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SPECTRAL ANALYSIS OF HYPOELLIPTIC RANDOM WALKS

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Abstract We study the spectral theory of a reversible Markov chain associated with a hypoelliptic random walk on a manifold M. This random walk depends on a parameter $h \in]0, h_0]$ which is roughly the size of each step of the walk. We prove uniform bounds with respect to h on the rate of convergence to equilibrium, and the convergence when $h \to 0$ to the associated hypoelliptic diffusion.

Keywords: partial differential equations; global analysis; analysis on manifolds; probability theory and stochastic processes

1. Introduction and results

The purpose of this paper is to study the spectral theory of a reversible Markov chain associated with a hypoelliptic random walk on a manifold M. This random walk will depend on a parameter $h \in]0, h_0]$ which is roughly the size of each step of the walk. We are in particular interested, as in [5, 6], in getting uniform bounds with respect to h on the rate of convergence to equilibrium. The main tool in our approach is to compare the random walk on M with a natural random walk on a nilpotent Lie group. This idea was used by Rotschild and Stein [14] to prove sharp hypoelliptic estimates for some differential operators. (See also the article by Nagel, Stein, and Wainger [13] for the study of hypoelliptic geometries.)

We will also verify that, when $h \to 0$, this random walk converges to a continuous hypoelliptic diffusion. The discretization of a continuous hypoelliptic diffusion with applications to numerical simulations has been performed in particular in [2, 3].

Let M be a smooth, connected, compact manifold of dimension m, equipped with a smooth volume form $d\mu$ such that $\int_M d\mu = 1$. We denote by μ the associated probability on M. Let $\mathcal{X} = \{X_1, \dots, X_p\}$ be a collection of smooth vector fields on M. Denote by $\mathcal{G}$ the Lie algebra generated by $\mathcal{X}$. In all the paper we assume that the X_k are divergence free with respect to $d\mu$:

$$\forall k = 1, \dots, p, \quad \int_M X_k(f) d\mu = 0, \ \forall f \in C^\infty(M), \tag{1.1}$$

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and that they satisfy the Hörmander condition:

$$\forall x \in M, \quad \mathcal{G}_x = T_x M. \tag{1.2}$$

Let $\mathfrak{r} \in \mathbb{N}$ be the smallest integer such that, for any $x \in M$, $\mathcal{G}_x$ is generated by commutators of length at most $\mathfrak{r}$. For $k = 1, \dots, p$ and $x_0 \in M$, denote by $\mathbb{R} \ni t \mapsto \mathrm{e}^{tX_k} x_0$ the integral curve of X_k starting from x_0 at $t = 0$.

Let $h \in]0, h_0]$ be a small parameter. Let us consider the following simple random walk, $x_0, x_1, \dots, x_n, \dots$ on M, starting at $x_0 \in M$: at step n, choose $j \in \{1, \dots, p\}$ at random and $t \in [-h, h]$ at random (uniform), and set $x_{n+1} = \mathrm{e}^{tX_j} x_n$.

Due to the condition $div(X_j) = 0$, this random walk is reversible for the probability μ on M. It is easy to compute the Markov operator T_h associated with this random walk: for any bounded and measurable function $f : M \to \mathbb{R}$, define

$$T_{k,h} f(x) = \frac{1}{2h} \int_{-h}^{h} f(\mathrm{e}^{tX_k} x) \mathrm{d}t. \tag{1.3}$$

Since the vector fields X_k are divergence free, for any f, g, we have

$$\int_M T_{k,h} f(x) g(x) \mathrm{d}\mu = \int_M f(x) T_{k,h} g(x) \mathrm{d}\mu,$$

and the Markov operator associated with our random walk is

$$T_h f(x) = \frac{1}{p} \sum_{k=1}^{p} T_{k,h} f(x). \tag{1.4}$$

One has $T_h(1) = 1$, $\|T_h\|_{L^\infty \to L^\infty} = 1$, and T_h can be uniquely extended as a bounded self-adjoint operator on $L^2 = L^2(M, \mathrm{d}\mu)$ such that $\|T_h\|_{L^2 \to L^2} = 1$. In the following, we will denote by $t_h(x, \mathrm{d}y)$ the distribution kernel of T_h, and by t_h^n the kernel of T_h^n. Then, by construction, the probability for the walk starting at x_0 to be in a Borel set A after n step is equal to

$$P(x_n \in A) = \int_A t_h^n(x_0, \mathrm{d}y).$$

The goal of this paper is to study the spectral theory of the operator T_h and the convergence of $t_h^n(x_0, \mathrm{d}y)$ towards μ as n tends to infinity. Since T_h is Markov and self-adjoint, its spectrum is a subset of $[-1, 1]$. We shall denote by $g(h)$ the spectral gap of the operator T_h. It is defined as the best constant such that the following inequality holds true for all $u \in L^2$:

$$\|u\|_{L^2}^2 - \langle u, 1\rangle_{L^2}^2 \leqslant \frac{1}{g(h)} \langle u - T_h u, u\rangle_{L^2}. \tag{1.5}$$

The existence of a non-zero spectral gap means that 1 is a simple eigenvalue of T_h, and the distance between 1 and the rest of the spectrum is equal to $g(h)$. Our first result is the following.

Theorem 1.1. *There exist $h_0 > 0$, $\delta_1, \delta_2 > 0$, $A > 0$, and constants $C_i > 0$ such that, for any $h \in]0, h_0]$, the following holds true.*

(i) The spectrum of T_h is a subset of $[-1+\delta_1, 1]$, 1 is a simple eigenvalue of T_h, and $\mathrm{Spec}(T_h) \cap [1-\delta_2, 1]$ is discrete. Moreover, for any $0 \leqslant \lambda \leqslant \delta_2 h^{-2}$, the number of eigenvalues of T_h in $[1-h^2\lambda, 1]$ (with multiplicity) is bounded by $C_1(1+\lambda)^A$.

(ii) The spectral gap satisfies

$$C_2 h^2 \leqslant g(h) \leqslant C_3 h^2, \tag{1.6}$$

and the following estimate holds true for all integers n:

$$\sup_{x\in\Omega} \|t_h^n(x, \mathrm{d}y) - \mu\|_{TV} \leqslant C_4 \mathrm{e}^{-ng(h)}. \tag{1.7}$$

Here, for two probabilities on M, $\|\nu-\mu\|_{TV} = \sup_A |\nu(A) - \mu(A)|$, where the $\sup$ is over all Borel sets A, is the total variation distance between ν and μ.

Key ingredients in the proof of Theorem 1.1 are the decomposition of a given function f on M into its low-frequency and high-frequency parts with respect to the spectral theory of T_h, $f = f_L + f_H$, and the use of a Nash inequality, which is a Sobolev inequality, on the low-frequency part. We have already used these types of argument in [5, 6]. However, in the hypoelliptic setting, a new difficulty appears in the control of the Sobolev norms of the low-frequency part by the Dirichlet form associated with T_h (see Lemma 5.3). This forces us to prove a new result on the semi-classical analysis of a system of vector fields satisfying the hypoelliptic condition (see Proposition 4.1).

We describe now the spectrum of T_h near 1. Let $\mathcal{H}^1(\mathcal{X})$ be the Hilbert space

$$\mathcal{H}^1(\mathcal{X}) = \{u \in L^2(M),\ \forall j = 1, \dots, p,\ X_j u \in L^2(M)\}.$$

Let ν be the best constant such that the following Poincaré inequality holds true for all $u \in \mathcal{H}^1(\mathcal{X})$:

$$\|u\|_{L^2}^2 - \langle u, 1\rangle_{L^2}^2 \leqslant \frac{\mathcal{E}(u)}{\nu}, \tag{1.8}$$

where

$$\mathcal{E}(u) = \frac{1}{6p}\int_M \sum_{k=1}^{p} |X_k u|^2 \mathrm{d}\mu. \tag{1.9}$$

Let us recall that local Poincaré inequalities have been proven in the hypoelliptic case by Jerison, in [11]. By the hypoelliptic theorem of Hörmander (see [10, Vol. 3]), one has, for some $s > 0$, $\mathcal{H}^1(\mathcal{X}) \subset H^s(M) = \{u \in \mathcal{D}'(M), Pu \in L^2(M),\ \forall P \in \Psi^s\}$, where Ψ^s denotes the set of classical pseudodifferential operators on M of degree s. On the other hand, standard Taylor expansion in formula (1.3) shows that, for any fixed smooth function $g \in C^\infty(M)$, one has the following convergence in the space $C^\infty(M)$:

$$\lim_{h\to 0} \frac{1-T_h}{h^2} g = L(g), \tag{1.10}$$

where the operator $L = -\frac{1}{6p}\sum_k X_k^2$ is the positive Laplacian associated with the Dirichlet form $\mathcal{E}(u)$. It has a compact resolvent and spectrum $\nu_0 = 0 < \nu_1 = \nu < \nu_2 < \cdots$. Let m_j

be the multiplicity of ν_j. One has $m_0 = 1$ since Ker(L) is spanned by the constant function 1 thanks to the Chow theorem [4]. In fact, for any $x, y \in M$ there exists a continuous curve connecting x to y which is a finite union of pieces of trajectory of one of the fields X_j.

Theorem 1.2. *One has*

$$lim_{h\to 0} h^{-2} g(h) = \nu. \tag{1.11}$$

Moreover, for any $R > 0$ and $\varepsilon > 0$ such that the intervals $[\nu_j - \varepsilon, \nu_j + \varepsilon]$ are disjoint for $\nu_j \leqslant R$, there exists $h_1 > 0$ such that, for all $h \in]0, h_1]$,

$$\text{Spec}\left(\frac{1 - T_h}{h^2}\right) \cap]0, R] \subset \cup_{j\geqslant 1} [\nu_j - \varepsilon, \nu_j + \varepsilon], \tag{1.12}$$

and the number of eigenvalues of $\frac{1-T_h}{h^2}$ with multiplicities, in the interval $[\nu_j - \varepsilon, \nu_j + \varepsilon]$, is equal to m_j.

The paper is organized as follows.

In § 2, we recall some basic facts on nilpotent Lie groups, and we recall the Goodman version (see [9]) of one of the main results of the Rotschild and Stein paper.

In § 3, the main result is Proposition 3.1, which gives a lower bound on a suitable power T_h^P of T_h. This in particular allows us to get a first crude but fundamental bound on the L^∞ norms of eigenfunctions of T_h associated with eigenvalues close to 1.

Section 4 is devoted to the study of the Dirichlet form associated with our random walk. The fundamental result of this section is Proposition 4.1. It allows to separate clearly the spectral theory of T_h in low and high frequencies with respect to the parameter h. In order to prove Proposition 4.1, we construct suitable h-pseudodifferential cutoff operators adapted to the hypoelliptic setting. In the case of left invariant vector fields on a nilpotent Lie algebra, Lemma A.2 allows us to use only convolution operators. This construction is extended to the general case using in particular results from the Rotschild and Stein paper [14].

Section 5 is devoted to the proof of Theorems 1.1 and 1.2. With Propositions 3.1 and 4.1 in hand, the proof follows the general strategy of [5, 6]. This section also contains a paragraph on the Fourier analysis associated with T_h that will be useful in 6. In particular, Lemma 5.5 gives a precise Sobolev estimate for the eigenfunctions of the Markov operator T_h associated with eigenvalues in $[1 - c_4, 1]$, with $c_4 > 0$ small enough, and Proposition 5.6 extends, in our Markov setting, the classical fact that a function is smooth iff its Fourier coefficients are rapidly decreasing.

Section 6 is devoted to the proof of the convergence when $h \to 0$ of our Markov chain to the hypoelliptic diffusion on the manifold M associated with the generator $L = \frac{-1}{6p} \sum_k X_k^2$. This is probably a well-known result for specialists, but we have not succeeded in finding a precise reference. Since this convergence follows as a simple byproduct of our estimates, we decided to include it in the paper.

Finally, the appendix contains two lemmas. Lemma A.1 shows how to deduce from Proposition 4.1 a Weyl-type estimate on the eigenvalues of T_h in a neighbourhood of 1.

Lemma A.2 is an elementary cohomological lemma on the Schwartz space of the nilpotent Lie algebra $\mathcal{N}$.

Remark 1.3. It is likely that Theorems 1.1 and 1.2 remain true (with almost the same proof) in the case of a compact manifold M with boundary, if one assumes that the boundary ∂M is non-characteristic, i.e., if, for any point $x \in \partial M$, there exists j such that $X_j(x)$ is not tangent to ∂M. In that case, the associated walk near the boundary will be defined by a Metropolis-type algorithm: at step n, choose $j \in \{1, \dots, p\}$ at random and $t \in [-h, h]$ at random (uniform), and set $x_{n+1} = \mathrm{e}^{tX_j} x_n$ if $\mathrm{e}^{sX_j} x_n \in M$ for all $s \in [0, t]$, and $x_{n+1} = x_n$ otherwise. Then, in Theorem 1.2, the limit operator should be $L = \sum_{j=1}^{d} X_j^2$ with the Neumann boundary condition.

2. The lifted operator to a nilpotent Lie algebra

We will use the notation $\mathbb{N}_q = \{1, \dots, q\}$. For any family of vector fields $Z_1, \dots, Z_p$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_p^k$, denote by $|\alpha| = k$ the length of α, and let

$$Z^\alpha = H_\alpha(Z_1, \dots, Z_p) = [Z_{\alpha_1}, [Z_{\alpha_2}, \dots [Z_{\alpha_{k-1}}, Z_{\alpha_k}] \dots]. \tag{2.1}$$

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_p$ be a system of generators of the free Lie algebra with p generators $\mathcal{F}$, and let $\mathcal{A}^\infty$ be a set of multi-indexes such that $(\mathcal{Y}^\alpha)_{\alpha \in \mathcal{A}^\infty}$ is a basis of $\mathcal{F}$.

Let $\mathcal{N}$ be the free up to step $\mathfrak{r}$ nilpotent Lie algebra generated by p elements $Y_1, \dots, Y_p$, and let N be the corresponding simply connected Lie group. We have the decomposition

$$\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_{\mathfrak{r}}, \tag{2.2}$$

where $\mathcal{N}_1$ is generated by $Y_1, \dots, Y_p$ and $\mathcal{N}_j$ is spanned by the commutators $Y^\alpha = H_\alpha(Y_1, \dots, Y_p)$ with $|\alpha| = j$ for $2 \leqslant j \leqslant \mathfrak{r}$. Let $\mathcal{A} = \{\alpha \in \mathcal{A}^\infty,\ |\alpha| \leqslant \mathfrak{r}\}$ and $\mathcal{A}_r = \{\alpha \in \mathcal{A},\ |\alpha| = r\}$. The family $(Y^\alpha)_{\alpha \in \mathcal{A}}$ is a basis for $\mathcal{N}$, and, for any $r \in \mathbb{N}_{\mathfrak{r}}$, $\{Y^\alpha,\ \alpha \in \mathcal{A}_r\}$ is a basis of $\mathcal{N}_r$. We denote by $D = \sharp\mathcal{A}$ the dimension of $\mathcal{N}$. The action of $\mathbb{R}_+$ on $\mathcal{N}$ is given by

$$t.(v_1, v_2, \dots, v_r) = (tv_1, t^2 v_2, \dots, t^{\mathfrak{r}} v_{\mathfrak{r}}).$$

A homogeneous norm $\|v\|$ which is smooth in $\mathcal{N} \setminus o_{\mathcal{N}}$ is given by

$$\|v\| = \left(\sum_j |v_j|^{(2\mathfrak{r}!)/j} \right)^{1/(2\mathfrak{r}!)},$$

where $|v_j|$ is a Euclidian norm on $\mathcal{N}_j$, and

$$Q = \sum j \dim(\mathcal{N}_j)$$

is the quasi-homogeneous dimension of $\mathcal{N}$. We will identify the Lie algebra $\mathcal{N}$ with the Lie group N by the exponential map; i.e., the product law $a.b$ on $\mathcal{N}$ is given by $\exp(a.b) = \exp(a)\exp(b)$. In particular, one has with this identification $a^{-1} = -a$ for all $a \in \mathcal{N}$. To avoid notational confusion, we will sometimes use the notation $e = o_{\mathcal{N}}$, so that $a.e = e.a = a$ for all $a \in \mathcal{N}$. For $Y \in T_e\mathcal{N} \simeq \mathcal{N}$, we denote by $\tilde{Y}$ the left invariant vector field on $\mathcal{N}$ such that $\tilde{Y}(o_{\mathcal{N}}) = Y$; i.e.,

$$\tilde{Y}(f)(x) = \frac{\mathrm{d}}{\mathrm{d}s}(f(x.sY)|_{s=0}.$$

The right invariant vector field on $\mathcal{N}$ such that $Z(o_{\mathcal{N}}) = Y$ is defined by

$$Z(f)(x) = \frac{\mathrm{d}}{\mathrm{d}s}(f(sY.x)|_{s=0}.$$

Here, sY is the usual product of the vector $Y \in \mathcal{N}$ by the scalar $s \in \mathbb{R}$. For $a \in \mathcal{N}$, let τ_a be the diffeomorphism of $\mathcal{N}$ defined by $\tau_a(u) = a.u$. One has

$$\tilde{Y}(a) = \mathrm{d}\tau_a(e)(Y).$$

Example 2.1. The standard 3D-Heisenberg group is $\mathcal{N} = \mathbb{R}^3$, with the product law

$$(x, y, t).(x', y', t') = (x + x', y + y', t + t' + xy' - yx'),$$

and the left invariant vector fields associated respectively to the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are in that case

$$\tilde{Y}_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial t}, \quad \tilde{Y}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial t}, \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{1}{2}[\tilde{Y}_1, \tilde{Y}_2].$$

Remark 2.2. In general, for $x = (x_1, \dots, x_{\mathfrak{r}})$ and $y = (y_1, \dots, y_{\mathfrak{r}})$, $x_j, y_j \in \mathcal{N}_j$, the product law is given by

$$\left.\begin{aligned} &(x_1, \dots, x_{\mathfrak{r}}).(y_1, \dots, y_{\mathfrak{r}}) = (z_1, \dots, z_{\mathfrak{r}}), \\ &z_j = x_j + y_j + P_j(x_{<j}, y_{<j}), \end{aligned}\right\} \tag{2.3}$$

with the notation $x_{<j} = (x_1, \dots, x_{j-1})$, and where P_j is a polynomial of degree j with respect to the homogeneity on $\mathcal{N}$; i.e.,

$$P_j((t.x)_{<j}, (t.y)_{<j}) = t^j P_j(x_{<j}, y_{<j}),$$

which is compatible with the identity $t.(x.y) = (t.x).(t.y)$.

Let $\lambda : \mathcal{N} \to \mathcal{G}$ be the unique linear map such that, for any $\alpha \in \mathcal{A}$, $\lambda(Y^\alpha) = X^\alpha$. Then λ is a Lie homomorphism 'up to step $\mathfrak{r}$':

$$\lambda([Y^\alpha, Y^\beta]) = [X^\alpha, X^\beta] \tag{2.4}$$

for any multi-indexes α, β such that $|\alpha| + |\beta| \leqslant \mathfrak{r}$.

Let $x_0 \in M$. There exists a subset $\mathcal{A}_{x_0} \subset \mathcal{A}$ such that $(X^\alpha(x))_{\alpha \in \mathcal{A}_{x_0}}$ is a basis of $T_x M$ for any x close to x_0. Therefore, there exists a neighbourhood Ω_0 of the origin $o_{\mathcal{N}}$ in $\mathcal{N}$ and a neighbourhood V_0 of x_0 in M such that the map Λ

$$\Lambda : u = \sum_{\alpha \in \mathcal{A}} u_\alpha Y^\alpha \in \Omega_0 \mapsto \mathrm{e}^{\lambda(u)} x_0 = \mathrm{e}^{\sum_{\alpha \in \mathcal{A}} u_\alpha X^\alpha} x_0$$

is a submersion from Ω_0 onto V_0, and the map $W_{x_0} : \mathcal{C}^\infty(V_0) \to \mathcal{C}^\infty(\Omega_0)$ defined by $W_{x_0} f(u) = f(\mathrm{e}^{\lambda(u)} x_0)$ is injective. Since Λ is a submersion, there exists a system of coordinates $\theta : \mathbb{R}^m \times \mathbb{R}^n \to \mathcal{N}$ defined near $o_{\mathcal{N}}$, where $m + n = D$, such that $\Lambda\theta : \mathbb{R}^m \to M$ is a system of coordinates near x_0, and in these coordinates one has $\Lambda(x, y) = x$. We thus may assume that in these coordinates one has $\Omega_0 = V_0 \times U_0$, where U_0 is a neighbourhood of $0 \in \mathbb{R}^n$.

