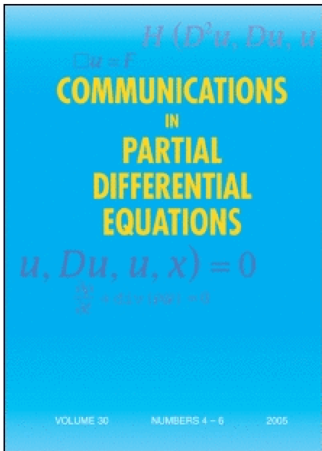

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Remarks on Non-Linear Schrödinger Equation with Magnetic Fields

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We study the nonlinear Schrödinger equation with time-depending magnetic field without smallness assumption at infinity. We obtain some results on the Cauchy problem, WKB asymptotics and instability.

Keywords Magnetic fields; Non-linear Schrödinger equation; WKB asymptotics.

Mathematics Subject Classification 35Q55; 35Q60.

1. Introduction

We consider the non-linear Schrödinger equation with magnetic field on $\mathbb{R}^n$, $n \geq 1$

$$i\partial_t u = H_{A(t)}u + b^\gamma f(x, u) \tag{1.1}$$

with initial condition

$$u_{|t=t_0} = \varphi. \tag{1.2}$$

Here

$$H_{A(t)} = \sum_{j=1}^{n}(i\partial_{x_j} - bA_j(t, x))^2, \quad t \in \mathbb{R}, \ \ x \in \mathbb{R}^n$$

is the time-depending Schrödinger operator associated to the magnetic potential $A(t, x) = (A_1(t, x), \ldots, A_n(t, x))$, the parameter $b > 0$ measures the strength of the magnetic field and $\gamma \geq 0$. We sometimes omit the space dependence and write $A(t)$ instead of $A(t, x)$. The first aim of this note is to study the Cauchy problem in the energy space. At the end of the paper we show how recent improvement in

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the qualitative study of non-linear Schrödinger equations can be adapted to the magnetic context. Let us begin with the general framework of our study.

We suppose that the magnetic potential is a smooth function $A \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n, \mathbb{R}^n)$ and that it satisfies the following assumption.

Assumption 1. There exists some constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^n$ such that

(1) $\forall \alpha \in \mathbb{N}^n \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha \partial_t A| \le C_\alpha$.
(2) $\forall |\alpha| \ge 1, \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha A| \le C_\alpha$.
(3) $\exists \epsilon > 0, \forall |\alpha| \ge 1, \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_x^\alpha B| \le C_\alpha \langle x\rangle^{-1-\epsilon}$

where $B(t, x)$ is the matrix defined by $B_{jk} = \partial_{x_j} A_k - \partial_{x_k} A_j$.

Note that compactly supported perturbations of linear (with respect to x) magnetic potentials satisfy the above hypothesis.

Under Assumption 1, the domain $D(H_{A(t)}) = \{u \in L^2(\mathbb{R}_x^n), H_{A(t)}u \in L^2(\mathbb{R}_x^n)\}$ does not depend on t. Indeed, for $t, t' \in \mathbb{R}$ one has

$$H_{A(t')} = H_{A(t)} + bW(t,t')(i\nabla_x - bA(t)) + b(i\nabla_x - bA(t))W(t,t') + b^2 W(t,t')^2 \quad (1.3)$$

with $x \mapsto W(t, t', x) = \int_t^{t'} \partial_s A(s, x)ds$. Moreover, W is bounded as well as its x-derivatives uniformly with respect to t, t' in any compact set. Therefore, the above identity shows that the space

$$H^\beta_{mg}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n), (1 + H_{A(t)})^{\beta/2} u \in L^2(\mathbb{R}^n)\}$$

does not depend on $t \in \mathbb{R}$. As $D(H_{A(t)}) = H^2_{mg}(\mathbb{R}^n)$, the above statement is straightforward. Moreover, the natural norms on this space are equivalent and this equivalence is uniform with respect to the parameter b for close times. More precisely, denoting $m_A = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |\partial_t A(t, x)|$, we have the following.

Proposition 1.1. *Suppose that Assumption 1 is satisfied and let $\beta > 0$ and $T > 0$. Then, for all $t, t' \in \mathbb{R}$ such that $|t - t'| \le b^{-1}T$ and all $u \in H^\beta_{mg}$ we have*

$$\|(H_{A(t')} + 1)^\beta u\|_{L^2} \le (1 + 2m_A T + m_A^2 T^2)^\beta \|(H_{A(t)} + 1)^\beta u\|_{L^2}.$$

Proof. It is a straightforward consequence of equation (1.3), Assumption 1 and the fact that $(i\nabla_x - bA(t))(H_{A(t)} + 1)^{-1}$ is bounded by 1 in L^2. □

For $\beta \in \mathbb{N}$ we define

$$\|u\|_{H^\beta_{A(t)}} = \|(i\nabla_x - bA(t))^\beta u\|_{L^2} + \|u\|_{L^2}. \quad (1.4)$$

This norm is clearly equivalent (uniformly with respect to b) to $\|(1 + H_{A(t)})^{\beta/2} u\|_{L^2}$. In view of Proposition 1.1, we define the magnetic Sobolev norm by

$$\|u\|_{H^\beta_{mg}} = \|u\|_{H^\beta_{A(t_0)}}.$$

Under Assumption 1 it is well-known (see [15], Th. 4.6, p. 143, or [18]) that for $\varphi \in H^1_{mg}$, the linear Schrödinger equation

$$i\partial_t u = H_{A(t)} u, \quad u_{|t=s} = \varphi \tag{1.5}$$

has a solution $U_0(t, s)\varphi$. The operator $U_0(t, s)$ is continuous from L^2 into L^2 and from H^1_{mg} into H^1_{mg}. Moreover, $U_0(t, s)\varphi$ is the unique H^1_{mg} valued solution of (1.5) and $U_0(t, s)$ is unitary on L^2.

The first aim of this paper is to solve the Cauchy problem for the non-linear equation in the most appropriate space. In the sequel we assume that $f : \mathbb{R}^n \times \mathbb{C} \to \mathbb{C}$ is a measurable function such that

Assumption 2.

(1) $f(x, 0) = 0$ for a.e. $x \in \mathbb{R}^n$.
(2) $\exists M \geq 0, \alpha \in [0, \frac{4}{n-2}[(\alpha \in [0, \infty[$ if $n = 1, 2)$ such that

$$|f(x, z_1) - f(x, z_2)[\leq M(1 + |z_1|^\alpha + |z_2|^\alpha)|z_1 - z_2|$$

for a.e. $x \in \mathbb{R}^n$ and for all $z_1, z_2 \in \mathbb{C}$.
(3) $\forall z \in \mathbb{C}, f(x, z) = (z/|z|)f(x, |z|)$.

These assumptions are often used in the case $A = 0$. More precisely, in the case $A = 0$, the second property of the above assumption corresponds to a subcritical non-linearity with respect to H^1.

