SPECTRAL ASYMPTOTICS FOR METROPOLIS ALGORITHM ON SINGULAR DOMAINS

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Abstract. We study the Metropolis algorithm on a bounded connected domain Ω of the euclidean space with proposal kernel localized at a small scale $h > 0$. We consider the case of a domain Ω that may have cusp singularities. For small values of the parameter h we prove the existence of a spectral gap $g(h)$ and study the behavior of $g(h)$ when h goes to zero. As a consequence, we obtain exponentially fast return to equilibrium in total variation distance.

Contents

1. Introduction

Let Ω be a bounded connected open subset of $\mathbb{R}^d$ and let $\rho(x)$ be a positive measurable function on $\bar{\Omega}$ such that

$$\forall x \in \Omega,\ m \leq \rho(x) \leq M \tag{1.1}$$

for some constants $m, M > 0$. We denote $\mu_\rho = \rho(x)dx$ the associated measure on Ω and we assume that $\mu_\rho(\Omega) = \int_\Omega \rho(x)dx = 1$. We consider the Metropolis algorithm associated to the density ρ defined as follows. For all $h \in]0, 1]$, we define the distribution kernel

$$k_{h,\rho}(x, y) = h^{-d}\phi(\frac{x-y}{h}) \min(\frac{\rho(y)}{\rho(x)}, 1) \tag{1.2}$$

where $\phi(z) = \frac{1}{\mathscr{V}_d}\mathbb{1}_{B(0,1)}(z)$, $B(0, 1)$ denotes the open unit ball in $\mathbb{R}^d$ and $\mathscr{V}_d$ is the volume of $B(0, 1)$. The Metropolis kernel, is then given by

$$t_{h,\rho}(x, dy) = m_{h,\rho}(x)\delta_{y=x} + k_{h,\rho}(x, y)dy \tag{1.3}$$

where $m_{h,\rho}(x) = 1 - \int_\Omega k_{h,\rho}(x,y)dy$. The kernel $t_{h,\rho}(x,dy)$ is clearly a Markov kernel on the state space Ω and the associated operator

$$T_{h,\rho}(u)(x) = m_{h,\rho}(x)u(x) + \int_\Omega k_{h,\rho}(x,y)u(y)dy \tag{1.4}$$

is a Markov operator. Throughout the paper, we sometimes omit the dependance of this operator with respect to ρ and write T_h instead of $T_{h,\rho}$ when there is no ambiguity. A straightforward computation shows that $T_{h,\rho}$ is self-adjoint on $L^2(\Omega, \rho(x)dx)$ which implies in particular that the measure μ_ρ is stationary for the kernel $t_{h,\rho}(x,dy)$. As a consequence, the iterated kernel $t^n_{h,\rho}(x,dy)$ converges to the measure μ_ρ as $n \to \infty$, which explains the use of this kernel to sample the measure μ_ρ.

Introduced in [8] to compute thermodynamical functionals by Monte-Carlo method, this algorithm has shown an impressive efficiency and is now used as a routine in many domains of science. From a theoretical point of view, the computation of the speed of convergence of the algorithm aroused many works in the setting of discrete state spaces (see [1], [4] for introduction to this topic and references). In [2], we obtained first results on a continuous state space in the limit $h \to 0$. More precisely, given a bounded domain Ω of $\mathbb{R}^d$ with Lipschitz boundary we proved that the operator T_h admits a spectral gap $g(h)$ of order h^2 and for smooth densities ρ, we did compute the limit of $h^{-2}g(h)$. Eventually, we obtained some total variation estimates

$$\sup_{x\in\Omega} \|t^n_{h,\rho}(x,dy) - d\mu_\rho(y)\|_{TV} \leq Ce^{-ng(h)} \tag{1.5}$$

for some constant $C > 0$ independent of h. In this approach the fact that $\partial\Omega$ has Lipschitz regularity plays a fundamental role at several stages. A natural question is then to explore situations where this regularity assumption on $\partial\Omega$ fails to be true. In the present paper, we consider the case where $\partial\Omega$ may have cuspidal singularities. More precisely we introduce the following assumption:

Assumption 1. *There exist a finite collection of open subsets of* $\mathbb{R}^d$, $(\omega_i)_{i\in I_r\cup I_c}$ *such that* $\partial\Omega \subset (\cup_{i\in I_r\cup I_c}\omega_i)$ *and*

i) *for all* $i \in I_r$, $\partial\Omega\cap\omega_i$ *has Lipschitz regularity,*
ii) *for all* $i \in I_c$, *there exists a closed submanifold* S_i *of* $\mathbb{R}^d$ *with dimension* d''_i, *and there exist* $\alpha_i > 1, r_i > 0, \epsilon_i > 0$ *and a coordinate system* $(x_1, x', x'') \in \mathbb{R}^d = \mathbb{R}\times\mathbb{R}^{d'_i}\times\mathbb{R}^{d''_i}$, *such that*

$$\Omega\cap\omega_i = \{(x_1,x',x''), 0 < x_1 < \epsilon_i,\ |x'|_{d'_i} < x_1^{\alpha_i},\ |x''|_{d''_i} < r_i\} \tag{1.6}$$

where $|\cdot|_k$ *stands for the euclidean norm on* $\mathbb{R}^k$.

Throughout the paper we will denote

$$\gamma = \max_{i\in I_c}(\alpha_i - 1)d'_i. \tag{1.7}$$

In our main results we need the cusp singularities to be not too sharp. We then introduce the following

Assumption 2. *The constant* γ *defined by* (1.7) *satisfies* $0 < \gamma < 2$.

Observe that as soon as I_c is non empty (that is there exists some cusps on the boundary), one has $\gamma > 0$. Under the above assumption one has the following rough localization of the spectrum $\sigma(T_h)$ of T_h. The proof of this result will be given in the next section.

Proposition 1.1. *Assume that Assumption 1 holds true. Then there exist* $\delta_1, \delta_2 > 0$ *and* $h_0 > 0$ *such that for all* $h \in]0, h_0]$, $\sigma(T_h) \subset [-1 + \delta_1 h^\gamma, 1]$ *and the essential spectrum satisfies* $\sigma_{ess}(T_h) \subset [-1 + \delta_1 h^\gamma, 1 - \delta_2 h^\gamma]$ *where* γ *is defined by* (1.7).

From the above result, it is clear that the spectrum of T_h in the interval $[1 - Ch^\gamma, 1]$ is made of eigenvalues of finite multiplicity. Our first main result will provide precise informations on the spectrum of T_h in a box $[1 - Ch^2, 1]$ under smoothness assumptions on the density ρ. For $\rho \in \mathcal{C}^1(\overline{\Omega})$, we introduce the associated diffusion operator L_ρ defined in a weak sense as follows. Given $u \in H^1(\Omega)$, let $\ell_u : H^1(\Omega) \to \mathbb{C}$ be defined by

$$\ell_u(v) = \int_\Omega \nabla \bar{u} \nabla v \, d\mu_\rho + \int_\Omega \bar{u} v \, d\mu_\rho$$

where we recall that $d\mu_\rho = \rho(x)dx$. We define the domain of L_ρ as the set of functions $u \in H^1$ such that ℓ_u is continuous for the L^2 topology:

$$D(L_\rho) = \{u \in H^1(\Omega),\ \exists C_u > 0,\ \forall v \in H^1(\Omega),\ |\ell_u(v)| \leq C_u \|v\|_{L^2}\}$$

Observe that $D(L_\rho)$ is not empty since it contains $\mathcal{C}_c^\infty(\Omega)$ (here we use the fact that ρ is $\mathcal{C}^1$). Since $H^1(\Omega)$ is dense in $L^2(\Omega)$ then for any $u \in D(L_\rho)$, ℓ_u can be extended as a continuous linear form on $L^2(\Omega)$ and by Riesz Theorem, there exists a unique $f \in L^2(\Omega)$ such that

$$\ell_u(v) = \langle f, v \rangle_{L^2(\rho)},\ \forall v \in H^1(\Omega).$$

We then set $L_\rho u = -u + f$. From Theorem 3.6 in [5], we know that $D(L_\rho)$ is dense in $H^1(\Omega)$ and that $\mathrm{Id} + L_\rho : D(L_\rho) \to L^2(\Omega)$ is bijective with bounded inverse. Now, it follows from Assumption 1 and the Theorem of section 8.3 in [7] that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and hence the resolvent $(\mathrm{Id} + L_\rho)^{-1}$ is compact. We introduce the sequence $\nu_0 < \nu_1 < \nu_2 < ...$ of the distinct eigenvalues of L_ρ with associated multiplicities m_j. Since, L_ρ is clearly non-negative and 0 is a simple eigenvalue, it follows $\nu_0 = 0$ and $m_0 = 1$.

Theorem 1.2. *Suppose that* $\rho \in \mathcal{C}^1(\bar{\Omega})$ *satisfies* (1.1). *Suppose that Assumptions 1 and 2 are verified. Let* $R > 0, \epsilon > 0$ *and* $J > 0$ *such that for all* $j \leq J$, $\nu_j < R$ *and for all* $j < J$, $\nu_{j+1} - \nu_j > 2\epsilon$. *Then there exists* $h_0 > 0$ *such that for all* $h \in]0, h_0]$,

$$\sigma\left(\frac{1 - T_h}{h^2}\right) \cap]0, R] \subset \cup_{j \geq 1} [\nu_j - \varepsilon, \nu_j + \varepsilon], \tag{1.8}$$

and the number of eigenvalues of $\frac{1-T_h}{h^2}$ *counted with multiplicities, in the interval* $[\nu_j - \varepsilon, \nu_j + \varepsilon]$, *is equal to* m_j.

Observe that this theorem is the analogous of Theorem 1.2 in [2]. Here we assume Assumption 2 to insure that there is no essential spectrum in the interval $[1 - Ch^2, 1]$. The case where $\gamma \geq 2$ seems more difficult to deal with since in this case the eigenvalues would be embedded in the essential spectrum.

If we drop the smoothness assumption on the density ρ we get the following results.

Theorem 1.3. *Assume that ρ is a measurable function satisfying* (1.1). *Suppose that Assumptions 1 and 2 are verified. Let $\delta_1, \delta_2 > 0$ be as in Prop. 1.1. There exists $C, h_0 > 0$ such that for any $h \in]0, h_0]$, the following hold true:*

i) The spectrum $\sigma(T_h)$ of T_h is contained in $[-1+\delta_1 h^\gamma, 1]$, 1 is a simple eigenvalue of T_h, and $\sigma(T_h) \cap [1-\delta_2 h^\gamma, 1]$ is discrete.

ii) The spectral gap $g(h) := \operatorname{dist}(1, \sigma(T_h) \setminus \{1\})$ satisfies

$$\frac{1}{C} h^2 \leq g(h) \leq C h^2. \tag{1.9}$$

As we shall see later, using (1.1) and comparaison of Dirichlet forms, this theorem is essentially a consequence of Theorem 1.2. From this spectral result we deduce estimates on the speed of convergence of the iterated kernel $(t^n_{h,\rho}(x, dy))$ towards the stationary measure μ_ρ. We recall that the total variation distance between two probability measures μ and ν is defined by $\|\mu - \nu\|_{TV} = \sup_{A \in \mathscr{B}} |\mu(A) - \nu(A)|$ where $\mathscr{B}$ denotes the set of Borel set. Moreover, one has

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sup_{f \in L^\infty, |f| \leq 1} |\int f d\mu - \int f d\nu|. \tag{1.10}$$

Theorem 1.4. *Assume that ρ is a measurable function satisfying* (1.1). *Suppose that Assumptions 1 and 2 are is verified. There exists $C, h_0 > 0$ such that for any $h \in]0, h_0]$ one has*

$$sup_{x \in \Omega} \|t^n_{h,\rho}(x, dy) - \mu_\rho\|_{TV} \leq C h^{-\gamma - \frac{d}{2}} e^{-ng(h)(1+O(h^{2-\gamma}))}. \tag{1.11}$$

for all $n \in \mathbb{N}$.

Compare to Theorem 1.1 in [2], the estimate (1.10) above suffers a loss of $h^{-\frac{d}{2}-\gamma}$ in front of the exponential. This loss is the natural loss when you go from convergence in L^2 sense (which follows from the spectral gap) to convergence in total variation. In [2], we used sophisticated tools (Nash estimates, Weyl asymptotics) to absorb this loss. In the present case, this strategy fails because of the cusp where nice estimates of eigenfunctions of T_h can not be obtained from (see Lemma 3.1). However, let us emphasize that this prefactor implies only a logarithmic loss in the time needed to reach equilibrium ($h^{-2} \log(h)$ instead of h^{-2}).

The proof of the above theorems follows the general strategy of [2]. In section 1, we prove Proposition 1.1. In order to prove Theorem 1.2 one uses minimax principle and quasimodes built from the eigenfunctions of L_ρ to prove that $h^{-2}(1 - T_h)$ has at least m_j eigenvalue near ν_j. The proof of the converse inequality is more difficult and requires to prove some regularity property of eigenfunctions of $1 - T_h$. This is done by mean of a dyadic decomposition of the cusp in section 3. Using these constructions we prove the main theorems in section 4. In a separate appendix we prove a gluing lemma of H^1 functions which is crucially used in the proof of the main result.

We conclude this introduction with some notations used in the sequel. On $\mathbb{R}^d$, we will denote by $|x|_d$ the euclidean norm of a vector x. When there is no ambiguity we will drop the index d and simply write $|x|$. Given a function $f : x = (x_1, x', x'') \in \mathbb{R}^{1+d'+d''} \mapsto f(x_1, x', x'') \in \mathbb{R}$ we will denote by $\nabla' f(x) \in \mathbb{R}^{d'}$ (resp. $\nabla'' f(x) \in \mathbb{R}^{d''}$) the gradient of f in the x' variable (resp. x'' variable). Given two quantities u_t, v_t depending on a parameter t, we denote $u \asymp v$ if there exists $C > 0$ such that $\frac{1}{C} u_t \leq v_t \leq C v_t$ for all t.

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2. Rough localization of the spectrum

In this section, we give a proof of Proposition 1.1. We first show that the operator $K_{h,\rho} : f \mapsto \int_\Omega k_{h,\rho}(x,y) f(y) \rho(y) dy$ is compact on $L^2(\Omega, \rho(y)dy)$. Let (ϕ_n) be a sequence of continuous functions such that $\phi \leq \phi_n \leq 1$ and (ϕ_n) converges to ϕ in $L^2(\rho(x)dx)$ when $n \to \infty$. Consider the sequence of kernels $k_{n,h,\rho} = h^{-d} \phi_n(\frac{x-y}{h}) \min(\frac{\rho(y)}{\rho(x)}, 1)$ and let $K_{n,h,\rho}$ be the associated operators. Then $(K_{n,h,\rho})$ converges to $K_{h,\rho}$ in $\mathcal{L}(L^2, L^2)$ when $n \to \infty$. On the other hand, since the kernels $k_{n,h,\rho}$ are continuous, the operators $(K_{n,h,\rho})$ are compact and hence $K_{h,\rho}$ is compact.

Let us prove that $\sigma_{ess}(T_h) \subset [-1, 1 - Ch^\gamma]$. Thanks to Weyl criterium and compactness of $K_{h,\rho}$ it is sufficient to prove that $\sup_{x \in \Omega} m_{h,\rho}(x) \leq 1 - Ch^\gamma$. Since

$$1 - m_{h,\rho}(x) \geq \frac{m h^{-d}}{M \mathscr{V}_d} \int_\Omega \mathbb{1}_{|x-y|<h} dy,$$

with m, M given by (1.1) the proof reduces to show that there exists $C, h_0 > 0$ such that

$$\forall h \in]0, h_0],\ \forall x \in \Omega,\ \theta_h(x) \geq C h^{d+\gamma} \tag{2.1}$$

where $\theta_h(x) := \int_\Omega \mathbb{1}_{|x-y|<h} dy$. Consider the family of subsets ω_i of Assumption 1 and let $\mathcal{O}_i = \Omega \cap \omega_i$. By a compactness argument, we can assume that there exists a family of open sets $(\tilde{\omega}_i')$, such that $\bar{\omega}_i' \subset \omega_i$ for all $i \in I_r \cup I_c$ and Assumption 1 holds true with the ω_i'. It follows that $\Omega = \cup_{i \in J}^m \mathcal{O}_i'$ with $\mathcal{O}_i' = \omega_i' \cap \Omega$ where $J = I_r \cup I_c \cup \{0\}$, and ω_0' is an open subset of Ω such that $d(\bar{\omega}_0', \partial\Omega) > 0$. Let us now estimate the function θ_h on each $\mathcal{O}_i'$.

We first observe that for $0 < h_0 < d(\bar{\omega}_0', \partial\Omega)$ and $h \in]0, h_0]$, one has $B(x,h) \subset \Omega$ for any $x \in \mathcal{O}_0'$ and hence $\theta_h(x) = h^d \mathscr{V}_d$ which establishes the bound (2.1) on $\mathcal{O}_0'$. Let us now study θ_h on $\mathcal{O}_i'$, $i \in I_r \cup I_c$. Taking $h_0 > 0$ sufficiently small, we can assume that for all $h \in]0, h_0]$ one has $\omega_i' + B(0,h) \subset \omega_i$ for all $i \in I_r \cup I_c$. Hence, if $\varphi : U_i \to \omega_i$ is a smooth local change of coordinates then for any $x \in \mathcal{O}_i'$, one has

$$\begin{aligned} \theta_h(x) &= \int_\Omega \mathbb{1}_{|x-y|<h} dy \geq \int_{\mathcal{O}_i} \mathbb{1}_{|\varphi(\varphi^{-1}(x))-y|<h} dy \\ &= \int_{U_i^+} J_\varphi(y) \mathbb{1}_{|\varphi(\varphi^{-1}(x))-\varphi(y)|<h} dy \end{aligned} \tag{2.2}$$

where $J_\varphi(y)$ denotes the Jacobian of φ and $U_i^+ = \varphi^{-1}(\mathcal{O}_i)$. On the other hand, since φ is a smooth function, there exists $C > 0$ such that for all $u, v \in U_i$, $|\varphi(u) - \varphi(v)| \leq C|u - v|$. Combined with (2.2), this implies

$$\theta_h(x) \geq \int_{U_i^+} J_\varphi(y)\mathbb{1}_{|\varphi^{-1}(x)-y|<h/C}dy \geq \tilde{C}\int_{U_i^+} \mathbb{1}_{|\varphi^{-1}(x)-y|<h/C}dy \tag{2.3}$$

for some positive constant $\tilde{C}$ such that $|J_\varphi| \geq \tilde{C}$ on U_i. This minoration shows that in order to get some lower bound on θ_h, we can suppose that we are in any suitable system of coordinates.

Suppose that $i \in I_r$. By a Lipschitz change of coordinates it is shown in [2] that there exists some constants $c_1, c_2 > 0$ such that

$$\theta_h(x) \geq c_1 \int_{x_1 \geq 0} \mathbb{1}_{|x-y|<h} dy \geq c_2 h^d \tag{2.4}$$

for all $x \in \mathcal{O}_i'$. Combined with the definition of θ_h, this shows that $(1 - m_h(x)) \geq c_3$ for some $c_3 > 0$ independent of h.