Example 2.3. Take for example the two vectors fields in $\mathbb{R}^2$, $X_1 = \partial_x$, $X_2 = x\partial_y$. Then $[X_1, X_2] = \partial_y$. Then take for $\mathcal{N}$ the 3D-Heisenberg group, and the map λ, with $T = 2\partial_t = [Y_1, Y_2]$, is given by

$$\lambda(u_1 Y_1 + u_2 Y_2 + u_3 T) = u_1 X_1 + u_2 X_2 + u_3 [X_1, X_2] = u_1 \partial_x + (u_3 + u_2 x)\partial_y.$$

Thus we get

$$\mathrm{e}^{\lambda(u)}(x, y) = \left(x + u_1,\ y + u_3 + u_2 x + \frac{1}{2} u_1 u_2\right). \tag{2.5}$$

Let $I_h = \{|u_1| < h, |u_2| < h, |u_3| < h^2\}$. One has $\mathrm{Vol}(I_h) = 8h^4$, and the set $\tilde{B}_{h,(x,y)} = \{\mathrm{e}^{\lambda(u)}(x, y), u \in I_h\}$, with (x, y) fixed and h small, has volume of order:
h^2 when $x \neq 0$, and h^3 when $x = 0$.

Let us now recall the notion of the order of a vector field used in [9, 14]. Denote by $\{\delta_t\}_{t>0}$ the one-parameter group of dilating automorphisms on $\mathcal{N}$:

$$\delta_t Y^\alpha = t^{|\alpha|} Y^\alpha.$$

Let Ω be a compact neighbourhood of $o_{\mathcal{N}}$ in $\mathcal{N}$. For any $m \in \mathbb{N}$, let

$$C_m^\infty = \{f \in C^\infty(\Omega, \mathbb{R}),\ f(u) = \mathcal{O}(\|u\|^m)\}.$$

We have the filtration $C^\infty(\Omega) = C_0^\infty \supseteq C_1^\infty \supseteq \dots$, and $C_m^\infty . C_n^\infty \subseteq C_{m+n}^\infty$. Let $T : C^\infty(\Omega) \to C^\infty(\Omega)$. We say that T is of order less than k at 0 if $T(C_m^\infty) \subseteq C_{m-k}^\infty$ for all integers $m \geqslant 0$. If ∂_α denotes differentiation in the direction Y^α, then a vector field $T = \sum_\alpha \varphi_\alpha \partial_\alpha$ is of order $\leqslant k$ iff $\varphi_\alpha \in C_{|\alpha|-k}^\infty$ for all α, with the convention $C_m^\infty = C_0^\infty$ for $m \leqslant 0$.

The following result is the Goodman version of one of the results of the article [14] by Rothschild and Stein.

Theorem 2.4. *For a sufficiently small Ω_0, there exist C^∞ vector fields $Z_1, \dots, Z_p$ on Ω_0 such that, for any $\alpha \in \mathcal{A}$, and with $Z^\alpha = H_\alpha(Z_1, \dots, Z_p)$ (see (2.1)), we have*

(i) $Z^\alpha W_{x_0} = W_{x_0} X^\alpha$.

(ii) $Z^\alpha = \widetilde{Y}^\alpha + R_\alpha$, *where R_α is a vector field of order $\leqslant |\alpha| - 1$ at 0.*

Observe that, in the previous coordinate system (x, y) on Ω_0, one can write, for $\alpha \in \mathcal{A}$,

$$X^\alpha = \sum_j a_{\alpha,j}(x) \frac{\partial}{\partial x_j}, \quad Z^\alpha = \sum_j a_{\alpha,j}(x) \frac{\partial}{\partial x_j} + \sum_l b_{\alpha,l}(x, y) \frac{\partial}{\partial y_l}. \tag{2.6}$$

As an obvious consequence of this theorem, we have the following, with $W = W_{x_0}$, and $\tilde{\lambda}(u) = \sum_{\alpha\in A} u_\alpha Z^\alpha$.

Proposition 2.5. *Let $f \in \mathcal{C}^0(V_0)$, and let $\omega_0 \subset\subset \Omega_0$ be a neighbourhood of $o_{\mathcal{N}}$. Then, there exists $r_0 > 0$ such that, for all $\|u\| \leqslant r_0$, and $v \in \omega_0$, we have*

$$(Wf)(e^{\tilde{\lambda}(u)}v) = W(f_u)(v), \tag{2.7}$$

where the function f_u is defined near x_0 by $f_u(x) = f(e^{\lambda(u)}x)$.

Using this proposition, we can easily compute the action of W on the operator T_h acting on functions with support close to x_0. We get immediately

$$WT_h = \widetilde{T}_h W, \quad \widetilde{T}_h = \frac{1}{p}\sum_{k=1}^{p} \widetilde{T}_{k,h}, \tag{2.8}$$

where, for $u \in \mathcal{N}$ small,

$$\widetilde{T}_{k,h} g(u) = \frac{1}{2h}\int_{-h}^{h} g(e^{tZ_k}u)dt. \tag{2.9}$$

Using the notation $T^\alpha = T_{\alpha_k,h}\dots T_{\alpha_1,h}$ for any multi-index $\alpha = (\alpha_1,\dots,\alpha_k)$, we get, for any $u \in \mathcal{N}$ close to $o_{\mathcal{N}}$ such that $\Lambda(u) = x$,

$$T^\alpha f(x) = W(T^\alpha f)(u) = \frac{1}{(2h)^k}\int_{[-h,h]^k} (Wf)(e^{t_1 Z_{\alpha_1}}\dots e^{t_k Z_{\alpha_k}}u)dt_1\dots dt_k. \tag{2.10}$$

3. Rough bounds on eigenfunctions

Let us recall from § 2 that, for $u = \sum_{\alpha\in\mathcal{A}} u_\alpha Y^\alpha \in \mathcal{N}$, the vector field $\lambda(u)$ on M is defined by $\lambda(u) = \sum_{\alpha\in\mathcal{A}} u_\alpha X^\alpha$. Let $\epsilon > 0$ and $I_{\epsilon,h}$ be the neighbourhood of $o_{\mathcal{N}}$ in $\mathcal{N}$ defined by

$$I_{\epsilon,h} = \left\{ u = \sum_{\alpha\in\mathcal{A}} u_\alpha Y^\alpha,\ u_\alpha \in]-\epsilon h^{|\alpha|}, \epsilon h^{|\alpha|}[\right\}.$$

For any $x \in M$, we define a positive measure $S_h^\epsilon(x, dy)$ on M by the formula

$$\forall f \in C^0(M), \quad \int f(y) S_h^\epsilon(x, dy) = h^{-Q}\int_{u\in I_{\epsilon,h}} f(e^{\lambda(u)}x)\, du, \tag{3.1}$$

where $du = \Pi_\alpha du_\alpha$ is the left (and right) invariant Haar measure on $\mathcal{N}$. Let us introduce the numerical sequence $(b_n)_{n\in\mathbb{N}^*}$ defined by $b_1 = 1$ and $b_{n+1} = 2b_n + 2$, so that, for all $n \in \mathbb{N}^*$, we have $b_n = 3.2^{n-1} - 2$.

Proposition 3.1. *For all $r = 1,\dots,\mathfrak{r}$, denote $a_r = \sharp\mathcal{A}_r = \dim\mathcal{N}_r$, and let $P = \sum_{r=1}^{\mathfrak{r}} a_r b_r$. There exist $\epsilon > 0$, $c > 0$, and $h_0 > 0$ such that, for all $h \in]0, h_0]$, $x \in M$,*

$$t_h^P(x, dy) = \rho_h(x, dy) + cS_h^\epsilon(x, dy), \tag{3.2}$$

where $\rho_h(x, dy)$ is a non-negative Borel measure on M for all $x \in M$.

Remark 3.2. As in [5], one can deduce from Proposition 3.1 that the inequality (3.2) holds true for $t_h^N(x, \mathrm{d}y)$ as soon as $N \geqslant P$, eventually with different constants $\epsilon > 0$, $c > 0$, and $h_0 > 0$ depending on N.

Before proving this proposition, let us give two simple but fundamental corollaries. Like in [5], these two corollaries will play a key role in the proofs of Theorems 1.1 and 1.2. Here, we use the same notation for a bounded measurable family in x of non-negative Borel measure $k(x, \mathrm{d}y)$ and the corresponding operator $f \mapsto K(f)(x) = \int f(y)k(x, \mathrm{d}y)$ acting on L^∞.

Corollary 3.3. *There exist $h_0 > 0$ and $\gamma < 1$ such that, for all $h \in]0, h_0]$ and all $x \in M$,*

$$\|\rho_h(x, \mathrm{d}y)\|_{L^\infty \to L^\infty} \leqslant \gamma < 1. \tag{3.3}$$

Proof. By definition, the non-negative measure ρ_h is given by $\rho_h(x, \mathrm{d}y) = t_h^P(x, \mathrm{d}y) - cS_h^\epsilon(x, \mathrm{d}y)$. Therefore

$$\left|\int_M f(x)\mathrm{d}\rho_h(x, \mathrm{d}y)\right| \leqslant \|f\|_{L^\infty} \int_M \mathrm{d}\rho_h(x, \mathrm{d}y) \leqslant \|f\|_{L^\infty} \left(1 - c \inf_{x \in M} \int_M S_h^\epsilon(x, \mathrm{d}y)\right), \tag{3.4}$$

since $t_h^P(x, \mathrm{d}y)$ is a Markov kernel. From (3.1), one has $\int_M S_h^\epsilon(x, \mathrm{d}y) = h^{-Q}\mathrm{meas}(I_{\epsilon,h}) = (2\epsilon)^D$. Combined with (3.4), this implies the result. □

Corollary 3.4. *Let $a \in]\gamma^{\frac{1}{P}}, 1]$ be fixed. There exists $C = C_a > 0$ such that, for any $\lambda \in [a, 1]$ and any $f \in L^2(M, \mathrm{d}\mu)$, we have*

$$T_h f = \lambda f \Longrightarrow \|f\|_{L^\infty} \leqslant Ch^{-\frac{Q}{2}}\|f\|_{L^2}. \tag{3.5}$$

Proof. Suppose that $T_h f = \lambda f$; then $T_h^P f = \lambda^P f$. Hence, $S_h^\epsilon f = \lambda^P f - \rho_h(f)$ and then

$$\|S_h^\epsilon f\|_{L^\infty} \geqslant \lambda^P \|f\|_{L^\infty} - \gamma\|f\|_{L^\infty} \geqslant c_a\|f\|_{L^\infty}, \tag{3.6}$$

with $c_a = a^P - \gamma$. On the other hand, since $u \mapsto \mathrm{e}^{\lambda(u)}x$ is a submersion from a neighbourhood of $o_{\mathcal{N}} \in \mathcal{N}$ onto a neighbourhood of $x \in M$, we get, by the Cauchy–Schwarz inequality,

$$|S_h^\epsilon f(x)| \leqslant h^{-Q}\mathrm{meas}(I_{\epsilon,h})^{1/2} \left(\int_{u \in I_{\epsilon,h}} |f(\mathrm{e}^{\lambda(u)}x)|^2 \,\mathrm{d}u\right)^{1/2} \leqslant Ch^{-Q/2}\|f\|_{L^2(M)}. \tag{3.7}$$

Putting together (3.6) and (3.7), we obtain the announced result. □

Let us now prove Proposition 3.1. We have to show that there exist $c, \epsilon > 0$ independent of h small such that, for any non-negative continuous function f on M, one has $T_h^P f(x) \geqslant cS_h^\epsilon f(x)$. Since M is compact and the operator T_h moves supports of functions at distance at most h, we can assume without loss of generality that f is supported near some point $x_0 \in M$ where we can apply the results of § 2. Recall that $\tilde{\lambda}(u) = \sum_{\alpha \in A} u_\alpha Z^\alpha$. From Proposition 2.5, one has $f(\mathrm{e}^{\lambda(u)}x) = W(f)(\mathrm{e}^{\tilde{\lambda}(u)}w)$ for any w close to $o_{\mathcal{N}}$ such that

$\Lambda(w) = x$. Using also (2.8), we are thus reduced to proving the existence of $c, \epsilon > 0$ independent of h small such that, for any non-negative continuous function g on $\mathcal{N}$ supported near $o_{\mathcal{N}}$, one has

$$\widetilde{T}_h^P g(w) \geqslant ch^{-Q} \int_{u \in I_{\epsilon,h}} g(e^{\tilde{\lambda}(u)} w)\, du. \tag{3.8}$$

For each possibly non-commutative sequence (A_k) of operators, we denote $\Pi_{k=1}^K A_k = A_K \dots A_1$ (i.e., A_1 is the first operator acting). Endowing $\mathcal{A}_r$ with the lexicographical order, we can write $\mathcal{A}_r = \{\alpha_1 < \cdots < \alpha_{a_r}\}$ and, for any non-commutative sequence (B_α) indexed by $\mathcal{A}$, we define $\Pi_{\alpha \in \mathcal{A}_r} B_\alpha = \Pi_{j=1}^{a_r} B_{\alpha_j}$ and $\Pi_{\alpha \in \mathcal{A}} B_\alpha = \Pi_{r=1}^{\mathfrak{r}} \Pi_{\alpha \in \mathcal{A}_r} B_\alpha$.

Let $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_p^k$, and let $t = (t_1, \dots, t_k) \in \mathbb{R}^k$ close to 0. One defines by induction on $|\alpha|$ a smooth diffeomorphism $\phi_\alpha(t)$ of $\mathcal{N}$ near $o_{\mathcal{N}}$, with $\phi_\alpha(0) = \mathrm{Id}$, by the following formulas.

If $|\alpha| = 1$ and $\alpha = j \in \{1, \dots, p\}$, set $\phi_\alpha(t)(w) = e^{tZ_j} w$. If $|\alpha| = k \geqslant 2$, set $\alpha = (j, \beta)$, with $\beta \in \mathbb{N}_p^{k-1}$ and $t = (t_1, t')$ with $t' \in \mathbb{R}^{k-1}$, and set

$$\phi_\alpha(t) = \phi_\beta^{-1}(t') e^{-t_1 Z_j} \phi_\beta(t') e^{t_1 Z_j}. \tag{3.9}$$

Observe that $\phi_\alpha(t) = \mathrm{Id}$ if one of the t_j is equal to 0. The map $(t, w) \mapsto \phi_\alpha(t)(w)$ is smooth, and one has, in local coordinates on $\mathcal{N}$, and for t close to 0,

$$\phi_\alpha(t)(w) = w + (\Pi_{1 \leqslant l \leqslant |\alpha|}\ t_l)\ Z^\alpha(w) + r_\alpha(t, w), \tag{3.10}$$

with $r_\alpha(t, w) \in (\Pi_{1 \leqslant l \leqslant |\alpha|}\ t_l) O(|t|)$. From (3.9), one easily gets by induction on k the following lemma.

Lemma 3.5. *For $2 \leqslant k \leqslant \mathfrak{r}$, there exist maps*

$$\epsilon_k : \{1, \dots, b_k\} \to \{\pm 1\}, \quad \ell_k : \{1, \dots, b_k\} \to \{1, \dots, k\}, \quad j_k : \{1, \dots, b_k\} \to \{1, \dots, p\},$$

such that $\epsilon_k(1) = 1$, $\epsilon_k(b_k/2) = -1$, $\ell_k(1) = 1$, $\ell_k(b_k/2) = 1$, $\sharp \ell_k^{-1}(j) = 2^j$ for $j \leqslant k-1$, $\sharp \ell_k^{-1}(k) = 2^{k-1}$, $j_k(m) = \alpha_{\ell_k(m)}$, and such that, for all $t = (t_1, \dots, t_k)$, one has

$$\phi_\alpha(t) = \prod_{m=1}^{b_k} e^{\epsilon_k(m) t_{\ell_k(m)} Z_{j_k(m)}}. \tag{3.11}$$

Since g is non-negative, one has

$$\widetilde{T}_h^P g(w) \geqslant \frac{1}{p^P} \prod_{\alpha \in \mathcal{A}} \prod_{k=1}^{b_{|\alpha|}} T_{j_{|\alpha|}(k), h}\, g(w). \tag{3.12}$$

Therefore, we are reduced to proving that there exist $\epsilon, c > 0$ independent of h small and w near $o_{\mathcal{N}}$ such that the following inequality holds true.

$$h^{-P} \int_{[-h,h]^P} g\left(\prod_{\alpha \in \mathcal{A}} \prod_{k=1}^{b_{|\alpha|}} e^{t_{|\alpha|,k} Z_{j_{|\alpha|}(k)}} w \right) dt \geqslant ch^{-Q} \int_{z \in I_{\epsilon,h}} g(e^{\tilde{\lambda}(z)} w)\, dz. \tag{3.13}$$

Let $\Phi_w : \mathbb{R}^P \longrightarrow \mathcal{N}$ be the smooth map defined for $s = (s_{\alpha,k})_{\alpha\in\mathcal{A},k=1,\dots,b_{|\alpha|}} \in \mathbb{R}^P$ by the formula

$$\Phi_w(s) = \left(\prod_{r=1}^{\mathfrak{r}} \prod_{\alpha\in\mathcal{A}_r} \prod_{k=1}^{b_r} \mathrm{e}^{s_{\alpha,k} Z_{j_{|\alpha|}(k)}}\right) w. \tag{3.14}$$

Since $(Z^\beta(w))_{\beta\in\mathcal{A}}$ is a basis of $T_w\mathcal{N}$, $u = (u_\beta)_{\beta\in\mathcal{A}} \mapsto \mathrm{e}^{\sum\limits_{\beta\in\mathcal{A}} u_\beta Z^\beta} w$ is a local coordinate system centred at $w \in \mathcal{N}$, and therefore, there exist smooth functions $U_{\beta,w}(s)$ such that

$$\Phi_w(s) = \mathrm{e}^{\sum_{\beta\in\mathcal{A}} U_{\beta,w}(s) Z^\beta} w. \tag{3.15}$$

Moreover, it follows easily from the Campbell–Hausdorff formula, that one has $U_{\beta,w}(s) \in O(s^{|\beta|})$ near $s = 0$. Let now $\kappa : \mathbb{R}^Q \longrightarrow \mathbb{R}^P$ be the map defined by

$$(t_{\alpha,l})_{\alpha\in\mathcal{A},l\in\mathbb{N}_{|\alpha|}} \mapsto (\epsilon_\alpha(k) t_{\alpha,\ell_{|\alpha|}(k)})_{\alpha\in\mathcal{A},k=1,\dots,b_{|\alpha|}}. \tag{3.16}$$

Then, from Lemma 3.5, we have the following identity for any $t = (t_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}^Q$:

$$\Phi_w \circ \kappa(t) = \Pi_{\alpha\in\mathcal{A}} \phi_\alpha(t_\alpha) w. \tag{3.17}$$

From (3.10) and the Campbell–Hausdorff formula, one gets

$$\left.\begin{aligned} &\Pi_{\alpha\in\mathcal{A}} \phi_\alpha(t_\alpha) w = \mathrm{e}^{\sum_{\beta\in\mathcal{A}} f_\beta(t) Z^\beta} w, \\ &f_\beta(t) = \Pi_{1\leqslant l\leqslant|\beta|} t_{\beta,l} + g_\beta((t_\gamma)_{|\gamma|<|\beta|}) + r_\beta(t), \end{aligned}\right\} \tag{3.18}$$

with g_β a homogeneous polynomial of degree $|\beta|$ depending only on $(t_\gamma)_{|\gamma|<|\beta|}$ and $r_\beta(t) \in O(|t|^{|\beta|+1})$. Let $\delta \in]\frac{1}{2}, 1[$, and define $\xi = (\xi_{\alpha,k})_{\alpha\in\mathcal{A},k\in\mathbb{N}_{|\alpha|}} \in \mathbb{R}^Q$ by $\xi_{\alpha,1} = 0$ and $\xi_{\alpha,k} = \delta h$ for $k = 2, \dots, |\alpha|$. Let $\zeta : \mathbb{R}^D \longrightarrow \mathbb{R}^Q$ be the map defined by the formula

$$\left.\begin{aligned} &s = (s_\alpha)_{\alpha\in\mathcal{A}} \mapsto (\zeta_{\alpha,k}(s))_{\alpha\in\mathcal{A},k\in\mathbb{N}_{|\alpha|}}, \\ &\zeta_{\alpha,1}(s) = s_\alpha, \quad \text{and} \quad \zeta_{\alpha,k}(s) = 0 \quad \forall k \geqslant 2, \end{aligned}\right\} \tag{3.19}$$

and let $\sigma : \mathbb{R}^{P-D} \longrightarrow \mathbb{R}^P$ be the map defined by the formula

$$\left.\begin{aligned} &v = (v_{\alpha,k})_{\alpha\in\mathcal{A},k=2,\dots,b_{|\alpha|}} \mapsto (\sigma_{\alpha,k}(v))_{\alpha\in\mathcal{A},k=1,\dots,b_{|\alpha|}}, \\ &\sigma_{\alpha,1}(v) = 0, \quad \text{and} \quad \sigma_{\alpha,k}(v) = v_{\alpha,k} \quad \forall k \neq 1. \end{aligned}\right\} \tag{3.20}$$

