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AROUND SUPERSYMMETRY FOR SEMICLASSICAL SECOND ORDER DIFFERENTIAL OPERATORS

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ABSTRACT. Let $P(h), h \in]0,1]$ be a semiclassical scalar differential operator of order 2. The existence of a supersymmetric structure given by a matrix $G(x;h)$ was exhibited by Hérau, Hitrik, and Sjöstrand (2011) under rather general assumptions. In this paper we give a sufficient condition on the coefficients of $P(h)$ so that the matrix $G(x;h)$ enjoys some nice estimates with respect to the semiclassical parameter.

1. INTRODUCTION

In many problems arising in physics there is interest in accurately computing the spectrum of some differential operators depending on a small parameter (that we shall denote by h throughout). In numerous situations, proving sharp results can be done by using a specific structure of the operator. For instance, in the setting of Schrödinger operators, geometric assumptions on the potential leads to sharp computation of the splitting between eigenvalues [4]. More recently, the computation of the low lying eigenvalues of the semiclassical Witten Laplacian was performed by using the specific structure of the operator [5], [3]. In these papers, the fact that the Witten Laplacian enjoys a supersymmetric structure (that is, can be written as a twisted Hodge Laplacian) is fundamental, and it doesn't seem possible to obtain the sharp results without using this property. Similarly, the existence of a supersymmetric structure was used in [1] to compute the spectrum of some semiclassical Markov operators and hence the rate of convergence to equilibrium of the associated random walk.

In a nonselfadjoint setting numerous results in the same spirit were obtained by Hérau, Hitrik, Sjöstrand [6–8]. In all these papers, the authors were led to compute a spectral gap for nonselfadjoint operators. Their approach was based on the fact that the underlying operator is supersymmetric for a convenient bilinear product and that then some tools developed for the study of the Witten Laplacian can be used (of course, one major additional difficulty comes from the fact that they are in a nonselfadjoint situation).

In most situations mentioned above, the supersymmetric structure of the operator is known in advance. Nevertheless, it can occur that the supersymmetric structure is hidden and has to be exhibited. This was for instance the case in [1] where the authors give a sufficient condition for selfadjoint pseudodifferential operators to be supersymmetric. Roughly speaking, the main assumption made in

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[1] is that the Weyl symbol of the operator is an even function of the ξ variable. The first motivation of the present work was hence to investigate what happens when this assumption fails to be true. As we shall see later, the pseudodifferential situation is quite intricate and we shall restrict our attention to the case of second order scalar differential operators. This issue was already addressed in [9], where the authors consider semiclassical and nonsemiclassical operators P. In both situation the author exhibit supersymmetric structure (under the assumption that the kernels of P and P^* contain some specific element), but they emphasize the fact that in the semiclassical situation, the factorization of the operator is done without control with respect to the semiclassical parameter. The goal of this paper is to give a sufficient condition in order to have a control with respect to h in the factorization and also to discuss the optimality of this condition. We decided to use the formalism of [9] that we recall in the next paragraph.

Let X be either $\mathbb{R}^n$ or an n-dimensional smooth connected compact manifold without boundary equipped with a smooth volume density $\omega(dx)$, and let $P = P(h)$, $h \in]0,1]$, denote a second order scalar semiclassical differential operator on X with real smooth coefficient. For $0 \leq k < n$, let $\Omega^k(X) = \mathcal{C}^\infty(X, \Lambda^k T^*X)$ and denote $d : \Omega^k(X) \to \Omega^{k+1}(X)$ the exterior derivative. For any $x \in X$, we recall the natural pairing $\langle ., . \rangle_{\Lambda,\Lambda^*}$ on $\Lambda^k TX \times \Lambda^k T^*X$ given by $\langle u, v^* \rangle_{\Lambda,\Lambda^*} = \det((v_i^*(u_j))_{i,j})$ for any $v^* = v_1^* \wedge \ldots \wedge v_k^*$ and $u = u_1 \wedge \ldots \wedge u_k$. It gives rise to a natural pairing on $\mathcal{C}^\infty(X, \Lambda^k T^*X) \times \mathcal{C}^\infty(X, \Lambda^k TX)$ by integrating the preceding formula against the volume form. Then, we let $\delta : \mathcal{C}^\infty(X, \Lambda^k TX) \to \mathcal{C}^\infty(X, \Lambda^{k-1} TX)$ be the adjoint of d for this pairing.