Suppose now that $i \in I_c$ and that ω_i is like in ii) of Assumption 1. Using a suitable change of coordinates, we can assume that there exist $\alpha > 1, r > 0, \epsilon > 0$ such that

$$\mathcal{O}_i = \Omega \cap \omega_i = \{(x_1, x', x''), 0 < x_1 < \epsilon,\ |x'|_{d'} < x_1^\alpha,\ |x''|_{d''} < r\},$$

where d', d'' are the local dimension appearing in Assumption 1 whose dependance with respect to the index i is omitted. Moreover, we can also assume that $\mathcal{O}_i' = \mathcal{O}_i \cap \{0 < x_1 < \epsilon/2\} \cap \{|x''| < r/2\}$. Endowing $\mathcal{O}_i$ with the equivalent norm

$$|(x_1, x', x'')|_\infty = \max\{|x_1|, |x'|_{d'}, |x''|_{d''}\}, \tag{2.5}$$

it is sufficient to find a lower bound for $\int_\Omega \mathbb{1}_{|x-y|_\infty < h} dx$ when x varies in $\mathcal{O}_i'$. For such x, one has

$$\begin{aligned}\int_\Omega \mathbb{1}_{|x-y|_\infty<h} dy &= \int_\Omega \mathbb{1}_{|(x_1,x',x'')-(y_1,y',y'')|_\infty<h} dy_1 dy' dy'' \\ &= \int_{|y''|_{d''}<r, |y'|_{d'}<y_1^\alpha, 0<y_1<\epsilon} \mathbb{1}_{|x''-y''|_{d''}<h} \\ &\qquad\qquad \mathbb{1}_{|x'-y'|_{d'}<h}\mathbb{1}_{|x_1-y_1|<h} dy_1 dy' dy'' \\ &\geq c h^{d''} W_h(x, x')\end{aligned} \tag{2.6}$$

where c is a positive constant and

$$W_h(x, x') := \int_{|y'|_{d'}<y_1^\alpha, 0<y_1<\epsilon} \mathbb{1}_{|y'-x'|_{d'}<h}\mathbb{1}_{|x_1-y_1|<h} dy' dy_1. \tag{2.7}$$

Denoting

$$\mathcal{C} = \{(y_1, y') \in \mathbb{R} \times \mathbb{R}^{d'},\ |y'|_{d'} < y_1^\alpha,\ 0 < y_1 < \epsilon\}$$

and

$$D_h(x_1, x') = \{y \in \mathcal{C},\ |y' - x'|_{d'} < h \text{ and } |x_1 - y_1| < h\},$$

we have $W_h(x_1, x') = vol(D_h(x_1, x'))$ and thanks to (2.6), one has to prove that $W_h(x_1, x') \geq c h^{\alpha d' + 1}$ for some uniform constant $c > 0$. We first observe that it holds true for $(x_1,x')=(0,0)$, since one has (using $\alpha>1$)

$$W_h(0,0)=\int_0^h\int_{\mathbb{R}^{d'}}\mathbb{1}_{B_{d'}(0,y_1^\alpha)}(y')dy'dy_1=\mathscr{V}_{d'}\int_0^h y_1^{\alpha d'}dy_1=ch^{\alpha d'+1}. \tag{2.8}$$

We now decompose the cusp into three zones that we treat differently: $\{0<x_1\leq h/2\}$, $\{h/2<x_1<(\delta h)^{\frac{1}{\alpha}}\}$ and $\{(\delta h)^{\frac{1}{\alpha}}<x_1<\epsilon\}$, where $\delta>0$ will be chosen sufficiently small.

- Suppose first that $x_1<\frac{h}{2}$, then since $\alpha>1$, one has for h small enough $D_h(x_1,x')\supset\{(y_1,y'),\ |y_1|<h/2\}\cap\mathcal{C}$. Combined with (2.8), this yields

$$W_h(x_1,x')\geq\int_{|y'|_{d'}<y_1^\alpha}\mathbb{1}_{|y_1|<h/2}dy_1dy'=W_{\frac{h}{2}}(0,0)=ch^{\alpha d'+1}$$

  which is the required lower bound on W_h.
- Suppose now that $h/2\leq x_1<(\delta h)^{\frac{1}{\alpha}}$, then

$$D_h(x_1,x')\supset\{x_1<y_1<x_1+h,\ |y'|<x_1^\alpha\}.$$

  Indeed, if $|y'|<x_1^\alpha$ and $y_1>x_1$ one gets immediately $(y_1,y')\in\mathcal{C}$ and since $|x'|<x_1^\alpha<\delta h$ then $|x'-y'|\leq|x'|+|y'|\leq 2\delta h<h$ for $0<\delta<\frac{1}{2}$. From the above inclusion, it follows

$$W_h(x_1,x')\geq\mathscr{V}_{d'}x_1^{\alpha d'}h\geq 2^{-\alpha d'}\mathscr{V}_{d'}h^{\alpha d'+1}.$$

- Eventually, suppose that $(\delta h)^{\frac{1}{\alpha}}\leq x_1<\epsilon$. We observe that the application $x'\mapsto W_h(x_1,x')$ is radial. Hence, it suffices to estimate from below the application $t\in[0,x_1^\alpha]\mapsto W_h(x_1,x'_t)$ with $x'_t=(t,0,\dots,0)$.
  - If $|t|<\delta h/2$ then the inclusion

$$D_h(x_1,x'_t)\supset\{x_1<y_1<x_1+h,\ |x'_t-y'|<\delta h/2\}$$

    implies $W_h(x_1,x'_t)\geq h(\delta h/2)^{d'}=ch^{d'+1}$.
  - If $\delta h/2\leq|t|<x_1^\alpha$ then

$$\{|y'-x'_{t_h}|<\delta h/4\}\subset\{|y'|<|t|\}\subset\{|y'|<x_1^\alpha\}$$

    where $t_h=t-\delta h/4$ and $x'_{t_h}=(t_h,0,\dots,0)$. Hence

$$D_h(x_1,x'_t)\supset\{x_1<y_1<x_1+h,\ |x'_{t_h}-y'|<\delta h/4\}$$

    which implies again that $W_h(x_1,x'_t)\geq ch^{d'+1}$.

  Summing up the above discussion, we have proved that for any $i\in I_c$, $W_h(x_1,x')\geq ch^{\alpha_i d'+1}$ uniformly on $\mathcal{O}'_i$. Combined with (2.6), this proves that $\theta_h(x)\geq ch^{(\alpha_i-1)d'}$ uniformly on $\mathcal{O}'_i$.

Since the boundary of Ω is compact, it follows from the above computations that there exists $c>0$ such that for all $x\in\Omega$ and all $h<h_0$, $m_h(x)\leq 1-Ch^\gamma$ with γ given by (1.7). This proves that $\sigma_{ess}(T_h)\subset[-1,1-Ch^\gamma]$.

We now prove that $\sigma(T_h)\subset[-1+Ch^\gamma,1]$ which is equivalent to show that

$$\langle u+T_hu,u\rangle_{L^2(\rho)}\geq Ch^\gamma||u||^2_{L^2(\rho)}$$

for all $u \in L^2(\Omega)$. For this purpose, we observe that thanks to [2], eq. (2.7), one has

$$\langle u + T_h u, u\rangle_{L^2(\rho)} \geq \frac{1}{2}\int_{\Omega\times\Omega} k_{h,\rho}(x,y)|u(x)+u(y)|^2\rho(x)dxdy.$$

Hence, it is sufficient to prove that there exist $C_0, h_0 > 0$ such that the following inequality holds true for all $h \in]0, h_0]$ and all $u \in L^2(\Omega)$:

$$\int_{\Omega\times\Omega} k_{h,\rho}(x,y)|u(x)+u(y)|^2\rho(x)dxdy \geq Ch^\gamma \|u\|^2_{L^2(\rho)}.$$

Since ρ is bounded from below, we can assume without loss of generality that $\rho = 1$. Following [2], we introduce a covering $(\nu_j)_j$ of Ω with $\nu_j \subset \Omega$ such that $\mathrm{diam}(\nu_j) < h$ and for some $C_1 > 0$ independent of h, the number of indices k such that $\nu_j \cap \nu_k \neq \emptyset$ is less than C_1. Moreover, since $\inf_\Omega \theta_h \geq Ch^{d+\gamma}$, we can also assume that there exists a constant $C_2 > 0$ such that $\mathrm{Vol}(\nu_j) \geq C_2 h^{d+\gamma}$ for any j. Working as in section 2 of [2], we get

$$\begin{aligned}
C_1 \int_{\Omega\times\Omega} & h^{-d}\phi(\frac{x-y}{h})|u(x)+u(y)|^2 dxdy \\
&\geq \sum_j \int_{\nu_j\times\nu_j} h^{-d}\phi(\frac{x-y}{h})|u(x)+u(y)|^2 dxdy \\
&\geq \sum_j h^{-d}\frac{1}{\mathscr{V}_d}\int_{\nu_j\times\nu_j} |u(x)+u(y)|^2 dxdy \\
&\geq \sum_j 2h^{-d}\frac{1}{\mathscr{V}_d}\mathrm{Vol}(\nu_j)||u||^2_{L^2(\nu_j)} \\
&\geq \frac{2C_2 h^\gamma}{\mathscr{V}_d}||u||^2_{L^2(\Omega)}.
\end{aligned}$$

This implies, $\langle u + T_h u, u\rangle \geq \tilde{C}h^\gamma ||u||^2$ and finally $\sigma(T_h) \subset [-1 + Ch^\gamma, 1]$. The proof of Proposition 1.1 is complete.

3. Regularity of eigenfunctions

The aim of this section is to prove regularity properties on families of eigenfunctions of T_h associated to eigenvalues in $[1 - Ch^\gamma, 1]$. Let us introduce the Dirichlet form of T_h

$$\mathcal{B}_{h,\rho}(f,g) := \langle (1 - T_h)f, g\rangle_{L^2(\rho)} \tag{3.1}$$

and $\mathcal{E}_{h,\rho}(f) = \mathcal{B}_{h,\rho}(f,f)$. One has

$$\mathcal{B}_{h,\rho}(f,g) = \frac{1}{2}\int_{\Omega\times\Omega} k_{h,\rho}(x,y)(f(x)-f(y))\overline{(g(x)-g(y))}\rho(x)dxdy$$

and denoting $d\mu^2_\rho = \min(\rho(x),\rho(y))dxdy$ we get

$$\mathcal{B}_{h,\rho}(f,g) = \frac{1}{2h^d\mathscr{V}_d}\int_{\Omega\times\Omega} \mathbb{1}_{|x-y|<h}(f(x)-f(y))\overline{(g(x)-g(y))}d\mu^2_\rho(x,y).$$

In particular, one has

$$\mathcal{E}_{h,\rho}(f) = \frac{1}{2h^d\mathscr{V}_d}\int_{\Omega\times\Omega} \mathbb{1}_{|x-y|<h}|f(x)-f(y)|^2 d\mu^2_\rho(x,y). \tag{3.2}$$

As mentioned before, we will sometimes drop index ρ in the notations when it is unambiguous. The following decomposition lemma is the key point in our analysis.

Lemma 3.1. *Let $(f_h)_{h\in]0,1]}$ be a family of function in $L^2(\Omega)$ such that $\|f_h\|_{L^2} \leq 1$ and $\mathcal{E}_h(f_h) \leq h^2$. Then, there exists $C, C_0, h_0 > 0$ such that for all $h \in]0, h_0]$, one has a decomposition $f_h = f_{h,\mathcal{C}} + f_{h,L} + f_{h,H}$ with*

- $\mathrm{supp}(f_{h,\mathcal{C}}) \subset \Gamma_{2h}$ *with* $\Gamma_h = \cup_{i\in I_c}\{x \in \omega_i,\ d(x, S_i) < Ch^{\frac{1}{\alpha_i}}\}$, ω_i, α_i, S_i *given by Assumption 1*
- $f_{h,L}$ *and* $f_{h,H}$ *are supported in* $\Omega \setminus \Gamma_h$ *and*

$$||\nabla f_{h,L}||_{L^2} \leq C_0 \text{ and } ||f_{h,H}||_{L^2} \leq C_0 h$$

This lemma is inspired from Lemma 2.2 in [2]. However, due to the presence of cusps there is an additional term in the decomposition of f_h for which we do not have nice estimates. Moreover, we have to face important complications in the proof. The next section is devoted to the proof of this lemma in the particular case where Ω is a model cusp.

3.1. **A model case.** In this section we consider the case where the domain Ω is an exact cusp

$$\Omega = \{(x_1, x', x''),\ 0 < x_1 < 1, |x'|_{d'} < x_1^\alpha, |x''|_{d''} < 1\}. \tag{3.3}$$

Since there is no ambiguity, Ω denotes the above domain in this section and a general domain in the rest of the paper. Since ρ is bounded from below and above by positive constant, we can assume that $\rho = 1$ without modifying the assumption $\mathcal{E}_h(f_h) = \mathcal{O}(h^2)$. One defines a dyadic partition $(\Omega_k)_{k\geq 0}$ of Ω in the following way:

$$\Omega_k := \Omega \cap \{\frac{1}{2^{k+1}} < x_1 < \frac{1}{2^k}\}, k \in \mathbb{N}.$$

For every $k \geq 0$, we define a change of variables

$$\begin{array}{rccc} \tau_k : & \Omega_k & \rightarrow & \Omega_0 \\ & (x_1, x', x'') & \mapsto & (2^k x_1, 2^{k\alpha} x', x'') \end{array} \tag{3.4}$$

whose jacobian is $j_k := \det d\tau_k = 2^{k(\alpha d'+1)}$. We also introduce the change of variable

$$\begin{array}{rccc} \hat{\tau}_k : & \Omega_k & \rightarrow & \Omega_1 \\ & (x_1, x', x'') & \mapsto & (2^{k-1} x_1, 2^{(k-1)\alpha} x', x'') \end{array} \tag{3.5}$$

and we observe that $\tau_k = \tau_1 \circ \hat{\tau}_k$.

3.1.1. *Sobolev space and dyadic decomposition of cusps.* Throughout the paper we will use the following notation. Given a set B a function $f \in H^1(B)$, and some parameters $h, \tilde{h}, \bar{h} > 0$, we denote

$$N_{\bar{h},\tilde{h},h}(f, B) = \Big(\|\bar{h}\partial_1 f\|^2_{L^2(B)} + \|\tilde{h}\nabla' f\|^2_{L^2(B)} + \|h\nabla'' f\|^2_{L^2(B)}\Big)^{\frac{1}{2}}. \tag{3.6}$$

In order to lighten the notation we introduce the parameter $\mathbf{h} = (\bar{h}, \tilde{h}, h)$ and we will often write

$$N_{\mathbf{h}}(f, B) = N_{\bar{h},\tilde{h},h}(f, B).$$

The following lemma gives an expression of Sobolev norms for dyadic decomposition of the domain Ω.

Lemma 3.2. *Let $f \in L^2(\Omega)$, then*

$$\|f\|^2_{L^2(\Omega)} = \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} \|f \circ \tau_k^{-1}\|^2_{L^2(\Omega_0)}$$

If one assume additionally that $f \in H^1(\Omega)$, then

$$\begin{aligned}\|\nabla f\|^2_{L^2(\Omega)} &= \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} \Big(\|2^k \partial_1 (f\circ\tau_k^{-1})\|^2_{L^2(\Omega_0)} \\ &\qquad + \|2^{k\alpha}\nabla'(f\circ\tau_k^{-1})\|^2_{L^2(\Omega_0)} + \|\nabla''(f\circ\tau_k^{-1})\|^2_{L^2(\Omega_0)}\Big) \\ &= \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} N_{2^k,2^{k\alpha},1}(f\circ\tau_k^{-1},\Omega_0)^2\end{aligned}$$

Proof. Use the partition $\Omega = \cup_{k\in\mathbb{N}}\Omega_k$, the change of variable τ_k and the chain rule. ☐

Let

$$\theta: \begin{array}{c} \mathbb{R}^*_+ \times \mathbb{R}^{d'} \times \mathbb{R}^{d''} \to \mathbb{R}^*_+ \times \mathbb{R}^{d'} \times \mathbb{R}^{d''} \\ (x_1, x', x'') \mapsto (x_1, x_1^{-\alpha}x', x'') \end{array} \tag{3.7}$$

and consider the open sets $B_j := \{\frac{1}{2^{j+1}} < x_1 < \frac{1}{2^j}, |x'| < 1, |x''| < 1\}$. Observe that θ is a C^1 diffeomorphism from Ω_j onto B_j. Hence, the maps

$$\sigma_k = \theta\circ\tau_k : \Omega_k \to B_0 \tag{3.8}$$

and

$$\hat{\sigma}_k = \theta\circ\hat{\tau}_k : \Omega_k \to B_1 \tag{3.9}$$

are also C^1 diffeomorphisms. Moreover, one has $\sigma_k = \check{\sigma}_1 \circ \hat{\sigma}_k$ where

$$\check{\sigma}_1 = B_1 \to B_0,\ \check{\sigma}_1 = \sigma_1\circ\theta^{-1} = \theta\circ\tau_1\circ\theta^{-1}. \tag{3.10}$$

The following lemma express L^2 and H^1 norm in terms of the dyadic decomposition.