Set $\hat{\kappa}_\xi(u, v) = \kappa(\zeta(u) + \xi) + \sigma(v)$, and let $\Psi_w : \mathbb{R}^D \times \mathbb{R}^{P-D} \to \mathcal{N}$ be defined by

$$\Psi_w(u, v) = \Phi_w(\hat{\kappa}_\xi(u, v)). \tag{3.21}$$

Then, it follows from (3.15) that there exist smooth maps $\hat{\varphi}_{\alpha,w}(u, v)$ such that

$$\Psi_w(u, v) = \mathrm{e}^{\sum_{\alpha\in\mathcal{A}} \hat{\varphi}_{\alpha,w}(u,v) Z^\alpha} w. \tag{3.22}$$

From (3.17), one has

$$\Psi_w(u,0)=\Phi_w(\kappa(\zeta(u)+\xi))=\Pi_{\alpha\in\mathcal{A}}\phi_\alpha(u_\alpha,\delta h,\dots,\delta h)w,$$

and therefore, from (3.18), we get, since $\hat\kappa_\xi(u,v)$ is linear in ξ,u,v,

$$\hat\varphi_{\alpha,w}(u,v)=u_\alpha(\delta h)^{|\alpha|-1}+g_{\alpha,w}((u_\gamma)_{|\gamma|<|\alpha|},\delta h)+p_{\alpha,w}(u,\delta h,v)+q_{\alpha,w}(u,\delta h,v),\tag{3.23}$$

where $g_{\alpha,w}(u,s)$ is a homogenous polynomial of degree $|\alpha|$ depending only on u_γ for $|\gamma|<|\alpha|$, $p_{\alpha,w}(u,s,v)$ is a homogenous polynomial of degree $|\alpha|$ in (u,s,v) such that $p_{\alpha,w}(u,s,0)=0$, and $q_{\alpha,w}(u,s,v)\in O((u,s,v)^{1+|\alpha|})$ near $(u,s,v)=(0,0,0)$. Moreover, from $\phi_\alpha(0,\delta h,\dots,\delta h)=\mathrm{Id}$, one gets $g_{\alpha,w}(0,s)=0$ and also $q_{\alpha,w}(0,s,0)=0$. Observe that w is just a smooth parameter in the above constructions. Thus, we will remove the dependence on w in what follows. Define now

$$\left.\begin{aligned}&\mathfrak{Q}:\mathbb{R}^P=\mathbb{R}^D\times\mathbb{R}^{P-D}\longrightarrow\mathbb{R}^P\\&(u,v)=((u_\alpha)_{\alpha\in\mathcal{A}},(v_{\alpha,k})_{\alpha\in\mathcal{A},k=2,\dots,b_{|\alpha|}})\mapsto((\hat\varphi_\alpha(u,v))_{\alpha\in\mathcal{A}},v),\end{aligned}\right\}\tag{3.24}$$

and, for $\eta,\epsilon>0$, let

$$\begin{aligned}\Delta_{\epsilon,\eta}=\{&(u,v)=((u_\alpha)_{\alpha\in\mathcal{A}},(v_{\alpha,k})_{\alpha\in\mathcal{A},k=2,\dots,b_{|\alpha|}})\in\mathbb{R}^P,\ |u_\alpha|<\epsilon h,\ \text{and}\\&|v_{\alpha,k}|<\eta h\text{ for all }\alpha,k\}.\end{aligned}$$

Lemma 3.6. *Let $\delta\in]\frac{1}{2},1[$ be fixed. There exist $0<\eta\ll\epsilon<1/2$ and $h_0>0$ such that the restriction $\mathfrak{Q}_{\epsilon,\eta}$ of $\mathfrak{Q}$ to $\Delta_{\epsilon,\eta}$ enjoys the following:*

1. *there exists $U_{\epsilon,\eta}$, open neighbourhood of $0\in\mathbb{R}^P$ such that $\mathfrak{Q}_{\epsilon,\eta}:\Delta_{\epsilon,\eta}\to U_{\epsilon,\eta}$ is a C^∞ diffeomorphism,*
2. *there exists some constant $C>0$ such that, for all $h\in]0,h_0]$ and all $(u,v)\in\Delta_{\epsilon,\eta}$,*
$$h^{Q-D}/C\leqslant \mathrm{J}\mathfrak{Q}_{\epsilon,\eta}(u,v):=|\det(D_{(u,v)}\mathfrak{Q}_{\epsilon,\eta})|\leqslant Ch^{Q-D},$$
3. *there exists $M\geqslant 1$ such that, for all $h\in]0,h_0]$, the set $U_{\epsilon,\eta}$ contains $I_{\epsilon/M,h}\times]-\eta h,\eta h[^{P-D}$, where $I_{\epsilon/M,h}=\prod_{\alpha\in\mathcal{A}}]-\epsilon h^{|\alpha|}/M,\epsilon h^{|\alpha|}/M[$.*

Proof. The proof is just a scaling argument. Set $u_\alpha=h\tilde u_\alpha$, $v_{\alpha,k}=h\tilde v_{\alpha,k}$, and $\hat\varphi_\alpha=h^{|\alpha|}z_\alpha$. Then the map $\mathfrak{Q}$ becomes after scaling $\tilde{\mathfrak{Q}}:(\tilde u,\tilde v)\mapsto(z,\tilde v)$, and from (3.23) one has

$$z_\alpha=\tilde u_\alpha\delta^{|\alpha|-1}+g_\alpha((\tilde u_\gamma)_{|\gamma|<|\alpha|},\delta)+p_\alpha(\tilde u,\delta,\tilde v)+h\tilde q_\alpha(\tilde u,\delta,\tilde v,h),$$

$p_\alpha(\tilde u,\delta,0)=0$, $\tilde q_\alpha(\tilde u,\delta,\tilde v,h)$ is smooth and vanishes at order $|\alpha|+1$ at 0 as a function of $(\tilde u,\delta,\tilde v)$, and $g_\alpha(0,\delta)=0$, $\tilde q_\alpha(0,\delta,0,h)=0$. From the triangular structure above, it is obvious that $\tilde{\mathfrak{Q}}$ is a smooth diffeomorphism at $0\in\mathbb{R}^P$, such that $\tilde{\mathfrak{Q}}(0)=0$. Thus, for $\eta\ll\epsilon$, $h\leqslant h_0$ small, and $M\gg1$, we get the inclusion $\{|z_\alpha|<\epsilon/M,|\tilde v_{\alpha,k}|<\eta\}\subset\tilde{\mathfrak{Q}}(\{|\tilde u_\alpha|<\epsilon,|\tilde v_{\alpha,k}|<\eta\})$. One has by construction $|\det(D_{(u,v)}\mathfrak{Q})|=h^{Q-D}|\det(D_{(\tilde u,\tilde v)}\tilde{\mathfrak{Q}})|$. The proof of Lemma 3.6 is complete. □

It is now easy to verify that (3.13) holds true. One has $\det D_{(u,v)}\hat{\kappa}_\xi = 1$ for all $(u, v) \in \mathbb{R}^P$, and for $\frac{1}{2} < \delta < 1$, and $0 < \eta \ll \epsilon < 1/2$, there exist some numbers $-1 < \alpha_i < \beta_i < 1$, $i = 1, \dots, P - D$ depending only on ϵ, η, δ and such that $\hat{\kappa}_\xi(\Delta_{\epsilon,\eta})$ is contained in the set $\widehat{\Delta}_{\epsilon,\eta} = \{(t, s),\ t \in [-\epsilon h, \epsilon h]^D, s \in \prod_{i=1}^{P-D}[\alpha_i h, \beta_i h]\}$. Using again the positivity of g and the change of variable $\hat{\kappa}$, we obtain, with a constant $c > 0$ changing from line to line,

$$\begin{aligned} h^{-P}\int_{[-h,h]^P} g(\Phi(t))\mathrm{d}t &\geqslant h^{-P}\int_{\widehat{\Delta}_{\epsilon,\eta}} g(\Phi(t))\mathrm{d}t \geqslant h^{-P}\int_{\hat{\kappa}_\xi(\Delta_{\epsilon,\eta})} g(\Phi(t))\mathrm{d}t \\ &\geqslant ch^{-P}\int_{\Delta_{\epsilon,\eta}} g(\Phi\circ\hat{\kappa}_\xi(u, v))\mathrm{d}u\mathrm{d}v = ch^{-P}\int_{\Delta_{\epsilon,\eta}} g(\Psi(u, v))\mathrm{d}u\mathrm{d}v. \end{aligned} \tag{3.25}$$

Thanks to Lemma 3.6, we can use the change of variable $\mathfrak{Q}_{\epsilon,\eta}$ to get

$$\begin{aligned} h^{-P}\int_{\Delta_{\epsilon,\eta}} g(\Psi(u, v))\mathrm{d}u\mathrm{d}v &\geqslant ch^{D-P-Q}\int_{U_{\epsilon,\eta}} g\left(\mathrm{e}^{\sum_{\alpha\in\mathcal{A}} z_\alpha Z^\alpha} w\right)\mathrm{d}z\mathrm{d}v \\ &\geqslant ch^{-Q}\int_{I_{\epsilon',\eta}} g(\mathrm{e}^{\sum_{\alpha\in\mathcal{A}} z_\alpha Z^\alpha} w)\mathrm{d}z = ch^{-Q}\int_{z\in I_{\epsilon',h}} g(\mathrm{e}^{\tilde{\lambda}(z)} w)\ \mathrm{d}z, \end{aligned} \tag{3.26}$$

with $\epsilon' = \epsilon/M$, and M is given by Lemma 3.6. The proof of Proposition 3.1 is complete.

4. Dirichlet form

Let $\mathcal{E}_h$ be the rescaled Dirichlet form associated with the Markov kernel T_h:

$$0 \leqslant \mathcal{E}_h(u) = \left(\frac{1 - T_h}{h^2} u | u\right)_{L^2}, \quad \forall u \in L^2(M, \mathrm{d}\mu). \tag{4.1}$$

The main result of this section is the following proposition.

Proposition 4.1. *Under the hypoelliptic hypothesis (1.2), there exist $C, h_0 > 0$ such that the following holds true for all $h \in]0, h_0]$: for all $u \in L^2(M, \mathrm{d}\mu)$ such that*

$$\|u\|^2_{L^2} + \mathcal{E}_h(u) \leqslant 1, \tag{4.2}$$

there exist $v_h \in \mathcal{H}^1(\mathcal{X})$ and $w_h \in L^2$ such that

$$u = v_h + w_h, \quad \|w_h\|_{L^2} \leqslant Ch, \quad \sup_{1\leqslant j\leqslant p} \|X_j v_h\|_{L^2} \leqslant C. \tag{4.3}$$

This proposition is easy to prove when the vector fields X_j span the tangent bundle at each point, by elementary Fourier analysis. Under the hypoelliptic hypothesis, the proof is more involved, and it will be done in several steps. In step 1, we reduce the problem to the construction of suitable operators acting on the Lie algebra $\mathcal{N}$ (see formula (4.11)). In step 2, we construct these operators in the special case of a system of left invariant vectors on $\mathcal{N}$. Finally, in step 3, this construction is extended to the general case.

Step 1: *Localization and reduction to the nilpotent Lie algebra.*

Let us first verify that, for all $\varphi \in C^\infty(M)$, there exists C_φ independent of $h \in]0, 1]$ such that

$$\mathcal{E}_h(\varphi u) \leqslant C_\varphi(\|u\|^2_{L^2} + \mathcal{E}_h(u)). \tag{4.4}$$

One has $1-T_h=\frac{1}{p}\sum_{k=1}^{p}(1-T_{k,h})$ and

$$2((1-T_{k,h})u|u)=\int_M \frac{1}{2h}\int_{-h}^{h}|u(x)-u(\mathrm{e}^{tX_k}x)|^2\mathrm{d}t\ \mathrm{d}\mu(x).$$

Since $\sup_{x\in M}|\varphi(x)-\varphi(\mathrm{e}^{tX_k}x)|\leqslant C|t|$, this implies that, for some constant C_φ and all k,

$$((1-T_{k,h})\varphi u|\varphi u)\leqslant C_\varphi(((1-T_{k,h})u|u)+h^2\|u\|^2_{L^2}),$$

and therefore (4.4) holds true. Thus, in the proof of Proposition 4.1, we may assume that $u\in L^2(M,\mathrm{d}\mu)$ is supported in a small neighbourhood of a given point $x_0\in M$ where Theorem 2.4 applies. More precisely, with the notation of § 2, we may assume in the coordinate system $\Lambda\theta$ centred at $x_0\simeq 0$ that u is supported in the closed ball $B^m_r=\{x\in\mathbb{R}^m,|x|\leqslant r\}\subset V_0$. Let $\chi(y)\in C_0^\infty(U_0)$ with support in $B^n_{r'}\subset U_0$, such that $\int\chi(y)\mathrm{d}y=1$. Set $g(x,y)=\chi(y)u(x)$. One has $g(x,y)=\chi(y)W_{x_0}(u)(x,y)$. By hypothesis, one has

$$\|u\|^2_{L^2}+\mathcal{E}_h(u)\leqslant 1,$$

which implies that, for all k,

$$2((1-T_{k,h})u|u)=\int_M \frac{1}{2h}\int_{-h}^{h}|u(x)-u(\mathrm{e}^{tX_k}x)|^2\mathrm{d}t\ \mathrm{d}\mu(x)\leqslant ph^2.$$

Thus, for any compact $K\subset U_0$, there exists C_K such that, for all k and $h\in]0,h_0]$, one has

$$\int_{V_0\times K}\frac{1}{2h}\int_{-h}^{h}|u(x)-u(\mathrm{e}^{tX_k}x)|^2\mathrm{d}t\ \mathrm{d}x\mathrm{d}y\leqslant C_Kh^2. \tag{4.5}$$

Here, h_0 is small enough so that $\mathrm{e}^{tX_k}x$ remains in V_0 for $|t|\leqslant h_0$ and $x\in B_r$. Let $\phi(x,y)=\chi(y)$. One has $\sup_{x,y}|\phi(x,y)-\phi(\mathrm{e}^{tZ_k}(x,y))|\leqslant C|t|$ and $\|g\|_{L^2}\leqslant C$. Thus, decreasing h_0, we get from (4.5) that there exists a constant C independent of k and $h\in]0,h_0]$ such that

$$\int_{V_0\times U_0}\frac{1}{2h}\int_{-h}^{h}|g(x,y)-g(\mathrm{e}^{tZ_k}(x,y))|^2\mathrm{d}t\ \mathrm{d}x\mathrm{d}y\leqslant Ch^2. \tag{4.6}$$

Therefore, there exists C_0 independent of $h\in]0,h_0]$ such that one has

$$\|g\|^2_{L^2(\mathcal{N})}+\sum_{j=1}^{p}h^{-2}\int_{V_0\times U_0}\frac{1}{2h}\int_{-h}^{h}|g(x,y)-g(\mathrm{e}^{tZ_k}(x,y))|^2\mathrm{d}t\ \mathrm{d}x\mathrm{d}y\leqslant C_0. \tag{4.7}$$

Lemma 4.2. *There exist $C_1,h_0>0$ such that, for all $h\in]0,h_0]$, any g with support in $B^m_r\times B^n_{r'}$ such that (4.7) holds true can be written in the form*

$$g=f_h+l_h,\quad \sum_{k=1}^{p}\|Z_kf_h\|_{L^2(V_0\times U_0)}\leqslant C_1,\quad \|l_h\|_{L^2(V_0\times U_0)}\leqslant C_1h.$$

Let us assume that Lemma 4.2 holds true. Then one can write $g = \chi(y)u(x) = f_h + l_h$. Let $\psi \in C_0^\infty(V_0 \times U_0)$ be equal to 1 near $B_r^m \times B_{r'}^n$. Set

$$v_h = \int \psi(x, y) f_h(x, y)\mathrm{d}y, \quad w_h = \int \psi(x, y) l_h(x, y)\mathrm{d}y.$$

One has $v_h + w_h = \int \psi(x, y)\chi(y)u(x)\mathrm{d}y = \int \chi(y)u(x)\mathrm{d}y = u(x)$ and $\|w_h\|_{L^2} \leqslant Ch$. Moreover, we get, from (2.6),

$$X_k(v_h) = \int \left(Z_k - \sum_l b_{k,l}(x, y) \frac{\partial}{\partial y_l} \right) \psi(x, y) f_h(x, y)\mathrm{d}y.$$

Since $f_h, Z_k(f_h) \in O_{L^2}(1)$ and $\int b \frac{\partial}{\partial y_l}(\psi f_h)\mathrm{d}y = -\int \frac{\partial}{\partial y_l}(b)\psi f_h \mathrm{d}y \in O_{L^2}(1)$, we get that (4.3) holds true. We are thus reduced to proving Lemma 4.2.

For any given k, the vector field Z_k is not singular; thus, decreasing V_0, U_0 if necessary, there exist coordinates $(z_1, \dots, z_D) = (z_1, z')$ such that $Z_k = \frac{\partial}{\partial z_1}$. Using a Fourier transform in z_1, we get that, if g satisfies (4.7), one has

$$2 \int \left(1 - \frac{\sin h\zeta_1}{h\zeta_1}\right) |\hat{g}(\zeta_1, z')|^2 \, \mathrm{d}\zeta_1 \mathrm{d}z' = \int \frac{1}{2h} \int_{-h}^{h} |1 - \mathrm{e}^{\mathrm{i}t\zeta_1}|^2 \mathrm{d}t |\hat{g}(\zeta_1, z')|^2 \, \mathrm{d}\zeta_1 \mathrm{d}z' \leqslant C_0' h^2. \tag{4.8}$$

Let $a > 0$ be small. There exists $c > 0$ such that $(1 - \frac{\sin h\zeta_1}{h\zeta_1}) \geqslant ch^2\zeta_1^2$ for $h|\zeta_1| \leqslant a$ and $(1 - \frac{\sin h\zeta_1}{h\zeta_1}) \geqslant c$ for $h|\zeta_1| > a$. Since

$$g(z_1, z') = \frac{1}{2\pi} \int_{h|\zeta_1| \leqslant a} \mathrm{e}^{\mathrm{i}z_1\zeta_1} \hat{g}(\zeta_1, z')\mathrm{d}\zeta_1 + \frac{1}{2\pi} \int_{h|\zeta_1| > a} \mathrm{e}^{\mathrm{i}z_1\zeta_1} \hat{g}(\zeta_1, z')\mathrm{d}\zeta_1 = v_{h,k} + w_{h,k},$$

we get from (4.8) that g satisfies, for some C_0 independent of $h \in]0, h_0]$,

$$\left.\begin{aligned} &\|g\|_{L^2(\mathcal{N})} \leqslant C_0, \quad \text{support}(g) \subset V_0 \times U_0 \\ &\forall k, \quad g = v_{h,k} + w_{h,k} \\ &\|Z_k v_{h,k}\|_{L^2(\mathcal{N})} \leqslant C_0, \quad \|w_{h,k}\|_{L^2(\mathcal{N})} \leqslant C_0 h, \end{aligned}\right\} \tag{4.9}$$

and we want to prove that the decomposition $g = v_{h,k} + w_{h,k}$ may be chosen independent of k, i.e., there exists $C > 0$ independent of h such that

$$\left.\begin{aligned} &g = v_h + w_h \\ &\forall k, \quad \|Z_k v_h\|_{L^2(\mathcal{N})} \leqslant C \\ &\|w_h\|_{L^2(\mathcal{N})} \leqslant Ch. \end{aligned}\right\} \tag{4.10}$$

In order to prove the implication (4.9) $\Rightarrow$ (4.10), we will construct operators Φ, C_j, $B_{k,j}$, R_l, depending on h, acting on L^2 functions with support in a small neighbourhood of $o_{\mathcal{N}}$ in $\mathcal{N}$, with values in $L^2(\mathcal{N})$, such that Φ, C_j, $B_{k,j}$, R_l, $C_j hZ_j$, $B_{k,j} hZ_k$ are uniformly in h bounded on L^2 and

$$\left.\begin{aligned} 1-\Phi &= \sum_{j=1}^{p} C_j hZ_j + hR_0 \\ Z_j \Phi &= \sum_{k=1}^{p} B_{k,j} Z_k + R_j \end{aligned}\right\}, \tag{4.11}$$

and then we set

$$v_h = \Phi(g), \quad w_h = (1-\Phi)(g).$$

With this decomposition of g, we get

$$w_h = \sum_{j=1}^{p} C_j hZ_j(v_{h,j} + w_{h,j}) + hR_0(g) \in O_{L^2}(h),$$

and

$$Z_k(v_h) = \sum_{j=1}^{p} B_{j,k} Z_j \left(v_{h,j} + h\frac{1}{h} w_{h,j} \right) + R_k(g) \in O_{L^2}(1).$$

We are thus reduced to proving the existence of the operators $\Phi, C_j, B_{k,j}, R_l$, with suitable bounds on L^2, and such that (4.11) holds true. This is a problem on the Lie algebra $\mathcal{N}$ with vector fields Z_j given by the Rothschild–Stein–Goodman theorem, Theorem 2.4. We will first do this construction in the special case where the vector fields Z_j are equal to the left invariant vector fields $\tilde{Y}_j$ on $\mathcal{N}$. In that special case, we will have $R_l = 0$ in formula (4.11). We will conclude in the general case by a suitable h-pseudodifferential calculus.

