R}^{n-2}.$$

- Conclude by stationnary phase method. The stationary points are given by the classical trajectories starting with initial momentum $\sqrt{\lambda_q}\omega$ in $t = -\infty$ and with asymptotic momentum $\sqrt{\lambda_q}\omega'$ in $t = +\infty$.