Suppose that $G(x) : T_x^*X \to T_xX$ is a linear mapping depending smoothly on $x \in X$. Then $\Lambda^k G$ maps $\Lambda^k T_x^*X$ into $\Lambda^k T_xX$ (by convention $\Lambda^0 G$ is the identity on $\mathbb{R}$) and we can define a bilinear product on $\mathcal{C}_c^\infty(X, \Lambda^k T^*X)$ by the formula

$$\langle u, v \rangle_G = \int_X \langle G(x)u(x), v(x) \rangle_{\Lambda,\Lambda^*} \omega(dx), \tag{1.1}$$

where for short, we write $G(x)$ instead of $\Lambda^k G(x)$. When $G(x)$ is invertible for any $x \in X$ we can define $d^{G,*} = (G^t)^{-1} \delta G^t$ and one checks easily that $d^{G,*}$ is the formal adjoint of d with respect to G,

$$\langle du, v \rangle_G = \langle u, d^{G,*} v \rangle_G, \ \forall u, v \in \mathcal{C}_c^\infty(X, \Lambda^k T^*X). \tag{1.2}$$

Notice that on 1-forms $d^{G,*}, \mathcal{C}^\infty(X, \Lambda^1 T^*X) \to \mathcal{C}^\infty(X, \mathbb{R})$ is given by $d^{G,*} = \delta \circ G^t$, which makes sense even if G is not invertible.

In the case where X is a compact Riemaniann manifold, we can identify T_x^*X and T_xX by means of the metric g, so that G can be considered as an operator acting on T_x^*X. When $X = \mathbb{R}^n$ is equipped with the Euclidean metric, then G will be identified with its matrix in the basis of canonical 1-forms.

Given $\varphi \in \mathcal{C}^\infty(X, \mathbb{R})$, the associated Witten complex is defined by the semiclassical weighted de Rham differentiation

$$d_{\varphi,h} = e^{-\varphi/h} \circ hd \circ e^{\varphi/h} = hd + d\varphi^\wedge$$

and its formal adjoint with respect to the bilinear form (1.1) is

$$d_{\varphi,h}^{G,*} = e^{\varphi/h} \circ hd^{G,*} \circ e^{-\varphi/h} = (G^t)^{-1} \circ (h\delta + d\varphi^\lrcorner) \circ G^t.$$

Let us now recall the definition of a supersymmetric structure used in [9].

Definition 1.1. Let $P = P(x, hD_x; h)$ be a second order scalar real semiclassical differential operator on X. We say that P has a supersymmetric structure if there exists a linear h-dependent map $G(x; h) : T_x^*X \to T_xX$, smooth with respect to $x \in X$, and functions $\varphi, \psi \in \mathcal{C}^\infty(X, \mathbb{R})$ such that

$$P = d_{\psi,h}^{G,*} d_{\varphi,h}$$

for all $h \in]0, h_0], h_0 > 0$.

Here we decided to consider phase functions φ, ψ which are independent of h in order to simplify. As noticed in [9], no control of $G(x; h)$ with respect to h is required in this definition. In order to get some bounds on $G(x; h)$, we first need to handle a metric on X. If $X = \mathbb{R}^n$ we consider g the Euclidean metric and if X is a compact manifold we take g to be any Riemaniann metric. From this metric we get a normed vector space structure on T_xX and T_x^*X. In the case where $X = \mathbb{R}^n$ we need to control the function at infinity. Given an order function a (in the sense of Def. 7.4 in [2]) we say that a function $f \in \mathcal{C}^\infty(X, \mathbb{R})$ belongs to $S(a)$ if

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0, \forall x \in X, \ |\partial^\alpha f(x)| \leq C_\alpha a(x). \tag{1.3}$$

Throughout, we will use the japanese bracket $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^n$. Given $m \in \mathbb{R}$, we will often use the order function ρ_m defined by $\rho_m(x) = \langle x \rangle^m$ when $X = \mathbb{R}^n$ and by $\rho_m = 1$ if X is compact. We shall denote $S_m = S(\rho_m)$. We introduce the following