Step 2: *The case of left invariant vector fields on* $\mathcal{N}$.

Let $f * u$ be the convolution on $\mathcal{N}$,

$$f * u(x) = \int_{\mathcal{N}} f(x.y^{-1})u(y)\mathrm{d}y = \int_{\mathcal{N}} f(z)u(z^{-1}.x)\mathrm{d}z.$$

Here, $\mathrm{d}y$ is the left (and right) invariant Haar measure on $\mathcal{N}$, which is simply equal to the Lebesgue measure $\mathrm{d}y_1 \ldots \mathrm{d}y_{\mathfrak{r}}$ in the coordinates used in formula (2.3). Then, for $u \in L^1(\mathcal{N})$, the map $f \mapsto f * u$ is bounded on $L^q(\mathcal{N})$ by $\|u\|_{L^1}$ for any $q \in [1,\infty]$. The vector fields $\tilde{Y}_j$ are divergence free for the Haar measure $\mathrm{d}y$.

If f is a function on $\mathcal{N}$, and $a \in \mathcal{N}$, let $\tau_a(f)$ be the function defined by $\tau_a(f)(x) = f(a^{-1}.x)$. One has, for any $a \in \mathcal{N}$ and $Y \in T_e\mathcal{N} \simeq \mathcal{N}$, $\tau_a \tilde{Y} = \tilde{Y}\tau_a$, and the following formula holds true:

$$\left.\begin{aligned} \tau_a(f) &= \delta_a * f \\ \tilde{Y} f &= f * \tilde{Y}\delta_e. \end{aligned}\right\} \tag{4.12}$$

Let us denote by $\mathcal{T}_h$ the scaling operator $\mathcal{T}_h(f)(x) = h^{-Q} f(h^{-1}.x)$. One has $h.(x^{-1}) = (h.x)^{-1}$ and $\mathcal{T}_h(f * g) = \mathcal{T}_h(f) * \mathcal{T}_h(g)$. The action of $\mathcal{T}_h$ on the space $\mathcal{D}'(\mathcal{N})$ of distributions on $\mathcal{N}$, compatible with the action on functions, is given by $\langle \mathcal{T}_h(T), \phi\rangle = \langle T, x \mapsto \phi(h.x)\rangle$. Thus one has $\mathcal{T}_h\delta_e = \delta_e$ and $\mathcal{T}_h(\tilde{Y}_j(\delta_e)) = h\tilde{Y}_j(\delta_e)$ for $j \in \{1, \dots, p\}$.

Let $\mathcal{S}(\mathcal{N})$ be the Schwartz space on $\mathcal{N}$, and let $\varphi \in \mathcal{S}(\mathcal{N})$, with $\int_{\mathcal{N}} \varphi(x)\mathrm{d}x = 1$. For $h \in]0, 1]$, let Φ_h be the operator defined by

$$\Phi_h(f) = f * \varphi_h, \quad \varphi_h(x) = h^{-Q}\varphi(h^{-1}.x) = \mathcal{T}_h(\varphi). \tag{4.13}$$

Since the Jacobian of the transformation $x \mapsto h.x$ is equal to h^Q, one has $\|\varphi_h\|_{L^1} = \|\varphi\|_{L^1}$ for all $h \in]0, 1]$, and therefore the operator Φ_h is uniformly bounded on L^2.

If we define the operators $B_{k,j,h}$ by $B_{k,j,h}(f) = f * \mathcal{T}_h(\varphi_{k,j})$, with $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$, the equation

$$\tilde{Y}_j\Phi_h = \sum_{k=1}^{p} B_{k,j,h}\tilde{Y}_k$$

is equivalent to finding the $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$ such that

$$\tilde{Y}_j\varphi = \sum_{k=1}^{p} \tilde{Y}_k\delta_e * \varphi_{k,j}. \tag{4.14}$$

One has $\int_{\mathcal{N}} \tilde{Y}_j(\varphi)(x)\mathrm{d}x = 0$, and, since $f \mapsto \tilde{Y}_k\delta_e * f$ is the right invariant vector field $\mathcal{Z}_k$ on $\mathcal{N}$ such that $\mathcal{Z}_k(o_{\mathcal{N}}) = Y_k$, (4.14) is solvable, thanks to Lemma A.2 in the appendix. Moreover, the operators Φ_h, $B_{k,j,h}$, and $B_{k,j,h}h\tilde{Y}_k$ are uniformly in $h \in]0, 1]$ bounded on L^2 (one has $B_{k,j,h}(h\tilde{Y}_k(f)) = f * \mathcal{T}_h(\tilde{Y}_k(\delta_e) * \varphi_{k,j})$ and $\tilde{Y}_k(\delta_e) * \varphi_{k,j} \in \mathcal{S}(\mathcal{N})$).

Let now $c_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$ be Schwartz for $\|x\| \geqslant 1$, and quasi-homogeneous of degree $-Q+1$ near $o_{\mathcal{N}}$ (i.e., $c_j(t.x) = t^{-Q+1}c_j(x)$ for $0 < \|x\| \leqslant 1$ and $t > 0$ small). Let $C_{j,h}$ be the operators defined by $C_{j,h}(f) = f * \mathcal{T}_h(c_j)$. Then the equation $1 - \Phi_h = \sum_j C_{j,h}h\tilde{Y}_j$ is equivalent to

$$\delta_e - \varphi = \sum_j \tilde{Y}_j\delta_e * c_j. \tag{4.15}$$

In order to solve (4.15), we denote by $E \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$ the (unique) fundamental solution, quasi-homogeneous of degree $-Q+2$ on $\mathcal{N}$, of the hypoelliptic equation (for the existence of E, we refer to [8, Theorem 2.1, p. 172])

$$\delta_e = \sum_{j=1}^{p} \mathcal{Z}_j^2(E), \quad \mathcal{Z}_j(f) = \tilde{Y}_j\delta_e * f.$$

Let $\psi \in C_0^\infty(\mathcal{N})$ with $\psi(x) = 1$ near $e = o_{\mathcal{N}}$. We will choose c_j of the form

$$c_j = \psi\mathcal{Z}_j(E) - d_j, \quad d_j \in \mathcal{S}(\mathcal{N}). \tag{4.16}$$

Then equation (4.15) is equivalent to

$$\varphi + \sum_{j=1}^{p} [\mathcal{Z}_j, \psi]\mathcal{Z}_j(E) = \varphi_0 = \sum_{j=1}^{p} \mathcal{Z}_j(d_j). \tag{4.17}$$

One has $\varphi_0 \in \mathcal{S}(\mathcal{N})$ and $\int_{\mathcal{N}} \varphi_0(x)\mathrm{d}x = 0$, since $\int_{\mathcal{N}} \varphi(x)\mathrm{d}x = 1$ and $\int_{\mathcal{N}} \sum_{j=1}^{p} [\mathcal{Z}_j, \psi]\mathcal{Z}_j(E)\mathrm{d}x$ $= -\int_{\mathcal{N}} \sum_{j=1}^{p} \psi \mathcal{Z}_j^2(E)\mathrm{d}x = -1$. Thus, (4.14) is solvable thanks to Lemma A.2. Moreover, since $c_j \in L^1(\mathcal{N})$, the operators $C_{j,h}$ are uniformly in h bounded on L^2. It remains to verify that the operators $C_{j,h} h\tilde{Y}_j$ are uniformly in h bounded on L^2. One has $C_{j,h} h\tilde{Y}_j(f) = f * \mathcal{T}_h(\mathcal{Z}_j(c_j))$. Since $\|T_h(f)\|_{L^2} = h^{-Q/2}\|f\|_{L^2}$, it is equivalent to prove that the operator $g \mapsto g * \mathcal{Z}_j(c_j)$ is bounded on L^2. By construction, one has $\mathcal{Z}_j(c_j) = \psi \mathcal{Z}_j^2(E) + l_j$, $l_j \in \mathcal{S}(\mathcal{N})$. With the terminology of [8], the distribution $Z_j^2(E)$ is homogeneous of degree 0 (i.e., quasi-homogeneous of degree $-Q$), and thus of the form $\mathcal{Z}_j^2(E) = a_j\delta_e + f_j$, where $f_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$, quasi-homogeneous of degree $-Q$, and such that $\int_{b<|u|<b'} f_j(u)du = 0$. Thus, by [8, Proposition 1.9, p. 167], the operator $g \mapsto g * \mathcal{Z}_j(c_j)$ is bounded on L^2.

Step 3: *A suitable h-pseudodifferential calculus on* $\mathcal{N}$.

Let Z^α be the smooth vector fields defined in a neighbourhood Ω of $o_{\mathcal{N}}$ in $\mathcal{N}$ given by the Goodman theorem, Theorem 2.4. In this last step, we will finally construct the operators such that (4.11) holds true. We first recall the construction of the map $\Theta(a, b)$, which play a crucial role in the construction of a parametrix for hypoelliptic operators in [14]. Let us recall that $(Y_a^\alpha = H_\alpha(Y_1, \dots, Y_p) \in T_e\mathcal{N}, \alpha \in \mathcal{A})$ is a basis of $T_e\mathcal{N}$. For $a \in \mathcal{N}$ close to e and $u = \sum_{\alpha\in\mathcal{A}} u_\alpha Y^\alpha \in T_e\mathcal{N}$ close to 0, let $\Lambda(u) = \sum_{\alpha\in\mathcal{A}} u_\alpha Z^\alpha$ and

$$\Phi(a, u) = \mathrm{e}^{\Lambda(u)}a. \tag{4.18}$$

Clearly, $(a, u) \mapsto (a, \Phi(a, u))$ is a diffeomorphism of a neighbourhood of $(e, 0)$ in $\mathcal{N} \times T_e\mathcal{N}$ onto a neighbourhood of (e, e) in $\mathcal{N} \times \mathcal{N}$, and $\Phi(a, 0) = a$. We denote by $\Theta(a, b)$ the map defined in a neighbourhood of (e, e) in $\mathcal{N} \times \mathcal{N}$ into a neighbourhood of $o_{\mathcal{N}}$ in $\mathcal{N} \simeq T_e\mathcal{N}$ by

$$\Phi(a, \Theta(a, b)) = b. \tag{4.19}$$

For $b = \Phi(a, u)$, one has $\Phi(b, -u) = \mathrm{e}^{\Lambda(-u)}(\mathrm{e}^{\Lambda(u)}a) = \mathrm{e}^{-\Lambda(u)}(\mathrm{e}^{\Lambda(u)}a) = a$. Thus one has the symmetry relation

$$\Theta(a, b) = -\Theta(b, a) = \Theta(b, a)^{-1}. \tag{4.20}$$

Observe that, in the special case $Z_j = \tilde{Y}_j$, $\Lambda(u)$ is equal to the left invariant vector field on $\mathcal{N}$ such that $\Lambda(u)(o_{\mathcal{N}}) = u$, i.e., $\Lambda(u) = \tilde{u}$ and $\Phi(a, u) = \mathrm{e}^{\tilde{u}}a = a.u$, and this implies in that case that

$$\Theta(a, b) = a^{-1}.b. \tag{4.21}$$

Let $\varphi \in \mathcal{S}(\mathcal{N})$, with $\int_{\mathcal{N}} \varphi(x)\mathrm{d}x = 1$. By step 2, there exist functions $\varphi_{k,j} \in \mathcal{S}(\mathcal{N})$, and $c_j \in C^\infty(\mathcal{N} \setminus \{o_{\mathcal{N}}\})$, Schwartz for $\|x\| \geqslant 1$, quasi-homogeneous of degree $-Q + 1$ near $o_{\mathcal{N}}$, such that the following hold true:

$$\left.\begin{aligned} \tilde{Y}_j(\varphi) &= \sum_{k=1}^{p} \mathcal{Z}_k(\varphi_{k,j}), \\ \delta_e - \varphi &= \sum_j \mathcal{Z}_j(c_j). \end{aligned}\right\} \tag{4.22}$$

Let $\omega_0 \subset\subset \omega_1$ be small neighbourhoods of $o_{\mathcal{N}}$ such that $\Theta(y,x)$ is well defined for $(y,x) \in \omega_0 \times \omega_1$, and $\chi \in C_0^\infty(\omega_1)$ be equal to 1 in a neighbourhood of $\overline{\omega}_0$. We define the operators Φ_h, $B_{k,j,h}$, and $C_{j,h}$ for $1 \leqslant j,k \leqslant p$ by the formulas

$$\left.\begin{aligned} \Phi_h(f)(x) &= \chi(x)\ h^{-Q} \int_{\mathcal{N}} \varphi(h^{-1}.\Theta(y,x)) f(y) \mathrm{d}y \\ B_{k,j,h}(f)(x) &= \chi(x)\ h^{-Q} \int_{\mathcal{N}} \varphi_{k,j}(h^{-1}.\Theta(y,x)) f(y) \mathrm{d}y \\ C_{j,h}(f)(x) &= \chi(x)\ h^{-Q} \int_{\mathcal{N}} c_j(h^{-1}.\Theta(y,x)) f(y) \mathrm{d}y. \end{aligned}\right\} \tag{4.23}$$

All these operators are of the form

$$A_h(f)(x) = h^{-Q} \int_{\mathcal{N}} g(x, h^{-1}.\Theta(y,x)) f(y) \mathrm{d}y, \tag{4.24}$$

where the function $g(x,.)$ is smooth in x, with compact support ω_1, and takes values in $L^1(\mathcal{N})$, i.e., $\sup_{x\in\omega_1} \|\partial_x^\beta g(x,.)\|_{L^1(\mathcal{N})} < \infty$ for all β. The function $A_h(f)$ is well defined for $f \in L^\infty(\mathcal{N})$ such that support$(f) \subset \omega_0$. We have introduced the cutoff $\chi(x)$ just to have $A_h(f)(x)$ defined for all $x \in \mathcal{N}$, and one has $A_h(f)(x) = 0$ for all $x \notin \omega_1$.

Lemma 4.3. *Let $g(x,.)$ be smooth in x with compact support in ω_1, with values in $L^1(\mathcal{N})$. Then the operator A_h defined by (4.24) is uniformly in $h \in]0,1]$ bounded from $L^q(\omega_0)$ into $L^q(\mathcal{N})$ for all $q \in [1,\infty]$.*

Proof. The proof is standard. By interpolation, it is sufficient to treat the two cases $q = \infty$ and $q = 1$. When $q = \infty$, the Jacobian of the change of coordinates $y \mapsto u = \Theta(y,x)$ is bounded by C for all $x \in \omega_1$, $y \in \omega_0$. Thus we get

$$|A_h(f)(x)| \leqslant C\|f\|_{L^\infty(\omega_0)} h^{-Q} \int_{\mathcal{N}} |g(x, h^{-1}.u)| du = C\|f\|_{L^\infty(\omega_0)} \|g(x,.)\|_{L^1}.$$

Since $x \mapsto g(x,.)$ is smooth in x with values in $L^1(\mathcal{N})$, one has $C_\infty = \sup_{x\in\omega_1} \|g(x,.)\|_{L^1} < \infty$. Thus we get $\|A_h(f)\|_{L^\infty} \leqslant C C_\infty \|f\|_{L^\infty(\omega_0)}$.

For $q = 1$, we first extend g as a smooth L-periodic function of $x \in \mathcal{N}$, with L large enough, $g(x,u) = \sum_{k\in\mathbb{Z}^D} g_k(u) \mathrm{e}^{2i\pi k.x/L}$, the equality being valid for $x \in \omega_1$. Observe that $\|g_k\|_{L^1(\mathcal{N})}$ is rapidly decreasing in k. Then one has

$$A_h(f)(x) = \sum_k A_{h,k}(f)(x) \mathrm{e}^{ik.x/L}, \quad A_{h,k}(f)(x) = h^{-Q} \int_{\mathcal{N}} g_k(h^{-1}.\Theta(y,x)) f(y) \mathrm{d}y.$$

The Jacobian of the change of coordinates $(x, y) \mapsto (u = \Theta(y, x), y)$ is bounded by C for all $(x, y) \in \omega_1 \times \omega_0$, and one has

$$\int_{\omega_1} |A_{h,k}(f)(x)|\mathrm{d}x \leqslant Ch^{-Q} \int_{\mathcal{N}} \int_{\omega_0} |g_k(h^{-1}.u)||f(y)|\mathrm{d}y du = C\|f\|_{L^1}\|g_k\|_{L^1}.$$

Thus we get $\sup_{h\in]0,1]} \|A_{h,k}\|_{L^1} = d_k$ with d_k rapidly decreasing in k, and this implies that $\sup_{h\in]0,1]} \|A_h\|_{L^1} \leqslant \sum_k d_k < \infty$. The proof of Lemma 4.3 is complete. □

Observe that, in the special case $Z_j = \tilde{Y}_j$, using (4.21), we get that the operators $\Phi_h, B_{k,j,h}, C_{j,h}$ defined by formula (4.23) are precisely equal, up to the factor $\chi(x)$, to the operators we have constructed in step 2.

In the general case, it remains to show that the following assertion hold true.

(i) The operators $R_{l,h}$ defined by

$$\left.\begin{aligned} R_{0,h} &= h^{-1}\Big(1 - \Phi_h - \sum_{j=1}^{p} C_{j,h} h Z_j\Big) \\ R_{j,h} &= Z_j \Phi_h - \sum_{k=1}^{p} B_{k,j,h} Z_k, \quad 1 \leqslant j \leqslant p \end{aligned}\right\} \tag{4.25}$$

are uniformly bounded in $h \in]0, 1]$ on L^2.

(ii) The operators $C_{j,h} h Z_j$ and $B_{k,j,h} h Z_k, k > 0$ are uniformly bounded in $h \in]0, 1]$ on L^2.

For the verification of (i) and ii), we just follow the natural strategy which is developed in [14]. If f is a function defined near $a \in \mathcal{N}$, let $\Phi_a(f)$ be the function defined near 0 in $\mathcal{N} \simeq T_e\mathcal{N}$ by $\Phi_a(f)(u) = f(\Phi(a, u))$. The following fundamental lemma is proven in [14, Theorem 5] and also in [9] (§ 5, 'Estimation of the error').

Lemma 4.4. *For all $j \in \{1, \dots, p\}$, and $a \in \mathcal{N}$ near e, the vector field $V_{j,a}$ defined near 0 in $\mathcal{N}$,*

$$V_{j,a}(g) = \Phi_a(Z_j(\Phi_a^{-1} g)) - \tilde{Y}_j(g), \tag{4.26}$$

is of order $\leqslant 0$ at 0. If we introduce the system of coordinates $(u_\alpha) = (u_{l,k})$ with $l(\alpha) = |\alpha|$ and $1 \leqslant k \leqslant a_l = dim(\mathcal{N}_l)$, we thus have

$$V_{j,a} = \sum_{l=1}^{\mathfrak{r}} \sum_{k=1}^{a_l} v_{j,l,k}(a, u) \frac{\partial}{\partial u_{l,k}}, \tag{4.27}$$

where the functions $v_{j,l,k}(a, u)$ are smooth and satisfy $v_{j,l,k}(a, u) \in O(\|u\|^l)$.

Let us denote by $A_h[g]$ an operator of the form (4.24). Recall that $g(x, u)$ is smooth in x with compact support in ω_1, with values in $L^1(\mathcal{N})$. More precisely, we have two cases to consider: (a) g is Schwartz in u, and (b) g is smooth in u in $\mathcal{N} \setminus \{o_{\mathcal{N}}\}$, Schwartz for $\|u\| \geqslant 1$, and quasi-homogeneous of degree $-Q+1$ near $o_{\mathcal{N}}$. We have to compute the kernel of the operators $Z_j A_h[g]$ and $A_h[g]Z_j$.