Lemma 3.3. *One has the following estimates*

$$\|f\|^2_{L^2(\Omega)} \asymp \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} \|f\circ\sigma_k^{-1}\|^2_{L^2(B_0)} \tag{3.11}$$

for any $f \in L^2(\tilde{\Omega})$ and

$$\|\nabla f\|^2_{L^2(\Omega)} \asymp \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} N_{2^k,2^{k\alpha},1}(f\circ\sigma_k^{-1}, B_0)^2 \tag{3.12}$$

for any $f \in H^1(\Omega)$. Conversely, assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence of functions of $H^1(B_0)$ such that

$$\sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)} (\|f_k\|^2_{L^2(B_0)} + N_{2^k,2^{k\alpha},1}(f_k, B_0)^2) < \infty \tag{3.13}$$

and $(f_k)_{|x_1=\frac{1}{2}} = (f_{k+1})_{|x_1=1}$*, where* $(f_k)_{|x_1=a}$ *denotes the trace of the* H^1 *function* f_k *on* $\{x_1 = a\}$*. Then the function* $f := \sum_{k=0}^{\infty} \mathbb{1}_{\Omega_k} f_k \circ \sigma_k$ *belongs to* $H^1(\Omega)$*. Moreover, for such functions, one has*

$$\|f\|_{H^1(\Omega)}^2 \asymp \sum_{k\in\mathbb{N}} 2^{-k(\alpha d'+1)}(\|f_k\|_{L^2(B_0)}^2 + N_{2^k,2^{k\alpha},1}(f_k, B_0)^2).$$

Proof. For any $j \geq 0$, θ defines a change of variable from Ω_j onto B_j. A standard computation shows that there exists $C > 1$ such that

$$\frac{1}{C}\|f \circ \theta^{-1}\|_{L^2(B_0)} \leq \|f\|_{L^2(\Omega_0)} \leq C\|f \circ \theta^{-1}\|_{L^2(B_0)},$$

and

$$\begin{aligned}
&\frac{1}{C}\|\nabla'(f \circ \theta^{-1})\|_{L^2(B_0)} \leq \|\nabla' f\|_{L^2(\Omega_0)} \leq C\|\nabla'(f \circ \theta^{-1})\|_{L^2(B_0)}\\
&\frac{1}{C}\|\nabla''(f \circ \theta^{-1})\|_{L^2(B_0)} \leq \|\nabla'' f\|_{L^2(\Omega_0)} \leq C\|\nabla''(f \circ \theta^{-1})\|_{L^2(B_0)}\\
&\frac{1}{C}\|\partial_1(f \circ \theta^{-1})\|_{L^2(B_0)} \leq \|\partial_1 f\|_{L^2(\Omega_0)} + \|\nabla' f\|_{L^2(\Omega_0)}\\
&\frac{1}{C}\|\partial_1 f\|_{L^2(\Omega_0)} \leq \|\partial_1(f \circ \theta^{-1})\|_{L^2(B_0)} + \|\nabla'(f \circ \theta^{-1})\|_{L^2(B_0)}
\end{aligned}$$

Combining these estimates with Lemma 3.2, we obtain (3.11) and (3.12). Conversely, assume that $f \in L^2(\Omega)$ is such that (3.13) holds true. In order to prove that $f \in H^1(\Omega)$, it suffices to show that f has no jump at $x_1 = 2^{-k}$. This exactly the condition $(f_k)_{|x_1=\frac{1}{2}} = (f_{k+1})_{|x_1=1}$. □

Remark 3.4. *If one splits the sums in the above lemma into even and odd terms, one gets*

$$\|f\|_{L^2(\tilde\Omega)}^2 \asymp \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)}\|f \circ \sigma_{2k}^{-1}\|_{L^2(B_0)}^2 + \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)}\|f \circ \sigma_{2k+1}^{-1}\|_{L^2(B_0)}^2$$

Using the identity $\sigma_{2k+1} = \sigma_1 \circ \theta^{-1} \circ \hat\sigma_{2k+1}$ *with* $\hat\sigma_{2k+1}$ *defined by* (3.9), (3.5) *and the fact that* $\theta \circ \sigma_1^{-1}$ *is a diffeomorphism from* B_0 *onto* B_1*, we get*

$$\|f\|_{L^2(\tilde\Omega)}^2 \asymp \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)}\|f \circ \sigma_{2k}^{-1}\|_{L^2(B_0)}^2 + \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)}\|f \circ \hat\sigma_{2k+1}^{-1}\|_{L^2(B_1)}^2$$

Similarly, we get the following identity for the norm of the gradient

$$\|\nabla f\|_{L^2(\tilde\Omega)}^2 \asymp \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)} N_{4^k,4^{k\alpha},1}(f \circ \sigma_{2k}^{-1}, B_0)^2 + \sum_{k\in\mathbb{N}} 4^{-k(\alpha d'+1)} N_{4^k,4^{k\alpha},1}(f \circ \hat\sigma_{2k+1}^{-1}, B_1)^2.$$

3.1.2. *Extension of the operator and comparison of Dirichlet forms.* Let $f \in L^2(\Omega)$ be such that $\|f\|_{L^2} = 1$ and $\mathcal{E}_h(f) = \mathcal{O}(h^2)$. Recall that $|.|$ denotes the euclidean norm on $\mathbb{R}^d$ and that $|.|_\infty$ is defined by (2.5). Since the norms $|.|$ and $|.|_\infty$ are equivalent, there exists a constant $C > 0$ such that

$$\mathcal{E}_{\infty,\frac{h}{C}}(f) \leq \mathcal{E}_h(f) \leq \mathcal{E}_{\infty,Ch}(f) \tag{3.14}$$

where

$$\mathcal{E}_{\infty,h}(f) := \frac{1}{2h^d\mathscr{V}_d}\int_{\Omega\times\Omega} \mathbb{1}_{|x-y|_\infty<h}|f(x)-f(y)|^2 dxdy. \tag{3.15}$$

Thanks to (3.14), one has

$$\begin{aligned}
\mathcal{E}_{Ch}(f) \geq \mathcal{E}_{\infty,h}(f) &= \frac{1}{2\mathscr{V}_d h^d}\int_{\Omega\times\Omega} \mathbb{1}_{|x-y|_\infty<h}|f(x)-f(y)|^2 dxdy\\
&\geq \sum_{k\geq 0}\frac{1}{2\mathscr{V}_d h^d}\int_{\Omega_k\times\Omega_k} \mathbb{1}_{|x-y|_\infty<h}|f(x)-f(y)|^2 dxdy\\
&= \sum_{k\geq 0}\frac{1}{2\mathscr{V}_d h^d}\int_{\Omega_0\times\Omega_0} \mathbb{1}_{|x_1-y_1|<2^k h,|x'-y'|<2^{k\alpha}h,|x''-y''|<h}\\
&\qquad |f\circ\tau_k^{-1}(x) - f\circ\tau_k^{-1}(y)|^2 j_k^{-1}(x) j_k^{-1}(y) dxdy\\
&= \sum_{k\geq 0}\frac{2^{-k(1+\alpha d')}}{2\mathscr{V}_d h^{d''}\tilde{h}_k^{d'}\bar{h}_k}\int_{\Omega_0\times\Omega_0} \mathbb{1}_{|x_1-y_1|<\bar{h}_k,|x'-y'|<\tilde{h}_k,|x''-y''|<h}\\
&\qquad |f\circ\tau_k^{-1}(x) - f\circ\tau_k^{-1}(y)|^2 dxdy
\end{aligned} \tag{3.16}$$

where $\bar{h}_k = 2^k h, \tilde{h}_k = 2^{k\alpha}h$. Given any domain $A \subset \mathbb{R}^d$, one then introduces the Dirichlet form defined on $L^2(A)$ by

$$\mathcal{E}^A_{\bar{h},\tilde{h},h}(g) = \frac{1}{2\mathscr{V}_d h^{d''}\tilde{h}^{d'}\bar{h}}\int_{A\times A} \mathbb{1}_{|x_1-y_1|<\bar{h},|x'-y'|<\tilde{h},|x''-y''|<h}|g(x)-g(y)|^2 dxdy.$$

Then the last inequality in (3.16) reads

$$\mathcal{E}_{Ch}(f) \geq \sum_{k\geq 0} 2^{-k(1+\alpha d')}\mathcal{E}^{\Omega_0}_{\bar{h}_k,\tilde{h}_k,h}(f\circ\tau_k^{-1}). \tag{3.17}$$

The next step in the computation is to compare the Dirichlet form $\mathcal{E}^{\Omega_0}_{\bar{h}_k,\tilde{h}_k,h}$ and $\mathcal{E}^{B_0}_{\bar{h}_k,\tilde{h}_k,h}$ associated respectively to the domains Ω_0 and B_0. As a preliminary step, we need the following result.

Lemma 3.5. *Let A be any open subset of $\mathbb{R}^d$ with Lipschitz boundary. For all $a, b, c > 1$, there exists $C_0, h_0 > 0$ such that for any $f \in L^2(A)$*

$$\mathcal{E}^A_{a\bar{h},b\tilde{h},ch}(f) \leq C_0\ \mathcal{E}^A_{\bar{h},\tilde{h},h}(f)$$

for all $h \in]0, h_0]$.

Proof. This is similar to the proof of Lemma 2.1 in [2]. We leave it to the reader. □

As for the Sobolev norm, we introduce the vectorial parameter $\mathbf{h} = (\bar{h}, \tilde{h}, h)$ and we denote by $\mathbf{h}\cdot x := (\bar{h}x_1, \tilde{h}x', hx'')$ the inhomogenous action of $\mathbf{h}$ on $x \in \mathbb{R}^d$. We will also denote $\mathbf{h}^{-1} = (\bar{h}^{-1}, \tilde{h}^{-1}, h^{-1})$, $\mathbf{h}^d = \bar{h}\tilde{h}^{d'}h^{d''}$ and $\mathcal{E}^A_{\mathbf{h}}(g) = \mathcal{E}^A_{\bar{h},\tilde{h},h}(g)$. With these notations, one has

$$\begin{aligned}\mathcal{E}^A_{\mathbf{h}}(g) &= \frac{1}{2\mathscr{V}_d\mathbf{h}^d}\int_{A\times A}\mathbb{1}_{|x_1-y_1|<\bar{h},|x'-y'|<\tilde{h},|x''-y''|<h}|g(x)-g(y)|^2dxdy\\ &= \frac{1}{2\mathscr{V}_d\mathbf{h}^d}\int_{A\times A}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(x-y)|_\infty<1}|g(x)-g(y)|^2dxdy\end{aligned}$$

Lemma 3.6. *There exists some constants $C>1$ and $h_0>0$ such that for any $f\in L^2(\Omega_0)$, one has*

$$\frac{1}{C}\mathcal{E}^{\Omega_0}_{\mathbf{h}}(f) \le \mathcal{E}^{B_0}_{\mathbf{h}}(f\circ\theta^{-1}) \le C\mathcal{E}^{\Omega_0}_{\mathbf{h}}(f)$$

for all $\mathbf{h} = (\bar{h},\tilde{h},h)$ such that $0<h,\tilde{h}<h_0$ and $0<\bar{h}\le\tilde{h}$.

Proof. Since the jacobian of θ^{-1} is bounded one has

$$\begin{aligned}\mathcal{E}^{\Omega_0}_{\mathbf{h}}(f) &= \frac{1}{2\mathscr{V}_d\mathbf{h}^d}\int_{\Omega_0\times\Omega_0}\mathbb{1}_{|x_1-y_1|<\bar{h},|x'-y'|<\tilde{h},|x''-y''|<h}|f(x)-f(y)|^2dxdy\\ &\le \frac{1}{2\mathscr{V}_d\mathbf{h}^d}\int_{B_0\times B_0}\mathbb{1}_{|x_1-y_1|<\bar{h},|x_1^\alpha x'-y_1^\alpha y'|<\tilde{h},|x''-y''|<h}\\ &\qquad\qquad |f\circ\theta^{-1}(x)-f\circ\theta^{-1}(y)|^2dxdy.\end{aligned}$$

On the other hand, since $\alpha\ge1$ then for $x,y\in B_0$ such that $|x_1-y_1|<\bar{h}\le\tilde{h}$, one has

$$|x_1^\alpha x'-y_1^\alpha y'|\ge|x_1|^\alpha|x'-y'|-|x_1^\alpha-y_1^\alpha||y'|\ge2^{-\alpha}|x'-y'|-C_\alpha\tilde{h}$$

for some constant $C_\alpha>0$. This implies that

$$\begin{aligned}\mathcal{E}^{\Omega_0}_{\mathbf{h}}(f) &\le \frac{1}{2\mathscr{V}_d\mathbf{h}^d}\int_{B_0\times B_0}\mathbb{1}_{|x_1-y_1|<\bar{h},|x'-y'|<M_\alpha\tilde{h},|x''-y''|<h}\\ &\qquad\qquad |f\circ\theta^{-1}(x)-f\circ\theta^{-1}(y)|^2dxdy\end{aligned}$$

with $M_\alpha = 2^\alpha(1+C_\alpha)$. Since ∂B_0 is Lipschitz, it follows from Lemma 3.5 that $\mathcal{E}^{\Omega_0}_{\mathbf{h}}(f)\le\tilde{C}_\alpha\mathcal{E}^{B_0}_{\mathbf{h}}(f\circ\theta^{-1})$, which proves the left inequality. The right inequality is proved similarly. ☐

Since for any $k\ge0$, one has $\bar{h}_k = 2^kh\le2^{k\alpha}h=\tilde{h}_k$, it follows from Lemma 3.6 and (3.17), that

$$\mathcal{O}(h^2) = \mathcal{E}_h(f)\ge\frac{1}{C}\sum_{k\ge0}2^{-k(1+\alpha d')}\mathcal{E}^{B_0}_{\bar{h}_k,\tilde{h}_k,h}(f\circ\sigma_k^{-1}). \tag{3.18}$$

Let $Q:=]0,1[^d$ and define the change of variable

$$\beta: B_0\to Q \tag{3.19}$$

given by $\beta(x_1,x',x'') = (2x_1-1,x',x'')$. Working as in Lemma 3.6, we show that there exists a constant $C>1$ such that

$$\frac{1}{C}\mathcal{E}^Q_{\mathbf{h}}(g\circ\beta^{-1})\le\mathcal{E}^{B_0}_{\mathbf{h}}(g)\le C\mathcal{E}^Q_{\mathbf{h}}(g\circ\beta^{-1}). \tag{3.20}$$

Combined with (3.18) this implies that there exists $C_0, h_0 > 0$ such that for $0 < h < h_0$

$$\mathcal{O}(h^2) = \mathcal{E}_h(f) \geq \frac{1}{C_0} \sum_{k \geq 0} 2^{-k(1+\alpha d')} \mathcal{E}^Q_{\bar{h}_k, \tilde{h}_k, h}(f \circ \sigma_k^{-1} \circ \beta^{-1}). \tag{3.21}$$

Let us now study the Dirichlet form on the cube Q. For any $i = 1, \dots, d$ let s_i denote the symmetry with respect to the hyperplane $\{x_i = 1\}$ and let G be the abelian group generated by the s_i. The group G acts on $]0, 2[^d$ and for every function $f \in L^2(Q)$, one can then define $g \in L^2(]0, 2]^d)$ by $g_{|]0,1[^d} = f$ and for all $s \in G$, $g \circ s = g$ (we do not specify the value of g on the hyperplanes $\{x_i = 1\}$ since they are negligible sets). Eventually, this permits to extend the function g (by means of translations) to a $(2\mathbb{Z})^d$-periodic function on $\mathbb{R}^d$. We then denote

$$\begin{gathered} E : L^2(Q) \to L^2(\mathbb{T}^d) \\ f \mapsto g \end{gathered}$$

where $\mathbb{T}^d = (\mathbb{R}/2\mathbb{Z})^d$. From the preceding discussion, E is continuous from $L^2(]0, 1[^d)$ into $L^2(\mathbb{T}^d)$ and from $H^1(]0, 1[^d)$ into $H^1(\mathbb{T}^d)$. Given $0 \leq a < b \leq 1$, we denote

$$\Pi^{d-1}_{]a,b[} =]a, b[\times \Pi^{d-1}.$$

We can perform a partial periodization by using only symmetries with respect to hyperplanes $\{x_i = 1\}$ with $i \geq 2$. We obtain an extension map

$$\tilde{E}_{]a,b[} : L^2(]a, b[\times]0, 1[^{d-1}) \to L^2(\Pi^{d-1}_{]a,b[}).$$

We also introduce the following restriction operators

$$\begin{aligned} & R : L^2(\Pi^d) \to L^2(]0, 1[^d) \\ & R_{]a,b[} : L^2(\Pi^d) \to L^2(]a, b[\times]0, 1[^{d-1}), \\ & R^1_{]a,b[} : L^2(\Pi^d) \to L^2(\Pi^{d-1}_{]a,b[}), \\ & \tilde{R}_{]a,b[} : L^2(\Pi^{d-1}_{]a,b[}) \to L^2(]a, b[\times]0, 1[^{d-1}), \end{aligned} \tag{3.22}$$

which satisfy the following relations:

$$RE = \mathrm{Id},\ \tilde{R}_{]a,b[} \tilde{E}_{]a,b[} = \mathrm{Id},\ R_{]a,b[} = \tilde{R}_{]a,b[} R^1_{]a,b[},\ R^1_{]a,b[} E = \tilde{E}_{]a,b[}. \tag{3.23}$$

Eventually, we observe that all these operators are continuous on H^1 and L^2 spaces. In order to get rid of boundary problems, the general idea is now to compare the Dirichlet form $\mathcal{E}^Q_{\bar{h}, \tilde{h}, h}$ with a suitable Dirichlet form on the torus. We first introduce the Metropolis operator on Π^d, defined by

$$\bar{T}_{\mathbf{h}}(g)(x) = \frac{1}{\mathscr{V}_{\infty,d} \mathbf{h}^d} \int_{\mathbb{T}^d} \mathbb{1}_{|x_1 - y_1| < \bar{h}, |x' - y'| < \tilde{h}, |x'' - y''| < h} g(y) dy \tag{3.24}$$

for any $g \in L^2(\mathbb{T}^d)$, where $\mathscr{V}_{\infty,d} = \int_{\mathbb{T}^d} \mathbb{1}_{|y|_\infty < 1} dy$. The associated Dirichlet form is

$$\begin{aligned} \bar{\mathcal{E}}_{\mathbf{h}}(g) &:= \langle (1 - \bar{T}_{\mathbf{h}})(g), g \rangle_{L^2(\mathbb{T}^d)} \\ &= \frac{1}{2 \mathscr{V}_{\infty,d} \mathbf{h}^d} \int_{\mathbb{T}^d \times \mathbb{T}^d} \mathbb{1}_{|\mathbf{h}^{-1} \cdot (x-y)|_\infty < 1} |g(x) - g(y)|^2 dx dy. \end{aligned}$$

Lemma 3.7. *There exists $C, h_0 > 0$ such that for all $0 < h, \bar{h}, \tilde{h} < h_0$ and all $f \in L^2(Q)$*

$$\mathcal{E}^Q_{\mathbf{h}}(f) \le \bar{\mathcal{E}}_{\mathbf{h}}(E(f)) \le C\mathcal{E}^Q_{\mathbf{h}}(f).$$

Proof. For any $f \in L^2(Q)$, one has

$$\begin{aligned}\bar{\mathcal{E}}_{\mathbf{h}}(E(f)) &= \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\int_{\mathbb{T}^d\times\mathbb{T}^d}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(x-y)|_\infty<1}|g(x)-g(y)|^2dxdy\\ &= \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\sum_{s,\tilde{s}\in G}\int_{s(Q)\times\tilde{s}(Q)}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(x-y)|_\infty<1}|g(x)-g(y)|^2dxdy\\ &= \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\sum_{s,\tilde{s}\in G}\int_{Q\times Q}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(s(x)-\tilde{s}(y))|_\infty<1}|g\circ s(x)-g\circ\tilde{s}(y)|^2dxdy\end{aligned}$$

and by definition of g it follows

$$\bar{\mathcal{E}}_{\mathbf{h}}(E(f)) = \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\sum_{s,\tilde{s}\in G}\int_{Q\times Q}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(s(x)-\tilde{s}(y))|_\infty<1} |f(x)-f(y)|^2dxdy. \tag{3.25}$$

Moreover, for all $s, \tilde{s} \in G$ and any $x, y \in Q$, one has

$$|\mathbf{h}^{-1}\cdot(s(x)-\tilde{s}(y))|_\infty < 1 \Longrightarrow |\mathbf{h}^{-1}\cdot(x-y)|_\infty < 2. \tag{3.26}$$

Indeed, since the elements of G are isometries of $\mathbb{R}^d$ for the norm $|\cdot|_\infty$, it suffices to prove (3.26) with $s = \mathrm{Id}$. If $\tilde{s} = \mathrm{Id}$, there is nothing to prove. Let us assume that $\tilde{s} \ne \mathrm{Id}$. Then, there exists $a \in \{0,1\}^d$ such that $\tilde{s} = \prod_{i=1}^d s_i^{a_i}$. Let us denote $I = \{i,\ a_i = 1\}$ and let $D = \cap_{i\in I}\{x_i = 1\}$. Since $x \in]0,1[^d$, $\tilde{s}(y) \notin]0,1[^d$ and $|\mathbf{h}^{-1}\cdot(x-\tilde{s}(y))|_\infty < 1$ then there exists $z \in D$ such that $|\mathbf{h}^{-1}\cdot(x-z)|_\infty < 1$ and $|\mathbf{h}^{-1}\cdot(z-\tilde{s}(y))|_\infty < 1$. Since $\tilde{s}(z) = z$, this last inequality implies that $|\mathbf{h}^{-1}\cdot(z-y)|_\infty < 1$ and hence

$$|\mathbf{h}^{-1}\cdot(x-y)|_\infty \le |\mathbf{h}^{-1}\cdot(x-z)|_\infty + |\mathbf{h}^{-1}\cdot(z-y)|_\infty < 2$$

which proves (3.26). Now, using (3.25) and (3.26), we obtain

$$\bar{\mathcal{E}}_{\mathbf{h}}(E(f)) \le \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\sum_{s,\tilde{s}\in G}\int_{Q\times Q}\mathbb{1}_{|\mathbf{h}^{-1}\cdot(x-y)|_\infty<2}|f(x)-f(y)|^2dxdy$$

and thanks to Lemma 3.5, there exists $C, h_0 > 0$ such that for all $0 < h < h_0$, one has

$$\bar{\mathcal{E}}_{\mathbf{h}}(E(f)) \le C\mathcal{E}^Q_{\mathbf{h}}(f).$$

This proves the right inequality. The left one is immediate. □

Remark 3.8. *The above proof can be easily adapted to show that given $0 \le a < b \le 1$, there exists $C > 0$ such that for all $0 < h, \bar{h}, \tilde{h} < h_0$ and all $f \in L^2(Q)$*

$$\bar{\mathcal{E}}^{]a,b[}_{\mathbf{h}}(\tilde{E}_{]a,b[}(f)) \le C\mathcal{E}^Q_{\mathbf{h}}(f)$$

where

$$\bar{\mathcal{E}}^{]a,b[}_{\mathbf{h}}(g) = \frac{1}{2\mathscr{V}_{\infty,d}\mathbf{h}^d}\int_{\Pi^{d-1}_{]a,b[}\times\Pi^{d-1}_{]a,b[}}\mathbb{1}_{|x_1-y_1|<\bar{h},|x'-y'|<\tilde{h},|x''-y''|<h} |g(x)-g(y)|^2dxdy.$$

3.1.3. *Fourier analysis of the Metropolis operator on the torus.* The following lemma gives an expression of the operator $\bar{T}_{\mathbf{h}}$ as a Fourier multiplier.