Definition 1.2. Let $P = P(x, hD_x; h)$ be a second order scalar real semiclassical differential operator on X. We say that P has a temperate supersymmetric structure if it has a supersymmetric structure (in the sense of the above definition) and if the map $G(x; h) : T_x^*X \to T_xX$ satisfies the following: there exist $m \in \mathbb{R}$ and some constants $C_\nu > 0$ such that

$$\|\partial_x^\nu G(x, h)\|_{T_x^*X \to T_xX} \leq C_\nu \rho_m(x), \ \forall x \in X, \tag{1.4}$$

for all $\nu \in \mathbb{N}^n, h \in]0, h_0]$.

Throughout the paper we shall call a "supersymmetric structure" any operator $G(x; h)$ as above. We shall say that G is temperate if it satisfies (1.4) for some $m \in \mathbb{R}$. Observe that the preceding definition doesn't depend on the choice of the metric g since if g_1 and g_2 are two metric on a compact manifold X, the corresponding norms on tangent and cotangent spaces are uniformly equivalent.

In the applications (e.g. for the analysis of the spectrum of the Witten Laplacian [3] or the Kramers-Fokker-Planck operator [8]), the supersymmetric structure is used to make a link between the spectrum of P and the spectrum of the associated operator on 1-forms. As we have seen before, this operator is well defined if the matrix $G(x; h)$ is invertible. It is then important to find a condition which ensures that G is invertible. We will come back to this issue at the end of section 2.

In practice, it is often useful to expand quantities in powers of the semiclassical parameter h. Given a function $f \in S(a)$, we shall say that it has a classical expansion if there exists a sequence $(f_k)_{k \in \mathbb{N}}$ in $S(a)$ such that for all $K \in \mathbb{N}$,

$$f - \sum_{k=0}^{K} h^k f_k \in S(h^{K+1} a).$$

We shall denote by $S_{cl}(a)$ the set of semiclassical functions having a classical expansion and $S_{m,cl} = S_{cl}(\rho_m)$. Let us now recall one of the results proved in [9].

Theorem 1.3. *Let $P = P(x, hD_x; h)$ be a second order scalar real semiclassical differential operator on X. Assume there exists $\varphi, \psi \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $P(e^{-\varphi/h}) = P^*(e^{-\psi/h}) = 0$, where P^* denotes the formal adjoint of P. Assume also that the δ complex is exact in degree 1 for smooth sections. Then P has a supersymmetric structure.*

Notice that the above theorem holds true in a very general context. For instance, if $X = \mathbb{R}^n$, any scalar second order differential operator such that $P(e^{-\varphi/h}) = P^*(e^{-\psi/h}) = 0$ admits a supersymmetric structure. Nevertheless, no control on the linear map $G(x; h)$ giving the supersymmetric structure is proved. This was noticed in the remark after Definition 1.1 of [8] and finding the condition that ensures control over $G(x; h)$ was raised as an open question. Moreover, the author emphasized the fact that the procedure of factorization of P runs in two separate ways. The factorization of the symmetric part of P is immediate and provides an explicit bound, whereas the antisymmetric part is obtained as a solution of a δ problem with exponential weights. This difficulty which concentrated on the antisymmetric part has to be linked with the general factorization result obtained for pseudodifferential operators in [1].

Let us now briefly recall this last result. We state the result in a slightly different class of symbols than the one the used in [1] so that it contains the second order differential operators. It is not difficult to see that the proof could be adapted to the present context. Let $p(x, \xi)$ be a symbol in the class $S(\langle\xi\rangle^2)$, and let $P = \mathrm{Op}_h^w(p)$ denote its Weyl quantization (we refer to [2] for the basics of pseudodifferential calculus). Assume that φ is a smooth function that behaves at most as $|x|$ at infinity such that $P(e^{-\varphi/h}) = 0$. The fundamental assumption made in [1] is the following:

$$\xi \mapsto p(x, \xi) \text{ is even for all } x \in \mathbb{R}^n. \tag{1.5}$$

From this assumption and the equation $P(e^{-\varphi/h}) = 0$ we get $P^*(e^{-\varphi/h}) = 0$, and the question of supersymmetry can be investigated. From Lemma 3.2 and Remark 3.3 in [1], it follows that there exists a matrix-valued pseudodifferential operator $Q_h(x, hD) \in \Psi(1)$ such that $P = d^*_{\varphi,h} Q_h(x, hD) d_{\varphi,h}$. In other word, P admits a temperate supersymmetric structure with the matrix $G(x; h)$ replaced by the pseudodifferential operator $Q_h(x, hD)$.