We first compute the kernel of $Z_j A_h(g)$. For any fixed y, perform the change of coordinates $x = \Phi_y(u)$ so that $\Theta(y,x) = u$. Denote by Z_j^x the vector field Z_j acting on the variable x. Using Lemma 4.4, we get

$$\begin{aligned} Z_j(A_h[g](f))(x) &= h^{-Q}\int_{\mathcal{N}} Z_j^x(g(x, h^{-1}.\Theta(y,x)))f(y)\mathrm{d}y \\ &= h^{-Q}\int_{\mathcal{N}} h^{-1}(\tilde{Y}_j^u g)(x, h^{-1}.\Theta(y,x))f(y)\mathrm{d}y \\ &\quad + h^{-Q}\int_{\mathcal{N}} (Z_j^x g)(x, h^{-1}.\Theta(y,x))f(y)\mathrm{d}y \\ &\quad + \sum_{l=1}^{\mathfrak{r}}\sum_{k=1}^{a_l} h^{-Q}\int_{\mathcal{N}} v_{j,l,k}(y, \Theta(y,x))h^{-l} \\ &\quad \times \frac{\partial g}{\partial u_{l,k}}(x, h^{-1}.\Theta(y,x))f(y)\mathrm{d}y. \end{aligned} \tag{4.28}$$

By Lemma 4.3, the second term in (4.28) is uniformly bounded in $h \in]0,1]$, from $L^2(\omega_0)$ into $L^2(\mathcal{N})$. The same holds true for the third term. To see this point, following the proof of Lemma 4.3, first write $v_{j,l,k}(y,u) = \sum_n v_{j,l,k,n}(u)\mathrm{e}^{2\mathrm{i}\pi n.y/L}$, with $v_{j,l,k,n}(u)$ rapidly decreasing in n and $O(\|u\|^l)$ near $u = o_{\mathcal{N}}$. We are then reduced to showing that an operator of the form

$$R_h(f) = h^{-Q}\int_{\mathcal{N}} h^{-l} G(\Theta(y,x))\frac{\partial g}{\partial u_{l,k}}(x, h^{-1}.\Theta(y,x))f(y)\mathrm{d}y,$$

with $G(u)$ smooth and $G(u) \in O(\|u\|^l)$, is uniformly bounded in $h \in]0,1]$ from $L^2(\omega_0)$ into $L^2(\mathcal{N})$ by a constant which depends linearly on a finite number of derivatives of G. Clearly, there exists such a constant C such that $h^{-l}|G(\Theta(y,x))| \leqslant C\|h^{-1}.\Theta(y,x)\|^l$. Thus the result follows from the proof of Lemma 4.3, since $\|u\|^l \frac{\partial g}{\partial u_{l,k}}(x,u)$ is L^1 in u in both case (a) and case (b) (the vector field $\|u\|^l \frac{\partial}{\partial u_{l,k}}$ is of order 0).

If we denote by R_h any operator uniformly bounded on L^2, we have thus proven that

$$Z_j A_h[g] = h^{-1}A_h[\tilde{Y}_j^u g] + R_h. \tag{4.29}$$

Let us now compute the kernel of $A_h[g]Z_j$. The basic observation is the following identity (recall that $u^{-1} = -u$ and $\mathcal{Z}_j(f) = \tilde{Y}_j(\delta_e) * f$ is the right invariant vector field such that $\mathcal{Z}_j(0) = Y_j$):

$$-\tilde{Y}_j(f(-u)) = \mathcal{Z}_j(f)(-u). \tag{4.30}$$

Let l_j be the smooth function such that ${}^t Z_j = -Z_j + l_j$. For any given x, perform the change of coordinates $y = \Phi_x(u)$. By (4.20), one has $\Theta(y,x) = -\Theta(x,y) = -u$. We thus get from Lemma 4.4 and (4.30) the following formula:

$$A_h[g](Z_j(f))(x) = h^{-Q}\int_{\mathcal{N}} g(x, h^{-1}.\Theta(y,x))Z_j(f)(y)\mathrm{d}y$$

$$
\begin{aligned}
&= h^{-Q} \int_{\mathcal{N}} (-Z_j^y + l_j(y))(g(x, h^{-1}.\Theta(y,x))) f(y)\mathrm{d}y \\
&= h^{-Q} \int_{\mathcal{N}} h^{-1} (\mathcal{Z}_j^u g)(x, h^{-1}.\Theta(y,x)) f(y)\mathrm{d}y \\
&\quad + h^{-Q} \int_{\mathcal{N}} g(x, h^{-1}.\Theta(y,x)) l_j(y) f(y)\mathrm{d}y \\
&\quad + \sum_{l=1}^{\mathfrak{r}} \sum_{k=1}^{a_l} h^{-Q} \int_{\mathcal{N}} v_{j,l,k}(x, -\Theta(y,x)) h^{-l} \\
&\quad \times \frac{\partial g}{\partial u_{l,k}}(x, h^{-1}.\Theta(y,x)) f(y)\mathrm{d}y. \qquad (4.31)
\end{aligned}
$$

As above, this gives the identity, with R_h uniformly bounded on L^2,

$$A_h[g]Z_j = h^{-1} A_h[\mathcal{Z}_j^u g] + R_h. \qquad (4.32)$$

Observe that formulas (4.22), (4.29), and (4.32) imply that (4.25) holds true. Moreover, from (4.32) and Lemma 4.3, the operators $B_{k,j,h} h Z_k$, $k > 0$ are uniformly bounded in $h \in]0,1]$ on L^2. In order to get from (4.32) the same uniform bounds for the operators $C_{j,h} h Z_j$, we just observe that, in the case where $g(x,u)$ is quasi-homogeneous in u of degree $-Q+1$ near $o_{\mathcal{N}}$, one has $\mathcal{Z}_j^u g(x,u) = C_j(x)\delta_e + f_j(x,u)$ with $\int_{b<|u|<b'} f_j(x,u)\mathrm{d}u = 0$, and we conclude as at the end of step 2 by Proposition 1.9 of [8].

The proof of Proposition 4.1 is complete.

5. Proof of Theorems 1.1 and 1.2

This section is devoted to the proof of Theorems 1.1 and 1.2. Let $\mathcal{B}_h$ be the bilinear form associated with the rescaled Dirichlet form $\mathcal{E}_h$:

$$\mathcal{B}_h(f,g) = \left(\frac{1 - T_h}{h^2} f | g \right)_{L^2}, \qquad f, g \in L^2(M, \mathrm{d}\mu). \qquad (5.1)$$

Proposition 5.1. *Let $f \in \mathcal{H}^1(\mathcal{X})$. Let $(r_h, \gamma_h) \in \mathcal{H}^1(\mathcal{X}) \times L^2$ be such that the sequence (r_h) converges weakly (when $h \to 0$) in $\mathcal{H}^1(\mathcal{X})$ to $r \in \mathcal{H}^1(\mathcal{X})$, and $\sup_h \|\gamma_h\|_{L^2} < \infty$. Then*

$$\lim_{h\to 0} \mathcal{B}_h(f, r_h + h\gamma_h) = \frac{1}{6p} \sum_{k=1}^{p} (X_k f | X_k r)_{L^2}. \qquad (5.2)$$

Proof. Write $r_h = r + r_h'$. The weak limit of r_h' in $\mathcal{H}^1(\mathcal{X})$ is 0. Since $\mathcal{B}_h(f, r_h) = \mathcal{B}_h(f, r) + \mathcal{B}_h(f, r_h')$, we have to prove the following two assertions:

$$\lim_{h\to 0} \mathcal{B}_h(f, r) = \frac{1}{6p} \sum_{k=1}^{p} (X_k f | X_k r)_{L^2}, \quad \forall f, r \in \mathcal{H}^1(\mathcal{X}), \qquad (5.3)$$

and, under the hypothesis that the weak limit of r_h in $\mathcal{H}^1(\mathcal{X})$ is 0,

$$\lim_{h\to 0} \left(\frac{1 - T_{k,h}}{h^2} f | r_h + h\gamma_h \right)_{L^2} = 0, \quad \forall k \in \{1, \dots, p\}. \qquad (5.4)$$

In order to verify (5.4), since M is compact, we may assume that f is supported in a small neighbourhood of a point $x_0 \in M$ where the Goodman theorem, Theorem 2.4, applies. With the notation of § 2, we may thus assume in the coordinate system $\Lambda\theta$ centred at $x_0 \simeq 0$ that f, r_h, γ_h are supported in the closed ball $B_r^m = \{x \in \mathbb{R}^m, |x| \leqslant r\} \subset V_0$. Let $\chi(y) \in C_0^\infty(U_0)$ with support in $B_{r'}^n \subset U_0$, such that $\int \chi(y)\mathrm{d}y = 1$, and write $\mathrm{d}\mu(x) = \rho(x)\mathrm{d}x$ with ρ smooth. For $u, v \in L^2(M)$ supported in B_r^m, one has

$$(u|v)_{L^2} = \int_{V_0} u(x)\overline{v}(x)\mathrm{d}\mu(x) = \int_{V_0\times U_0} u(x)\overline{\rho(x)\chi(y)v(x)}\mathrm{d}x\mathrm{d}y.$$

Set $\tilde{f}(x, y) = W_{x_0}(f)(x, y) = f(x), \tilde{r}_h(x, y) = \rho(x)\chi(y)r_h(x), \tilde{\gamma}_h(x, y) = \rho(y)\chi(y)\gamma_h(x)$. We get, from (2.8),

$$\left(\frac{1 - T_{k,h}}{h^2} f | r_h + h\gamma_h\right)_{L^2} = \int_{V_0\times U_0} \left(\frac{1 - \tilde{T}_{k,h}}{h^2} \tilde{f}\right)\overline{\tilde{r}_h + h\tilde{\gamma}_h}\, \mathrm{d}x\mathrm{d}y. \tag{5.5}$$

Observe that $\tilde{\gamma}_h$ is bounded in $L^2(V_0 \times U_0)$. Since the injection $\mathcal{H}^1(\mathcal{X}) \subset L^2(M)$ is compact, r_h converges strongly to 0 in L^2, and therefore $\tilde{r}_h$ converges strongly to 0 in $L^2(V_0 \times U_0)$. Moreover, $Z_k(\tilde{r}_h)$ converges weakly to 0 in $L^2(V_0 \times U_0)$. Finally, since $\tilde{T}_{k,h}$ increases the support of at most $\simeq h$, we may replace $\tilde{f}$ by $F = \theta(y)\tilde{f}$ with $\theta \in C_0^\infty$ equal to 1 near the support of χ. Then F is compactly supported in $V_0 \times U_0$ and satisfies $F \in L^2$ and $Z_k F \in L^2$. Since the vector field Z_k is not singular, decreasing V_0, U_0 if necessary, there exist coordinates $(z_1, \dots, z_D) = (z_1, z')$ such that $Z_k = \frac{\partial}{\partial z_1}$. One has $\mathrm{d}x\mathrm{d}y = q(z)\mathrm{d}z$ with $q > 0$ smooth. Set $q\tilde{r}_h = R_h, q\tilde{\gamma}_h = Q_h$. Using a Fourier transform in z_1, it remains to show that

$$\left.\begin{aligned} &\lim_{h\to 0} I_h = 0, \quad I_h = h^{-2} \int \left(1 - \frac{\sin(h\xi_1)}{h\xi_1}\right) \hat{F}(\xi_1, z')\overline{\hat{R}_h(\xi_1, z')}\mathrm{d}\xi_1\mathrm{d}z' \\ &\lim_{h\to 0} J_h = 0, \quad J_h = h^{-1} \int \left(1 - \frac{\sin(h\xi_1)}{h\xi_1}\right) \hat{F}(\xi_1, z')\overline{\hat{Q}_h(\xi_1, z')}\mathrm{d}\xi_1\mathrm{d}z'. \end{aligned}\right\} \tag{5.6}$$

Recall that Q_h is bounded in L^2, R_h converges strongly to zero in L^2, $\partial_{z_1} R_h$ converges weakly to zero in L^2, and $F, \partial_{z_1} F \in L^2$. We write the first integral in (5.6) in the form

$$I_h = \int \psi(h\xi_1)\xi_1 \hat{F}(\xi_1, z')\overline{\xi_1 \hat{R}_h(\xi_1, z')}\mathrm{d}\xi_1\mathrm{d}z',$$

with $\psi(x) = x^{-2}(1 - \frac{\sin(x)}{x})$. One has $\psi \in C^\infty(\mathbb{R})$ and $|\psi(x)| \leqslant C\frac{1}{1+x^2}$. Then we write $I_h = I_{1,h} + I_{2,h}$ with $I_{1,h}$ defined by the integral over $|\xi_1| \leqslant M$ and $I_{2,h}$ defined by the integral over $|\xi_1| > M$. Since $\xi_1 \hat{R}_h(\xi_1, z')$ is bounded in L^2, and $\psi \in L^\infty$, we get, by the Cauchy–Schwarz inequality,

$$|I_{2,h}| \leqslant C \left(\int_{|\xi_1|>M} |\xi_1 \hat{F}(\xi_1, z')|^2 \mathrm{d}\xi_1\mathrm{d}z'\right)^{1/2} \to 0 \quad \text{when } M \to \infty.$$

On the other hand, one has $\psi(x) = \psi(0) + \tau(x)$ with $\psi(0) = 1/6$ and $\sup_{x\in\mathbb{R}} \tau(x)/x \leqslant C_0$. Thus we get

$$I_{1,h} = \frac{1}{6} \int_{|\xi_1|\leqslant M} \xi_1 \hat{F}(\xi_1, z')\overline{\xi_1 \hat{R}_h(\xi_1, z')}\mathrm{d}\xi_1\mathrm{d}z' + \int_{|\xi_1|\leqslant M} \tau(h\xi_1)\xi_1 \hat{F}(\xi_1, z')\overline{\xi_1 \hat{R}_h(\xi_1, z')}\mathrm{d}\xi_1\mathrm{d}z'. \tag{5.7}$$

For any fixed M, the first term in (5.7) goes to 0 when $h \to 0$ since $\xi_1 \hat{R}_h(\xi_1, z')$ converges weakly to 0 in L^2 and $\xi_1 \hat{F}(\xi_1, z') \in L^2$. Since $\xi_1 \hat{R}_h(\xi_1, z')$ is bounded in L^2 by say A, by the Cauchy–Schwarz inequality, the second term is bounded by $C_0 hMA \|\partial_{z_1} F\|_{L^2}$. Thus one has $\lim_{h\to 0} I_h = 0$.

We proceed exactly in the same way to prove that $\lim_{h\to 0} J_h = 0$: one has, with $x\psi = \phi$,

$$J_h = \int \phi(h\xi_1)\xi_1 \hat{F}(\xi_1, z')\overline{\hat{Q}_h(\xi_1, z')}\mathrm{d}\xi_1 \mathrm{d}z',$$

and we use the fact that $\phi \in L^\infty$, $\hat{Q}_h(\xi_1, z')$ is bounded in L^2, $\phi(0) = 0$, and $\phi(x)/x \in L^\infty(\mathbb{R})$.

Let us now verify (5.3). From (1.10), this is obvious if f is smooth and $r \in \mathcal{H}^1(\mathcal{X})$. Standard smoothing arguments show that $C^\infty(M)$ is dense in $\mathcal{H}^1(\mathcal{X})$. Let now $f \in \mathcal{H}^1(\mathcal{X})$, and choose $f_h \in C^\infty(M)$ converging strongly to f in $\mathcal{H}^1(\mathcal{X})$. Then $\lim_{h\to 0}(X_k f_h|X_k r)_{L^2} = (X_k f|X_k r)_{L^2}$, and from (5.4) one has also $\lim_{h\to 0} \mathcal{B}_h(f_h, r) = \lim_{h\to 0} \mathcal{B}_h(r, f_h) = \mathcal{B}_h(f, r)$.

The proof of Proposition 5.1 is complete. □

5.1. Proof of Theorem 1.1

Let $|\triangle_h|$ be the rescaled (non-negative) Laplacian associated with the Markov kernel T_h:

$$|\triangle_h| = \frac{1 - T_h}{h^2}. \tag{5.8}$$

From Proposition 4.1 and Lemma A.1, there exist $h_0 > 0$ and $C_4, C_5 > 0$ independent of $h \in]0, h_0]$, such that $\mathrm{Spec}(|\triangle_h|) \cap [0, \lambda]$ is discrete for all $\lambda \leqslant C_4 h^{-2}$, and one has the Weyl-type estimate

$$\#(\mathrm{Spec}(|\triangle_h|) \cap [0, \lambda]) \leqslant C_5 \langle\lambda\rangle^{\dim(M)/2s}, \quad \forall \lambda \leqslant C_4 h^{-2}. \tag{5.9}$$

In particular, since $T_h(1) = 1$, 1 is an isolated eigenvalue of T_h. Let us verify that 1 is a simple eigenvalue of T_h. Let $f \in L^2 = L^2(M, \mathrm{d}\mu)$ such that $T_h(f) = f$. One has, for any $g \in L^2$,

$$((1 - T_h)g|g)_{L^2} = \frac{1}{2}\iint |g(x) - g(y)|^2 t_h(x, \mathrm{d}y)\mathrm{d}\mu(x). \tag{5.10}$$

Thus we get, for all $k \in \{1, \ldots, p\}$,

$$\int_M \int_{-h}^{h} |f(x) - f(\mathrm{e}^{tX_k}x)|^2 \,\mathrm{d}t\mathrm{d}\mu(x) = 0.$$

This gives $f(x) - f(\mathrm{e}^{tX_k}x) = 0$ for almost all $(x, t) \in M \times]-h, h[$. Therefore, one has $X_k f = 0$ in $\mathcal{D}'(M)$ for all k, and this implies that $f = Cte$ thanks to the Hörmander and Chow theorems. We can also give a more direct argument: we have $T_h^P(f) = f$, and therefore if we use (5.10) with the Markov kernel T_h^P and Proposition 3.1, we get

$$\int_M \int_{u\in I_{\epsilon,h}} |f(x) - f(\mathrm{e}^{\lambda(u)}x)|^2 \,\mathrm{d}u\mathrm{d}\mu(x) = 0.$$

Since $u \mapsto e^{\lambda(u)}x$ is a submersion, this implies that $f(x) - f(y) = 0$ for almost all (x, y) in a neighbourhood of the diagonal in $M \times M$, and therefore $f = Cte$.

Let us now verify that there exists $\delta_1 > 0$ such that, for all $h \in]0, h_0]$, the spectrum of T_h is a subset of $[-1 + \delta_1, 1]$. It is sufficient to prove that the same holds true for an odd power T_h^{2N+1} of T_h. We are thus reduced to proving the existence of $h_0, C_0 > 0$ such that the following inequality holds true for all $h \in]0, h_0]$ and all $f \in L^2(\Omega)$:

$$(f + T_h^{2N+1} f | f)_{L^2} = \frac{1}{2} \int_{M\times M} t_h^{2N+1}(x, dy)|f(x) + f(y)|^2 d\mu(x) \geqslant C_0 \|f\|_{L^2}^2. \tag{5.11}$$

Take N large enough such that Proposition 3.1 applies for T_h^{2N+1}, i.e., $t_h^{2N+1}(x, dy) \geqslant cS_h^{\epsilon}(x, dy)$. Then we are reduced to proving the existence of C independent of h such that

$$\int_{M\times M} S_h^{\epsilon}(x, dy)|f(x) + f(y)|^2 d\mu(x) \geqslant C\|f\|_{L^2}^2. \tag{5.12}$$

From definition (3.1) of S_h^{ϵ}, we get

$$\int_{M\times M} S_h^{\epsilon}(x, dy)|f(x) + f(y)|^2 d\mu(x) = \int_M h^{-Q} \int_{u\in I_{\epsilon,h}} |f(x) + f(e^{\lambda(u)}x)|^2 du d\mu(x) = B.$$

Define A by the formula

$$A = \int_M h^{-2Q} \int_{u\in I_{\epsilon/2,h}} \int_{v\in I_{\epsilon/2,h}} |f(e^{\lambda(v)}y) + f(e^{\lambda(u)}y)|^2 du dv d\mu(y).$$

Since $\lambda(v)$ is divergence free as a linear combination with constant coefficients of commutators of the vector fields X_k, the change of variables $e^{\lambda(v)}y = x$ gives

$$A = \int_M h^{-2Q} \int_{u\in I_{\epsilon/2,h}} \int_{v\in I_{\epsilon/2,h}} |f(x) + f(e^{\lambda(u-v)}x)|^2 du dv d\mu(x).$$

Therefore, one has, for some constant $c_\epsilon > 0$ independent of h, $B \geqslant c_\epsilon A$. Clearly, one has

$$\int_M Re\Big(\int_{u\in I_{\epsilon/2,h}} \int_{v\in I_{\epsilon/2,h}} f(e^{\lambda(v)}y)\overline{f}(e^{\lambda(u)}y) du dv\Big) d\mu(y) \geqslant 0,$$

and this implies, still using the change of variables $e^{\lambda(v)}y = x$, that

$$\begin{aligned} A &\geqslant 2 \int_M h^{-2Q} \int_{u\in I_{\epsilon/2,h}} \int_{v\in I_{\epsilon/2,h}} |f(e^{\lambda(v)}y)|^2 du dv d\mu(y) \\ &= 2\epsilon^D \int_M h^{-Q} \int_{v\in I_{\epsilon/2,h}} |f(e^{\lambda(v)}y)|^2 dv d\mu(y) = 2\epsilon^{2D} \int_M |f(x)|^2 d\mu(x). \end{aligned} \tag{5.13}$$

From (5.13) and $B \geqslant c_\epsilon A$, we get that (5.12) holds true.