Let us introduce some energy functional associated to these nonlinerarities. We define

$$F(x, z) = \int_0^{|z|} f(x, s)ds, \quad G(u) = \int_{\mathbb{R}^n} F(x, u(x))dx$$

and for $t \in \mathbb{R}$ and $u \in H^1_{mg}$ we define the energy

$$E(b, t, u) = \int_{\mathbb{R}^n} \frac{1}{2}|(i\nabla_x - bA(t, x))u(x)|^2 dx + b^\gamma G(u).$$

Formally, it is straightforward to see that any sufficiently regular solution of (1.1), (1.2), enjoys the following energy evolution law:

$$E(b, t, u) = E(b, 0, \varphi) - b\,\mathrm{Re}\int_0^t \langle \partial_s A(s)u(x), (i\nabla - bA(s))u(s)\rangle ds,$$

where $\langle , \rangle$ denotes the L^2 scalar product. Therefore, a natural space to solve (1.1)–(1.2) is H^1_{mg}.

Theorem 1. *Suppose that Assumptions* 1 *and* 2 *are satisfied and let* $\varphi \in H^1_{mg}$. *Then, there exists* $T_b, T^b > 0$ *and a unique* $u \in C(]-T_b, T^b[, H^1_{mg}) \cap C^1(]-T_b, T^b[, H^{-1}_{mg})$ *solution of* (1.1).

Moreover, either $T_b = \infty$ *(resp.* $T^b = \infty$*), or* $\lim_{t\to -T_b} \|u(t)\|_{H^1_{mg}} = \infty$ *(resp.* $\lim_{t\to T^b} \|u(t)\|_{H^1_{mg}} = \infty$*) and*

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2}, \tag{1.6}$$

$$E(b, t, u) = E(b, 0, \varphi) - b\,\mathrm{Re}\textstyle\int_0^t \langle \partial_s A(s)u(x), (i\nabla - bA(s))u(s)\rangle_{L^2} ds, \tag{1.7}$$

for all $t \in]-T_b, T^b[$. *In addition, there exists* $\epsilon > 0$ *such that, for all* $b > 0$ *and* $\varphi \in H^1_{mg}$ *such that* $\|\varphi\|_{H^1_{mg}} \leq Cb$, *we have* $T_b, T^b \geq \epsilon b^{-\delta}$ *with* $\delta > 0$ *depending only on* α, γ, n.

Let us make a few remarks on this result. The Cauchy problem for non-linear Schrödinger equation has a long story. In absence of magnetic field there are numerous results; see for instance [5, 9, 10].

In presence of magnetic field, the behavior of A when $|x|$ becomes large plays an important role. In the case where the magnetic potential A is bounded, the spaces H^1_{mg} and H^1 coincide and the Cauchy problem can be solved in H^1 using standard techniques. If the magnetic field is unbounded, it is not possible to solve the Cauchy problem in H^1 since the product $u \mapsto Au$ is not bounded on L^2.

To overcome this difficulty a possible strategy is to work in the weighted Sobolev space $\Sigma = \{u \in H^1(\mathbb{R}^n),\ (1+|x|)u \in L^2,\}$ (see for instance [7, 14]). In particular, a decay of the initial data at infinity is required.

In [7], a decay is required because the author uses dispersive properties for the Laplacian instead of $H_{A(t)}$. In [14] the authors use magnetic Strichartz estimates but their method is based on fixed-point theorem and is not adapted to the magnetic context.

On the other hand, there exists also a result of Cazenave and Esteban [4] dealing with the special case where the magnetic field B is constant (and hence, A does not depend on t and is linear with respect to x). In one way, this paper is more satisfactory as they only require u_0 to belong to the energy space. Nevertheless, their result applies only to constant magnetic field.

Our theorem is a generalization of the above results. Before going further, let us remark that for unbounded A, the spaces H^1, H^1_{mg} and Σ are different. First, it is evident that Σ is contained in $H^1 \cap H^1_{mg}$. Let us give an example where Σ is strictly contained in H^1_{mg}. For this purpose, we restrict ourselves to the case of dimension $n = 2$ and consider the magnetic potential $A(x, y) = (y, x)$. Let $g \in H^1(\mathbb{R}^2)$ be such that $|x|g \notin L^2$: simple computations show that $f(x, y) = g(x, y)e^{-ixy}$ belongs to $H^1_{mg} \backslash \Sigma$.

In the case of defocusing non-linearities the energy law implies the following result.

Corollary 1. *Suppose that* $f(x, z) \geq 0$ *for all* x, z, *then* $T_b, T^b = +\infty$.

Proof. For $f \geq 0$, we deduce from (1.7) and Cauchy–Schwarz inequality, that

$$\|(i\nabla - bA(t))u(t)\|_{L^2} \leq C_1 + C_2 \int_0^t \|(i\nabla - bA(s))u(s)\|_{L^2} ds$$

for some fixed constant $C_1, C_2 > 0$. Hence, Gronwall Lemma shows that $\|(i\nabla - bA(t))u(t)\|_{L^2}$ remains bounded on any bounded time-interval. Using (1.6) and the characterization of T_b, we obtain the result. □

The next section contains the proof of Theorem 1. In Section 3 we give some qualitative results on the solution of (1.1) in the limit $b \to \infty$. More precisely, we construct WKB solutions and prove instability results with respect to the initial data and to the parameter b.

2. Cauchy Problem in the Energy Space

The proof of Theorem 1 relies on the Strichartz estimates proved in [18] for the problem

$$i\partial_t u = H_{A(t)}u + g(t), \quad u_{|t=s} = \varphi \tag{2.1}$$

In the following, we denote $2^* = \frac{2n}{n-2}$ if $n \geq 3$ and $2^* = +\infty$ if $n = 1, 2$.

Theorem 2 (Yajima). *Let I be a finite real interval, (q, r) and (γ_j, ρ_j), $j = 1, 2$ be such that $r, \rho_j \in [2, 2^*[$, $q, r_j \in]2, +\infty]$, $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$ and $\frac{2}{\gamma_j} = n(\frac{1}{2} - \frac{1}{\rho_j})$. Let $g_j \in L^{\gamma_j'}(I, L^{\rho_j'}(\mathbb{R}_x^n))$, $j = 1, 2$, where γ_j', ρ_j' are the conjugate exponents of γ_j, ρ_j. Then the solution u to* (2.1) *with $g = g_1 + g_2$ satisfies*

$$\|u\|_{L^q(I,L^r(\mathbb{R}_x^n))} \leq C(\|g_1\|_{L^{\gamma_1'}(I,L^{\rho_1'}(\mathbb{R}_x^n))} + \|g_2\|_{L^{\gamma_2'}(I,L^{\rho_2'}(\mathbb{R}_x^n))} + \|\varphi\|_{L^2(\mathbb{R}^n)}) \tag{2.2}$$

where the constant C depends only on the length of I and the constants $(C_\alpha)_{\alpha\in\mathbb{N}^n}$ of Assumption 1.