In order to discuss the preceding results and state our first theorem, we need to write the operator P in a specific form. It is not hard to verify that any second order scalar real semiclassical differential operator on X can be written in a unique way under the form

$$P(x, hD_x, h) = h\delta \circ A(x; h) \circ hd + U(x; h) \circ hd + v(x; h), \tag{1.6}$$

where A, U and v have the following properties:

- $A(x; h) : T_x^* X \to T_x X$ and $U(x; h) : T_x^* X \to \mathbb{R}$ are linear and $v(x; h) \in \mathbb{R}$.
- Identifying $T_x^{**} X$ and $T_x X$, $A(x; h)$ is symmetric.
- A, U and v belong to S_m for some $m \in \mathbb{R}$.

Observe that $U(x; h) \in T_x^{**} X$ for any $x \in X$. Again using the canonical identification $T_x^{**} X \simeq T_x X$, it can be seen as an element of $T_x X$. In local coordinates, (1.6) reads

$$P = -\sum_{i,j=1}^{n} h\partial_{x_i} \circ a_{i,j}(x;h) \circ h\partial_{x_j} + \sum_{k=1}^{n} u_k(x;h) \circ h\partial_{x_k} + v(x;h) \tag{1.7}$$

for some real symmetric matrix $A = (a_{ij}(x;h))$, some vector $U = (u_k(x;h))$ and $v(x;h) \in \mathbb{R}$. So that we can compare the results in [1] and [9], we shall rewrite the assumptions in a pseudodifferential way. Assume that we work on $X = \mathbb{R}^n$ and that P is given by (1.6). Then one has $P = \mathrm{Op}_h^w(p)$, $p = p_{even} + p_{odd}$ with

$$p_{even}(x,\xi) = \xi^t A(x)\xi + v(x) + \frac{h}{2}\,\mathrm{div}(U) + \frac{h^2}{4}\sum_{i,j} \partial_i\partial_j a_{ij}(x) \text{ and } p_{odd}(x,\xi) = iU(x)\xi, \tag{1.8}$$

where we dropped the dependence of the functions with respect to h in order to lighten the notation. Suppose that $U(x) = 0$ and $P(e^{-\varphi/h}) = 0$. Then we can apply the results of [1] and the operator P admits a temperate supersymmetric structure $P = d^*_{\varphi,h} Q d_{\varphi,h}$, where the operator Q obtained from [1] is a priori a pseudodifferential operator. Nevertheless, a careful look at the proof shows that the operator Q is in fact a multiplication by a temperate matrix. In the case where $\varphi = \psi$ this gives an improvement of the conclusion of Theorem 1.3. In the following we try to find a sharp assumption to make on the antisymmetric part of P in order to prove temperate supersymmetry.

Let us recall the general framework. We consider a second order scalar semiclassical differential operator written under the form (1.6) and we assume that there exists φ, ψ such that $P(e^{-\varphi/h}) = P^*(e^{-\psi/h}) = 0$. We wonder if P admits a temperate supersymmetric structure. A simple computation shows that $P(e^{-\varphi/h}) = 0$ if and only if the following eikonal equation holds true:

$$d\varphi \lrcorner (A(x)d\varphi) + U(x)d\varphi - v(x) + h\delta(A(x)d\varphi) = 0. \tag{1.9}$$

On the other hand, since A is symmetric, we have $P^* = h\delta \circ A(x) \circ hd - U \circ hd + h\delta(U) + v(x)$, and we see that $P^*(e^{-\psi/h}) = 0$ is equivalent to a second eikonal equation:

$$d\psi \lrcorner (A(x)d\psi) - U(x)d\psi - v(x) - h\delta(U) + h\delta(A(x)d\psi) = 0, \tag{1.10}$$

where $U(x) : T_x^*X \to \mathbb{R}$ is sometimes seen as an element of T_xX.