Lemma 5.2. *There exist $C_2, C_3 > 0$ such that the spectral gap of T_h satisfies*

$$C_2 h^2 \leqslant g(h) \leqslant C_3 h^2. \tag{5.14}$$

Proof. The right inequality in (5.14) is an obvious consequence of the min–max principle, since for any $f \in C^\infty(M)$ one has $\lim_{h\to 0} \frac{1-T_h}{h^2} f = L(f)$. From (5.9), we get that, for any $a \in]0,1]$, $m_a = \sharp(\mathrm{Spec}(T_h) \cap [1-ah^2, 1[)$ is bounded by a constant independent of h small, and we have to verify that there exist $h_0 > 0$ and $a > 0$ independent of $h \in]0, h_0]$ such that $m_a = 0$. If this is not true, there exist two sequences $\epsilon_n, h_n \to 0$ and a sequence $f_n \in L^2$, with $\|f_n\|_{L^2} = 1$ and $(f_n|1)_{L^2} = \int_M f_n \mathrm{d}\mu = 0$ such that

$$T_{h_n} f_n = (1 - h_n^2 \epsilon_n) f_n.$$

This implies that $\mathcal{E}_{h_n}(f_n) = \epsilon_n$. Using Proposition 4.1, we get $f_n = v_n + h_n \gamma_n$ with $\sup_n \|\gamma_n\|_{L^2} < \infty$ and $\|v_n\|_{\mathcal{H}^1(\mathcal{X})} \leqslant C$. The hypoelliptic theorem of Hörmander implies the existence of $s > 0$ such that one has $\mathcal{H}^1(\mathcal{X}) \subset H^s(M)$; hence the injection $\mathcal{H}^1(\mathcal{X}) \subset L^2(M)$ is compact. As a direct byproduct, we get (up to extraction of a subsequence) that the sequence f_n converges strongly in L^2 to some $f \in \mathcal{H}^1(\mathcal{X})$, and v_n converges weakly in $\mathcal{H}^1(\mathcal{X})$ to f. Set $v_n = f + r_n$. Then r_n converges weakly to 0 in $\mathcal{H}^1(\mathcal{X})$, $f_n = f + r_n + h_n \gamma_n$, and one has

$$\mathcal{E}_{h_n}(f_n) = \mathcal{E}_{h_n}(f) + 2Re(\mathcal{B}_{h_n}(f, r_n + h\gamma_n)) + \mathcal{E}_{h_n}(r_n + h_n\gamma_n).$$

Since one has $\mathcal{E}_h(.) \geqslant 0$, Proposition 5.1 implies that

$$\frac{1}{6p} \sum_{k=1}^{p} \|X_k f\|_{L^2}^2 = \lim_{n\to\infty} \mathcal{E}_{h_n}(f) \leqslant \liminf_{n\to\infty} \mathcal{E}_{h_n}(f_n) = 0, \tag{5.15}$$

and therefore $f = Cte$. But since f_n converges strongly in L^2 to f, one has $\|f\|_{L^2} = 1$ and $(f|1)_{L^2} = \int_M f \mathrm{d}\mu = 0$. This is a contradiction. The proof of Lemma 5.2 is complete. □

To conclude the proof of Theorem 1.1, it remains to prove the total variation estimate (1.7). Let Π_0 be the orthogonal projector in $L^2(M, \mathrm{d}\mu)$ onto the space of constant functions

$$\Pi_0(f)(x) = \int_M f \mathrm{d}\mu. \tag{5.16}$$

Then

$$2 \sup_{x\in M} \|t_h^n(x, \mathrm{d}y) - \mu\|_{TV} = \|T_h^n - \Pi_0\|_{L^\infty \to L^\infty}. \tag{5.17}$$

Thus, we have to prove that there exist C_0, h_0, such that, for any n and any $h \in]0, h_0]$, one has

$$\|T_h^n - \Pi_0\|_{L^\infty \to L^\infty} \leqslant C_0 \mathrm{e}^{-ng(h)}. \tag{5.18}$$

Observe that, since $g(h) \simeq h^2$, and $\|T_h^n - \Pi_0\|_{L^\infty \to L^\infty} \leqslant 2$, in the proof of (5.18), we may assume that $n \geqslant Ch^{-2}$ with C large. Let $E_{h,L}$ be the (finite-dimensional) subspace of $L^2(M, \mathrm{d}\mu)$ spanned by the eigenvectors $e_{j,h}$ of $|\Delta_h|$, associated with eigenvalues $\lambda_{j,h} \leqslant C_4 h^{-2}$, with $C_4 > 0$ small enough. Here, the subscript L means 'low frequencies'. Recall from (5.9) that $\dim(E_{h,L}) \leqslant Ch^{-\dim(M)/2s}$. We will denote by J_h the set of indices

$$J_h = \{j, \lambda_{j,h} \leqslant C_4 h^{-2}\}. \tag{5.19}$$

Lemma 5.3. *There exist $p > 2$ and C independent of $h \in]0, h_0]$ such that, for all $u \in E_{h,L}$, the following inequality holds true:*

$$\|u\|^2_{L^p(M)} \leqslant C(\mathcal{E}_h(u) + \|u\|^2_{L^2}). \tag{5.20}$$

Proof. We denote by $C > 0$ a constant independent of h, changing from line to line. Let $u \in E_{h,L}$ such that $\mathcal{E}_h(u) + \|u\|^2_{L^2} \leqslant 1$. From Proposition 4.1, one has $u = v_h + w_h$ with $\|v_h\|_{\mathcal{H}^1(\mathcal{X})} \leqslant C$ and $\|w_h\|_{L^2} \leqslant Ch$. From the continuous imbedding $\mathcal{H}^1(\mathcal{X}) \subset H^s(M) \subset L^q(M)$ with $s > 0, q > 2, s = \dim(M)(1/2 - 1/q)$, we get

$$\|v_h\|_{L^q} \leqslant C.$$

One has $u = \sum_{\lambda_{j,h} \leqslant C_4 h^{-2}} z_{j,h} e_{j,h}$ with $\sum_{\lambda_{j,h} \leqslant C_4 h^{-2}} |z_{j,h}|^2 \leqslant 1$. From Corollary 3.4, one has, for $C_4 > 0$ small enough, $\|e_{j,h}\|_{L^\infty} \leqslant Ch^{-Q/2}$. Therefore, by the Cauchy–Schwarz inequality, we get

$$\|u\|_{L^\infty} \leqslant Ch^{-Q/2} \left(\sum_{\lambda_{j,h} \leqslant C_4 h^{-2}} |z_{j,h}|^2 \right)^{1/2} (\dim(E_{h,L}))^{1/2} \leqslant Ch^{-Q/2 - \dim(M)/4s}. \tag{5.21}$$

From the proof of Proposition 4.1 (see Lemma 4.3), one has $\|v_h\|_{L^\infty} \leqslant C\|u\|_{L^\infty}$. Thus we get $\|w_h\|_{L^\infty} \leqslant \|u\|_{L^\infty} + \|v_h\|_{L^\infty} \leqslant Ch^{-Q/2 - \dim(M)/4s}$. Since $\|w_h\|_{L^2} \leqslant Ch$, we get by interpolation that there exists $q' > 2$ such that

$$\|w_h\|_{L^{q'}} \leqslant C.$$

Then (5.20) holds true with $p = \min(q, q') > 2$. The proof of Lemma 5.3 is complete. □

We are now ready to prove (5.18), essentially following the strategy of [5], but with some simplifications. We split T_h in two pieces, according to spectral theory. We write $T_h - \Pi_0 = T_{h,1} + T_{h,2}$, with

$$T_{h,1}(x, y) = \sum_{\lambda_{1,h} \leqslant \lambda_{j,h} \leqslant C_4 h^{-2}} (1 - h^2 \lambda_{j,h}) e_{j,h}(x) e_{j,h}(y). \tag{5.22}$$

One has $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$, and we will get the bound (5.18) for each of the two terms. We start with very rough bounds. From $\|e_{j,h}\|_{L^\infty} \leqslant Ch^{-Q/2}$, $|(1 - h^2 \lambda_{j,h})| \leqslant 1$, we get, with $A = Q/2 + \dim(M)/4s$, as in the proof of (5.21), with C independent of $n \geqslant 1$ and h,

$$\|T_{h,1}^n\|_{L^\infty \to L^\infty} \leqslant \|T_{h,1}^n\|_{L^2 \to L^\infty} \leqslant Ch^{-A}. \tag{5.23}$$

Since T_h^n is bounded by 1 on L^∞, we get, from $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$,

$$\|T_{h,2}^n\|_{L^\infty \to L^\infty} \leqslant Ch^{-A}. \tag{5.24}$$

Let P be the integer defined at the beginning of § 3. Let M_h be the Markov operator $M_h = T_h^P$. Write $n = kP + r$, with $0 \leqslant r < P$. From Proposition 3.1 and Corollary 3.3, one has $M_h = \rho_h + R_h$, with

$$\left.\begin{aligned} &\|\rho_h\|_{L^\infty \to L^\infty} \leqslant \gamma < 1, \\ &\|R_h\|_{L^2 \to L^\infty} \leqslant C_0 h^{-Q/2}. \end{aligned}\right\} \tag{5.25}$$

From this, we deduce that, for any $k = 1, 2, \dots$, one has $M_h^k = A_{k,h} + B_{k,h}$, with $A_{1,h} = \rho_h$, $B_{1,h} = R_h$, and the recurrence relation $A_{k+1,h} = \rho_h A_{k,h}$, $B_{k+1,h} = \rho_h B_{k,h} + R_h M_h^k$. Thus one gets, since M_h^k is bounded by 1 on L^2,

$$\left.\begin{aligned} &\|A_{k,h}\|_{L^\infty\to L^\infty} \leqslant \gamma^k, \\ &\|B_{k,h}\|_{L^2\to L^\infty} \leqslant C_0 h^{-Q/2}(1+\gamma+\dots+\gamma^k) \leqslant C_0 h^{-Q/2}/(1-\gamma). \end{aligned}\right\} \tag{5.26}$$

Let $\theta = 1 - C_4 < 1$ so that $\|T_{h,2}\|_{L^2\to L^2} \leqslant \theta$. Then one has

$$\|T_{h,2}^n\|_{L^\infty\to L^2} \leqslant \|T_{h,2}^n\|_{L^2\to L^2} \leqslant \theta^n. \tag{5.27}$$

For $m \geqslant 1$, $k \geqslant 1$, and $0 \leqslant r < P-1$, one gets, using the fact that T_h is bounded by 1 on L^∞, and (5.24), (5.26), and (5.27),

$$\begin{aligned} \|T_{h,2}^{kP+r+m}\|_{L^\infty\to L^\infty} = \|T_h^r M_h^k T_{h,2}^m\|_{L^\infty\to L^\infty} &\leqslant \|M_h^k T_{h,2}^m\|_{L^\infty\to L^\infty} \\ &\leqslant \|A_{k,h} T_{h,2}^m\|_{L^\infty\to L^\infty} + \|B_{k,h} T_{h,2}^m\|_{L^\infty\to L^\infty} \\ &\leqslant C h^{-A}\gamma^k + C_0 h^{-Q/2}\theta^m/(1-\gamma). \end{aligned} \tag{5.28}$$

Thus we get that there exist $C > 0$, $\mu > 0$, and a large constant $B \gg 1$, such that

$$\|T_{h,2}^n\|_{L^\infty\to L^\infty} \leqslant C\mathrm{e}^{-\mu n}, \quad \forall h,\ \forall n \geqslant B\log(1/h), \tag{5.29}$$

and thus the contribution of $T_{h,2}^n$ is far smaller than the bound we have to prove in (5.18). It remains to study the contribution of $T_{h,1}^n$.

From Lemma 5.3, using the interpolation inequality $\|u\|_{L^2}^2 \leqslant \|u\|_{L^p}^{\frac{p}{p-1}} \|u\|_{L^1}^{\frac{p-2}{p-1}}$, we deduce the Nash inequality, with $1/d = 2 - 4/p > 0$:

$$\|u\|_{L^2}^{2+1/d} \leqslant C(\mathcal{E}_h(u) + \|u\|_{L^2}^2)\|u\|_{L^1}^{1/d}, \quad \forall u \in E_{h,L}. \tag{5.30}$$

For $\lambda_{j,h} \leqslant C_4 h^{-2}$, one has $h^2\lambda_{j,h} \leqslant 1$, and thus, for any $u \in E_{h,L}$, one gets $\mathcal{E}_h(u) \leqslant \|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2$, and thus we get, from (5.30),

$$\|u\|_{L^2}^{2+1/d} \leqslant C h^{-2}(\|u\|_{L^2}^2 - \|T_h u\|_{L^2}^2 + h^2\|u\|_{L^2}^2)\|u\|_{L^1}^{1/d}, \quad \forall u \in E_{h,L}. \tag{5.31}$$

From (5.29) and $T_h^n - \Pi_0 = T_{h,1}^n + T_{h,2}^n$, we get that there exists C_2 such that, for all h and all $n \geqslant B\log(1/h)$, one has $\|T_{1,h}^n\|_{L^\infty\to L^\infty} \leqslant C_2$, and thus, since $T_{1,h}$ is self-adjoint on L^2, $\|T_{1,h}^n\|_{L^1\to L^1} \leqslant C_2$. Fix $p \simeq B\log(1/h)$. Take $g \in L^2$ such that $\|g\|_{L^1} \leqslant 1$, and consider the sequence $c_n, n \geqslant 0$ defined by

$$c_n = \|T_{h,1}^{n+p} g\|_{L^2}^2. \tag{5.32}$$

Then, $0 \leqslant c_{n+1} \leqslant c_n$, and, from (5.31) and $T_{h,1}^{n+p} g \in E_{h,L}$, we get

$$\begin{aligned} c_n^{1+\frac{1}{2d}} &\leqslant C h^{-2}(c_n - c_{n+1} + h^2 c_n)\|T_{h,1}^{n+p} g\|_{L^1}^{1/d} \\ &\leqslant C C_2^{1/d} h^{-2}(c_n - c_{n+1} + h^2 c_n). \end{aligned} \tag{5.33}$$

Thus there exists A which depends only on C, C_2, d, such that, for all $0 \leqslant n \leqslant h^{-2}$, one has $c_n \leqslant (\frac{Ah^{-2}}{1+n})^{2d}$ (this is the key point in the argument; for a proof of this estimate, see [7]). Thus, for all $0 \leqslant n \leqslant h^{-2}$, and with $p \simeq B\log(1/h)$, one has

$$\|T_{h,1}^{n+p} g\|_{L^2} \leqslant \left(\frac{Ah^{-2}}{1+n}\right)^d \|g\|_{L^1}, \tag{5.34}$$

and since $T_{1,h}$ is self-adjoint on L^2, we get, by duality,

$$\|T_{h,1}^{n+p} g\|_{L^\infty} \leqslant \left(\frac{Ah^{-2}}{1+n}\right)^d \|g\|_{L^2}. \tag{5.35}$$

Thus there exists C_0 such that, for $N \simeq h^{-2}$, one has

$$\|T_{h,1}^{N+p} g\|_{L^\infty} \leqslant C_0 \|g\|_{L^2}, \tag{5.36}$$

and so we get, for any $m \geqslant 0$, and with $N \simeq h^{-2}$,

$$\|T_{h,1}^{N+p+m} g\|_{L^\infty} \leqslant C_0 (1 - h^2 \lambda_{1,h})^m \|g\|_{L^2}. \tag{5.37}$$

Thus, for $n \geqslant h^{-2} + N + p$, since $h^2 \lambda_{1,h} = g(h)$ and $0 \leqslant (1-r)^m \leqslant \mathrm{e}^{-mr}$ for $r \in [0, 1]$, we get

$$\|T_{h,1}^{n}\|_{L^\infty \to L^\infty} \leqslant C_0 \mathrm{e}^{-(n-(N+p))g(h)} = C_0 \mathrm{e}^{(N+p)g(h)} \mathrm{e}^{-ng(h)} \leqslant C_0' \mathrm{e}^{-ng(h)}. \tag{5.38}$$

The proof of Theorem 1.1 is complete.

5.2. Proof of Theorem 1.2

The proof of Theorem 1.2 is exactly the same that the one given in [6]. Let $R > 0$ be fixed. If $\nu_h \in [0, R]$ and $u_h \in L^2(M)$ satisfy $|\Delta_h| u_h = \nu_h u_h$ and $\|u_h\|_{L^2} = 1$, then, thanks to Proposition 4.1, u_h can be decomposed as $u_h = v_h + w_h$ with $\|w_h\|_{L^2} = O(h)$ and v_h bounded in $\mathcal{H}^1(\mathcal{X})$. Hence (extracting a subsequence if necessary) it may be assumed that v_h weakly converges in $\mathcal{H}^1(\mathcal{X})$ to a limit v and that ν_h converges to a limit ν. Hence u_h converges strongly in L^2 to v. It now follows from Proposition 5.1 that, for any $f \in C^\infty(M)$,

$$\begin{aligned} \nu(f|v) &= \lim_{h\to 0} (f|\nu_h u_h) = \lim_{h\to 0} (|\Delta_h|(f)|u_h) \\ &= \lim_{h\to 0} \mathcal{B}_h(f, v_h + w_h) = \frac{1}{6p} \sum_{k=1}^{p} (X_k f | X_k v)_{L^2} = (f|Lv). \end{aligned} \tag{5.39}$$

Since f is arbitrary, it follows that $(L - \nu)v = 0$. By the Weyl-type estimate (5.9), the number of eigenvalues $|\Delta_h|$ in the interval $[0, R]$ is uniformly bounded. Moreover, the dimension of an orthonormal basis is preserved by strong limit. So the above argument proves that, for any $\epsilon > 0$ small, there exists $h_\epsilon > 0$ such that, for $h \in]0, h_\epsilon]$, one has

$$\mathrm{Spec}(|\Delta_h|) \cap [0, R] \subset \cup_j [\nu_j - \epsilon, \nu_j + \epsilon] \tag{5.40}$$

and

$$\sharp\mathrm{Spec}(|\Delta_h|) \cap [\nu_j - \epsilon, \nu_j + \epsilon] \leqslant m_j. \tag{5.41}$$

The fact that one has equality in (5.41) for ϵ small follows exactly like in the proof of Theorem 2(iii) in [6]: this uses only Proposition 5.1, the min–max principle, and a compactness argument. The proof of Theorem 1.2 is complete.

Remark 5.4. Observe that estimate (5.14) on the spectral gap is a direct consequence of Theorem 1.2, and moreover observe that in the proof of Theorem 1.2 we only use Proposition 5.1 in the special case $f \in C^\infty(M)$, and that, for $f \in C^\infty(M)$, Proposition 5.1 is obvious. However, we think that the fact that Proposition 5.1 holds true for any function $f \in \mathcal{H}^1(\mathcal{X})$ is interesting by itself, and, since it is an easy byproduct of Proposition 4.1, we decided to include it in the paper.

5.3. Elementary Fourier analysis

We conclude this section by collecting some basic results on Fourier analysis theory (uniformly with respect to h) associated with the spectral decomposition of T_h. These results are consequences of the preceding estimates. We start with the following lemma, which gives an honest L^∞ estimate of the eigenfunction $e_{j,h} \in E_{h,L}$. Recall that $\langle x\rangle = (1+x^2)^{1/2}$.