Lemma 3.9. *For $0<\bar{h},\tilde{h},h<1$, one has*

$$\bar{T}_{\mathbf{h}} = \Gamma_1(\bar{h}^2\partial_1^2)\Gamma_{d'}(\tilde{h}^2\Delta_{x'})\Gamma_{d''}(h^2\Delta_{x''})$$

with

$$\Gamma_n(|\xi|^2) := G_n(\xi) := \frac{1}{\mathscr{V}_n}\int_{\mathbb{R}^n} \mathbb{1}_{|z|<1} e^{i\pi z\cdot\xi}dz$$

Proof. First, observe that since $\mathscr{V}_{\infty,d} = \mathscr{V}_1\mathscr{V}_{d'}\mathscr{V}_{d''}$, one has $\bar{T}_{\bar{h},\tilde{h},h} = M_{1,\bar{h}}M_{d',\tilde{h}}M_{d'',h}$ where for any $\hbar>0$ we set

$$M_{n,\hbar}g(x) = \frac{1}{\mathscr{V}_n\hbar^n}\int_{\mathbb{T}^n}\mathbb{1}_{|x-y|<\hbar}g(y)dy.$$

On the other hand, if one denotes $e_k := \frac{1}{2^{n/2}}e^{i\pi z\cdot k}$ for all $k\in\mathbb{Z}^n$, then (e_k) is an orthonormal basis of $L^2((\mathbb{R}/2\mathbb{Z})^n)$. Moreover, for $0<\hbar<1$, the map $y\mapsto x+\hbar y$ is a change of variable from $B(0,1)$ onto $B(x,h)$ in Π^n and we get

$$M_{n,\hbar}(e_k) = e_k(x)\frac{1}{\mathscr{V}_n\hbar^n}\int_{\mathbb{T}^n}\mathbb{1}_{|x-y|<\hbar}e^{i\pi k\cdot(y-x)}dy = G_n(\hbar k)e_k.$$

Since the function G_n is radial, this proves the announced result. □

From the discussion below (1.6) in [6] one knows that G_n is a smooth functions on $\mathbb{R}^n$ such that $|G_n|\leq 1$, $|G_n(\xi)|=1$ iff $\xi=0$ and

$$G_n(\xi) = 1-\frac{1}{2(n+2)}|\xi|^2+\mathcal{O}(|\xi|^4). \tag{3.27}$$

With the notation (3.6), we have the following

Lemma 3.10. *There exists $C>0$ such that for all $0<h,\bar{h},\tilde{h}<1$ and all $g\in L^2(\Pi^d)$ such that $\|g\|_{L^2}\leq 1$ and $\bar{\mathcal{E}}_{\mathbf{h}}(g)\leq h^2$, there exists a decomposition $g=g_L+g_H$ such that*

$$\|g_L\|^2_{L^2(\Pi^d)} + h^{-2}N_{\mathbf{h}}(g_L,\Pi^d)^2 \leq C$$

and

$$||g_H||^2_{L^2(\Pi^d)}\leq Ch^2.$$

Remark 3.11. *Let $\lambda_1=\bar{h}/h$, $\lambda_2=\tilde{h}/h$. One has*

$$\|g_L\|^2_{L^2(\Pi^d)} + h^{-2}N_{\mathbf{h}}(g_L,\Pi^d)^2 = \|g_L\|^2_{H^1_{\lambda_1,\lambda_2,1}(\Pi^d)}$$

where the semiclassical Sobolev spaces $H^1_{\lambda_1,\lambda_2,1}$ are defined in appendix.

Proof. Denote $\alpha_n = \frac{1}{2(n+2)}$ and let $1/\Upsilon_1 = \frac14\min(\alpha_1,\alpha_{d'},\alpha_{d''})>0$. Let $g\in L^2(\Pi^d)$ be such that $\bar{\mathcal{E}}_{\bar{h},\tilde{h},h}(g)\leq h^2$. From Lemma 3.9, one knows that for any $0<h,\bar{h},\tilde{h}<1$, one has

$$\begin{aligned} h^2 &\geq \langle(1-\bar{T}_{\mathbf{h}})g,g\rangle_{L^2(\Pi^d)}\\ &\geq \langle(1-G_1(\bar{h}\partial_1)G_{d'}(\tilde{h}\nabla')G_{d''}(h\nabla''))g,g\rangle_{L^2(\Pi^d)}\end{aligned} \tag{3.28}$$

On the other hand, it follows from (3.27) that for all $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that for all $|\xi| < \delta_n$, one has

$$0 < G_n(\xi) \leq 1 - \frac{\alpha_n}{2}|\xi|^2.$$

Hence, for all $\xi = (\xi_1, \xi', \xi'') \in \mathbb{R}^d$ such that $|\xi| < \delta := \min(\delta_1, \delta_{d'}, \delta_{d''})$, one has

$$\begin{aligned} 1 - G_1(\xi_1)G_{d'}(\xi')G_{d''}(\xi'') &\geq \frac{\alpha_1}{2}|\xi_1|^2 + \frac{\alpha_{d'}}{2}|\xi'|^2 + \frac{\alpha_{d''}}{2}|\xi''|^2 + \mathcal{O}(|\xi|^4) \\ &\geq \frac{2}{\Upsilon_1}|\xi|^2 + \mathcal{O}(|\xi|^4). \end{aligned}$$

Decreasing δ as much as necessary, we obtain

$$1 - G_1(\xi_1)G_{d'}(\xi')G_{d''}(\xi'') \geq \frac{1}{\Upsilon_1}|\xi|^2 \tag{3.29}$$

for any $|\xi| < \delta$. On the other hand, since G_n is bounded by 1 and goes to zero at infinity and $1 - G_n$ vanishes only at the origin, there exists $\Upsilon_2 > 0$ such that for all $|\xi| \geq \delta$,

$$1 - G_1(\xi_1)G_{d'}(\xi')G_{d''}(\xi'') \geq \frac{1}{\Upsilon_2}. \tag{3.30}$$

Let us decompose g in the Fourier basis (e_k), $g = \sum_{k\in\mathbb{Z}^d} \hat{g}(k)e_k$ and let

$$g_L := \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| < \delta} \hat{g}(k)e_k, \quad g_H := 1 - g_L = \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| \geq \delta} \hat{g}(k)e_k.$$

From (3.28), (3.29) and (3.30), one deduces

$$\begin{aligned} h^2 &\geq \sum_{k\in\mathbb{Z}^d} (1 - G_1(\bar{h}k_1)G_{d'}(\tilde{h}k')G_{d''}(hk''))|\hat{g}(k)|^2 \\ &= \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| < \delta} (1 - G_1(\bar{h}k_1)G_{d'}(\tilde{h}k')G_{d''}(hk''))|\hat{g}(k)|^2 \\ &\qquad + \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| \geq \delta} (1 - G_1(\bar{h}k_1)G_{d'}(\tilde{h}k')G_{d''}(hk''))|\hat{g}(k)|^2 \\ &\geq \frac{1}{\Upsilon_1} \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| < \delta} (|\bar{h}k_1|^2 + |\tilde{h}k'|^2 + |hk''|^2)|\hat{g}(k)|^2 \\ &\qquad + \frac{1}{\Upsilon_2} \sum_{|(\bar{h}k_1, \tilde{h}k', hk'')| \geq \delta} |\hat{g}(k)|^2. \end{aligned}$$

From standard Fourier analysis, we deduce

$$h^2 \geq \frac{1}{\Upsilon_1}\Big(\|\bar{h}\partial_1 g_L\|^2 + \|\tilde{h}\nabla' g_L\|^2 + \|h\nabla g_L\|^2\Big) + \frac{1}{\Upsilon_2}\|g_H\|^2$$

Taking $C = \max(\Upsilon_1, \Upsilon_2)$ we get the announced result. □

3.1.4. *Decomposition Lemma in the cusp.* The main result of this section is the following

Lemma 3.12. *Assume that* Ω *has the particular form* (3.3)*, then the conclusion of Lemma 3.1 holds true.*

Proof. Throughout C denotes a positive constant independent of f and h that may change from line to line and $h \in]0, h_0[$ where $h_0 > 0$ is supposed sufficiently small in order that the conclusions of the preceding lemmas hold true. Let $f \in L^2(\Omega)$ be such that $\mathcal{E}_h(f) \leq h^2$ and $||f||_{L^2} \leq 1$. It follows from (3.21) that

$$\sum_{k\geq 0} 2^{-k(1+\alpha d')} \mathcal{E}^Q_{\bar{h}_k, \tilde{h}_k, h}(f \circ \sigma_k^{-1} \circ \beta^{-1}) = \mathcal{O}(h^2)$$

where β is defined by (3.19). Denoting

$$\bar{g}_k = E(f \circ \sigma_k^{-1} \circ \beta^{-1}), \tag{3.31}$$

it follows from Lemma 3.7 that

$$\bar{\mathcal{E}}_{\bar{h}_k, \tilde{h}_k, h}(\bar{g}_k) \leq C \mathcal{E}^Q_{\bar{h}_k, \tilde{h}_k, h}(f \circ \sigma_k^{-1} \circ \beta^{-1}) \tag{3.32}$$

and hence

$$\sum_{k\geq 0} 2^{-k(1+\alpha d')} \bar{\mathcal{E}}_{\bar{h}_k, \tilde{h}_k, h}(\bar{g}_k) = \mathcal{O}(h^2). \tag{3.33}$$

From now, given $D \subset \mathbb{R}^p \times \Pi^q$, $p, q \in \mathbb{N}^*$, $g \in L^2(D)$ and $\mathbf{h} = (\bar{h}, \tilde{h}, h)$ we denote

$$\mathcal{V}^D_{\mathbf{h}}(g) = \mathcal{V}^D_{\bar{h}, \tilde{h}, h}(g) := \|g\|^2_{L^2(D)} + h^{-2} \mathcal{E}^D_{\mathbf{h}}(g) \tag{3.34}$$

and for shortness we denote $\mathcal{V}_k(f, h) = \mathcal{V}^{B_0}_{\bar{h}_k, \tilde{h}_k, h}(f \circ \sigma_k^{-1})$. We also denote $\mathbf{h}_k = (\bar{h}_k, \tilde{h}_k, h)$. Thanks to (3.11) and (3.33), one has

$$\sum_{k=0}^{\infty} 2^{-k(1+\alpha d')} \mathcal{V}_k(f, h) = \mathcal{O}(1), \tag{3.35}$$

and (3.20) and (3.32) implies

$$\|\bar{g}_k\|^2_{L^2(\Pi^d)} + h^{-2} \bar{\mathcal{E}}_{\mathbf{h}_k}(\bar{g}_k) \leq C \mathcal{V}_k(f, h).$$

This estimate combined with Lemma 3.10 shows that there exists a new constant $C > 0$ such that for any $k \in \mathbb{N}$ and $h > 0$ such that $\tilde{h}_k < h_0$, there exists $\bar{g}_{k,L} \in H^1(\Pi^d)$ and $\bar{g}_{k,H} \in L^2(\Pi^d)$ such that $\bar{g}_k = \bar{g}_{k,L} + \bar{g}_{k,H}$ with

$$\|\bar{g}_{k,L}\|^2_{L^2(\Pi^d)} + h^{-2} N_{\mathbf{h}_k}(\bar{g}_{k,L}, \Pi^d)^2 \leq C \mathcal{V}_k(f, h)$$

and

$$\|\bar{g}_{k,H}\|^2_{L^2} \leq C h^2 \mathcal{V}_k(f, h).$$

Since the restriction operator $R^1_{]0,1[}$ (defined in (3.22)) is continuous, it follows from the above estimates that

$$\|R^1_{]0,1[}(\bar{g}_{k,L})\|^2_{L^2(\Pi^{d-1}_{]0,1[})} + h^{-2} N_{\mathbf{h}_k}(R^1_{]0,1[}(\bar{g}_{k,L}), \Pi^{d-1}_{]0,1[})^2 \leq C \mathcal{V}_k(f, h) \tag{3.36}$$

and

$$||R^1_{]0,1[}(\bar{g}_{k,H})||^2_{L^2(\Pi^{d-1}_{]0,1[})} \leq C h^2 \mathcal{V}_k(f, h) \tag{3.37}$$

Combined with (3.35), this implies

$$\sum_{k\geq 0} 2^{-k(1+\alpha d')}\|R^1_{]0,1[}(\bar{g}_{k,H})\|^2_{L^2(\Pi^{d-1}_{]0,1[})} \leq Ch^2 \tag{3.38}$$

and

$$\sum_{k\geq 0} 2^{-k(1+\alpha d')}\Big(\|R^1_{]0,1[}(\bar{g}_{k,L})\|^2_{L^2(\Pi^{d-1}_{]0,1[})} + h^{-2}N_{\mathbf{h}_k}(R^1_{]0,1[}(\bar{g}_{k,L}),\Pi^{d-1}_{]0,1[})^2\Big) \leq C.$$

Since $N_{\mathbf{h}_k}(.,.) = hN_{2^k,2^{k\alpha},1}(.,.)$, this later equation implies

$$\sum_{k\geq 0} 2^{-k(1+\alpha d')}\Big(\|R^1_{]0,1[}(\bar{g}_{k,L})\|^2_{L^2(\Pi^{d-1}_{]0,1[})} + N_{2^k,2^{k\alpha},1}(R^1_{]0,1[}(\bar{g}_{k,L}),\Pi^{d-1}_{]0,1[})^2\Big) = \mathcal{O}(1). \tag{3.39}$$

In view of Lemma 3.3, estimates (3.38) and (3.39) almost imply the conclusion by considering the restriction of $R^1(\bar{g}_k)$ to B_0. The main issue to get the conclusion is that nothing insures that the no-jump condition of the lemma $R^1_{]0,1[}(\bar{g}_{k,L})_{|x_1=\frac{1}{2}} = R^1_{]0,1[}(\bar{g}_{k+1,L})_{|x_1=1}$ holds true (observe here that the interface $x_1 = 2^{-k-1}$ in the orginial variable corresponds to $x_1 = \frac{1}{2}$ for $\bar{g}_{k,L}$ and to $x_1 = 1$ for $\bar{g}_{k+1,L}$). The end of the proof consists to modify slightly the above decomposition in order to satisfy the assumptions of Lemma 3.3.

Let us explain briefly the idea of this modification before entering into technical details. As already said, it follows from the above estimates that we can decompose the functions $f_{|\Omega_{2k}}$ and $f_{|\Omega_{2k+1}}$ as the sum of a H^1 function and a small function in L^2. The idea is that we can do an analogous decomposition with quadriadic decomposition so that the function f restricted to $\Omega_{2k}\cup\Omega_{2k+1}$ admits also such a decomposition. Then we can apply Lemma 5.2 of the appendix to glue smoothly $f^L_{|\Omega_{2k}}$ and $f^L_{|\Omega_{2k+1}}$ up to a small error in L^2. In order to get a global estimate, we need to prove estimates uniform with respect to the dyadic parameter k which makes the computation a bit more heavy.

Let us now enter into the details. We first observe that thanks to (3.23), one has

$$f = \sum_{k\geq 0} \mathbb{1}_{\Omega_k} f_k\circ\sigma_k = \sum_{k\geq 0}\mathbb{1}_{\Omega_k} R(\bar{g}_k)\circ\beta\circ\sigma_k \tag{3.40}$$

with $f_k = f\circ\sigma_k^{-1} = R(\bar{g}_k)\circ\beta$ ($\bar{g}_k$ given by (3.31), β given by (3.19) and R given by (3.22)). We introduce the following functions defined on $\Pi^{d-1}_{]\frac{1}{2},1[}$:

$$\check{g}_k = R^1_{]0,1[}(\bar{g}_k)\circ\beta,\ \check{g}_{k,L} = R^1_{]0,1[}(\bar{g}_{k,L})\circ\beta,\ \check{g}_{k,H} = R^1_{]0,1[}(\bar{g}_{k,H})\circ\beta \tag{3.41}$$

which of course verify

$$\check{g}_K = \check{g}_{k,L} + \check{g}_{k,H} \tag{3.42}$$

thanks to the above construction. First observe that thanks to (3.23), one has for any $k\in\mathbb{N}$, $f_k = R(\bar{g}_k)\circ\beta = \tilde{R}_{]0,1[}(R^1_{]0,1[}(\bar{g}_k))\circ\beta = \tilde{R}_{]\frac{1}{2},1[}(R^1_{]0,1[}(\bar{g}_k)\circ\beta)$ and hence

$$f_k = \tilde{R}_{]\frac{1}{2},1[}(\check{g}_k) \tag{3.43}$$

where we recall that $\tilde{R}_{]a,b[} : L^2(\Pi^{d-1}_{]a,b[}) \to L^2(]a,b[\times]0,1[^{d-1})$ denotes the restriction operator in the (x', x'') variable. Splitting (3.40) into even and odd terms and using (3.43), we get

$$f = \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k}} \tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k}) \circ \sigma_{2k} + \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k+1}} \tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k+1}) \circ \check{\sigma}_1 \circ \check{\sigma}_1^{-1} \circ \sigma_{2k+1} \tag{3.44}$$

with $\check{\sigma}_1$ given by (3.10). Since this change of variable is simply given by $\check{\sigma}_1(x) = (2x_1, x', x'')$ we have for any ψ

$$\tilde{R}_{]\frac{1}{2},1[}(\psi) \circ \check{\sigma}_1 = \tilde{R}_{]\frac{1}{4},\frac{1}{2}[}(\psi \circ \check{\sigma}_1)$$

where with a slight abuse of notation we use the symbol $\check{\sigma}_1$ to denote the above dilation defined from $\Pi^{d-1}_{]\frac{1}{4},\frac{1}{2}[}$ into $\Pi^{d-1}_{]\frac{1}{2},1[}$. Combined with (3.44) and the identity $\check{\sigma}_1^{-1} \circ \sigma_{2k+1} = \hat{\sigma}_{2k+1}$ (see (3.5), (3.9) for the definition of $\hat{\sigma}_k$), this implies

$$f = \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k}} \tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k}) \circ \sigma_{2k} + \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k+1}} \tilde{R}_{]\frac{1}{4},\frac{1}{2}[}(\check{g}_{2k+1} \circ \check{\sigma}_1) \circ \hat{\sigma}_{2k+1}. \tag{3.45}$$

Denote $D_k = \Omega_{2k} \cup \Omega_{2k+1}$ for any $k \in \mathbb{N}$ and let $\nu_k : \ D_k \to B_0 \cup B_1$ be defined by $\nu_k(x) = \theta(4^k x_1, 4^{\alpha k} x', x'')$. Since $(\nu_k)_{|\Omega_{2k}} = \sigma_{2k}$ and $(\nu_k)_{|\Omega_{2k+1}} = \hat{\sigma}_{2k+1}$, equation (3.45) becomes

$$f = \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k}} \tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k}) \circ \nu_k + \sum_{k\geq 0} \mathbb{1}_{\Omega_{2k+1}} \tilde{R}_{]\frac{1}{4},\frac{1}{2}[}(\check{g}_{2k+1} \circ \check{\sigma}_1) \circ \nu_k. \tag{3.46}$$

We now relate the quadriadic decomposition of the function f to the dyadic decomposition.