For any $\phi \in \mathcal{C}^\infty(X)$ and $N \in \mathbb{N}$, let $E_\phi^N \subset \mathcal{C}^\infty(X, \Lambda^2 TX)$ denote the subspace of $\mathcal{C}^\infty(X, \Lambda^2 TX)$ given by

$$E_\phi^N = \{\theta = \sum_{\mathit{finite}} (\alpha_j \circ \phi)\,\theta_j,\ \alpha_j \in S(\langle t\rangle^N),\, \theta_j \in \ker(\delta)\}$$

and also define the corresponding classical set by

$$E_{\phi,cl}^N = \{\theta \in E_\phi,\ \alpha_j \in S_{cl}(\langle t\rangle^N)\}.$$

We are now in a position to state our first result.

Theorem 1.4. *Let P be as in* (1.6) *with coefficients A, U, v belonging to S_{m_1} for some $m_1 \in \mathbb{R}$. Assume that there exists $\varphi, \psi \in S_{m_2}$ for some $m_2 \in \mathbb{R}$ such that* (1.9) *holds true and assume that*

$$U + d(\varphi - \psi)^{\lrcorner} \circ A \in \delta(E_{\varphi+\psi}^N). \tag{1.11}$$

Then P admits a temperate supersymmetric structure given by some $G(x;h)$: $T_x^*X \to T_xX$.

If additionally, A and v are classical functions and $U+d(\varphi-\psi)^\lrcorner \circ A \in \delta(E^N_{\varphi+\psi,cl})$, then the linear map $G(x;h)$ has a classical expansion.

Observe that the conclusion of the above theorem implies that the second eikonal equation (1.10) holds true. This could look surprising since we did not specifically require (1.10) in our assumptions. In fact, one can easily prove that (1.9) and (1.11) imply (1.10). It is natural to wonder if assumption (1.11) is necessary in order to have a temperate supersymmetric structure. In the last section of this paper we give a partial answer to this question.

2. Proof of Theorem 1.4

For any antisymmetric $H(x) : T_x^*X \to T_xX$, define $\delta(H) : T_x^*X \to \mathbb{R}$ by $\delta \circ H \circ d = \delta(H) \circ d$ (since H is antisymmetric, this operator is indeed an homogeneous first order differential operator). For any $G = G(x;h) : T_x^*X \to T_xX$, one has on the 0-forms

$$\begin{aligned} d^{G,*}_{\psi,h} d_{\varphi,h} &= h\delta \circ G^t \circ hd + d\psi^\lrcorner \circ G^t \circ hd + h\delta \circ G^t(d\varphi) + d\psi^\lrcorner \circ G^t \circ d\varphi^\wedge \\ &= h\delta \circ G^t \circ hd + d\psi^\lrcorner \circ G^t \circ hd - d\varphi^\lrcorner \circ G \circ hd + h\delta(G^t d\varphi) + d\psi \lrcorner G^t(d\varphi) \\ &= h\delta \circ \frac{G^t+G}{2} \circ hd + h\frac{\delta(G^t-G)}{2} \circ hd + (d\psi^\lrcorner \circ G^t - d\varphi^\lrcorner \circ G) \circ hd \\ &\qquad + h\delta(G^t d\varphi) + d\psi \lrcorner G^t(d\varphi). \end{aligned}$$

Let us introduce the symmetric and antisymmetric part of G:

$$G^s = \frac{1}{2}(G+G^t) \text{ and } G^a = \frac{1}{2}(G-G^t).$$

Then we get

$$\begin{aligned} d^{G,*}_{\psi,h} d_{\varphi,h} &= h\delta \circ G^s \circ hd - h\delta(G^a) \circ hd + (d(\psi-\varphi)^\lrcorner \circ G^s - d(\varphi+\psi)^\lrcorner \circ G^a) \circ hd \\ &\qquad + h\delta(G^s d\varphi) - h\delta(G^a d\varphi) + d\psi \lrcorner (G^s d\varphi) - d\psi \lrcorner (G^a d\varphi). \end{aligned} \tag{2.1}$$