Proof. In the case $g = 0$ it is exactly Theorem 1 of [18]. In the general case it suffices to follow the proof of Proposition 2.15 of [2] using a celebrated result of Christ and Kiselev [6]. The fact that the constant C depends only on the C_α is a direct consequence of the construction of Yajima [18]. □

Remark 2.1. In the case where the magnetic potential is not regular, recent results of Stefanov [16] and Georgiev and Tarulli [8] provide Strichartz estimates under smallness assumption on the magnetic fields. This should lead to the corresponding existence and uniqueness result for NLS in the case of small magnetic field. This could also have consequences on the well-posedness of the Schrödinger–Maxwell system (see [12, 13, 17] for results on this topics).

It is important to notice that Theorem 1 is not a straightforward consequence of the above Strichartz estimate. Indeed, if we apply a fixed point method to equation (1.1), a difficulty occurs when one aims at controlling the nonlinearity in the H^1_{mg} norm. Consider for instance the case $f(u) = |u|^2 u$, then we have

$$(i\nabla_x - bA(t))(|u|^2 u) = |u|^2(i\nabla_x - bA(t))(u) + ui\nabla_x(|u^2|).$$

The first term of the right hand side of this equality will be controlled by $\|u\|_{H^1_{mg}}$, whereas in the second term, as $A(t, x)$ is not bounded with respect to x, there is no chance to control $i\nabla_x(|u^2|)$ by $(i\nabla_x - bA(t))(|u^2|)$. For the same reason it does not seem easy to solve the Cauchy problem in magnetic Sobolev spaces of high degree.

To overcome this difficulty, we work as in [5, 4] and approximate the solution of (1.1) by the solution of a non-linear Schrödinger equation with a non-linearity which is linear at infinity. In the work of Cazenave and Weissler, the main tool to justify the approximation is an energy conservation. In our case, the Hamiltonian depends on time, so that the energy is not conserved. Nevertheless, the error term is controlled by the H^1_{mg}-norm so that it is possible to implement the same strategy. Another difference involved by the dependence with respect to time of the

Hamiltonian is that the usual techniques to solve the Cauchy problem with regular initial data and suitable non-linearities do not apply in our context. Therefore, in addition to the approximation of the non-linearity, we introduce an approximation of the magnetic field itself and justify the convergence to the initial problem.

Let us introduce the approximated non-linearities used in the sequel. Following [5], we decompose $f = \tilde{f}_1 + \tilde{f}_2$ with

$$\tilde{f}_1(x, z) = 1_{\{|z|\leq 1\}} f(x, z) + 1_{\{|z|>1\}} f(x, 1)z \tag{2.3}$$

and

$$\tilde{f}_2(x, z) = 1_{\{|z|>1\}}(f(x, z) - f(x, 1)z). \tag{2.4}$$

Next, we define $f_m = \tilde{f}_1 + \tilde{f}_{2,m}$ where

$$\tilde{f}_{2,m}(x, z) = 1_{\{|z|\leq m\}} \tilde{f}_2(x, z) + 1_{\{|z|>m\}} \tilde{f}_2(x, m)\frac{z}{m} \tag{2.5}$$

Remark that these functions satisfy Assumption 2. We consider also the energy functional associated to these approximated non-linearities. We define

$$F_m(x, z) = \int_0^{|z|} f_m(x, s)ds, \quad G_m(u) \int_{\mathbb{R}^n} F_m(x, u(x))dx \tag{2.6}$$

and for $t \in \mathbb{R}$ and $u \in H^1_{mg}$ we define

$$E_m(b, t, u) = \int_{\mathbb{R}^n} \frac{1}{2}|(i\nabla_x - bA(t, x))u(x)|^2 dx + b^{\gamma} G_m(u). \tag{2.7}$$

Without loss of generality, it suffices to prove Theorem 1 for $t_0 = 0$. For simplicity we prove Theorem 1 in the particular case $b = 1$. To get the general case it suffices to keep track of b along the proof. We will also restrict our study to $t \geq 0$, the case of negative times being treated by reversing time in the equation.

2.1. *Preliminary Results*

In the sequel, we need Sobolev embeddings in the magnetic context. In this subsection, $A(t, x)$ is a magnetic potential satisfying Assumption 1. We also suppose that $t \in [0, T_0]$ with $T_0 < 1/m_A$ to be chosen.

Lemma 2.2. *Let $0 < s < \frac{n}{2}$ and $p_s = \frac{2n}{n-2s}$, then H^s_{mg} is continuously embedded in $L^p(\mathbb{R}^n)$ for all $p \in [2, p_s]$ and there exists $C > 0$ independent of A such that for all $t \in [0, T_0]$*

$$\|u\|_{L^p} \leq C\|u\|_{H^s_{mg}} \tag{2.8}$$

Proof. From the diamagnetic inequality (see [1]), we know that almost everywhere we have

$$|u| = |(H_{A(0)} + 1)^{-\frac{s}{2}}(H_{A(0)} + 1)^{\frac{s}{2}} u| \leq (-\Delta + 1)^{-\frac{s}{2}} |(H_{A(0)} + 1)^{\frac{s}{2}} u|.$$

Taking the L^p norm, the result follows from standard Sobolev inequalities. □

Until the end of this section, we suppose that $\rho_1 = 2$, $\rho_2 = \alpha + 2$ and for $k = 1, 2$ $2/\gamma_k = n(1/2 - 1/\rho_k)$. We also denote, ρ_k', γ_k' the conjugate exponents of ρ_k, γ_k.

Lemma 2.3. *Let $M > 0$, then*

(1) *the sequence $(\tilde{f}_{2,m}(., u))_{m\in\mathbb{N}^*}$ converges to $\tilde{f}_2(., u)$ in $L^{\rho_2'}(\mathbb{R}^n)$ uniformly with respect to $u \in H^1_{mg}$ such that $\|u\|_{H^1_{mg}} \le M$.*
(2) *there exists $C(M) > 0$ independent of A such that for all $m \in \mathbb{N}^*$ and for all $u, v \in H^1_{mg}$ with $\max(\|u\|_{H^1_{mg}}, \|v\|_{H^1_{mg}}) \le M$ we have*

$$\|\tilde{f}_1(., u) - \tilde{f}_1(., v)\|_{L^{\rho_1'}(\mathbb{R}^n)} \le C(M)\|u - v\|_{L^{\rho_1}}$$

and

$$\|\tilde{f}_{2,m}(., u) - \tilde{f}_{2,m}(., v)\|_{L^{\rho_2'}(\mathbb{R}^n)} + \|\tilde{f}_2(., u) - \tilde{f}_2(., v)\|_{L^{\rho_2'}(\mathbb{R}^n)} \le C(M)\|u - v\|_{L^{\rho_2}(\mathbb{R}^n)}.$$

Proof. We follow the method of Example 3 in [5]. Taking χ as the characteristic function of the set $\{x \in \mathbb{R}^n \mid |u(x)| > m\}$ and using Assumption 2, we have

$$\|\tilde{f}_2(u) - \tilde{f}_{2,m}(u)\|_{L^{\rho_2'}(\mathbb{R}^n)} \le 2\|\chi|u|^{\alpha+1}\|_{L^{\rho_2'}} = 2\|\chi u\|^{\alpha+1}_{L^{\alpha+2}}. \tag{2.9}$$

On the other hand, using Lemma 2.2 we get for $p \in]\alpha + 2, 2^*[$,

$$\|u\|_{H^1_{mg}} \ge C\|\chi u\|_{L^p} \ge Cm^{1-\frac{\alpha+2}{p}}\|\chi u\|^{\frac{\alpha+2}{p}}_{L^{\alpha+2}}. \tag{2.10}$$

As $1 - \frac{\alpha+2}{p} > 0$, combining Equations (2.9) and (2.10), we obtain the first point of Lemma 2.3.