Sub-lemma 3.13. *Let $\check{f}_k := \tilde{E}_{]\frac{1}{4},1[}(f \circ \nu_k^{-1})$ and denote*

$$\tilde{\mathcal{V}}_k(f,h) = \mathcal{V}^{B_0 \cup B_1}_{\tilde{h}_{2k} \tilde{h}_{2k}, h}(f \circ \nu_k^{-1}) \tag{3.47}$$

where the functional $\mathcal{V}$ is defined by (3.34). *One has*

$$\check{f}_k = \mathbb{1}_{\frac{1}{4} < x_1 < \frac{1}{2}} \check{g}_{2k+1} \circ \check{\sigma}_1 + \mathbb{1}_{\frac{1}{2} < x_1 < 1} \check{g}_{2k} \tag{3.48}$$

and

$$\sum_{k=0}^{\infty} 4^{-k(1+\alpha d')} \tilde{\mathcal{V}}_k(f,h) = O(1). \tag{3.49}$$

Moreover, one has the decompositions $\check{g}_\bullet = \check{g}_{\bullet,L} + \check{g}_{\bullet,H}$ *and* $\check{f}_k = \check{f}_{k,L} + \check{f}_{k,H}$ *with*

$$(3.50)\quad \begin{aligned} &\|\check{g}_{2k+1,L}\circ\check{\sigma}_1\|^2_{L^2(\Pi^{d-1}_{]\frac14,\frac12[})} + h^{-2}N_{\mathbf{h}_{2k}}(\check{g}_{2k+1,L}\circ\check{\sigma}_1,\Pi^{d-1}_{]\frac14,\frac12[})^2 \\ &\qquad\qquad\leq C\tilde{\mathcal{V}}_k(f,h) \\ &\|\check{g}_{2k,L}\|^2_{L^2(\Pi^{d-1}_{]\frac12,1[})} + h^{-2}N_{\mathbf{h}_{2k}}(\check{g}_{2k,L},\Pi^{d-1}_{]\frac12,1[})^2 \leq C\tilde{\mathcal{V}}_k(f,h)\end{aligned}$$

and

$$(3.51)\quad \|\check{g}_{2k+1,H}\circ\check{\sigma}_1\|^2_{L^2(\Pi^{d-1}_{]\frac14,\frac12[})} + \|\check{g}_{2k,H}\|^2_{L^2(\Pi^{d-1}_{]\frac12,1[})} \leq Ch^2\tilde{\mathcal{V}}_k(f,h)$$

and

$$(3.52)\quad \|\check{f}_{k,L}\|^2_{L^2(\Pi^{d-1}_{]\frac14,1[})} + h^{-2}N_{\mathbf{h}_{2k}}(\check{f}_{k,L},\Pi^{d-1}_{]\frac14,1[})^2 \leq C\tilde{\mathcal{V}}_k(f,h)$$

and

$$(3.53)\quad \|\check{f}_{k,H}\|^2_{L^2(\Pi^{d-1}_{]\frac14,1[})} \leq Ch^2\tilde{\mathcal{V}}_k(f,h).$$

where C is a positive constant independent of h.

Proof. By definition, one has $f = \sum_{k\geq 0} \mathbb{1}_{D_k} f\circ\nu_k^{-1}\circ\nu_k$, which combined to (3.46) proves that

$$(3.54)\quad f\circ\nu_k^{-1} = \mathbb{1}_{B_0}\tilde{R}_{]\frac12,1[}(\check{g}_{2k}) + \mathbb{1}_{B_1}\tilde{R}_{]\frac14,\frac12[}(\check{g}_{2k+1}\circ\check{\sigma}_1).$$

Applying $\tilde{E}_{[\frac14,1]}$ on both sides of this identity, we get (3.48).

We now observe that the analysis of Lemma 3.7,3.9, 3.10 can be performed with the "quadriadic" decomposition of the cusp induced by the change of variable ν_k. This yields

$$(3.55)\quad h^2 \geq \mathcal{E}_h(f) \geq \frac{1}{C}\sum_{k\geq 0} 4^{-k(1+\alpha d')}\mathcal{E}^{B_0\cup B_1}_{4^k h,4^{k\alpha}h,h}(f\circ\nu_k^{-1}).$$

Dividing by h^2 and adding the L^2 norm, this implies

$$\sum_{k=0}^{\infty} 4^{-k(1+\alpha d')}\tilde{\mathcal{V}}_k(f,h) = O(1)$$

which is exactly (3.49). Moreover, it follows from Remark 3.8, (3.43), (3.54) and the inclusion $B_0\times B_0\cup B_1\times B_1 \subset \check{(}B_0\cup B_1)^2$ that

$$(3.56)\quad \begin{aligned}\mathcal{E}^{B_0\cup B_1}_{4^k h,4^{k\alpha}h,h}(f\circ\nu_k^{-1}) &\geq \mathcal{E}^{B_0}_{\bar{h}_{2k},\tilde{h}_{2k},h}(f\circ\sigma_{2k}^{-1}) + \mathcal{E}^{B_1}_{\bar{h}_{2k},\tilde{h}_{2k},h}(f\circ\hat{\sigma}_{2k+1}^{-1}) \\ &\geq \frac{1}{C}\Big(\bar{\mathcal{E}}^{]\frac12,1[}_{\bar{h}_{2k},\tilde{h}_{2k},h}(\check{g}_{2k}) + \bar{\mathcal{E}}^{]\frac14,\frac12[}_{\bar{h}_{2k},\tilde{h}_{2k},h}(\check{g}_{2k+1}\circ\check{\sigma}_1)\Big)\end{aligned}$$

for some constant $C>0$. Observe that $\bar{h}_{2k+1} = 2\bar{h}_{2k}, \tilde{h}_{2k+1} = 2^\alpha\tilde{h}_{2k}$. Hence (3.56) proves that there exists $C>0$ such that

$$(3.57)\quad \mathcal{V}_{2k}(f,h) + \mathcal{V}_{2k+1}(f,h) \leq C\tilde{\mathcal{V}}_k(f,h).$$

Combined with (3.36), (3.37), (3.41), this proves (3.50) and (3.51). On the other hand, using Lemma 3.10 and (3.55) we have also a decomposition

$$E(f\circ\nu_k^{-1}) = E(f\circ\nu_k^{-1})_L + E(f\circ\nu_k^{-1})_H$$

with suitable bounds on the right hand side. Restricting this decomposition to $\frac{1}{4} \leq x_1 \leq 1$, we get $\check{f}_k = \check{f}_{k,L} + \check{f}_{k,H}$ with $\check{f}_{k,L}, \check{f}_{k,H}$ which satisfy (3.52) and (3.53). This completes the proof of the sub-lemma. □

Let us now apply Lemma 5.2 with $A_0 = \Pi^{d-1}_{]\frac{1}{4},\frac{1}{2}[}$, $A_1 = \Pi^{d-1}_{]\frac{1}{2},1[}$, $A_2 = \Pi^{d-1}_{]\frac{1}{4},1[}$, $\phi_0 = \check{g}_{2k+1,L} \circ \check{\sigma}_1$ and $\phi_1 = \check{g}_{2k,L}$. Let $w_k := \mathbb{1}_{A_0}\phi_0 + \mathbb{1}_{A_1}\phi_1$ and denote $r_0 = \mathbb{1}_{A_0}\check{g}_{2k+1,H} \circ \check{\sigma}_1$, $r_1 = \mathbb{1}_{A_1}\check{g}_{2k,H}$. Thanks to (3.42) and (3.48), w_k satisfies

$$w_k = \mathbb{1}_{A_0}\check{g}_{2k+1} \circ \check{\sigma}_1 + \mathbb{1}_{A_1}\check{g}_{2k} - r_0 - r_1 = \mathbb{1}_{A_2}\check{f}_k - r_0 - r_1. \tag{3.58}$$

Hence, $w_k = \phi_2 + r_2$ with

$$\phi_2 = \mathbb{1}_{A_2}\check{f}_{k,L} \text{ and } r_2 = \mathbb{1}_{A_2}\check{f}_{k,H} - r_0 - r_1. \tag{3.59}$$

Moreover, thanks to (3.51), (3.52) and (3.53), one has

$$||r_2||^2_{L^2(A_2)} \leq Ch^2\tilde{\mathcal{V}}_k \tag{3.60}$$

and

$$N_{4^k,4^{k\alpha},1}(\phi_2, A_2) \leq C\tilde{\mathcal{V}}_k.$$

where we write for shortness $\tilde{\mathcal{V}}_k = \tilde{\mathcal{V}}_k(f,h)$. From Lemma 5.2 with $\lambda_1 = 2^{2k}$, $\lambda' = 2^{2k\alpha}$ and $\lambda'' = 1$, there exists $\Upsilon_1 > 0$ and $h_1 > 0$ such that for any k such that $\bar{h}_{2k} \leq h_1$ (that is $2^{-2k} > h/h_1$), there exists a function ψ_{2k} supported in $\Pi^{d-1}_{]\frac{1}{4},\frac{1}{2}[} \cap \{\frac{1}{2} \leq x_1 \leq \frac{1}{2} + \bar{h}_{2k}\}$ such that $(\psi_{2k})_{|x_1=\frac{1}{2}} = (\phi_0)_{|x_1=\frac{1}{2}} - (\phi_1)_{|x_1=\frac{1}{2}}$ and

$$N_{4^k,4^{k\alpha},1}(\psi_{2k}, A_1)^2 \leq \Upsilon_1\tilde{\mathcal{V}}_k$$

and

$$\|\psi_{2k}\|^2_{L^2(A_1)} \leq \Upsilon_1 h^2\tilde{\mathcal{V}}_k.$$

From now, we suppose that $\tilde{h}_{2k} < h_2 := \min(h_0, h_1)$ with h_0 given by Lemmas 3.6 and 3.7 and h_1 by Lemma 5.2. We then rewrite $\check{g}_{2k}$ as $\check{g}_{2k} = \check{g}^{mod}_{2k,L} + \check{g}^{mod}_{2k,H}$ with $\check{g}^{mod}_{2k,L} = \check{g}_{2k,L} + \psi_{2k}$ and $\check{g}^{mod}_{2k,H} = \check{g}_{2k,H} - \psi_{2k}$. By construction, we have

$$\begin{cases} N_{4^k,4^{k\alpha},1}(\check{g}^{mod}_{2k,L}, \Pi^{d-1}_{]\frac{1}{2},1[})^2 \leq C\tilde{\mathcal{V}}_k \text{ and} \\ \|\check{g}^{mod}_{2k,H}\|^2_{L^2(\Pi^{d-1}_{]\frac{1}{2},1[})} \leq Ch^2\tilde{\mathcal{V}}_k \\ (\check{g}^{mod}_{2k,L})_{x_1=\frac{1}{2}} = (\check{g}_{2k+1,L} \circ \check{\sigma}_1)_{x_1=\frac{1}{2}}. \end{cases} \tag{3.61}$$

Let $K(h) \in \mathbb{N}$ be the largest integer such that $4^{\alpha K(h)} \leq h_2/h$. Then for $k \leq K(h)$ the functions $\check{g}^{mod}_{2k,L}$ and $\check{g}^{mod}_{2k,H}$ are well-defined and we can introduce the decomposition $f := f_L + f_H + f_{\mathcal{C}}$ with

$$f_L = \sum_{k=0}^{K(h)} \Big(\mathbb{1}_{\Omega_{2k}}\tilde{R}_{]\frac{1}{2},1[}(\check{g}^{mod}_{2k,L}) \circ \sigma_{2k} + \mathbb{1}_{\Omega_{2k+1}}\tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k+1,L}) \circ \sigma_{2k+1}\Big),$$

$$f_H = \sum_{k=0}^{K(h)} \Big(\mathbb{1}_{\Omega_{2k}}\tilde{R}_{]\frac{1}{2},1[}(\check{g}^{mod}_{2k,H}) \circ \sigma_{2k} + \mathbb{1}_{\Omega_{2k+1}}\tilde{R}_{]\frac{1}{2},1[}(\check{g}_{2k+1,H}) \circ \sigma_{2k+1}\Big),$$

and

$$f_{\mathcal{C}} = f - f_L - f_H.$$

It follows from (3.46) and the definition of $K(h)$ that $f_{\mathcal{C}}$ is supported in $\{0 < x_1 < (h/h_2)^{1/\alpha}\}$ which is the required property on $f_{\mathcal{C}}$. On the other hand, we deduce from (3.50) and (3.61) that

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-k(\alpha d'+1)} N_{2^k,2^{k\alpha},1}(f_L \circ \sigma_k^{-1} \circ \theta^{-1}, B_0)^2 \\ \leq C \sum_{k=0}^{K(h)} 4^{-k(\alpha d'+1)} N_{4^k,4^{k\alpha},1}(\check{g}_{2k,L}^{mod}, \Pi^{d-1}_{]\frac{1}{2},1[})^2 \\ + C \sum_{k=0}^{K(h)} 4^{-k(\alpha d'+1)} N_{4^k,4^{k\alpha},1}(\check{g}_{2k+1,L} \circ \check{\sigma}_1, \Pi^{d-1}_{]\frac{1}{4},\frac{1}{2}[})^2 \\ \leq C \sum_{k=0}^{\infty} 4^{-k(\alpha d'+1)} \tilde{\mathcal{V}}_k \leq C'\end{aligned}$$

where the last inequality follows from (3.49) and C' is a positive constant. Hence, f_L satisfies

$$\sum_{k=0}^{\infty} 2^{-k(\alpha d'+1)} N_{2^k,2^{k\alpha},1}(f_L \circ \sigma_k^{-1}, \Omega_0)^2 = \mathcal{O}(1) \tag{3.62}$$

and thanks to (3.61) the functions $\mathbb{1}_{\Omega_{2k+1}} f_L$ and $\mathbb{1}_{\Omega_{2k}} f_L$ have the same trace on $x_1 = 2^{-2k-1}$. Working similarly near $x_1 = 2^{-2k}$, we can modify $(f_L)_{\Omega_{2k}}$ in order that $\mathbb{1}_{\Omega_{2k}} f_L$ and $\mathbb{1}_{\Omega_{2k-1}} f_L$ have the same trace on $x_1 = 2^{-2k}$. Moreover, this new modification is supported in $2^{-2k}[1-h_2, 1]$. Hence, for $h_2 > 0$ small enough, it doesn't intersect the support of the modification ψ_{2k} which is contained in $2^{-2k}[\frac{1}{2}, \frac{1}{2} + h_2]$. Eventually, we modify also the function $\check{g}_{2K(h)+1}$ in order that $(f_L)_{|x_1=4^{-K(h)-1}} = 0$. Consequently, the fonction f_L that we obtain satisfies the assumptions of Lemma 3.3 and it follows that $f_L \in H^1(\Omega)$ and $\|f_L\|_{H^1(\Omega)} = \mathcal{O}(1)$. The fact that $\|f_H\|_{L^2(\Omega)} = \mathcal{O}(h)$ follows immediately from (3.49), (3.51), (3.61) and Lemma 3.3. □

3.2. **The general case.** Suppose that $(f_h)_{h\in]0,1]}$ is a family of functions in $L^2(\Omega)$ such that $\|f_h\|_{L^2} = 1$ and $\mathcal{E}_h(f_h) = \mathcal{O}(h^2)$. Let $J = I_c \cup I_r \cup \{0\}$ and for all $j \in J$, let $\mathcal{O}_j = \omega_j \cap \Omega$ where the ω_j, $j \in I_c \cup I_r$ are defined in Assumption 1 and ω_0 is a relatively compact open subset of Ω such that $\Omega \subset \cup_{j\in J} \mathcal{O}_j$. Since J is finite (independent of h), there exists $C > 0$ such that for any $f \in L^2$ one has

$$\mathcal{E}_h(f) \geq \frac{1}{C} \sum_{i\in J} \mathcal{E}_h^{\mathcal{O}_i}(f) \tag{3.63}$$

with

$$\mathcal{E}_h^{\mathcal{O}_i}(f) := \frac{1}{2h^d} \iint_{\mathcal{O}_i\times\mathcal{O}_i} \mathbb{1}_{|x-y|<h} |f(x) - f(y)|^2 d\mu_\rho^2(x, y).$$

Let $(\chi_i)_{i\in J}$ be a family of non negative smooth functions such that $\mathrm{supp}(\chi_i) \subset \omega_i$ for all $i \in J$ and $\sum_{i\in J} \chi_i = 1$ near Ω. For all $i \in J$, denote $f_{i,h} = (f_h)_{|\mathcal{O}_i}$

and observe that

$$f_h = \sum_{i\in J} \chi_i f_{i,h}.$$

Moreover, since $\mathcal{E}_h(f_h) = \mathcal{O}(h^2)$ it follows from (3.63) that for all $i \in J$, $\mathcal{E}_h^{\mathcal{O}_i}(f_{i,h}) = \mathcal{O}(h^2)$. Suppose first that $i \in I_c$. In a suitable coordinate system, $\mathcal{O}_i$ has the form (1.6) and we can apply Lemma 3.12 to get the decomposition

$$f_{i,h} = \varphi_{i,h} + g_{i,h} + r_{i,h} \tag{3.64}$$

with $\|g_{i,h}\|_{L^2(\mathcal{O}_i)} = \mathcal{O}(h)$, $\mathrm{supp}(r_{i,h}) \subset \Gamma_h$ (where Γ_h is defined in Lemma 3.1) and $(\varphi_{i,h})_{h\in]0,1]}$ bounded in $H^1(\mathcal{O}_i)$. On the other hand, it follows from Lemma 2.2 in [2] that for any $i \in I_r \cup \{0\}$, (3.64) holds true with $r_{i,h} = 0$. As a consequence, we get a global decomposition $f_h = \varphi_h + g_h + r_h$ with

$$\varphi_h = \sum_{i\in J} \chi_i \varphi_{i,h}, \quad g_h = \sum_{i\in J} \chi_i g_{i,h}, \quad r_h = \sum_{i\in I_c} \chi_i r_{i,h}.$$

The functions g_h and r_h satisfy trivially the required properties. Since χ_i is supported in $\mathcal{O}_i$, one has the identity

$$\nabla(\chi_i\varphi_{i,h}) = \varphi_{i,h}\nabla\chi_i + \chi_i\nabla\varphi_{i,h}$$

which permits easily to show that (φ_h) is bounded in H^1.