Identifying (2.1) and (1.6), we see that $P = d^{G,*}_{\psi,h} d_{\varphi,h}$ if and only if

$$\left\{ \begin{array}{l} G^s(x) = A(x;h) \\ U(x;h) + d(\varphi-\psi)^\lrcorner \circ G^s = -h\delta(G^a) - d(\varphi+\psi) \lrcorner G^a \\ v(x;h) = h\delta(G^s d\varphi) - h\delta(G^a d\varphi) + d\psi \lrcorner (G^s d\varphi) - d\psi \lrcorner (G^a d\varphi). \end{array} \right. \tag{2.2}$$

Looking for G under the form $G = A + B$ with B antisymmetric, (2.2) becomes

$$\left\{ \begin{array}{l} U(x;h) + d(\varphi-\psi)^\lrcorner \circ A = -h\delta(B) - d(\varphi+\psi)^\lrcorner \circ B \\ v(x;h) = h\delta(Ad\varphi) - h\delta(Bd\varphi) + d\psi \lrcorner (Ad\varphi) - d\psi \lrcorner (Bd\varphi). \end{array} \right. \tag{2.3}$$

Suppose now that the first equation of the above system is solved. Then,

$$U(d\varphi) = d\psi \lrcorner (Ad\varphi) - d\varphi \lrcorner (Ad\varphi) - h\delta(Bd\varphi) + d\psi \lrcorner Bd\varphi,$$

and using (1.9), we get easily the second one. Hence, we are reduced to find a map $B \in \mathcal{C}^\infty(X, \Lambda^2 TX)$ which is temperate and solves

$$U + d(\varphi-\psi)^\lrcorner \circ A = -h\delta(B) - d\phi \lrcorner B, \tag{2.4}$$

where $\phi = \varphi + \psi$. Thanks to assumption (1.11), there exists $\theta_1, \dots, \theta_K \in \mathcal{C}^\infty(X, \Lambda^2 TX)$ and $\alpha_1, \dots \alpha_K \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $\delta\theta_k = 0$ for all k and

$$U + d(\varphi - \psi)^\lrcorner \circ A = \delta\theta$$

with $\theta = \sum_{k=1}^K (\alpha_k \circ \phi)\, \theta_k$. Hence (2.4) is equivalent to

$$\delta\theta = -h\delta(B) - d\phi \lrcorner B. \tag{2.5}$$

On the other hand, for any k, we have

$$\delta((\alpha_k \circ \phi)\theta_k) = (\alpha_k \circ \phi)\delta\theta_k - d(\alpha_k \circ \phi)\lrcorner\theta_k = -(\alpha_k' \circ \phi) d\phi \lrcorner \theta_k,$$

and hence (2.5) is equivalent to

$$h\delta(B) + d\phi\lrcorner B = \sum_{k=1}^K (\alpha_k' \circ \phi) d\phi\lrcorner\theta_k. \tag{2.6}$$

Since this is a linear equation, it suffices find some temperate B_k such that

$$h\delta(B_k) + d\phi\lrcorner B_k = (\alpha_k' \circ \phi) d\phi\lrcorner\theta_k$$

for all k. In order to lighten the notation, we will drop the index k in the following lines. Setting $\tilde{B} = e^{-\phi/h} B$, the above equation is equivalent to

$$h\delta(\tilde{B}) = e^{-\phi/h}(\alpha' \circ \phi) d\phi\lrcorner\theta. \tag{2.7}$$

Our aim is to find a solution $\tilde{B}$ of this equation such that $B = e^{\phi/h}\tilde{B}$ is temperate. For this purpose, simply observe that

$$e^{-\phi/h}(\alpha' \circ \phi) d\phi = d(\beta \circ \phi)$$

with $\beta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$\beta(t) = -\int_t^{m_\infty} \alpha'(s) e^{-s/h} ds \tag{2.8}$$

with $m_\infty = +\infty$ when $X = \mathbb{R}^n$ and $m_\infty = 1 + \sup\phi$ when X is a compact manifold. Hence (2.7) becomes

$$h\delta(\tilde{B}) = d(\beta \circ \phi)\lrcorner\theta = -\delta((\beta \circ \phi)\theta).$$