The second assertion follows, as in Example 3 in [5], from Hölder's inequality, Assumption 2 and Lemma 2.2. The fact that the constant $C(M)$ is independent of the magnetic field follows from the uniformity of the constant in Lemma 2.2. □

Lemma 2.4. *For $M > 0$ there exists a constant $C(M)$ independent of A, such that the following hold true:*

(1) *for all $t \in \mathbb{R}$ and $u, v \in H^1_{A(t)}$ with $\max(\|u\|_{H^1_{A(t)}}, \|v\|_{H^1_{A(t)}}) \le M$ we have*

$$|G(u) - G(v)| + |G_m(u) - G_m(v)| \le C(M)(\|v - u\|_{L^2} + \|v - u\|^{\nu}_{L^2}),$$

with $\nu = 1 - \frac{2}{\gamma_2}$.
(2) *for all $0 < T < T_0$ and for all $u, v \in L^\infty([0, T], H^1_{mg})$, we have*

$$\|\tilde{f}_1(., u) - \tilde{f}_1(., v)\|_{L^{\gamma_1'}([0,T],L^{\rho_1'}(\mathbb{R}^n))} \le C(M)(T + T^{1/2})\|u - v\|_{L^{\gamma_1}([0,T],L^{\rho_1}(\mathbb{R}^n))}.$$

and

$$\begin{aligned}\|\tilde{f}_{2,m}(., u) - \tilde{f}_{2,m}(., v)\|_{L^{\gamma_2'}([0,T],L^{\rho_2'}(\mathbb{R}^n))} + \|\tilde{f}_2(., u) - \tilde{f}_2(., v)\|_{L^{\gamma_2'}([0,T],L^{\rho_2'}(\mathbb{R}^n))}\\ \le C(M)(T + T^{\nu})(\|u - v\|_{L^{\gamma_2}([0,T],L^{\rho_2}(\mathbb{R}^n))} + \|u - v\|_{L^\infty([0,T],L^2(\mathbb{R}^n))})\end{aligned}$$

Moreover, $G_m \to G$ as $m \to \infty$ uniformly on bounded sets of H^1_{mg}.

Proof. Noting that $G(u) = \int_0^1 \langle f(x, su), u\rangle_{L^2} ds$ and $G_m(u) = \int_0^1 \langle f_m(x, su), u\rangle_{L^2} ds$, we mimic the proof of Lemma 3.3 in [5], replacing classical Sobolev inequalities by Lemma 2.2 and using Lemma 2.3. □

We are now in position to prove the uniqueness part of Theorem 1.

Proposition 2.5. *Let $T > 0$ and $u, v \in C([0, T[, H^1_{mg}) \cap C^1([0, T[, H^{-1}_{mg})$ be solutions of* (1.1). *Then $u = v$.*

Proof. Let $u, v \in C([0, T[, H^1_{mg}) \cap C^1([0, T[, H^{-1}_{mg})$ be solutions of (1.1), and define $w = v - u$. Then $w(0) = 0$ and

$$i\partial_t w - H_{A(t)} w = \widetilde{f}_1(u) - \widetilde{f}_1(v) + \widetilde{f}_2(u) - \widetilde{f}_2(v).$$

Let $r \in [2, 2^*[$ and $q \in]2, +\infty]$ such that $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$. Applying Theorem 2 together with Lemma 2.4, we get

$$\|w\|_{L^q([0,T[,L^r)} \leq C(T + T^\nu)(\|w\|_{L^\infty([0,T[,L^2)} + \|w\|_{L^{\gamma_2}([0,T[,L^{\rho_2})}).$$

As we can alternatively take (q, r) to be equal to $(2, \infty)$ and (γ_2, ρ_2), we conclude by summing the resulting inequalities and by taking $T > 0$ small enough. □

2.2. *Autonomous Case*

In this section we sketch how to solve the Cauchy problem in H^1_{mg} when the magnetic field $A(t, x) = A(x)$ is time independent. In this context, the functional E does not depend on time and formally we have the following conservation of energy: assume that u is solution of (1.1) then

$$E(u(t)) = E(\varphi), \quad \forall t.$$

Moreover, in that case the norms $\|.\|_{mg}$ and $\|.\|_{H^1_A}$ coincide.

Proposition 2.6. *Let $M > 0$ and let A be time independent and satisfying Assumption* 1 *with some constants $(C_\alpha)_{\alpha\in\mathbb{N}^n}$. Then, there exists $T > 0$ depending only on M and the (C_α)s such that for all $\varphi \in H^1_A$ such that $\|\varphi\|_{H^1_A} \leq M$, there exists a unique $u \in C^0([0, T[, H^1_A) \cap C^1([0, T[, H^{-1}_A)$ maximal solution of*

$$i\partial_t u = H_A u + f(x, u)$$

with initial condition $u_{|t=0} = \varphi$. Moreover, for all $t \in [0, T[$ we have

$$E(u(t)) = E(\varphi).$$

In addition, if $T < \infty$ then $\lim_{t\to T} \|u\|_{H^1_A} = \infty$.

The proof is slight adaption of [5, 4] to our context. We need also to investigate the dependence of the existence time with respect to the magnetic field. However, the scheme of proof is the same and consists to consider an approximate problem and justify convergence on fixed time intervals. Let us give the main steps of the proof.

Step 1. Let f_m be defined by (2.3)–(2.5) and let A be a magnetic field satisfying the above hypotheses. Consider the problem

$$i\partial_t u = H_A u + f_m(x, u), \quad u_{t=0} = \varphi \tag{2.11}$$

with $\varphi \in H^1_A$. We have the following

Lemma 2.7. *Let $\varphi \in H^1_A$, then there exists $\tau_{m,A} > 0$ such that there exists $u_m \in C([0, \tau_{m,A}[, H^1_A) \cap C^1([0, \tau_{m,A}[, H^{-1}_A)$ solution of* (2.11). *Moreover for any $t \in [0, \tau_{m,A}[$, we have*

$$E_m(u_m) = E_m(\varphi) \tag{2.12}$$

and

$$\|u_m(t)\|_{L^2} = \|\varphi\|_{L^2}. \tag{2.13}$$

Proof. The proof is the same as in Lemma 3.5 of [5], replacing usual derivatives by magnetic derivatives. □

Step 2. We show that the existence time $\tau_{m,A}$ can be bounded from below uniformly with respect to $m \in \mathbb{N}$ and A satisfying the assumptions of Proposition (2.6).