4. Spectral analysis.

4.1. **Weak convergence of Dirichlet forms.** We start this section with a lemma giving estimates of the Dirichlet form $\mathcal{E}_h$ on H^1 fonctions. Given a subset U of Ω, we use the notation

$$\mathcal{E}_h^U(u) := \frac{1}{2h^d}\iint_{U\times U} \mathbb{1}_{|x-y|<h}|u(x)-u(y)|^2 d\mu_\rho^2(x,y).$$

Lemma 4.1. *Suppose that the domain Ω satisfies Assumption 1. There exists $C > 0$ and $h_0 > 0$ such that for any subset $U \subset \Omega$ and any $u \in H^1(\Omega)$, one has for all $h \in]0, h_0]$*

$$\mathcal{E}_h^U(u) \leq Ch^2\|\nabla u\|^2_{L^2(U+B(0,Ch))}.$$

Proof. From Theorem 2, p 27 in [7], we know that $\mathcal{C}^\infty(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$ for any open set Ω. Hence, we can assume that $u \in C^\infty(\Omega)$. Let $(\omega_i)_{i\in J}$, $J = I_c \cup I_r \cup \{0\}$ be a covering of Ω as in section 3.2. For any $i \in J$, we denote $\omega_i^h = \omega_i + B(0,h)$. We have

$$\begin{aligned}\mathcal{E}_h^U(u) &:= \sum_{j\in J}\frac{1}{2h^d}\iint_{U\cap\omega_j\times U}\mathbb{1}_{|x-y|<h}|u(x)-u(y)|^2 d\mu_\rho^2(x,y)\\ &= \sum_{j\in J}\frac{1}{2h^d}\iint_{U\cap\omega_j\times U\cap\omega_j^h}\mathbb{1}_{|x-y|<h}|u(x)-u(y)|^2 d\mu_\rho^2(x,y)\\ &\leq \sum_{j\in J}\mathcal{E}_h^{U\cap\omega_j^h}(u)\end{aligned} \tag{4.1}$$

For any $j \in J$, since ρ is bounded, one has

$$\mathcal{E}_h^{U\cap\omega_j^h}(u) \leq Ch^{-d} \iint_{x,y\in U\cap\omega_j^h} \mathbb{1}_{|x-y|<h}|u(x)-u(y)|^2 dxdy.$$

and using the change of variable $y = x + hz$ this implies

$$\mathcal{E}_h^{U\cap\omega_j^h}(u) \leq C \int_{x\in U\cap\omega_j^h} \int_{z\in D_h} |u(x)-u(x+hz)|^2 dxdz$$

where $D_h = \{z \in B(0,1),\ x + hz \in U \cap \omega_j^h\}$. Using local coordinates in ω_j^h one sees that there exists a piecewise smooth path $\gamma_{x,z,h} : [0,1] \to \Omega$ joining x to $x + hz$ in ω_j^h such that in local coordinates $\gamma_{x,z,h}$ is the union of two straight lines from $x = (x_1, x')$ to $(x_1 + hz_1, x')$ and from $(x_1 + hz_1, x')$ to $(x_1 + hz_1, x' + hz')$. In particular there exist $C > 0$ independent of x, z, h such that $|\dot\gamma_{x,z,h}(t)| \leq Ch$ for all $t \in [0,1]$ and $d_x\gamma_{x,z,h}(t) = Id + O(h)$ uniformly with respect to z and t. Hence, for any $t \in [0,1]$, $z \in B(0,1)$ and $h > 0$ small enough, the map $\kappa_{t,z,h} : x \mapsto \gamma_{x,z,h}(t)$ is a change of variable from $\{x \in U \cap \omega_j^h,\ x + thz \in U \cap \omega_j^h\}$ onto a subset V_j^h of $U \cap \omega_j^h + B(0,h)$. By the fundamental theorem of analysis, it follows that

$$\mathcal{E}_h^{U\cap\omega_j^h}(u) \leq C \int_{x\in U\cap\omega_j^h} \int_{z\in D_h} |\int_0^1 \dot\gamma_{x,z,h}(t) \cdot \nabla u(\gamma_{x,z,h}(t)dt|^2 dxdz$$

and thanks to the bound $|\dot\gamma_{x,z,h}(t)| \leq Ch$ we get

$$\mathcal{E}_h^{U\cap\omega_j^h}(u) \leq Ch^2 \int_{x\in U\cap\omega_j^h} \int_{z\in D_h} \int_0^1 |\nabla u(\gamma_{x,z,h}(t))|^2 dtdzdx$$

and using the change of variable $y = \kappa_{t,z,h}(x)$ it follows that

$$\begin{aligned}\mathcal{E}_h^{U\cap\omega_j^h}(u) &\leq Ch^2 \int_0^1 \int_{z\in B(0,1)} \int_{y\in V_j^h} |\nabla u(y)|^2 dydzdt \\ &\leq Ch^2 \int_{y\in U\cap\omega_j^h+B(0,h)} |\nabla u(y)|^2 dy.\end{aligned}$$

Plugging this inequality in the last inequality of (4.1) and since J is finite, we get the result. □

From now, given $r > 0$, we denote

$$\Omega_r = \{x \in \Omega,\ d(x, \partial\Omega) \leq r\}. \tag{4.2}$$

Since we do not use in this section the notation Ω_k related to the dyadic decomposition of the cusp, there is no ambiguity.

Corollary 4.2. *Suppose that* (u_h) *is a family of functions which is bounded in* $H^1(\Omega)$. *Then* $\mathcal{E}_h(u_h) = O(h^2)$. *Moreover, if* $u \in H^1(\Omega)$ *is a fixed function independent of* h, *one has*

$$\mathcal{E}_h(u) = \mathcal{E}_h^{\Omega\setminus\Omega_h}(u) + o(h^2).$$

Proof. The first estimate is a direct consequence of the preceding lemma with $U = \Omega$. To get the second estimate observe that

$$\mathcal{E}_h(u) = \mathcal{E}_h^{\Omega\setminus\Omega_h}(u) + \mathcal{E}_h^{\Omega_h}(u) + R_h$$

with

$$R_h = \frac{1}{h^d}\iint_{\Omega_h\times\Omega\setminus\Omega_h} \mathbb{1}_{|x-y|<h}|u(x)-u(y)|^2 d\mu_\rho^2(x,y) \leq 2\mathcal{E}_h^{\Omega_{2h}}(u).$$

From the preceding lemma with $U = \Omega_{2h}$, it follows that

$$h^{-2}\mathcal{E}_h^{\Omega_{2h}}(u) \leq C\int \mathbb{1}_{\Omega_{3h}}(x)|\nabla u(x)|^2 dx$$

which goes to 0 as $h \to 0$ by the dominated convergence theorem (since u doesn't depend on h). □

Recall that $\mathcal{B}_h$ and $\mathcal{B}$ denote the Dirichlet forms associated to $1 - T_h$ and L_ρ respectively. One has the following

Lemma 4.3. *Let Ω be an open set satisfying Assumption 1. Suppose that $(u_h)_{h\in]0,1]}$ is a bounded family of functions in $L^2(\Omega)$ and assume there exists a decomposition $u_h = \varphi_h + v_h + r_h$ such that the following assumptions hold true*

- *(φ_h) converges weakly in $H^1(\Omega)$ towards a limit φ when $h \to 0$.*
- *$\|v_h\|_{L^2} = \mathcal{O}(h)$ when $h \to 0$.*
- *$\mathrm{supp}(r_h) \subset \Omega_{c_0 h}$ for some $c_0 > 0$*
- *there exists $C > 0$ such that $\mathcal{E}_h(r_h) \leq Ch^2$ for all $h \in]0,1]$.*

Then for all $\theta \in H^1(\Omega)$, one has

$$\lim_{h\to 0} h^{-2}\mathcal{B}_h(u_h,\theta) = \mathcal{B}(\varphi,\theta).$$

Proof. Let us denote $\widetilde{\mathcal{B}}_h = h^{-2}\mathcal{B}_h$ and let $\theta \in H^1(\Omega)$. We have to prove that

i) $\lim_{h\to 0}\widetilde{\mathcal{B}}_h(r_h,\theta) = 0$
ii) $\lim_{h\to 0}\widetilde{\mathcal{B}}_h(\varphi_h,\theta) = \mathcal{B}(\varphi,\theta)$
iii) $\lim_{h\to 0}\widetilde{\mathcal{B}}_h(v_h,\theta) = 0$.

Let $M \geq 1$ denote a parameter to be fixed later and let $\Omega_{Mh}^c = \Omega\setminus\Omega_{Mh}$ with Ω_{Mh} defined by (4.2). Given two subset A, B of Ω we denote

$$\widetilde{\mathcal{B}}_h^{A,B}(u,v) = \frac{1}{2h^{d+2}\mathscr{V}_d}\iint_{x\in A, y\in B} \mathbb{1}_{|x-y|<h}(u(x)-u(y))\overline{(v(x)-v(y))}d\mu_\rho^2(x,y)$$

and when $A = B$ we denote $\widetilde{\mathcal{B}}_h^{A,B}(u,v) = \widetilde{\mathcal{B}}_h^{A}(u,v)$. By Cauchy-Schwarz inequality, one has

$$\widetilde{\mathcal{B}}_h^{A,B}(u,v) \leq h^{-2}\sqrt{\mathcal{E}_h^A(u)\mathcal{E}_h^B(v)}. \tag{4.3}$$

Since (φ_h) is bounded in H^1, it follows from Corollary 4.2 that $\mathcal{E}_h(\varphi_h) = O(h^2)$. On the other hand, $\|v_h\|_{L^2} = O(h)$ implies $\mathcal{E}_h(v_h) = O(h^2)$ and hence $\mathcal{E}_h(u_h) = O(h^2)$. Suppose now that u_h, θ are as above. We claim that

$$\widetilde{\mathcal{B}}_h(u_h,\theta) = \widetilde{\mathcal{B}}_h^{\Omega_{Mh}^c}(u_h,\theta) + o(1). \tag{4.4}$$

Indeed, one has

$$\widetilde{\mathcal{B}}_h(u_h,\theta) = \widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega_{Mh}}(u_h,\theta) + \widetilde{\mathcal{B}}_h^{\Omega_{Mh}^c,\Omega_{Mh}}(u_h,\theta) + \widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega_{Mh}^c}(u_h,\theta) + \widetilde{\mathcal{B}}_h^{\Omega_{Mh}^c,\Omega_{Mh}^c}(u_h,\theta),$$

and it follows from (4.3) that

$$
\begin{aligned}
|\widetilde{\mathcal{B}}_h(u_h,\theta) - \widetilde{\mathcal{B}}_h^{\Omega^c_{Mh},\Omega^c_{Mh}}(u_h,\theta)| &\leq h^{-2}\sqrt{\mathcal{E}_h^{\Omega_{Mh}}(u_h)\mathcal{E}_h^{\Omega_{Mh}}(\theta)} \\
&\quad + h^{-2}\sqrt{\mathcal{E}_h^{\Omega^c_{Mh}}(u_h)\mathcal{E}_h^{\Omega_{Mh}}(\theta)} + |\widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega^c_{Mh}}(u_h,\theta)| \\
&\leq 2h^{-2}\sqrt{\mathcal{E}_h(u_h)\mathcal{E}_h^{\Omega_{Mh}}(\theta)} + |\widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega^c_{Mh}}(u_h,\theta)|.
\end{aligned}
\tag{4.5}
$$

Since the operator T_h localizes at scale h one has

$$
|\widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega^c_{Mh}}(u_h,\theta)| = |\widetilde{\mathcal{B}}_h^{\Omega_{Mh},\Omega^c_{Mh}\cap\Omega_{(M+1)h}}(u_h,\theta)| \leq h^{-2}\sqrt{\mathcal{E}_h(u_h)\mathcal{E}_h^{\Omega_{(M+1)h}}(\theta)}.
$$

Combining this estimate with (4.5) and using the bound $\mathcal{E}_h(u_h) = O(h^2)$, we obtain

$$
|\widetilde{\mathcal{B}}_h(u_h,\theta) - \widetilde{\mathcal{B}}_h^{\Omega^c_{Mh},\Omega^c_{Mh}}(u_h,\theta)| \leq Ch^{-1}\sqrt{\mathcal{E}_h^{\Omega_{(M+1)h}}(\theta)}.
$$

By Corollary 4.2, one knows that $\mathcal{E}_h^{\Omega_{(M+1)h}}(\theta) = o(h^2)$ which proves (4.4).

Since $\mathcal{E}_h(r_h) = O(h^2)$, (4.4) implies $\widetilde{\mathcal{B}}_h(r_h,\theta) = \widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(r_h,\theta) + o(1)$ and since for $M > c_0$, $r_h = 0$ on Ω^c_{Mh}, we get directly i).

Let us now prove ii). Using a partition of unity, we can write $\theta = \sum_{j\in J}\theta_j$ with θ_j supported in ω_j for all $j \in J$. Since both side of the equality in ii) are linear with respect to θ we can assume from now that θ is supported in a small chart ω_j. Using the change of variable $y = x + hz$, one has

$$
\widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(\varphi_h,\theta) = \frac{1}{2h^2\mathscr{V}_d}\int_{\Omega^c_{Mh}}\int_{z\in D_{x,h}}(\varphi_h(x) - \varphi_h(x+hz)) \\
(\theta(x) - \theta(x+hz))w_h(x,z)dzdx
$$

where $D_{x,h} = \{z \in \mathbb{R}^d,\ |z| < 1 \text{ and } x + hz \in \Omega\}$. Since $M \geq 1$, for any $x \in \Omega^c_{Mh}$, one has $D_h = D = \{|z| < 1\}$ and for any $t \in [0,1]$ we get $x + thz \in \Omega$. Using this path and the argument of Lemma 4.1, we can write

$$
\widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(\varphi_h,\theta) = \frac{1}{2\mathscr{V}_d}\int_{\Omega^c_{Mh}}\int_{|z|<1}\Big(\int_0^1 z\cdot\nabla\varphi_h(x+shz))ds\Big) \\
\Big(\int_0^1 z\cdot\nabla\theta(x+thz)dt\Big)w_h(x,z)dzdx.
$$

Since ρ is $\mathcal{C}^1$, then $w_h(x,z) = \rho(x) + \mathcal{O}(h)$ and hence

$$
\widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(\varphi_h,\theta) = \frac{1}{2\mathscr{V}_d}\int_{|z|<1}\int_{\Omega^c_{Mh}}\Big(\int_0^1 z\cdot\nabla\varphi_h(x+shz))ds\Big) \\
\Big(\int_0^1 z\cdot\nabla\theta(x+thz)dt\Big)\rho(x)dxdz + O(h).
$$

Using the change of variable $\kappa_{s,h,z} : x \mapsto x - shz$, this implies

$$
\widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(\varphi_h,\theta) = \frac{1}{2\mathscr{V}_d}\int_0^1\int_0^1\int_{|z|<1}\int_{x\in V_{s,h,z}}\Big(z\cdot\nabla\varphi_h(x))\Big) \\
\Big(z\cdot\nabla\theta(x+(t-s)hz)\Big)\rho(y)dxdzdsdt + O(h)
\tag{4.6}
$$

where $V_{s,h,z}=\kappa_{s,h,z}^{-1}(\Omega_{Mh}^c)$. We claim that

$$\int_0^1\int_0^1\int_{|z|<1}\int_{V_{s,h,z}}|\nabla\theta(x+(t-s)hz)-\nabla\theta(x)|^2dxdzdtds=o(1). \tag{4.7}$$

Indeed by density of $C^\infty(\Omega)\cap H^1(\Omega)$ in $H^1(\Omega)$ we can assume that $\theta\in C^\infty(\Omega)$. Let us fix $\epsilon>0$. Since $\theta\in H^1(\Omega)$, there exists $r>0$ such $\int_{\Omega_r}|\nabla\theta(x)|^2dx\leq\epsilon^2$. Moreover, since $\nabla\theta$ is uniformly continuous on $\overline{\Omega}_r^c$, there exists $h_0>0$ such that for all $h\in]0,h_0]$

$$\int_0^1\int_0^1\int_{|z|<1}\int_{\Omega_r^c\cap V_{s,h,z}}|\nabla\theta(x+(t-s)hz)-\nabla\theta(x)|^2dxdzdtds<\epsilon^2.$$

Combining these two estimates, we obtain (4.7). Combined to (4.6), it implies

$$\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}(\varphi_h,\theta)=\frac{1}{2\mathscr{V}_d}\int_0^1\int_{|z|<1}\int_{y\in\Omega}\Big(z\cdot\nabla\varphi_h(y))\Big)\Big(\mathbb{1}_{V_{s,h,z}}(y)z\cdot\nabla\theta(y)\Big)\rho(y)dydzds+o(1)$$

Moreover, since φ_h is bounded in H^1, $\theta\in H^1$ and $\mathbb{1}_{\Omega\setminus V_{s,h,z}}\to0$ pointwise, it follows from Cauchy-Schwarz inequality and dominated convergence theorem that

$$\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}(\varphi_h,\theta)=\frac{1}{2\mathscr{V}_d}\int_{|z|<1}\int_{y\in\Omega}\Big(z\cdot\nabla\varphi_h(y))\Big)\Big(z\cdot\nabla\theta(y)\Big)\rho(y)dydz+o(1)$$

and since φ_h converges weakly to φ in H^1, we obtain

$$\begin{aligned}\lim_{h\to0}\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}(\varphi_h,\theta)&=\frac{1}{2\mathscr{V}_d}\int_\Omega\int_{|z|<1}(z\cdot\nabla\varphi(x))(z\cdot\nabla\theta(x))dz\rho(x)dx\\&=\frac{1}{2\mathscr{V}_d}\sum_{i,j=1}^d\int_\Omega\int_{|z|<1}z_i\partial_i\varphi(x)z_j\partial_j\theta(x)dz\rho(x)dx.\end{aligned}$$

For parity reason the terms associated to $i\neq j$ vanish and using (4.4), we get

$$\lim_{h\to0}\widetilde{\mathscr{B}}_h(\varphi_h,\theta)=\lim_{h\to0}\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}(\varphi_h,\theta)=\sum_{i=1}^d a_j\int_\Omega\partial_i\varphi(x)\partial_i\theta(x)\rho(x)dx$$

with

$$a_j=\frac{1}{2\mathscr{V}_d}\int_{|z|<1}z_i^2dz=\frac{1}{2(d+2)}.$$

This proves ii).