Lemma 5.5. *There exists C independent of h such that, for any eigenfunction $e_{j,h} \in E_{h,L}$, $\|e_{j,h}\|_{L^2} = 1$, associated with the eigenvalue $1 - h^2\lambda_{j,h}$ of T_h, the following inequality holds true:*

$$\|e_{j,h}\|_{L^\infty} \leqslant C\langle\lambda_{j,h}\rangle^d. \tag{5.42}$$

Proof. This is a byproduct of the preceding estimate (5.35). Apply this inequality to $g = e_{j,h}$. This gives

$$(1 - h^2\lambda_{j,h})^{n+p}\|e_{j,h}\|_{L^\infty} \leqslant \left(\frac{Ah^{-2}}{1+n}\right)^d. \tag{5.43}$$

Thus we get, with $n \simeq h^{-2}\langle\lambda_{j,h},\rangle^{-1}$

$$\|e_{j,h}\|_{L^\infty} \leqslant \left(\frac{Ah^{-2}}{h^{-2}\langle\lambda_{j,h}\rangle^{-1}}\right)^d (1 - h^2\lambda_{j,h})^{-h^{-2}\langle\lambda_{j,h}\rangle^{-1} - B\log(1/h)} \leqslant C\langle\lambda_{j,h}\rangle^d. \tag{5.44}$$

The proof of Lemma 5.5 is complete. □

Let $h_0 > 0$ be a small given real number. We will use the following notation. If X is a Banach space, we denote by X_h the space $L^\infty(]0,h_0], X)$, i.e., the space of functions $h \mapsto x_h$ from $h \in]0,h_0]$ into X such that $\sup_{h\in]0,h_0]}\|x_h\|_X < \infty$. For $a \geqslant 0$, the notation $x_h \in O_X(h^a)$ means that there exists C independent of h such that $\|x_h\|_X \leqslant Ch^a$, and $x_h \in O_X(h^\infty)$ means that $x_h \in O_X(h^a)$ for all a. We denote $C_h^\infty = \cap_{k\geqslant 0} C_h^k(M)$.

Let $\Pi_{h,L}$ be the L^2-orthogonal projection on $E_{h,L}$, and denote $\Pi_{h,2} = \mathrm{Id} - \Pi_{h,L}$. Let $(e_{j,h})_{j\in J_h}$ be an orthonormal basis of $E_{h,L}$ with $T_h(e_{j,h}) = (1 - h^2\lambda_{j,h})e_{j,h}$. For $f \in L^2$ we denote by $c_{j,h}(f) = (f|e_{j,h})$ the corresponding Fourier coefficient of f. Recall that J_h is defined in (5.19).

Proposition 5.6. *Let $f_h \in C_h^\infty$. For all integers N, the following holds true:*

$$|\triangle_h|^N f_h \in C_h^\infty \quad \textit{and} \quad \exists\, C_N, \quad \sup_{h\in]0,h_0]} \sum_{j\in J_h} \lambda_{j,h}^N |c_{j,h}(f_h)|^2 \leqslant C_N. \tag{5.45}$$

Moreover, one has the following estimates:

$$\Pi_{h,L}(f_h) \in O_{L^\infty(M)}(1) \tag{5.46}$$

and

$$\Pi_{h,2}(f_h) \in O_{L^\infty(M)}(h^N). \tag{5.47}$$

Proof. Let X be a vector field on M, and let $f \in C^\infty(M)$. The smooth function $F(t,x) = f(\mathrm{e}^{tX}x)$ satisfies the transport equation

$$\partial_t F = X(f), \quad F(0,x) = f(x).$$

Thus, one has, by Taylor expansion at $t = 0$, and for any integer N,

$$F(t,x) = \sum_{n\leqslant N} \frac{t^n}{n!} X^n(f)(x) + t^{N+1} r_N(t,x),$$

with $r_N(t,x)$ smooth. From the definition of T_h, we thus get

$$T_h f(x) \sum_{n \text{ even } \leqslant N} \frac{h^n}{(n+1)!} \left(\frac{1}{p} \sum_{k=1}^{p} X_k^n(f)(x) \right) + h^{N+1} \tilde{r}_N(h,x),$$

with $\tilde{r}_N(h,x) \in C_h^\infty$. This implies, for $f_h \in C_h^\infty$, that

$$|\triangle_h| f_h = L(f_h) + h^2 g_h, \quad g_h \in C_h^\infty.$$

Therefore, one has $|\triangle_h| f_h \in C_h^\infty$, and hence by induction $|\triangle_h|^N f_h \in C_h^\infty$ for all N. The second assertion of (5.45) follows from $\sup_{h\in]0,h_0]} \|g_h\|_{L^2} < \infty$ for any $g_h \in C_h^\infty$ and the fact that

$$\sum_{j\in J_h} \lambda_{j,h}^N |c_{j,h}(f_h)|^2 = \|\Pi_{h,L} |\triangle_h|^N f_h\|_{L^2}^2 \leqslant \| |\triangle_h|^N f_h\|_{L^2}^2.$$

For the proof of (5.46), we just write

$$\Pi_{h,L}(f_h) = \sum_{j\in J_h} c_{j,h}(f_h) e_{j,h},$$

and we use estimate (5.42) of Lemma 5.5 to get the bound

$$\begin{aligned} \|\Pi_{h,L}(f_h)\|_{L^\infty} &\leqslant C \sum_{j\in J_h} |c_{j,h}(f_h)| \langle \lambda_{j,h} \rangle^d \\ &\leqslant C \left(\sum_{j\in J_h} |c_{j,h}(f_h)|^2 \langle \lambda_{j,h} \rangle^{2d+2N} \right)^{1/2} \left(\sum_{j\in J_h} \langle \lambda_{j,h} \rangle^{-2N} \right)^{1/2}. \end{aligned}$$

From the Weyl-type estimate (5.9), there exist N and C independent of h such that

$$\left(\sum_{j\in J_h}\langle\lambda_{j,h}\rangle^{-2N}\right)^{1/2}\leqslant C,$$

and therefore (5.46) follows from (5.45). It remains to prove the estimate (5.47). We first prove the weaker estimate,

$$\Pi_{h,2}(f_h)\in O_{L^2(M)}(h^N). \tag{5.48}$$

Observe that $\Pi_{h,2}(f_h)$ satisfies, for all $N\geqslant 1$, the equation

$$h^{2N}\Pi_{h,2}(|\Delta_h|^N f_h)=(h^2|\Delta_h|)^N\Pi_{h,2}(f_h)=(\mathrm{Id}-T_h\Pi_{h,2})^N\Pi_{h,2}(f_h). \tag{5.49}$$

By (5.27), the operator $\mathrm{Id}-T_h\Pi_{h,2}=\mathrm{Id}-T_{h,2}$ is invertible on L^2 with inverse bounded by $(1-\theta)^{-1}$. Since $|\Delta_h|^N f_h\in C_h^\infty$, we get, from (5.49), $\Pi_{h,2}(f_h)\in O_{L^2}(h^{2N})$.

Set $g_h=\Pi_{h,2}(f_h)$. One has $|\Delta_h|^N f_h=\Pi_{h,L}(|\Delta_h|^N f_h)+|\Delta_h|^N g_h$. From (5.45) and (5.46), one has $\Pi_{h,L}(|\Delta_h|^N f_h)\in O_{L^\infty}(1)$. Thus we get $|\Delta_h|^N g_h\in O_{L^\infty}(1)$, for any N. Let $M_h=T_h^P$, and $|\tilde\Delta_h|=(\mathrm{Id}+T_h+\cdots+T_h^{P-1})|\Delta_h|$. Then g_h satisfies the equation

$$h^2|\tilde\Delta_h|g_h=g_h-M_hg_h. \tag{5.50}$$

As in (5.25), write $M_h=\rho_h+R_h$. Since T_h is bounded by 1 on L^∞, one gets

$$g_h-\rho_hg_h=h^2r_h+R_hg_h,\quad r_h=|\tilde\Delta_h|g_h\in O_{L^\infty}(1). \tag{5.51}$$

By the second line of (5.25) and (5.48), one has $R_hg_h\in O_{L^\infty}(h^\infty)$, and by the first line of (5.25), the operator $\mathrm{Id}-\rho_h$ is invertible on L^∞ with inverse bounded by $(1-\gamma)^{-1}$. Thus we get, from (5.51), $g_h\in O_{L^\infty}(h^2)$. Since $|\tilde\Delta_h|g_h=\Pi_{h,2}(|\tilde\Delta_h|f_h)$ and $|\tilde\Delta_h|f_h\in C_h^\infty$, the same estimates shows that $|\tilde\Delta_h|g_h=r_h\in O_{L^\infty}(h^2)$. Then (5.51) implies that $g_h\in O_{L^\infty}(h^4)$. By induction, we get $g_h\in O_{L^\infty}(h^{2N})$ for all N. The proof of Proposition 5.6 is complete. □

Let $F_k=\mathrm{Ker}(L-\nu_k)$. Recall that $m_k=\dim(F_k)$ is the multiplicity of the eigenvalue ν_k of L. Let us denote by $\mathcal{J}_k$ the set of indices j such that, for h small, $\lambda_{j,h}$ is close to ν_k, and $F_{h,k}=\mathrm{span}(e_{j,h},\, j\in\mathcal{J}_k)$. By Theorem 1.2 and its proof, the set $\mathcal{J}_k$ is independent of $h\in]0,h_k]$ for h_k small, and one has $\sharp(\mathcal{J}_k)=\dim(F_{h,k})=k$ for $h\in]0,h_k]$. Let Π_{F_k} and $\Pi_{F_{h,k}}$ be the L^2-orthogonal projectors on F_k and $F_{h,k}$.

Lemma 5.7. *For all $f\in F_k$, one has*

$$\lim_{h\to 0}\|f-\Pi_{F_{h,k}}(f)\|_{L^\infty}=0. \tag{5.52}$$

Proof. For $f\in F_k$, and h small, one has

$$f-\Pi_{F_{h,k}}(f)=\sum_{j\in J_h\setminus\mathcal{J}_k}c_{j,h}(f)e_{j,h}+\Pi_{h,2}(f). \tag{5.53}$$

One has $f\in C_h^\infty$, and thus, by (5.47), we get

$$\Pi_{h,2}(f)\in O_{L^\infty}(h^\infty). \tag{5.54}$$

Since $f \in F_k$, for any given $j \in J_h \setminus \mathcal{J}_k$, one has $\lim_{h\to 0} c_{j,h}(f) = \lim_{h\to 0}(f|e_{j,h})_{L^2} = 0$. Therefore, it remains to prove that

$$\lim_{N\to\infty} \sup_{h\in]0,h_0]} \sum_{j\in J_h, j\geqslant N} |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} = 0. \tag{5.55}$$

Let $N \gg \nu_k$. From (5.42), the Cauchy–Schwarz inequality, (5.45), and the Weyl-type estimate (5.9), there exist N_0 and a constant $C(f)$ independent of h such that one has the estimate

$$\begin{aligned}\sum_{j\in J_h, j\geqslant N} |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} &\leqslant C \sum_{j\in J_h, j\geqslant N} |c_{j,h}(f)| \langle\lambda_{j,h}\rangle^d \\ &\leqslant C \left(\sum_{j\in J_h} |c_{j,h}(f)|^2 \langle\lambda_{j,h}\rangle^{2d+2N_0}\right)^{1/2} \left(\sum_{j\in J_h, j\geqslant N} \langle\lambda_{j,h}\rangle^{-2N_0}\right)^{1/2} \\ &\leqslant C(f) \sup_{h\in]0,h_0]} \left(\sum_{j\in J_h, j\geqslant N} \langle\lambda_{j,h}\rangle^{-2N_0}\right)^{1/2} \longrightarrow 0 \quad (N\to\infty).\end{aligned} \tag{5.56}$$

In fact, since by (5.9) one has $\sharp\{j, \lambda_{j,h} \leqslant m\} \leqslant C_5 \langle m\rangle^{\dim(M)/2s}$, one can choose $N_0 = 1 + \dim(M)/4s$. Then one has

$$\sup_{h\in]0,h_0]} \sum_{j\in J_h, j\geqslant N} \langle\lambda_{j,h}\rangle^{-2N_0} \leqslant C_5 \sum_{m\geqslant m(N)} \langle m\rangle^{-2N_0} \langle m+1\rangle^{\dim(M)/2s},$$

with $m(N)$ the bigger integer such that $\lambda_{N,h} \geqslant m(N)$ for any $h \in]0, h_0]$. Observe that (5.9) implies that $\lim_{N\to\infty} m(N) = \infty$. The proof of Lemma 5.7 is complete. □

6. The hypoelliptic diffusion

We refer to the paper of Bismut [1] and references therein for a construction of the hypoelliptic diffusion associated with the generator L.

For a given $x_0 \in M$, let $X_{x_0} = \{\omega \in C^0([0,\infty[, M),\ \omega(0) = x_0\}$ be the set of continuous paths from $[0,\infty[$ to M, starting at x_0, equipped with the topology of uniform convergence on compact subsets of $[0,\infty[$, and let $\mathcal{B}$ be the Borel σ-field generated by the open sets in X_{x_0}. We denote by W_{x_0} the Wiener measure on X_{x_0} associated with the hypoelliptic diffusion with generator L. Let $p_t(x,y)\mathrm{d}\mu(y)$ be the heat kernel, i.e., the kernel of the self-adjoint operator $\mathrm{e}^{-tL}, t \geqslant 0$. Then W_{x_0} is the unique probability on $(X_{x_0}, \mathcal{B})$, such that, for any $0 < t_1 < t_2 < \cdots < t_k$ and any Borel sets $A_1, \ldots, A_k$ in M, one has

$$\begin{aligned}&W_{x_0}(\omega(t_1) \in A_1, \omega(t_2) \in A_2, \ldots, \omega(t_k) \in A_k) \\ &\quad = \int_{A_1\times A_2\times\cdots\times A_k} p_{t_k-t_{k-1}}(x_k, x_{k-1}) \ldots p_{t_2-t_1}(x_2, x_1) \\ &\qquad \times p_{t_1}(x_1, x_0)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\ldots\mathrm{d}\mu(x_k).\end{aligned} \tag{6.1}$$

Let us first introduce some notation. Let $Y = \{1, \dots, p\} \times [-1, 1]$, and let ρ be the uniform probability on Y, which means that, for any function $g(k, s)$ on Y, one has

$$\int_Y g \mathrm{d}\rho = \frac{1}{2p} \sum_{k=1}^{p} \int_{-1}^{+1} g(k, s) \mathrm{d}s. \tag{6.2}$$

We denote by $Y^{\mathbb{N}}$ the infinite product space $Y^{\mathbb{N}} = \{\underline{y} = (y_1, y_2, \dots, y_n, \dots), \ \ y_j \in Y\}$. Equipped with the product topology, it is a compact metrisable space, and we denote by $\rho^{\mathbb{N}}$ the product probability on $Y^{\mathbb{N}}$. Let $M^{\mathbb{N}}$ be the infinite product space $M^{\mathbb{N}} = \{\underline{x} = (x_1, x_2, \dots, x_n, \dots), \ \ x_j \in M\}$. Equipped with the product topology, $M^{\mathbb{N}}$ is a compact metrisable space. For $h \in]0, 1]$, and $x_0 \in M$, let $\pi_{x_0,h}$ be the continuous map from $Y^{\mathbb{N}}$ into $M^{\mathbb{N}}$ defined by

$$\pi_{x_0,h}((k_j, s_j)_{j \geqslant 1}) = (x_j)_{j \geqslant 1}, \quad x_j = \mathrm{e}^{s_j h X_{k_j}} \dots \mathrm{e}^{s_2 h X_{k_2}} \mathrm{e}^{s_1 h X_{k_1}} x_0. \tag{6.3}$$

We will use the notation $X^n_{h,x_0} = (\pi_{x_0,h})_n$. This means that X^n_{h,x_0} is the position after n steps of the random walk starting at x_0. Let $\mathcal{P}_{x_0,h}$ be the probability on $M^{\mathbb{N}}$ defined by $\mathcal{P}_{x_0,h} = (\pi_{x_0,h})_*(\rho^{\mathbb{N}})$. Then, by construction, one has, for all Borel sets $A_1, \dots, A_k$ in M,

$$\begin{aligned} &\mathcal{P}_{x_0,h}(x_1 \in A_1, x_2 \in A_2, \dots, x_k \in A_k) \\ &\quad = \int_{A_1 \times A_2 \times \cdots \times A_k} t_h(x_{k-1}, \mathrm{d}x_k) \dots t_h(x_1, \mathrm{d}x_2) t_h(x_0, \mathrm{d}x_1). \end{aligned} \tag{6.4}$$

Let us recall that $x_{j+1} = \mathrm{e}^{s_{j+1} h X_{k_{j+1}}} x_j$. Then $t \in [0, h^2] \mapsto \mathrm{e}^{\frac{t}{h^2} s_{j+1} h X_{k_{j+1}}} x_j$ is a smooth curve connecting x_j and x_{j+1}. Let $j_{x_0,h}$ be the map from $Y^{\mathbb{N}}$ into X_{x_0} defined by, with $\underline{y} = ((k_j, s_j)_{j \geqslant 1})$,

$$j_{x_0,h}(\underline{y}) = \omega \iff \forall j \geqslant 0, \quad \forall t \in [0, h^2], \quad \omega(jh^2 + t) = \mathrm{e}^{\frac{t}{h^2} s_{j+1} h X_{k_{j+1}}} x_j \tag{6.5}$$

with $x_j = (\pi_{x_0,h}(\underline{y}))_j$ if $j \geqslant 1$. Let $P_{x_0,h}$ be the probability on X_{x_0} defined as the image of $\rho^{\mathbb{N}}$ by the continuous map $j_{x_0,h}$. Our aim is to prove the following theorem of weak convergence of $P_{x_0,h}$ to the Wiener measure W_{x_0} when $h \to 0$.

Theorem 6.1. *For any bounded continuous function $\omega \mapsto f(\omega)$ on X_{x_0}, one has*

$$\lim_{h \to 0} \int f \mathrm{d}P_{x_0,h} = \int f \mathrm{d}W_{x_0}. \tag{6.6}$$

Observe that the proof below shows that our study of the Markov kernel T_h on M is also a way to prove the existence of the Wiener measure W_{x_0} associated with the hypoelliptic diffusion. Let g be a Riemannian distance on M, and let d_g the associated distance. We start by proving that the family of probability $P_{x_0,h}$ is tight, and hence is compact by the Prohorov theorem.

Proposition 6.2. *For any $\varepsilon > 0$, there exists $h_\varepsilon > 0$ such that the following holds true for any $T > 0$:*

$$\lim_{\delta \to 0} \left(\sup_{h \in]0, h_\varepsilon]} P_{x_0,h} \left(\max_{|s-t| \leqslant \delta, \ 0 \leqslant s,t \leqslant T} d_g(\omega(s), \omega(t)) > \varepsilon \right) \right) = 0. \tag{6.7}$$

Proof. We start with the following lemma.