Lemma 2.8. *Let $M > 0$ and let A satisfy Assumption 1 with some constants $(C_\alpha)_{\alpha\in\mathbb{N}^n}$. Then, there exists $T_1 > 0$ depending only on M and the (C_α)'s such that for all $\varphi \in H^1_A$ such that $\|\varphi\|_{H^1_A} \leq M$, we have*

$$\|u_m\|_{L^\infty([0,T_1],H^1_A)} \leq 2\|\varphi\|_{H^1_A}.$$

Proof. The proof is exactly the same as in Lemma 3.6 of [5], using Lemma 2.7 (in particular, we use strongly the conservation of energy) and Lemma 2.3 to get uniformity with respect to A. □

Step 3. The final step is to prove the convergence of the u_m to a solution of the initial problem. First, we prove convergence in L^2.

Lemma 2.9. *Let $M > 0$ and let A satisfy Assumption 1 with some constants $(C_\alpha)_{\alpha\in\mathbb{N}^n}$. Then, there exists $T_2 > 0$ depending only on M and the (C_α)'s such that for all $\varphi \in H^1_A$ such that $\|\varphi\|_{H^1_A} \leq M$, $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $C([0, T_2], L^2)$.*

Proof. The proof is the same as in [5], using Theorem 2, Lemmas 2.3, 2.4 and 2.7. □

We complete the proof of Proposition 2.6. We denote by u the limit of u_m in $C([0, T_2], L^2)$. From Lemma 2.8, it follows that $u \in L^\infty([0, T_2], H^1_A)$ and by Lemma 2.2, u_m converges to u in $C([0, T_2], L^r)$ for all $r \in [2, 2^*[$. Hence, it follows from Lemma 2.3 that $f_m(u_m)$ converges to $f(u)$ in $C([0, T_2], H^{-1}_A)$ and u solves (1.1) in $L^\infty([0, T_2], H^{-1}_A)$. Moreover, combining Lemmas 2.4 and 2.7 we get

$$E(u(t)) = E(\varphi).$$

This shows that $u \in C([0, T_2], H^1_A)$ and hence $u \in C^1([0, T_2], H^{-1}_A)$.

2.3. *Cauchy Problem in the Time-Depending Case*

We suppose now that $A(t, x)$ satisfies Assumption 1. The strategy of proof is the same as in the autonomous case and we first consider the problem

$$i\partial_t u = H_{A(t)}u + f_m(x, u), \quad u_{t=0} = \varphi. \tag{2.14}$$

At least formally, we can see that the energy of the solution of this equation satisfies the following identity

$$E(t, u) = E(0, \varphi) - \mathrm{Re}\int_0^t \langle \partial_s A(s)u(s), (i\nabla_x - A(s))u(s)\rangle ds. \tag{2.15}$$

This replaces the energy conservation in our approach. On the other hand another problem occurs if we try to mimic the proof of [5]. Indeed, the natural first step would be to obtain a generalization of Lemma 2.7 in the time depending framework. Following the proof of Lemma 3.5 in [5], we should then regularize the initial data and solve the Cauchy problem in H^2_{mg}. In time-depending context the difficulty is that contrary to the autonomous case, the existence of smooth solution is not easy to prove. Indeed, the key point in the approach of [5] is that for any $g \in C([0, T], H^1)$ being Lipschitz continuous with respect to time, the function $v(t) = \int_0^t U_0(t, s)g(s)ds$ is also Lipschitz continuous with respect to time. Such a result is easily proved in the autonomous case as the identity $U_0(t+h, s) = U_0(t, s-h)$ permits to use the assumption on g. This fails to be true in the time-depending case. For this reason, we prove the existence in H^1_{mg} in a direct way.

2.3.1. Existence of Solution for Approximated Problem.

Proposition 2.10. *Let $\varphi \in H^1_{mg}$, then there exists $\widetilde{T} > 0$ such that there exists $u_m \in C([0, \widetilde{T}[, H^1_{mg}) \cap C^1([0, \widetilde{T}[, H^{-1}_{mg})$ solution of* (2.14). *Moreover for any $t \in [0, \widetilde{T}[$, we have*

$$E_m(t, u_m) = E_m(t, \varphi) - \mathrm{Re}\int_0^t \langle \partial_s A(s)u(s), (i\nabla_x - A(s))u(s)\rangle ds. \tag{2.16}$$

and

$$\|u_m(t)\|_{L^2} = \|\varphi\|_{L^2}. \tag{2.17}$$

Proof. The method consists in approximating the magnetic potential $A(t, x)$ by potentials which are piecewise constant with respect to time. For this purpose, we first notice that, thanks to Assumption 1 and Proposition 2.6, for all $M > 0$ there exists $T_2 = T_2(M) \in]0, T_0]$ such that for all $t_0 \in [0, T_2]$ the Cauchy problem

$$i\partial_t u = H_{A(t_0)}u(t) + f_m(u(t)), \quad u_{|t=t_0} = \varphi$$

can be solved in $C([t_0, t_0 + T_2[, H^1_{A(t_0)})$ for all initial data such that $\|\varphi\|_{H^1_{A(t_0)}} \leq M$.

Let $\varphi \in H^1_{mg}$ be such that $\|\varphi\|_{H^1_{A(0)}} \leq \frac{M}{4}$ and let $T \in]0, T_2]$. For $n \in \mathbb{N}^*$, $k \in \{0, \ldots, n\}$ we define $t_n^k = \frac{kT}{n}$ and

$$A_n(t, x) = \sum_{k=0}^{n} 1_{[t_n^k, t_n^{k+1}[}(t) A(t_n^k, x), \quad \forall t \in [0, T]$$

Next, we define the Hamiltonian $H_n = (i\nabla_x - A_n)^2$ and we look for solutions $u_{n,m}$ of

$$i\partial_t u = H_n u + f_m(u), \quad u_{|t=0} = \varphi. \tag{2.18}$$

From uniqueness in the autonomous case, such a function is necessarily given by

$$u_{n,m}(t, x) = \sum_{k=0}^{n-1} 1_{[t_n^k, t_n^{k+1}[}(t) v_{k,n,m}(t, x), \tag{2.19}$$

where the functions $v_{k,n,m}(t, x)$ are defined as follows. We choose $v_{0,n,m}$ to be a solution of

$$\begin{cases} i\partial_t v_{0,n,m} = (i\nabla_x - A(t_n^0, x))^2 v_{0,n,m} + f_m(v_{0,n,m}) \\ v_{0,n,m}(t_n^0, x) = \varphi(x) \end{cases} \tag{2.20}$$

and for $k \geq 1$, $v_{k,n,m}(t, x)$ is the solution of

$$\begin{cases} i\partial_t v_{k,n,m} = (i\nabla_x - A(t_n^k, x))^2 v_{k,n,m} + f_m(v_{k,n,m}) \\ v_{k,n,m}(t_n^k, x) = v_{k-1,n,m}(t_n^k, x). \end{cases} \tag{2.21}$$