It remains to prove iii). As before we can work with the functional $\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}$ instead of $\widetilde{\mathscr{B}}_h$. One has

$$\widetilde{\mathscr{B}}_h^{\Omega_{Mh}^c}(v_h,\theta)=\frac{1}{2h^2\mathscr{V}_d}\int_\Omega\int_{|z|<1}(v_h(x)-v_h(x+hz))(\theta(x)-\theta(x+hz)w_h(x,z)dzdx$$

Splitting the difference $(v_h(x) - v_h(x+hz))$ in two different integrals and making the change of variable $x \mapsto x - hz$ in the term corresponding to $v_h(x+hz)$ we get

$$\widetilde{\mathcal{B}}_h^{\Omega^c_{Mh}}(v_h,\theta) = \widetilde{\mathcal{B}}_h^{+}(v_h,\theta) + \widetilde{\mathcal{B}}_h^{-}(v_h,\theta)$$

with

$$\mathcal{B}_h^{\pm}(v_h,\theta) = \frac{1}{2h^2\mathscr{V}_d}\int_{\Omega^c_{Mh}}\int_{|z|<1} v_h(x)(\theta(x) - \theta(x \pm hz))w_h(x,z)dzdx,$$

where the integration domain in the variable z is the unit disc for the same reason as before. We show how to estimate $\widetilde{\mathcal{B}}_h^{+}$, the case of $\widetilde{\mathcal{B}}_h^{-}$ is similar. The same computation as above shows that

$$\widetilde{\mathcal{B}}_h^{+}(v_h,\theta) = \frac{1}{2h\mathscr{V}_d}\int_{|z|<1}\int_{\Omega^c_{Mh}} v_h(x) \Big(\int_0^1 z\cdot\nabla\theta(x+thz)dt\Big)\rho(x)dxdz + O(h)$$

where we used again $w_h(x,z) = \rho(x) + O(h)$. Since $\theta \in H^1$, $\|v_h\|_{L^2} = O(h)$ and $\mathbb{1}_{\Omega\setminus\Omega^c_{Mh}} \to 0$ pointwise, we get as in the proof of ii) that

$$\widetilde{\mathcal{B}}_h^{+}(v_h,\theta) = \frac{1}{2h\mathscr{V}_d}\int_{\Omega}\int_{|z|<1} v_h(x)\Big(z\cdot\nabla\theta(x)\Big)\rho(x)dzdx + o(1),$$

and since $\int_{|z|<1}(z\cdot\nabla\theta(x))dz = 0$, we obtain $\widetilde{\mathcal{B}}_h^{+}(v_h,\theta) = o(1)$ which proves iii). □

4.2. **Case of smooth densities.** In this section we prove Theorem 1.2. We follow the proof of Theorem 1.2 in [3]. Let $|\triangle_h|$ be the rescaled (non negative) Laplacien associated to the Markov kernel T_h

$$|\triangle_h| = \frac{1-T_h}{h^2}. \tag{4.8}$$

Let $R > 0$ be fixed. If $\nu_h \in [0,R]$ and $u_h \in L^2(\Omega)$ satisfy $|\triangle_h|u_h = \nu_h u_h$ and $\|u_h\|_{L^2} = 1$, then thanks to Lemma 3.1, u_h can be decomposed as $u_h = \varphi_h + v_h + r_h$ with $\|v_h\|_{L^2} = O(h)$, φ_h bounded in $H^1(\Omega)$ and r_h supported in $\Gamma_h \subset \Omega_{c_0 h}$ for some $c_0 > 0$. Moreover, we claim that $\mathcal{E}_h(r_h) = O(h^2)$. Indeed, since $r_h = u_h - \varphi_h - v_h$ and $\mathcal{E}_h(u_h) = h^2\nu_h$ with ν_h bounded, it suffices to show that $\mathcal{E}_h(v_h)$ and $\mathcal{E}_h(\varphi_h)$ are $O(h^2)$. The bound on $\mathcal{E}_h(v_h)$ follows directly from the fact that $\|r_h\|_{L^2} = \mathcal{O}(h)$ and that $1-T_h$ is bounded on L^2. The bound on $\mathcal{E}_h(\varphi_h)$ is obtained from the fact that φ_h is bounded in H^1 and Corollary 4.2. Consequently, (extracting a subsequence if necessary) we can assume that (φ_h) weakly converges in $H^1(\Omega)$ to a limit φ and that (ν_h) converges to a limit ν. Hence (u_h) converge strongly in L^2 to φ, and it now follows from Lemma 4.3 that for any $\theta \in H^1(\Omega)$,

$$\nu\langle\varphi,\theta\rangle = \lim_{h\to 0}\nu_h\langle u_h,\theta\rangle = \lim_{h\to 0} h^{-2}\mathcal{B}_h(u_h,\theta) = \mathcal{B}(\varphi,\theta). \tag{4.9}$$

Since θ is arbitrary in H^1 this shows that $\varphi \in D(L_\rho)$ and that $(L_\rho - \nu)\varphi = 0$. Hence ν is an eigenvalue of L_ρ. Moreover, the dimension of an orthonormal basis is preserved by strong limit. So the above argument proves that for any $\epsilon > 0$ small, there exists $h_\epsilon > 0$ such that for $h \in]0, h_\epsilon]$, one has

$$\sigma(|\Delta_h|) \cap [0, R] \subset \cup_j [\nu_j - \epsilon, \nu_j + \epsilon] \tag{4.10}$$

and

$$\sharp\sigma(|\Delta_h|) \cap [\nu_j - \epsilon, \nu_j + \epsilon] \leq m_j. \tag{4.11}$$

In order to show that one has equality in (4.11) for ϵ small enough, observe that for any $\psi \in H^1(\Omega)$ independent of h, one has

$$\lim_{h\to 0} h^{-2}\mathcal{E}_h(\psi) = \mathcal{B}(\psi, \psi)$$

thanks to Lemma 4.3. In particular, if $\psi \in D(L_\rho)$ satisfies $L_\rho\psi = \nu\psi$ for some $\nu > 0$, then $\lim_{h\to 0} h^{-2}\mathcal{E}_h(\psi) = \nu\|\Psi\|^2$. Hence, we can mimic the proof of Theorem 2 iii) in [3] to get the result.

4.3. **Case of measurable densities.** In this section we assume that ρ is a measurable function satisfying (1.1) and we prove Theorem 1.3. We first apply Theorem 1.2 with $\rho_0 = 1$. It follows that 1 is a simple eigenvalue of T_{h,ρ_0}. Moreover, denoting $(\mu_{k,\rho_0}(h))_{k\in\mathbb{N}}$ the decaying sequence of positive eigenvalues of T_{h,ρ_0}, one has $1 = \mu_{0,\rho_0} > \mu_{1,\rho_0}(h)$ and $\mu_{1,\rho_0}(h) = h^2\nu_1 + o(h^2)$ where we recall that $\nu_1 > 0$ is the lowest positive eigenvalue of the Neumann Laplacian on Ω. Moreover, one has $\ker(T_{h,\rho_0} - 1) = \mathrm{Span}(1)$. Combined to the spectral theorem, this implies that for all $u \in \mathrm{Span}(1)^\perp$, we have

$$\langle (1 - T_{h,\rho_0})u, u\rangle_{L^2(\rho_0)} \geq Ch^2\|u\|^2_{L^2(\rho_0)}. \tag{4.12}$$

On the other hand, from (3.2) one has

$$\langle (1 - T_{h,\rho_0})u, u\rangle_{L^2(\rho_0)} = \frac{1}{2h^d\mathscr{V}_d}\int_{\Omega\times\Omega} \mathbb{1}_{|x-y|<h}(f(x) - f(y))^2 d\mu^2_{\rho_0}(x, y),$$

and since $m \leq \rho \leq M$, then

$$\langle (1 - T_{h,\rho_0})u, u\rangle_{L^2(\rho_0)} \leq \frac{1}{m}\langle (1 - T_{h,\rho})u, u\rangle_{L^2(\rho)}$$

and $\|u\|^2_{L^2(\rho_0)} \geq \frac{1}{M}\|u\|^2_{L^2(\rho)}$. Combined with (4.12), this implies that there exists a new positive constant C such that for all $u \in \mathrm{Span}(1)^\perp$, we have

$$\langle (1 - T_{h,\rho})u, u\rangle_{L^2(\rho)} \geq Ch^2\|u\|^2_{L^2(\rho)}.$$

This proves i) and the lower bound on $g(h)$. The upper bound is proved in the same way, using the equivalence of Dirichlet forms.

4.4. **Total variation estimates.** This section is devoted to the proof of Theorem 1.4. Thanks to (1.10), we have

$$\sup_{x\in\Omega} \|t^n_{h,\rho}(x, dy) - \mu_\rho\|_{TV} = \frac{1}{2}\|T^n_{h,\rho} - \Pi_0\|_{L^\infty\to L^\infty}$$

where Π_0 denotes the orthogonal projection on $\mathrm{Span}(1)$ in $L^2(\rho)$. Throughout this section, we drop the dependance with respect to ρ in the notations. For any $p \in \mathbb{N}$, one has $T^p_h = A_p + B_p$ with $A_1 = m_h$, $B_1 = K_h$ and for any $p \geq 1$ $A_{p+1} = m_hA_p$, $B_{p+1} = m_hB_p + K_hT^p_h$. Since $\|m_h\|_{L^\infty\to L^\infty} \leq 1 - Ch^\gamma$

and $\|K_h\|_{L^2\to L^\infty} \leq Ch^{-\frac{d}{2}}$, it follows from (2.49) and (2.50) in [2] that for any $p \in \mathbb{N}$

$$\begin{aligned} \|A_p\|_{L^\infty\to L^\infty} &\leq (1 - Ch^\gamma)^p \\ \|B_p\|_{L^2\to L^\infty} &\leq Ch^{-\gamma-\frac{d}{2}}. \end{aligned} \tag{4.13}$$

Suppose now that $p, n \in \mathbb{N}$. Since $T_h\Pi_0 = \Pi_0$ we get

$$\begin{aligned} \|T_h^{p+n+1} - \Pi_0\|_{L^\infty\to L^\infty} \leq \|A_p\|_{L^\infty\to L^\infty}\|T_h^{n+1} - \Pi_0\|_{L^\infty\to L^\infty} \\ + \|B_p(T_h^{n+1} - \Pi_0)\|_{L^\infty\to L^\infty}. \end{aligned}$$

Taking $p = \lfloor Mnh^{2-\gamma}\rfloor$ with $M > 0$ to be chosen large enough (here we denote $\lfloor n\rfloor$ the integer part of $n \in \mathbb{N}$), we deduce from (4.13) that

$$\|A_p\|_{L^\infty\to L^\infty} \leq e^{-nMCh^2}$$

where C is a positive constant independent of h and M. Since T_h is markovian, T_h and Π_0 are bounded by 1 on L^∞ and conseqently

$$\|T_h^{p+n+1} - \Pi_0\|_{L^\infty\to L^\infty} \leq Ce^{-nMCh^2} + \|B_p(T_h^{n+1} - \Pi_0)\|_{L^\infty\to L^\infty}. \tag{4.14}$$

We shall now estimate the second term in the above right hand side. One has

$$\|B_p(T_h^{n+1} - \Pi_0)\|_{L^\infty\to L^\infty} \leq \|B_p\|_{L^2\to L^\infty}\|T_h^n - \Pi_0\|_{L^2\to L^2}\|T_h\|_{L^\infty\to L^2}$$

and from Proposition 1.1 and Theorem 1.3, we know that $\sigma(T_h) \setminus \{1\} \subset [-1 + Ch^\gamma, 1 - g(h)]$ with $h^2/C \leq g(h) \leq Ch^2$ and $\gamma < 2$. Hence it follows from the spectral theorem, that for h small enough

$$\|T_h^n - \Pi_0\|_{L^2\to L^2} \leq (1 - g(h))^n.$$

Combined with (4.13) and the estimate $\|T_h\|_{L^\infty\to L^2} \leq \|T_h\|_{L^\infty\to L^\infty} = 1$, it follows that

$$\|B_p(T_h^{n+1} - \Pi_0)\|_{L^\infty\to L^\infty} \leq Ch^{-\gamma-\frac{d}{2}}(1 - g(h))^n \leq Ch^{-\gamma-\frac{d}{2}}e^{-ng(h)}.$$

Together with (4.14), this implies

$$\|T_h^{p+n+1} - \Pi_0\|_{L^\infty\to L^\infty} \leq Ce^{-nMCh^2} + Ch^{-\gamma-\frac{d}{2}}e^{-ng(h)}.$$

Since $h^2/C \leq g(h) \leq Ch^2$, it follows that for $M > 0$ large enough one has

$$\|T_h^{p+n+1} - \Pi_0\|_{L^\infty\to L^\infty} \leq Ch^{-\gamma-\frac{d}{2}}e^{-ng(h)}.$$

Taking advantage of $p = \lfloor Mnh^{2-\gamma}\rfloor$, this can be written

$$\|T_h^n - \Pi_0\|_{L^\infty\to L^\infty} \leq Ch^{-\gamma-\frac{d}{2}}e^{-ng(h)(1+O(h^{2-\gamma}))}$$

which proves (1.11).

5. Appendix

Let $\Pi^d=(\mathbb{R}/2\mathbb{Z})^d$, $d=1+d'+d''$. For $f\in L^2(\Pi^d)$ and for any $k=(k_1,k',k'')\in\mathbb{Z}\times\mathbb{Z}^{d'}\times\mathbb{Z}^{d''}$, we denote by

$$\hat{f}(k)=\mathcal{F}f(k):=\frac{1}{2^{d/2}}\int_{\Pi^d}e^{-i\pi\langle x,k\rangle}f(x)dx. \tag{5.1}$$

the Fourier coefficients of the function f. The map $\mathcal{F}$ is an isometry from $L^2(\Pi^d)$ onto $\ell^2(\mathbb{Z}^d)$ and we denote by $\bar{\mathcal{F}}$ its adjoint:

$$\bar{\mathcal{F}}(a)=\frac{1}{2^{d/2}}\sum_{k\in\mathbb{Z}^d}a_k e^{i\pi\langle x,k\rangle} \tag{5.2}$$

for any $a=(a_k)_{k\in\mathbb{Z}^d}$. Let also $\lambda_1,\lambda',\lambda''>0$ be some parameters and denote $\lambda=(\lambda_1,\lambda',\lambda'')$. We recall that for any $\xi=(\xi_1,\xi',\xi'')\in\mathbb{R}^{1+d'+d''}$ we denote $\lambda\cdot\xi=(\lambda_1\xi_1,\lambda'\xi',\lambda''\xi'')$. For any $s\in\mathbb{R}$, we define the λ-Sobolev space as the space of functions ϕ such that $\|\phi\|_{H^s_\lambda}<\infty$ where

$$\|\phi\|_{H^s_\lambda}=\|(\langle\lambda\cdot k\rangle^s\mathcal{F}f(k))_k\|_{\ell^2(\mathbb{Z}^d)}. \tag{5.3}$$

We define similarly the partial Fourier coefficients $\mathcal{F}_{x',x''}:L^2(\Pi^{1+d'+d''})\to\ell^2(\mathbb{Z}^{d'+d''},L^2(\Pi))$, $\mathcal{F}_{x_1}:L^2(\Pi^{1+d'+d''})\to\ell^2(\mathbb{Z},L^2(\Pi^{d'+d''}))$ and their adjoint $\bar{\mathcal{F}}_{x',x''}$, $\overline{\mathcal{F}}_{x_1}$. Consider the hypersurface $\Sigma_a=\{x_1=a\}\times\Pi^{d-1}\subset\Pi^d$. We define the trace operator $\gamma_a^\Pi:H^1_\lambda(\Pi^d)\to H^{1/2}_\lambda(\Sigma_a)$ by

$$\begin{aligned}\gamma_a^\Pi\phi(x',x'')&=\frac{1}{\sqrt{2}}\bar{\mathcal{F}}_{x',x''}\Big(\sum_{k_1\in\mathbb{Z}}e^{i\pi k_1 a}\mathcal{F}\phi(k_1,k',k'')\Big)(x',x'')\\&=\frac{1}{\sqrt{2}}\sum_{k_1\in\mathbb{Z}}e^{i\pi k_1 a}\mathcal{F}_{x_1}\phi(k_1,x',x'').\end{aligned} \tag{5.4}$$

Lemma 5.1. *Let $s>\frac12$. There exists $C>0$ such that for any $\lambda_1,\lambda',\lambda''>0$ and any $\phi\in H^s_\lambda(\Pi^d)$ such that $\mathcal{F}_{x_1}\phi(0,x',x'')=0$, one has*

$$\|\gamma_a^\Pi\phi\|_{H^{s-\frac12}_{\lambda',\lambda''}(\Sigma_a)}\leq C\lambda_1^{-\frac12}\|\phi\|_{H^s_\lambda(\Pi^d)}. \tag{5.5}$$

Proof. We may assume without loss of generality that $a=0$. By a density argument, it is sufficient to prove (5.5) for $\phi\in C^\infty(\Pi^d)$ such that $\mathcal{F}_{x_1}\phi(0,x',x'')=0$. For such functions, the sum in (5.4) is over $k_1\in\mathbb{Z}^*$ and it follows from Cauchy-Schwarz inequality that

$$\begin{aligned}|\mathcal{F}_{x',x''}(\gamma_a\phi)(k',k'')|&=\frac{1}{\sqrt{2}}|\sum_{k_1\in\mathbb{Z}^*}e^{i\pi k_1 a}\mathcal{F}\phi(k)|\\&\leq\frac{1}{\sqrt{2}}\Big(\sum_{k_1\in\mathbb{Z}^*}\langle\lambda\cdot k\rangle^{-2s}\Big)^{\frac12}\Big(\sum_{k_1\in\mathbb{Z}^*}\langle\lambda\cdot k\rangle^{2s}|\mathcal{F}\phi(k)|^2\Big)^{\frac12}.\end{aligned} \tag{5.6}$$

We claim that there exists a constant $C>0$ such that for any $(k',k'')\in\mathbb{Z}^{d'+d''}$ and any $\lambda',\lambda''>0$, one has

$$\sum_{k_1\in\mathbb{Z}^*}\langle\lambda\cdot k\rangle^{-2s}\leq C\lambda_1^{-1}\langle(\lambda'k',\lambda''k'')\rangle^{1-2s}. \tag{5.7}$$

Indeed, since the function $m : t \mapsto (1 + |\lambda_1 t|^2 + |\lambda' k'|^2 + |\lambda'' k''|^2)^{-s}$ is decreasing and integrable on $\mathbb{R}$, one has

$$\sum_{k_1 \in \mathbb{Z}^*} \langle \lambda \cdot k \rangle^{-2s} = \sum_{k_1 \in \mathbb{Z}^*} m(k_1) \leq \int_{\mathbb{R}} m(t) dt. \tag{5.8}$$

Using the change of variable $t \mapsto \frac{\langle (\lambda' k', \lambda'' k'') \rangle}{\lambda_1} t$ one gets $\int_{\mathbb{R}} m(t)dt = C_1 \lambda_1^{-1} \langle (\lambda' k', \lambda'' k'') \rangle^{1-2s}$ for some universal constant C_1. Combined with (5.8), this proves (5.7).

Now, using (5.7) and (5.6), we get

$$|\mathcal{F}_{x',x''}(\gamma_a \phi)(k', k'')|^2 \leq C \lambda_1^{-1} \langle (\lambda' k', \lambda'' k'') \rangle^{1-2s} \sum_{k_1 \in \mathbb{Z}^*} \langle \lambda \cdot k \rangle^{2s} |\mathcal{F}\phi(k)|^2$$

and hence

$$\begin{aligned} \|\gamma_a^{\Pi} \phi\|^2_{H^{s-\frac{1}{2}}_{\lambda',\lambda''}} &= \sum_{k',k''} \langle (\lambda' k', \lambda'' k'') \rangle^{2s-1} |\mathcal{F}_{x',x''}(\gamma_a \phi)(k', k'')|^2 \\ &\leq C \lambda_1^{-1} \sum_{k',k''} \sum_{k_1 \neq 0} \langle \lambda \cdot k \rangle^{2s} |\mathcal{F}\phi(k)|^2 = C \lambda_1^{-1} \|\phi\|^2_{H^s_\lambda(\Pi^d)} \end{aligned}$$

which proves the result. □

Given $0 < a < b < 2$, the restriction operator defined by (3.22) acts on H^1 functions $R_{]a,b[} : H^1(\Pi^d) \to H^1(]a, b[\times \Pi^{d-1})$ and one defines the trace operator

$$\gamma_a : H^1(]a, b[\times \Pi^{d-1}) \to H^{\frac{1}{2}}(\Sigma_a) \tag{5.9}$$

by $\gamma_a f = \gamma_a^{\Pi} \tilde{f}$ for any $\tilde{f} \in H^1(\Pi^d)$ such that $R_{]a,b[} \tilde{f} = f$. Throughout we write $\gamma_a f = f_{|x_1=a}$

Suppose now that $a < b < c$ are some fixed real numbers and let $A_0, A_1, A_2 \subset \mathbb{R} \times \Pi^{d-1}$ be defined by $A_0 =]a, b[\times \Pi^{d-1}$, $A_1 =]b, c[\times \Pi^{d-1}$, $A_2 =]a, c[\times \Pi^{d-1}$.