Lemma 6.3. *Let $f \in C^\infty(M)$. There exists C such that, for all $h \in]0, h_0]$, one has*

$$\forall \delta \in [0,1], \qquad \sup_{nh^2 \leqslant \delta} \|T_h^n(f) - f - nh^2|\Delta_h| f\|_{L^\infty} \leqslant C\delta^2. \tag{6.8}$$

Proof. We may assume that $\delta > 0$ and $n \geqslant 1$. Then $nh^2 \leqslant \delta$ implies that $h \leqslant \sqrt{\delta}$. With the notation of § 5, one has

$$\left.\begin{aligned} &T_h^n(f) - f - nh^2|\Delta_h| f = \textstyle\sum_{j\in J_h} c_{j,h}(f)\Big((1-h^2\lambda_{j,h})^n - 1 - nh^2\lambda_{j,h}\Big)e_{j,h} + R(n,h) \\ &R(n,h) = T_h^n \Pi_{h,2}(f) - \Pi_{h,2}(f + nh^2|\Delta_h| f). \end{aligned}\right\} \tag{6.9}$$

One has $|\Delta_h| f \in C_h^\infty$, by (5.45), T_h is bounded by 1 on L^∞, and $nh^2 \leqslant \delta \leqslant 1$. Thus, from (5.47), we get

$$\sup_{nh^2 \leqslant \delta} \|R(n,h)\|_{L^\infty} \in O(h^\infty) \subset O(\delta^\infty). \tag{6.10}$$

For all $j \in J_h$, one has $h^2\lambda_{j,h} \in [0,1]$, and, for all $x \in [0,1]$,

$$|(1-x)^n - 1 - nx| \leqslant \frac{n(n-1)}{2}x^2.$$

Therefore, we get

$$\left\| \sum_{j\in J_h} c_{j,h}(f)\Big((1-h^2\lambda_{j,h})^n - 1 - nh^2\lambda_{j,h}\Big)e_{j,h} \right\|_{L^\infty} \leqslant \frac{n^2h^4}{2} \sum_{j\in J_h} \lambda_{j,h}^2 |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty}. \tag{6.11}$$

By the Weyl-type estimate (5.9), (5.42), and (5.45), there exists a constant C such that

$$\sup_{h\in]0,h_0]} \sum_{j\in J_h} \lambda_{j,h}^2 |c_{j,h}(f)| \|e_{j,h}\|_{L^\infty} \leqslant C.$$

Therefore (6.8) is consequence of (6.10) and (6.11). The proof of Lemma 6.3 is complete. □

The proof of Proposition 6.2 is now standard, and it proceeds as follows. Let $\varepsilon_0 > 0$ be small with respect to the injectivity radius of the Riemannian manifold (M, g), and let $\varepsilon \in]0, \varepsilon_0]$ be fixed. One has

$$\rho^{\mathbb{N}}(d_g(X_{h,x_0}^n, x_0) > \varepsilon) = \int_{d_g(y,x_0)>\varepsilon} t_h^n(x_0, \mathrm{d}y) = T_h^n(1_{d_g(y,x_0)>\varepsilon})(x_0). \tag{6.12}$$

Let $\varphi(r) \in C^\infty([0,\infty[)$ be a nondecreasing function equal to 0 for $r \leqslant 3/4$ and equal to 1 for $r \geqslant 1$, and set

$$\varphi_{x_0,\varepsilon}(x) = \varphi\left(\frac{d_g(x,x_0)}{\varepsilon}\right). \tag{6.13}$$

Then $\varphi_{x_0,\varepsilon}$ is a smooth function, and, from $\mathbb{1}_{d_g(y,x_0)>\varepsilon} \leqslant \varphi_{x_0,\varepsilon} \leqslant 1$, we get, since T_h is Markovian,

$$0 \leqslant T_h^n(\mathbb{1}_{d_g(y,x_0)>\varepsilon}) \leqslant T_h^n(\varphi_{x_0,\varepsilon}). \tag{6.14}$$

Since T_h moves the support at distance $\leqslant ch$, one has $\varphi_{x_0,\varepsilon}(x_0) + nh^2(|\Delta_h|\varphi_{x_0,\varepsilon})(x_0) = 0$ for $ch \leqslant \varepsilon/2$. From Lemma 6.3, we thus get that there exist $h_\varepsilon > 0$ and C_ε such that

$$\sup_{h\in]0,h_\varepsilon]} \sup_{nh^2\leqslant\delta} T_h^n(\varphi_{x_0,\varepsilon})(x_0) \leqslant C_\varepsilon \delta^2. \tag{6.15}$$

Since M is compact, it is clear from the proof of Lemma 6.3 that we may assume C_ε to be independent of $x_0 \in M$. From (6.12), (6.14), and (6.15) we get

$$\sup_{x_0\in M} \sup_{h\in]0,h_\varepsilon]} \sup_{nh^2\leqslant\delta} \rho^{\mathbb{N}}(d_g(X^n_{h,x_0}, x_0) > \varepsilon) \leqslant C_\varepsilon \delta^2. \tag{6.16}$$

Let $T > 0$ be given. One has, for $h \in]0, h_\varepsilon]$, the following inequalities.

$$\begin{aligned}
&\rho^{\mathbb{N}}(\exists j \langle l \leqslant h^{-2}T, (l-j)h^2 \leqslant \delta, d_g(X^j_{h,x_0}, X^l_{h,x_0}) \rangle 4\varepsilon) \\
&\leqslant \frac{C}{\delta} \sup_{y_0\in M} \rho^{\mathbb{N}}(\exists j \langle l \leqslant h^{-2}\delta, \quad d_g(X^j_{h,y_0}, X^l_{h,y_0}) \rangle 4\varepsilon) \\
&\leqslant \frac{C}{\delta} \sup_{y_0\in M} \rho^{\mathbb{N}}(\exists j \leqslant h^{-2}\delta, d_g(X^j_{h,y_0}, y_0) > 2\varepsilon) \\
&\leqslant \frac{2C}{\delta} \sup_{z_0\in M, nh^2\leqslant\delta} \rho^{\mathbb{N}}(d_g(X^n_{z_0}, z_0) > \varepsilon) \\
&\text{(by (6.16))} \leqslant 2CC_\varepsilon\delta.
\end{aligned} \tag{6.17}$$

In fact, for the first inequality in (6.17), we just use the fact that the interval $[0, T]$ is a union of $\simeq C/\delta$ intervals of length $\delta/2$. The second inequality is obvious, since the event $\{\exists j \langle l \leqslant h^{-2}\delta, d_g(X^j_{h,y_0}, X^l_{h,y_0})\rangle 4\varepsilon\}$ is a subset of $\{\exists j \leqslant h^{-2}\delta,\ d_g(X^j_{h,y_0}, y_0) > 2\varepsilon\}$. For the third one, we use the fact that the event $A = \{\exists j \leqslant h^{-2}\delta,\ d_g(X^j_{h,y_0}, y_0) > 2\varepsilon\}$ is contained in $B \cup_{j<k} (C_j \cap D_j)$ with $B = \{d_g(X^k_{h,y_0}, y_0) > \varepsilon\}$ (k is the greatest integer $\leqslant \delta h^{-2}$), $C_j = \{d_g(X^j_{h,y_0}, X^k_{h,y_0}) > \varepsilon\}$, $D_j = \{d_g(X^j_{h,y_0}, y_0) > 2\varepsilon$ and $d_g(X^l_{h,y_0}, y_0) \leqslant 2\varepsilon$ for $l < j\}$, and the fact that C_j and D_j are independent and the D_j are disjoints.

Since $P_{x_0,h} = (j_{x_0,h})_*(\rho^{\mathbb{N}})$, (6.7) follows easily from (6.17) and definition (6.5) of the map $j_{x_0,h}$. The proof of Proposition 6.2 is complete. □

With the result of Proposition 6.2, the proof of Theorem 6.1 follows now the classical proof of weak convergence of a sequence of random walks in the Euclidian space $\mathbb{R}^d$ to Brownian motion on $\mathbb{R}^d$, for which we refer to [12, Chapter 2.4]. We have to prove that any weak limit P_{x_0} of a sequence P_{x_0,h_k}, $h_k \to 0$, is equal to the Wiener measure W_{x_0}. We denote by $\omega_h(t)$ the map from $Y^{\mathbb{N}}$ into M defined by $\omega_h(t)(\underline{y}) = j_{x_0,h}(\underline{y})(t)$. By Theorem 4.15 of [12], it is sufficient to show that, for any $m \geqslant 1$, any $0 < t_1 < \cdots < t_m$, and any continuous function $f(x_1, \dots, x_m)$ defined on the space M^m, one has

$$\lim_{h\to 0} \int_{Y^{\mathbb{N}}} f(\omega_h(t_1), \dots, \omega_h(t_m)) \mathrm{d}\rho^{\mathbb{N}} = \int f(x_1, \dots, x_m) p_{t_m - t_{m-1}}(x_m, x_{m-1}) \dots p_{t_2-t_1}(x_2, x_1) \times p_{t_1}(x_1, x_0) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2) \dots \mathrm{d}\mu(x_m). \tag{6.18}$$

As in [12], we may assume that $m = 2$. For a given $t \geqslant 0$, let $n(t, h) \in \mathbb{N}$ be the greatest integer such that $h^2 n(t, h) \leqslant t$. By (6.5), one has, for some $c > 0$ independent of h and $\underline{y} \in Y^{\mathbb{N}}$, $d_g(\omega_h(t), X_{h,x_0}^{n(t,h)}) \leqslant ch$. Since f is uniformly continuous on M^m, we are reduced to proving that

$$\lim_{h\to 0} \int f(X_{h,x_0}^{n(t_1,h)}, X_{h,x_0}^{n(t_2,h)}) \mathrm{d}\rho^{\mathbb{N}} = \int f(x_1, x_2) p_{t_2-t_1}(x_2, x_1) p_{t_1}(x_1, x_0) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2). \tag{6.19}$$

From (6.4), one has

$$\int f(X_{h,x_0}^{n(t_1,h)}, X_{h,x_0}^{n(t_2,h)}) \mathrm{d}\rho^{\mathbb{N}} = \int f(x_1, x_2) t_h^{n(t_2,h)-n(t_1,h)}(x_1, \mathrm{d}x_2) t_h^{n(t_1,h)}(x_0, \mathrm{d}x_1). \tag{6.20}$$

By (6.19), (6.20), we have to show that, for any continuous function $f(x_1, x_2)$ on the product space $M \times M$, one has

$$\lim_{h\to 0} \int_{M\times M} f(x_1, x_2) t_h^{n(t_2,h)-n(t_1,h)}(x_1, \mathrm{d}x_2) t_h^{n(t_1,h)}(x_0, \mathrm{d}x_1) = \int_{M\times M} f(x_1, x_2) p_{t_2-t_1}(x_2, x_1) p_{t_1}(x_1, x_0) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2), \tag{6.21}$$

or, equivalently,

$$\lim_{h\to 0} T_h^{n(t_1,h)} \left(T_h^{n(t_2,h)-n(t_1,h)}(f(x_1, .))(x_1) \right)(x_0) = \mathrm{e}^{-t_1 L} \left(\mathrm{e}^{-(t_2-t_1)L}(f(x_1, .))(x_1) \right)(x_0). \tag{6.22}$$

Since $\|T_h^{n(t,h)}\|_{L^\infty} \leqslant 1$ and $\|\mathrm{e}^{-tL}\|_{L^\infty} \leqslant 1$, the following 'central limit' theorem will conclude the proof of Theorem 6.1.

Lemma 6.4. *For all $f \in C^0(M)$, and all $t > 0$, one has*

$$\lim_{h\to 0} \|\mathrm{e}^{-tL}(f) - T_h^{n(t,h)}(f)\|_{L^\infty} = 0. \tag{6.23}$$

Since one has $\|T_h^{n(t,h)}\|_{L^\infty} \leqslant 1$ and $\|\mathrm{e}^{-tL}\|_{L^\infty} \leqslant 1$, it is sufficient to prove that (6.23) holds true for $f \in \mathcal{D}$, with $\mathcal{D}$ a dense subset of the space $C^0(M)$, and therefore we may assume that $f \in F_k$ is an eigenvector of L associated with the eigenvalue ν_k. We set $n = n(t, h)$, and we use the notation of § 5. One has

$$T_h^n(f) = \sum_{j \in \mathcal{J}_k} c_{j,h}(f)(1 - h^2 \lambda_{j,h})^n e_{j,h} + R_{t,h}(f), \tag{6.24}$$

with

$$R_{t,h}(f) = \sum_{j\in J_h\setminus\mathcal{J}_k} c_{j,h}(f)(1-h^2\lambda_{j,h})^n e_{j,h} + T_h^n\Pi_{h,2}(f). \tag{6.25}$$

One has $|(1-h^2\lambda_{j,h})^n| \leqslant 1$, and T_h is bounded by 1 on L^∞. By (5.54) and (5.55), we thus get

$$\lim_{h\to 0} \|R_{t,h}(f)\|_{L^\infty} = 0.$$

One has $\lim_{h\to 0}(1-h^2\lambda_{j,h})^{n(t,h)} = \mathrm{e}^{-t\nu_k}$, for all $j\in\mathcal{J}_k$. Moreover, one has $\sharp\mathcal{J}_k = m_k$ and $\sup_{h\in]0,h_0]}\sup_{j\in\mathcal{J}_k}\|e_{j,h}\|_{L^\infty} < \infty$, by Lemma 5.5. Therefore, Lemma 5.7 and $\mathrm{e}^{-tL}(f) = \mathrm{e}^{-t\nu_k}f$ imply that

$$\lim_{h\to 0}\left\|\sum_{j\in\mathcal{J}_k} c_{j,h}(f)(1-h^2\lambda_{j,h})^n e_{j,h} - \mathrm{e}^{-tL}(f)\right\|_{L^\infty} = 0.$$

The proof of Lemma 6.4 is complete. □

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A.

Let $P = P(x,\partial_x)$ be an elliptic second-order differential operator on M, with smooth coefficients, such that $P = P^* \geqslant \mathrm{Id}$, where P^* is the formal adjoint on $L^2(M,\mu) = L^2$. Let $(e_j)_{j\geqslant 1}$ be an orthonormal basis of eigenfunctions of P in L^2, and let $1\leqslant \nu_1\leqslant \nu_2\ldots$ be the associated eigenvalues. By the classical Weyl formula, one has

$$\#\{j,\, \nu_j^{1/2}\leqslant r\} \simeq r^{\dim(X)}. \tag{A 1}$$

For $s\in\mathbb{R}$ and $f = \sum_j f_j e_j$ in the Sobolev space $H^s(M)$, we set

$$\|v\|_{H^s}^2 = \sum_j \nu_j^s |f_j|^2 = (P^s f|f)_{L^2}.$$

Let us recall that this H^s-norm depends on P, but another choice for P gives an equivalent norm. The following elementary lemma is useful for us.

Lemma A.1. *Let $s>0$, and let $A_h = A_h^* \geqslant 0$, $h\in]0,1]$, be a family of non-negative self-adjoint bounded operators acting on $L^2(M,\mu)$. Assume that there exists a constant $C_0>0$ independent of h such that, for all $u\in L^2(M,\mu)$, the following holds true:*

$$((\mathrm{Id}+A_h)u|u)\leqslant 1 \Rightarrow \exists (v,w)\in H^s\times L^2 \text{ such that } u = v+w,\ \|v\|_{H^s}\leqslant C_0,\ \|w\|_{L^2}\leqslant C_0 h. \tag{A 2}$$

Let $C_1 < \frac{1}{4C_0^2}$. *There exists* $C_2 > 0$ *independent of* h *such that* $\mathrm{Spec}(A_h) \cap [0, \lambda - 1]$ *is discrete for all* $\lambda \leqslant C_1 h^{-2}$, *and*

$$\#(\mathrm{Spec}(A_h) \cap [0, \lambda - 1]) \leqslant C_2 \langle \lambda \rangle^{\dim(M)/2s}, \quad \forall \lambda \leqslant C_1 h^{-2}. \tag{A 3}$$

Here, $\#(\mathrm{Spec}(A_h) \cap [0, r])$ *is the number of eigenvalues of* A_h *in the interval* $[0, r]$ *with multiplicities, and* $\langle \lambda \rangle = \sqrt{1 + \lambda^2}$.

Proof. Let $B_h = \mathrm{Id} + A_h$. Let C_h be the bounded operator on L^2 defined by

$$C_h \left(\sum_j u_j e_j \right) = \sum_j \min(h^{-1}, \nu_j^{s/2}) u_j e_j.$$

For $u = v + w$, one has

$$\|C_h u\|_{L^2}^2 \leqslant 2\|C_h v\|_{L^2}^2 + 2\|C_h w\|_{L^2}^2 \leqslant 2(\|v\|_{H^s}^2 + h^{-2}\|w\|_{L^2}^2).$$

From (A 2), we get, for all $u \in L^2$,

$$\|C_h u\|_{L^2}^2 \leqslant 4C_0^2 (B_h u | u). \tag{A 4}$$

For any non-negative self-adjoint bounded operator T on L^2, set, for $j \geqslant 1$,

$$\lambda_j(T) = \min_{\dim(F)=j} \left(\max_{u \in F, \|u\|_{L^2}=1} (Tu|u) \right).$$

It is well known that, if $\#\{j, \lambda_j(T) \in [0, a[\} < \infty$, the spectrum of T in $[0, a[$ is discrete, and, in that case, the $\lambda_j(T) \in [0, a[$ are the eigenvalues of T in $[0, a[$ with multiplicities. From (A 4), we get, for all $j \geqslant 1$, the inequality

$$\lambda_j(B_h) \geqslant \frac{1}{4C_0^2} \lambda_j(C_h^2). \tag{A 5}$$

For all j such that $\nu_j^s < h^{-2}$, one has $\lambda_j(C_h^2) = \nu_j^s$, and, therefore, for all $\lambda < h^{-2}$, we get from (A 1), $\#\{j, \lambda_j(C_h^2) \leqslant \lambda\} \leqslant C\langle \lambda \rangle^{\dim(M)/2s}$. Therefore, the spectrum of B_h in $[0, h^{-2}/4C_0^2[$ is discrete, and (A 3) follows from (A 5) and $\mathrm{Spec}(A_h) = Spec(B_h) - 1$. The proof of Lemma A.1 is complete. □

Lemma A.2. *Let* $\mathcal{N} = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_\mathfrak{r}$ *be the free up to rank* $\mathfrak{r}$ *nilpotent Lie algebra with* p *generators. Let* $(Y_1, \ldots, Y_p)$ *be a basis of* $\mathcal{N}_1$, *and let* $(\mathcal{Z}_1, \ldots, \mathcal{Z}_p)$ *be the right invariant vector fields on* $\mathcal{N}$ *such that* $\mathcal{Z}_j(0) = Y_j$. *Let* $\mathcal{S}(\mathcal{N})$ *be the Schwartz space of* $\mathcal{N}$. *Let* $\varphi \in \mathcal{S}(\mathcal{N})$ *be such that* $\int_{\mathcal{N}} \varphi \mathrm{d}x = 0$. *Then there exists* $\varphi_k \in \mathcal{S}(\mathcal{N})$ *such that*

$$\varphi = \sum_{k=1}^{p} \mathcal{Z}_k(\varphi_k). \tag{A 6}$$

Proof. Let $Y^\alpha = H_\alpha(Y_1, \ldots, Y_p)$, and let $\mathcal{Z}^\alpha$ be the right invariant vector fields on $\mathcal{N}$ such that $\mathcal{Z}^\alpha(0) = Y^\alpha$. Let $u_\alpha, \alpha \in \mathcal{A}$ be the coordinates on $\mathcal{N}$ associated with the basis

$(Y^\alpha, \alpha \in \mathcal{A})$ of $\mathcal{N}$. Let ∂_α be the derivative in the direction of u_α. Let $\varphi \in \mathcal{S}(\mathcal{N})$ such that $\int_{\mathcal{N}} \varphi \mathrm{d}x = 0$. Using the Fourier transform in coordinates (u_α), and $\hat{\varphi}(0) = 0$, one easily gets that there exist functions $\psi_\alpha \in \mathcal{S}(\mathcal{N})$ such that

$$\varphi = \sum_{\alpha \in \mathcal{A}} \partial_\alpha(\psi_\alpha). \tag{A 7}$$

By (2.3), the vector field $\mathcal{Z}^\alpha$ is of the form

$$\mathcal{Z}^\alpha = \partial_\alpha + \sum_{|\beta|>|\alpha|} p_{\alpha,\beta}(u_{\langle|\beta|})\, \partial_\beta = \partial_\alpha + \sum_{|\beta|\rangle|\alpha|} \partial_\beta \;\, p_{\alpha,\beta}(u_{<|\beta|}),$$

where the $p_{\alpha,\beta}$ are polynomials in u depending only on $(u_1, \dots, u_j)$ with $j < |\beta|$. Therefore, there exist polynomials $q_{\alpha,\beta}$ such that

$$\partial_\alpha = \mathcal{Z}^\alpha + \sum_{|\beta|>|\alpha|} \mathcal{Z}^\beta q_{\alpha,\beta}.$$

Since the Schwartz space $\mathcal{S}(\mathcal{N})$ is stable by multiplication by polynomials, we get from (A 7) that there exists $\phi_\alpha \in \mathcal{S}(\mathcal{N})$ such that

$$\varphi = \sum_{\alpha \in \mathcal{A}} \mathcal{Z}^\alpha(\phi_\alpha). \tag{A 8}$$

For $|\alpha| > 1$, there exist $j \in \{1, \dots, p\}$ and β with $|\beta| = |\alpha| - 1$ such that $\mathcal{Z}^\alpha = \mathcal{Z}_j \mathcal{Z}^\beta - \mathcal{Z}^\beta \mathcal{Z}_j$. By induction on $|\alpha|$, since the Schwartz space $\mathcal{S}(\mathcal{N})$ is stable by the vector fields $\mathcal{Z}_j$, this shows that, for any α and $\phi \in \mathcal{S}(\mathcal{N})$, there exists $\phi_j \in \mathcal{S}(\mathcal{N})$ such that $\mathcal{Z}^\alpha(\phi) = \sum_{j=1}^{p} \mathcal{Z}_j(\phi_j)$. Thus (A 6) follows from (A 8). The proof of Lemma A.2 is complete. □

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