Let us show that the functions $v_{k,n,m}, k = 0, \ldots, n-1$ are well-defined. As $\|\varphi\|_{H^1_{A(0)}} < M/4$ it follows from Proposition 2.6 that one can solve the problem (2.20). Moreover, for any $k \in \{1, \ldots, n-1\}$, to prove that $v_{k,n,m}$ is well defined, it suffices to show that $\|v_{k-1,n,m}\|_{H^1_{A(t_n^k)}} \leq M$. Let $k_1 \in \{1, \ldots, n-1\}$ be the greatest integer such that the preceding inequality holds true. Then, the function $u_{n,m}$ given by (2.19) is well-defined for $t \in [0, t_n^{k_1+1}[$ and is continuous with values in H^1_{mg}. For $w \in H^1_{mg}(\mathbb{R}^n)$ we define

$$E_{n,m}(t, w) = \frac{1}{2} \int_{\mathbb{R}^n} |(i\nabla_x - A_n(t, x)) w(x)|^2 dx + G_m(w).$$

Then, for all $k \in \{1, \ldots, k_1\}$ and $t \in [t_n^k, t_n^{k+1}[$, it follows from Proposition 2.6 that

$$E_{n,m}(t, v_{k,n,m}(t)) = E_{n,m}(t_n^k, v_{k,n,m}(t_n^k)).$$

Let us write $A(t_n^k, x) = A(t_n^{k-1}, x) + W_{n,k}(x)$ with $W_{n,k}(x) = \int_{t_n^{k-1}}^{t_n^k} \partial_s A(s, x) ds$ and use $v_{k,n,m}(t_n^k, x) = v_{k-1,n,m}(t_n^k, x)$, then

$$\begin{aligned} E_{n,m}(t_n^k, u_{n,m}(t_n^k)) = E_{n,m}&(t_n^{k-1}, u_{n,m}(t_n^{k-1})) \\ &- \int_{t_n^{k-1}}^{t_n^k} \mathrm{Re}\langle (i\nabla_x - A(t_n^{k-1})) u_{n,m}(t_n^k), \partial_s A(s, x) u_{n,m}(t_n^k) \rangle ds \\ &+ \|W_{n,k} u_{n,m}(t_n^k)\|_{L^2}^2 \end{aligned} \tag{2.22}$$

for all $k \in \{1, \ldots, k_1\}$. Thanks to Assumption 1 and the conservation of the mass, we have

$$\|W_{n,k}u_{n,m}(t_n^k)\|_{L^2}^2 = O\left(\frac{\|\varphi\|_{L^2}^2}{n^2}\right)$$

uniformly with respect to k, n, m. For $t \in [0, t_n^{k_1+1}[$ denote $k_0 = [\frac{nt}{T}]$. Taking the sum of equations (2.22) for $k = 1, \ldots, k_0$, and using the fact that the energy is constant on $[t_n^{k_0}, t_n^{k_0+1}[$ we get for $t \in [t_n^{k_0}, t_n^{k_0+1}[$

$$\begin{aligned} E_{n,m}(t, u_{n,m}(t)) = E_{n,m}(0, \varphi) & \\ - \sum_{k=1}^{k_0} \int_{t_n^{k-1}}^{t_n^k} \mathrm{Re}\langle (i\nabla_x - A(t_n^{k-1}))u_{n,m}(t_n^k), \partial_s A(s, x)u_{n,m}(t_n^k)\rangle ds & \\ + O\left(\frac{1}{n}\|\varphi\|_{L^2}^2\right). & \qquad (2.23) \end{aligned}$$

On the other hand, thanks to Proposition 1.1, Lemmas 2.2 and 2.3 and the equation satisfied by $u_{n,m}$, there exists $K(M) > 0$ independent of $n, m \in \mathbb{N}$, such that

$$\|\partial_t u_{n,m}\|_{H_{mg}^{-1}} \le K(M), \quad \forall n, m \in \mathbb{N}, \ \forall t \in [0, t_n^{k_1+1}[$$

and consequently,

$$\|u_{n,m} - \varphi\|_{L^2}^2 \le 2MK(M)t, \quad \forall t \in [0, t_n^{k_1+1}[. \qquad (2.24)$$

Moreover, it follows from (2.23) that

$$\begin{aligned} &\frac{1}{2}\|(i\nabla_x - A_n(t))u_{n,m}(t)\|_{L^2}^2 \\ &\quad = \frac{1}{2}\|(i\nabla_x - A(0))\varphi\|_{L^2}^2 - G_m(u_{n,m}) + G_m(\varphi) \\ &\quad - \sum_{k=1}^{n} \int_{t_n^{k-1}}^{t_n^k} \mathrm{Re}\langle (i\nabla_x - A(t_n^k))u_{n,m}(t_n^k), \partial_s A(s, x)u_{n,m}(t_n^{k-1})\rangle ds \\ &\quad + O\left(\frac{t}{n}\|\varphi\|_{L^2}^2\right). \qquad (2.25) \end{aligned}$$

As $\partial_t A$ is bounded, the fourth term of the right hand side of (2.25) is bounded by CtM^2. Moreover it follows from Lemma 2.4 and estimate (2.24) that

$$|G_m(u_{n,m}) - G_m(\varphi)| \le C(M)(t^{1/2} + t^{\nu/2}).$$

Combining these estimates with Proposition 1.1 we get

$$\|u_{n,m}(t)\|_{H_{mg}^1}^2 \le \frac{M^2}{4} + C(M)\big(T_0^{1/2} + T_0^{\nu/2}\big), \qquad (2.26)$$

for any $t \in [0, t_n^{k_1+1}[$. Taking T_0 sufficiently small, the right hand side of (2.26) is smaller than M^2. This proves that $v_{k,n,m}$ is well defined for all $k \in \{1, \dots, n\}$ and that for all $T \in [0, T_0]$ we have

$$\|u_{n,m}\|_{L^\infty([0,T],H^1_{mg})} \leq M, \quad \forall n, m \in \mathbb{N}. \tag{2.27}$$

For $m \in \mathbb{N}$ fixed we claim that $(u_{n,m})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(L^2)$. Indeed, for $p, p' \in \mathbb{N}$, we have

$$\begin{cases} i\partial_t(u_{p,m} - u_{p',m}) = H_p(u_{p,m} - u_{p',m}) + R_{p,p',m} + f_m(u_{p,m}) - f_m(u_{p',m}) \\ (u_{p,m} - u_{p',m})_{|t=0} = 0, \end{cases}$$

where

$$\begin{aligned} R_{p,p',m}(t) = ((A_{p'} - A_p)(t)(i\nabla - A(0)) + (i\nabla - A(0))(A_{p'} - A_p)(t) \\ + (A_p^2 - A_{p'}^2)(t) + 2A(0)(A_{p'} - A_p)(t))u_{p',m}(t). \end{aligned}$$