Lemma 5.2. *Let $(\phi_j)_{j=0,1,2} \in H^1_\lambda(A_j)$ and $r_2 \in L^2(A_2)$ be some functions depending one some parameters $\lambda = (\lambda_1, \lambda', \lambda'') \in]0, +\infty[^3$ and $h > 0$. Let $f \in L^2(A_2)$ given by $f = \mathbb{1}_{A_0} \phi_0 + \mathbb{1}_{A_1} \phi_1$ and assume that $f = \phi_2 + r_2$ with*

$$\|\phi_j\|_{H^1_\lambda(A_j)} \leq 1 \text{ and } \|r_2\|_{L^2(A_2)} \leq h$$

for all $j = 0, 1, 2$. Then there exists $h_1 > 0$ and $\Upsilon > 0$ such that for $0 < \lambda_1 h < h_1$, there exists $\psi \in H^1_\lambda(A_1)$ supported in $b \leq x_1 < b + h\lambda_1$ and such that $\psi_{|x_1=b} = (\phi_0)_{|x_1=b} - (\phi_1)_{|x_1=b}$ and

$$\|\psi\|_{H^1_\lambda(A_1)} \leq \Upsilon \text{ and } ||\psi||_{L^2(A_1)} \leq \Upsilon h \tag{5.10}$$

Proof. Throughout C denotes a positive constant independent of h and λ that may change from line to line. First observe that the statement of the lemma is invariant by translation and dilation in the variable x_1. Hence we can assume without loss of generality that $a = -1, b = 0$ and $c = 1$. We denote $\Sigma = \{x_1 = 0\} \times \Pi^{d-1} \subset]a, c[\times \Pi^{d-1}$ and we let $\sigma :]a, c[\times \Pi^{d-1} \to]a, c[\times \Pi^{d-1}$ denote the symmetry with respect to Σ. We define $g_0 = \mathbb{1}_{A_0} \phi_0 + \mathbb{1}_{A_1} \phi_0 \circ \sigma$ and $g_1 = \mathbb{1}_{A_0} \phi_1 \circ \sigma + \mathbb{1}_{A_1} \phi_1$. We denote

$$\theta = (\phi_0)_{|\Sigma} - (\phi_1)_{|\Sigma} := \gamma_0(g_0 - g_1) \tag{5.11}$$

with γ_0 defined by (5.9). We claim that there exists $C_0 > 0$ independent of the ϕ_i such that

$$\|\theta\|_{L^2(\Sigma)} \leq C_0\sqrt{\frac{h}{\lambda_1}} \tag{5.12}$$

and

$$\|\theta\|_{H^{1/2}_{\lambda',\lambda''}(\Sigma)} \leq \frac{C_0}{\sqrt{\lambda_1}}. \tag{5.13}$$

In order to prove (5.12), let $\varepsilon > 0$ a constant to be fixed later and let

$$I_\epsilon = I_\epsilon(x') := \int_{-\epsilon}^{0} f(x_1, x')dx_1 - \int_0^{\epsilon} f(x_1, x')dx_1 \tag{5.14}$$

wich is well defined for $|\epsilon| < h_1 := \min(b-a, c-b)$. By Taylor expansion, one has $\phi_i(x) = \phi_i(0, x') + \int_0^{x_1} \partial_1\phi_i(t, x')dt$ and hence

$$\begin{aligned} I_\epsilon(x') &= \int_{-\epsilon}^{0} \phi_0(x_1, x')dx_1 - \int_0^{\epsilon} \phi_1(x_1, x')dx_1 \\ &= \epsilon\theta(x') + \int_{-\epsilon}^{0}\int_0^{x_1} \partial_1\phi_0(t, x')dtdx_1 \\ &\quad - \int_0^{\epsilon}\int_0^{x_1} \partial_1\phi_1(t, x')dtdx_1. \end{aligned} \tag{5.15}$$

Moreover, one has

$$\begin{aligned} &\Big\| \int_{-\epsilon}^{0}\int_0^{x_1} \partial_1\phi_0(t, x')dtdx_1\Big\|_{L^2(\Pi^{d-1})} \\ &\qquad \leq \Big\| \int_{-\epsilon}^{0} \sqrt{|x_1|}\lambda_1^{-1}\|\lambda_1\partial_1\phi_0\|_{L^2(]a,b[)}dx_1\Big\|_{L^2(\Pi^{d-1})} \\ &\qquad \leq \lambda_1^{-1}\int_{-\epsilon}^{0} \sqrt{|x_1|}\|\lambda_1\partial_1\phi_0\|_{L^2(A_0)}dx_1 \\ &\qquad \leq \frac{2}{3}\epsilon^{\frac{3}{2}}\lambda_1^{-1}\|\phi_0\|_{H^1_\lambda(A_0)} \leq C\epsilon^{\frac{3}{2}}\lambda_1^{-1}, \end{aligned} \tag{5.16}$$

and of course an estimate similar to (5.16) holds true for ϕ_1. On the other hand, since $f = \phi_2 + r_2$, then

$$\begin{aligned} I_\epsilon(x') = &\int_{-\epsilon}^{0} r_2(x_1, x')dx_1 - \int_0^{\epsilon} r_2(x_1, x')dx_1 + \int_{-\epsilon}^{0} \phi_2(0, x')dx' - \int_0^{\epsilon} \phi_2(0, x')dx' \\ &+ \int_{-\epsilon}^{0}\int_0^{x_1} \partial_1\phi_2(t, x')dtdx_1 - \int_0^{\epsilon}\int_0^{x_1} \partial_1\phi_2(t, x')dtdx_1 \\ = &\int_{-\epsilon}^{0} r_2(x_1, x')dx_1 - \int_0^{\epsilon} r_2(x_1, x')dx_1 \\ &+ \int_{-\epsilon}^{0}\int_0^{x_1} \partial_1\phi_2(t, x')dtdx_1 - \int_0^{\epsilon}\int_0^{x_1} \partial_1\phi_2(t, x')dtdx_1. \end{aligned}$$

The two last terms of the above identity are estimated as above. It follows that

$$\|I_\epsilon(x')\|_{L^2(\Pi^{d-1})} \leq \int_{-\epsilon}^{\epsilon} \Big\|r_2(x_1, x')\Big\|_{L^2(\Pi^{d-1})}dx_1 + C\epsilon^{\frac{3}{2}}\lambda_1^{-1}.$$

Using Cauchy-Schwarz and the assumption on r_2, we get

$$\|I_\epsilon(x')\|_{L^2(\Pi^{d-1})} \leq C(h\sqrt{\epsilon} + \epsilon^{\frac{3}{2}}\lambda_1^{-1}).$$

Combining this estimate, (5.15) and (5.16), we get

$$\|\theta\|_{L^2(\Sigma)} \leq C(\frac{\sqrt{\epsilon}}{\lambda_1} + \frac{h}{\sqrt{\epsilon}}).$$

Minimizing the right hand side by taking $\epsilon = h\lambda_1$we get $||\theta||_{L^2(\Sigma)}= \mathcal{O}(\sqrt{h/\lambda_1})$ which proves (5.12).

Next we want to prove (5.13). We recall that θ is defined by (5.11) and we decompose $g_0 - g_1 = \delta + \bar{g}$ with

$$\bar{g}(x) = \int_{-1}^{1} (g_0 - g_1)(t, x', x'')dt. \tag{5.17}$$

The function $\bar{g}$ is independant of x_1, hence it is defined as a function on A_2 and as a function on Σ

Sub-lemma 5.3. *One has $\bar{g} \in H^1_\lambda(A_2)$ and $\bar{g} \in H^{\frac{1}{2}}_{\lambda',\lambda''}(\Sigma)$. Moreover*

$$\|\bar{g}\|_{H^1_\lambda(A_2)} \leq \|\phi_0\|_{H^1_\lambda(A_0)} + \|\phi_1\|_{H^1_\lambda(A_1)} \tag{5.18}$$

and there exists a constant $C > 0$ such that

$$\|\bar{g}\|_{H^{\frac{1}{2}}_{\lambda',\lambda''}(\Sigma)} \leq \frac{C}{\sqrt{\lambda_1}} \tag{5.19}$$

Proof. By Cauchy-Schwarz inequality, one has $\|\bar{g}\|_{L^2(A_2)} \leq C(\|\phi_0\|_{L^2} + \|\phi_1\|_{L^2})$, $\|\lambda'\partial_{x'}\bar{g}\|_{L^2(A_2)} \leq C(\|\lambda'\partial_{x'}\phi_0\|_{L^2} + \|\lambda'\partial_{x'}\phi_1\|_{L^2})$ and a similar estimate for derivative in the variable x''. Moreover, $\lambda_1\partial_{x_1}\bar{g} = 0$. Hence we have $\bar{g} \in H^1_\lambda(A_2)$ and (5.18) holds true. By a classical trace theorem, it follows that $\bar{g} \in H^{\frac{1}{2}}_{\lambda',\lambda''}(\Sigma)$ and it remains to prove (5.19). We can assume $\lambda_1 \geq 1$. One has

$$\|\bar{g}\|^2_{H^{\frac{1}{2}}_{\lambda',\lambda''}(\Sigma)} = \sum_{\tilde{k}\in\mathbb{Z}^{d'+d''}} \langle\tilde{\lambda}\cdot\tilde{k}\rangle|\mathcal{F}_{x',x''}\bar{g}(\tilde{k})|^2$$

where we denote $\tilde{\lambda} = (\lambda', \lambda'')$ and $\tilde{k} = (k', k'')$. Splitting the sum in two parts we get $\|\bar{g}\|^2_{H^{\frac{1}{2}}_{\lambda',\lambda''}(\Sigma)} = S_{\leq}(\lambda_1) + S_{>}(\lambda_1)$ where

$$S_{>}(\lambda_1) = \sum_{\langle\tilde{\lambda}\cdot\tilde{k}\rangle>\lambda_1} \langle\tilde{\lambda}\cdot\tilde{k}\rangle|\mathcal{F}_{x',x''}\bar{g}(\tilde{k})|^2.$$

One has

$$S_{>}(\lambda_1) \leq \frac{1}{\lambda_1}\sum_{\langle\tilde{\lambda}\cdot\tilde{k}\rangle>\lambda_1} \langle\tilde{\lambda}\cdot\tilde{k}\rangle^2|\mathcal{F}_{x',x''}\bar{g}(\tilde{k})|^2 \leq \frac{1}{\lambda_1}\|\bar{g}\|^2_{H^1_\lambda(A_2)} \leq \frac{C}{\lambda_1} \tag{5.20}$$

thanks to (5.18). In order to estimate the low frequencies, we observe that

$$S_{\leq}(\lambda_1) \leq \lambda_1 \sum_{\langle\tilde{\lambda}\cdot\tilde{k}\rangle\leq\lambda_1} |\mathcal{F}_{x',x''}\bar{g}(\tilde{k})|^2 \leq \lambda_1\|\bar{g}\|^2_{L^2(\Sigma)}. \tag{5.21}$$

We claim that

$$\|\bar{g}\|^2_{L^2(\Sigma)} \leq (h^2 + \frac{1}{\lambda_1^2}) \tag{5.22}$$

Indeed, by Cauchy-Schwarz inequality and thanks to the symetric form of g_0 and g_1, one has

$$\begin{aligned}\|\bar{g}\|_{L^2(\Sigma)} &\leq C\|\phi_0 - \phi_1 \circ \sigma\|_{L^2(A_0)} = C\|f - f\circ\sigma\|_{L^2(A_2)}\\ &\leq C\|\phi_2 - \phi_2\circ\sigma\|_{L^2(A_2)} + C\|r_h\|_{L^2(A_2)}\end{aligned}$$

Moreover, since $\phi_2(x) - \phi_2\circ\sigma(x) = \int_{-x_1}^{x_1} \partial_1\phi_2(t, x', x'')dt$, we get

$$\|\bar{g}\|_{L^2(\Sigma)} \leq \frac{C}{\lambda_1}\|\phi_2\|_{H^1_\lambda} + C\|r_h\|_{L^2(A_2)} \leq \frac{C}{\lambda_1} + Ch \tag{5.23}$$

which proves (5.22). Now combining (5.21) and (5.22) we get $S_{\leq}(\lambda_1) \leq \frac{C}{\lambda_1} + Ch$ which combined with (5.20) proves the result since $h\lambda_1$ is bounded. □

We are now in position to estimate θ in $H^{\frac{1}{2}}$. One has $\theta = \bar{g} + \gamma_0(\delta)$ with $\delta = g_0 - g_1 - \bar{g}$ and from Sub-lemma 5.3 we know that

$$\|\bar{g}\|_{H^{1/2}_{\lambda',\lambda''}(\Sigma)} \leq C/\sqrt{\lambda_1}. \tag{5.24}$$

Moreover, by construction, one has $\|g_j\|_{H^1_\lambda(A_2)} \leq C$ for $j = 0, 1$ and by Sub-lemma 5.3 one has also $\|\bar{g}\|_{H^1_\lambda(A_2)} \leq C$. Hence $\|\delta\|_{H^1_\lambda(A_2)} \leq C$ and since $\int_{A_2} \delta(x_1, x', x'')dx_1 = 0$, Lemma 5.1 implies $\|\delta\|_{H^{1/2}_{\lambda',\lambda''}(\Sigma)} \leq C/\sqrt{\lambda_1}$. Combined with (5.24), this proves (5.13).

We are now in position to define the function ψ. Let $\rho \in C^\infty(\mathbb{R}_+)$ be such that $\rho(0) = 1$ and $\mathrm{supp}(\rho) \subset [0, \frac{1}{2}]$. We define ψ via its partial Fourier coefficients in the variables (x', x''). For $x_1 \in]0, 1[$ and $\tilde{k} = (k', k'') \in \mathbb{Z}^{d'+d''}$, let

$$\hat{\psi}(x_1, \tilde{k}) = \begin{cases} \rho\Big(\frac{x_1}{h\lambda_1}\Big)\hat{\theta}(\tilde{k}) \text{ if } \langle\tilde{k}\rangle \leq h^{-1}\\ \rho\Big(\frac{x_1\langle\tilde{k}\rangle}{\lambda_1}\Big)\hat{\theta}(\tilde{k}) \text{ if } \langle\tilde{k}\rangle \geq h^{-1}\end{cases}$$

where for sake of shortness we denote $\hat{u} = \mathcal{F}_{\lambda',\lambda''}(u)$. Of course, $\psi_{|\Sigma} = \theta$ since one has $\hat{\psi}(0, \tilde{k}) = \hat{\theta}(\tilde{k})$. Moreover, since ρ is supported in $[0, \frac{1}{2}]$, then ψ is supported in $0 < x_1 \leq h\lambda_1$. Let us now estimate its L^2 and H^1 norms. Denoting $\|u\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})} = \sum_{k\in\mathbb{Z}^{d-1}} \|u(., k)\|^2_{L^2(]0,1[)}$, we have

$$\begin{aligned}||\psi||^2_{L^2(A_1)} &= \|\hat{\psi}(x_1, \tilde{k})\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})}\\ &= \sum_{\langle\tilde{k}\rangle\leq h^{-1}} |\hat{\theta}(\tilde{k})|^2 \int_0^\infty |\rho(\frac{x_1}{h\lambda_1})|^2 dx_1 + \sum_{\langle\tilde{k}\rangle\geq h^{-1}} |\hat{\theta}(\tilde{k})|^2 \int_0^\infty |\rho(\frac{x_1\langle\tilde{k}\rangle}{\lambda_1})|^2 dx_1\\ &\leq \|\rho\|^2_{L^2}\Big(\sum_{\langle\tilde{k}\rangle\leq h^{-1}} h\lambda_1|\hat{\theta}(\tilde{k})|^2 + \sum_{\langle\tilde{k}\rangle\geq h^{-1}} \frac{\lambda_1}{\langle\tilde{k}\rangle}|\hat{\theta}(\tilde{k})|^2\Big)\\ &\leq h\lambda_1\|\rho\|^2_{L^2(\mathbb{R}_+)}\|\theta\|^2_{L^2(\Sigma)} \leq Ch^2\end{aligned}$$

thanks to (5.12). This proves the second part of (5.10). To prove the H^1 estimate, we observe that

$$\|\psi\|^2_{H^1_\lambda(A_1)} = \|\psi\|^2_{L^2(A_1)} + \|\lambda_1\partial_1\hat{\psi}\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})} + \|\langle\tilde{k}\rangle\hat{\psi}\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})} \tag{5.25}$$

and we estimate separately each term of the right hand side. First, we have

$$\begin{aligned}
\|\lambda_1\partial_1\hat{\psi}\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})} &= \sum_{\langle\tilde{k}\rangle\le h^{-1}} h^{-2}|\hat{\theta}(\tilde{k})|^2\int_0^\infty|\rho'(\frac{x_1}{h\lambda_1})|^2dx_1 \\
&\quad + \sum_{\langle\tilde{k}\rangle\ge h^{-1}} \langle\tilde{k}\rangle^2|\hat{\theta}(\tilde{k})|^2\int_0^\infty|\rho'(\frac{x_1\langle\tilde{k}\rangle}{\lambda_1})|^2dx_1 \\
&= \|\rho'\|^2_{L^2}\Big(\frac{\lambda_1}{h}\sum_{\langle\tilde{k}\rangle\le h^{-1}}|\hat{\theta}(\tilde{k})|^2 + \lambda_1\sum_{\langle\tilde{k}\rangle\ge h^{-1}}\langle\tilde{k}\rangle|\hat{\theta}(\tilde{k})|^2\Big) \\
&\le \|\rho'\|^2_{L^2}\big(\frac{\lambda_1}{h}\|\theta\|^2_{L^2} + \lambda_1\|\theta\|^2_{H^{\frac12}_{\lambda',\lambda''}(\Sigma)}\big) \le C\|\rho'\|^2_{L^2}
\end{aligned}$$

thanks to (5.12) and (5.13). Let us now estimate the last term in (5.25). We have

$$\begin{aligned}
\|\langle\tilde{k}\rangle\hat{\psi}\|^2_{L^2(]0,1[\times\mathbb{Z}^{d-1})} &= \sum_{\langle\tilde{k}\rangle\le h^{-1}} |\hat{\theta}(\tilde{k})|^2\langle\tilde{k}\rangle^2\int_0^\infty|\rho(\frac{x_1}{h\lambda_1})|^2dx_1 \\
&\quad + \sum_{\langle\tilde{k}\rangle\ge h^{-1}} |\hat{\theta}(\tilde{k})|^2\langle\tilde{k}\rangle^2\int_0^\infty|\rho(\frac{x_1\langle\tilde{k}\rangle}{\lambda_1})|^2dx_1 \\
&= \|\rho\|^2_{L^2}\Big(h\lambda_1\sum_{\langle\tilde{k}\rangle\le h^{-1}}|\hat{\theta}(\tilde{k})|^2\langle\tilde{k}\rangle^2 + \lambda_1\sum_{\langle\tilde{k}\rangle\ge h^{-1}}|\hat{\theta}(\tilde{k})|^2\langle\tilde{k}\rangle\Big) \\
&\le \|\rho\|^2_{L^2}\lambda_1\|\theta\|^2_{H^{\frac12}_{\lambda',\lambda''}(\Sigma)} \le C\|\rho\|^2_{L^2}
\end{aligned}$$

thanks again to (5.13). This achieves to prove that $\|\psi\|^2_{H^1_\lambda(A_1)} = \mathcal{O}(1)$. □

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