Thanks to Theorem 2 and Lemma 2.4, for $\widetilde{T} \in]0, T]$, $r \in [2, 2^*[$ and $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$, we have

$$\begin{aligned} \|u_{p,m} - u_{p',m}\|_{L^q([0,\widetilde{T}],L^r(\mathbb{R}^n))} &\leq \|R_{p,p',m}\|_{L^\infty([0,\widetilde{T}],L^2(\mathbb{R}^n))} \\ &\quad + C(M)(\widetilde{T} + \widetilde{T}^\nu)\big(\|u_{p,m} - u_{p',m}\|_{L^\infty([0,\widetilde{T}],L^2(\mathbb{R}^n))} \\ &\quad + \|u_{p,m} - u_{p',m}\|_{L^{\gamma_2}([0,\widetilde{T}],L^{\rho_2}(\mathbb{R}^n))}\big). \end{aligned} \tag{2.28}$$

On the other hand, $\epsilon > 0$ being fixed, we have

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^n} |A_p - A_{p'}| \leq \rho\epsilon,$$

where $\rho > 0$ can be taken as small as we want provided p, p' are large enough. Hence,

$$\|R_{p,p',m}\|_{L^\infty([0,\widetilde{T}],L^2(\mathbb{R}^n))}\| \leq 2\rho\epsilon\|u_{p',m}\|_{H^1_{mg}} + C\,\rho\epsilon\|u_{p',m}\|_{L^2} \leq CM\,\rho\epsilon. \tag{2.29}$$

Combining (2.28) and (2.29), we obtain for p, p' large enough

$$\begin{aligned} \|u_{p,m} - u_{p',m}\|_{L^q([0,\widetilde{T}],L^r(\mathbb{R}^n))} &\leq C(M)(\widetilde{T} + \widetilde{T}^\nu)\big(\|u_{p,m} - u_{p',m}\|_{L^\infty([0,\widetilde{T}],L^2(\mathbb{R}^n))} \\ &\quad + \|u_{p,m} - u_{p',m}\|_{L^{\gamma_2}([0,\widetilde{T}],L^{\rho_2}(\mathbb{R}^n))}\big) + \epsilon. \end{aligned}$$

This estimate is available, both for $(q, r) = (\infty, 2)$ or $(q, r) = (\gamma_2, \rho_2)$. Summing the two inequalities obtained and taking $\widetilde{T} > 0$ small enough, we get

$$\|u_{p,m} - u_{p',m}\|_{L^q([0,\widetilde{T}],L^r(\mathbb{R}^n))} \leq 2\epsilon.$$

Therefore, the sequence $(u_{n,m})_{n\in\mathbb{N}}$ converges, as n goes to infinity, to a limit $u_m \in L^2$ which is solution of (2.14). Moreover, since $(u_{n,m})_{n\in\mathbb{N}}$ is bounded in H^1_{mg} we can assume without loss of generality that it converges weakly to u_m in H^1_{mg}. Combining these properties with equation (2.23) it is straightforward that

$$E_{n,m}(t, u_{n,m}) - E_{n,m}(0, \varphi) \longrightarrow -\mathrm{Re} \int_0^t \langle \partial_s A(s) u_m(s), (i\nabla_x - A(s)) u_m(s) \rangle ds,$$

as n goes to infinity. From Lemma 2.3 and weak lower semicontinuity of the magnetic Sobolev norm $\|(i\nabla - A(t))\,.\|_{L^2}$ it follows that

$$E_m(t, u_m) \le E_m(0, \varphi) - \mathrm{Re} \int_0^t \langle \partial_s A(s) u(s), (i\nabla_x - A(s)) u(s) \rangle ds. \tag{2.30}$$

Finally, $t > 0$ being fixed, consider $v_{n,m}(s) = u_{n,m}(t - s)$, which is solution of

$$i\partial_s v_{n,m} = -H_{A(t-s)} v_{n,m} + g_m(v_{n,m})$$

with initial data $v_{n,m}(s = 0) = u_{n,m}(t)$. Then we perform the same computations as above to get the converse inequality to (2.30) and hence (2.16) is proved. □

2.3.2. Convergence to the Initial Problem. In this section, we show that the sequence u_m converges to a solution of (1.1) when m goes to infinity..

Lemma 2.11. *There exists $\widetilde{T}_2 > 0$ depending only on $\|\varphi\|_{H^1_{mg}}$ such that $(u_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $C([0, \widetilde{T}_2], L^2)$.*

Proof. The proof is the same as in [5], using Theorem 2, Lemmas 2.3, 2.4 and Proposition 2.10. □

Now, we complete the proof of Theorem 1. This is the same as in [5] and we recall it for reader's convenience. Let u be the limit of u_m in $C([0, \widetilde{T}_2], L^2)$. From estimate (2.27), it follows that $u \in L^\infty([0, \widetilde{T}_2], H^1_{mg})$ and by Lemma 2.2, u_m converges to u in $C([0, \widetilde{T}_2], L^r)$ for all $r \in [2, 2^*[$. Hence, it follows from Lemma 2.3 that $f_m(u_m)$ converges to $f(u)$ in $C([0, \widetilde{T}_2], H^{-1}_{mg})$ and u solves (1.1) in $L^\infty([0, \widetilde{T}_2], H^{-1}_{mg})$. Moreover, combining Lemma 2.4 and Proposition 2.10 we prove that

$$E(t, u) = E(0, \varphi) - \mathrm{Re} \int_0^t \langle \partial_s A(s) u(s), (i\nabla_x - A(s)) u(s) \rangle ds.$$

This shows that $u \in C([0, \widetilde{T}_2[, H^1_{mg})$ and hence $u \in C^1([0, \widetilde{T}_2[, H^{-1}_{mg})$.

3. WKB Approximation

In this section we justify WKB approximation for solution of (1.1) when the strength of the magnetic field b goes to infinity and we prove instability results. We focus our attention on the case where the magnetic field and the non-linearity both have the same strength; that is we consider the case $\gamma = 2$ and search approximate solution for

$$\begin{cases} i\partial_s u = H_{A(s)} u + b^2 u g(|u|^2)) \\ u_{|s=0} = a_0(x) e^{ibS(x)} \end{cases} \tag{3.1}$$

where g does not depend on x. Note that with the previous notations, $f = ug(|u|^2)$. In this section we still assume that f satisfies Assumption 2 and we require additionally.

Assumption 3. $g \in C^\infty(\mathbb{R}_+, \mathbb{R})$ with $g' > 0$.

Remark that if we assume that $a_0 \in H^1$ and $\nabla S + A(0) \in L^2$ then the initial data satisfies $\|a_0(x)e^{ibS(x)}\|_{H^1_{mg}} = O(b)$. Therefore, under Assumptions 1, 2 and 3 it follows from Theorem 1 that there exists a unique solution of (3.1) in $C(-T_b, T^b], H^1_{mg})$ with $T_b, T^b \geq Cb^{-\delta}$, $\delta > 0$. In fact this solution takes a particular form.

Theorem 3. *Let $\sigma > \frac{n}{2} + 2$ and suppose that Assumptions 1, 2 and 3 are satisfied. In addition, assume that $\partial_t A$ belongs to $H^{\sigma-1}(\mathbb{R}^n)$ for all $t \in \mathbb{R}$ and let a_0 be in $H^\sigma(\mathbb{R}^n)$ and S such that $\nabla S + A(t=0)$ belongs to $H^{\sigma-1}(\mathbb{R}^n)$. Then, there exist $T > 0$ and α_b, ϕ_b in $C([0, T[, H^\sigma(\mathbb{R}^n)) \cap C^1([0, T[, H^{\sigma-1}(\mathbb{R}^n))$ such that $u(t,x) = \alpha_b(bt,x)e^{ib(S(x)+\phi_b(bt,x))}$ is a solution of (3.1) on $[0, b^{-1}T]$.*

Proof. We start the proof by a time rescaling that leads to a semiclassical feature. We define $h = b^{-1} > 0$ and set $u(s) = v(bs)$. Then equation (3.1) is equivalent to

$$\begin{cases} ih\partial_t v = (ih\nabla_x - A(ht))^2 v + vg(|v(t)|^2) \\ v_{|t=0} = a_0(x)e^{ih^{-1}S(x)} \end{cases} \tag{3.2}$$

We follow the general method initiated by Grenier [11] for the semiclassical Schrödinger equation and look for a phase and an amplitude depending on the parameter h. Putting $v(t,x) = \alpha_h(t,x)e^{ih^{-1}\phi_h(t,x)}$ in the Equations (3.2) we get

$$\begin{cases} \partial_t \phi_h + |\nabla_A \phi_h|^2 + g(|\alpha_h|)^2 = 0 \\ \partial_t \alpha_h + \nabla_A \phi_h . \nabla \alpha_h + div(\nabla_A \phi_h)\alpha_h = ih\Delta\alpha_h \end{cases} \tag{3.3}$$

where $\nabla_A \phi = (\nabla_x \phi + A(ht))$. Next we let $\varphi_h(t,x) = \nabla_A \phi_h(t,x) \in \mathbb{R}^n$ and differentiate the above eikonal equation with respect to x. We obtain

$$\begin{cases} \partial_t \varphi_h + 2\varphi_h . \nabla \varphi_h + 2g'(|\alpha_h|^2)\,\mathrm{Re}(\overline{\alpha_h}\nabla\alpha_h) = h\partial_t A(ht,x) \\ \partial_t \alpha_h + \varphi_h . \nabla \alpha_h + div(\varphi_h)\alpha_h = ih\Delta\alpha_h \end{cases} \tag{3.4}$$

Separating real and imaginary parts of $\alpha_h = \alpha_{1,h} + i\alpha_{2,h}$, (3.4) becomes

$$\partial_t w_h + \sum_{j=1}^n A_j(w_h)\partial_{x_j} w_h = hLw_h + v_h \tag{3.5}$$

with

$$w_h = \begin{pmatrix} \alpha_{1,h} \\ \alpha_{2,h} \\ \varphi_{1,h} \\ \vdots \\ \varphi_{n,h} \end{pmatrix}, \quad v_h = \begin{pmatrix} 0 \\ 0 \\ h\partial_t A_1(ht,x) \\ \vdots \\ h\partial_t A_n(ht,x) \end{pmatrix} \tag{3.6}$$

$$L = \begin{pmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & 0_{n\times n} \end{pmatrix} \tag{3.7}$$

and

$$A_j(w) = \begin{pmatrix} \varphi_{j,h} & 0 & \alpha_1 & \dots & \alpha_1 \\ 0 & \varphi_{j,h} & \alpha_2 & \dots & \alpha_2 \\ 2g'\alpha_1 & 2g'\alpha_2 & v_j & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ 2g'\alpha_1 & 2g'\alpha_2 & 0 & 0 & \varphi_{j,h} \end{pmatrix} \tag{3.8}$$

This system has the same form as in [3, 11] except the source term v_h in right hand side of (3.5) and the initial data. Thanks to the assumptions, v_h belongs to $H^{\sigma-1}(R^n)$. Moreover the initial condition in (3.2) yields

$$w_h(t=0) = \begin{pmatrix} \mathrm{Re}\, a_o \\ \mathrm{Im}\, a_0 \\ \partial_{x_1} S + A_1(0) \\ \vdots \\ \partial_{x_n} S + A_n(0) \end{pmatrix} \tag{3.9}$$

which belongs to $H^{\sigma-1}(\mathbb{R}^n)$.

In addition, thanks to the assumption on g', the system (3.5) can be symmetrized by

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{g'} I_n \end{pmatrix} \tag{3.10}$$

which is symmetric and positive. It follows from general theory of hyperbolic systems that problem (3.5) together with initial condition (3.9) has a unique solution $w_h \in L^\infty([0, T_h], H^{\sigma-1})$ for some $T_h > 0$.

Hence, we have to bound T_h from below by a constant independent of h. This is done by computing classical energies estimates as in [3, 11], and using the fact that $\partial_t A$, $\nabla_x S + A(0)$ belong to $H^{\sigma-1}$.

Finally we define α_h and ϕ_h by $\alpha_h = w_{1,h} + i w_{2,h}$ and

$$\phi_h = S(x) - \int_0^t |\varphi_h|^2 + f(|\alpha_h|^2)\, ds.$$

By construction, ϕ_h belongs to L^2. Moreover, a simple calculus shows that $\nabla_x \phi_h = \varphi_h - A(ht)$ belongs to $H^{\sigma-1}$ so that ϕ_h is in fact in H^σ. Going back to the equation on α_h and making energies estimates we show that $\alpha_h \in H^\sigma$. Finally, straightforward computation shows that (α_h, ϕ_h) defined above solves (3.3). □

Remark 3.1. The above solution belongs to the magnetic Sobolev space H^1_{mg}. Indeed,

$$(i\nabla_x - bA)\big(\alpha_b e^{ib\phi_b}\big) = (i\nabla\alpha_b - b(\nabla\phi_b + A)\alpha_b)e^{ib\phi_b}$$

belongs to L^2. Therefore the solution obtained in Theorem 3 coincides with the solution of Theorem 1.

With Theorem 3 it is easy to prove instability results:

Proposition 3.2. *Let* $\sigma > \frac{n}{2} + 2$ *and let* A *satisfy the assumptions of Theorem* 3. *Suppose that* S *is such that* $\nabla S + A(t = 0)$ *belongs to* $H^{\sigma-1}(\mathbb{R}^n)$. *Then, there exists* a_0 *and* $\widetilde{a}_{0,b}$ *in* $H^\sigma(\mathbb{R}^n)$ *and* $0 < t_b < Cb^{-1}$ *such that*

$$\|a_0 - \widetilde{a}_{0,b}\|_{L^2} \to 0 \quad \text{as } b \to \infty$$

and the solutions u_b *(resp.* $\widetilde{u}_b$*) associated to* (3.1) *with initial data* $a_0 e^{ibS(x)}$ *(resp.* $\widetilde{a}_0 e^{ibS(x)}$*) satisfy*

$$\|u_b - \widetilde{u}_b\|_{L^\infty([0,t_b],L^2)} \geq 1.$$

Proof. It is a straightforward consequence of Theorem 3 and the methods of Carles [3]. □

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