A solution is trivially given by $\tilde{B} = -\frac{1}{h}(\beta \circ \phi)\theta$, that is,

$$\begin{aligned} B(x) &= \Big(\frac{1}{h} e^{\phi/h} \int_{\phi(x)}^{m_\infty} \alpha'(s) e^{-s/h} ds\Big)\theta(x) \\ &= \frac{1}{h}\Big(\int_0^{m_\infty - \phi(x)} \alpha'(s + \phi(x)) e^{-s/h} ds\Big)\theta(x). \end{aligned} \tag{2.9}$$

It remains to check that B is temperate. For this purpose it suffices to observe that $m_\infty - \phi(x) \geq 0$, and hence we necessarily have $e^{-s/h} \leq 1$ in the above integral. In the case where X is compact, this shows immediately that $\partial^\nu B = \mathcal{O}(\rho_{Nm_2})$ for all $\nu \in \mathbb{N}^n$. In the case where $X = \mathbb{R}^n$, using the fact that α has at most polynomial growth and performing integration by parts we similarly obtain $\partial^\nu B = \mathcal{O}(\rho_{Nm_2})$.

Suppose now that A, v have a classical expansion and that $U + d(\varphi - \psi)^\lrcorner \circ A \in \delta(E^N_{\varphi+\psi,cl})$. In order to show that G has a classical expansion, it suffices to do so for $B(x)$ above. A simple change of variable shows that

$$B(x) = -\Big(\int_0^{(m_\infty - \phi(x))/h} \alpha'(hs + \phi(x)) e^{-s} ds\Big)\theta(x).$$

In the case where $X = \mathbb{R}^n$, $m_\infty = +\infty$ and a simple Taylor expansion in the above integral gives the result. Suppose now that X is a compact manifold. The Taylor expansion of $\alpha'(hs + \phi(x))$ reduces the proof to expand terms of the form

$$\int_0^{(m_\infty - \phi(x))/h} (hs)^k e^{-s} ds,$$

which is easily obtained by integration by parts. This concludes the proof. $\square$

Remark 2.1. From the preceding proof, we see that if α satisfies additional properties, then B can be computed explicitly. For instance, if α is a polynomial, integration by parts leads to an explicit formula for B.

In the conclusion of this section we shall discuss the invertibility of $G(x; h)$. In order to simplify the discussion, we suppose that $X = \mathbb{R}^n$. Assume that there exists $\theta \in E_{\varphi+\psi}$ such that $U + d(\varphi - \psi)^\lrcorner \circ A = \delta\theta$. From the proof above, one has $G = A + B$ with A real symmetric defined by (1.6) and B real antisymmetric. Assume that A is uniformly positive definite, that is,

$$\exists C > 0,\ \forall x \in \mathbb{R}^n,\ \forall \xi \in T_x^* X,\ \langle A(x;h)\xi, \xi\rangle \geq C|\xi|^2.$$

One checks easily that G enjoys the same estimate and hence is invertible with G^{-1} bounded by C^{-1}.

Let us now consider the case where $A(x; h)$ is only positive (not necessary definite). Assume that there exist some orthogonal subspaces E, F independent of (x, h) such that $\mathbb{R}^n = E \oplus F$ with $E = \ker A(x; h)$ for all $x \in \mathbb{R}^n$. In some specific situations we can ensure that G is invertible. For instance, if $\ker B = F$ and $B_{|E}$ and $A_{|F}$ are invertible with inverse uniformly bounded, then G has a uniformly bounded inverse. Another interesting situation is a generalization of the Kramers-Fokker-Planck operator. Recall that the Kramers-Fokker-Planck operator is defined on $\mathbb{R}^{2n}$ by

$$K(h) = y \cdot h\nabla_x - \nabla_x V(x) \cdot h\nabla_y - h^2\Delta_y + y^2 - hn, \tag{2.10}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ denotes the space variable and V is a smooth function. This operator admits a supersymmetric structure $K(h) = d_{\varphi,h}^{G,*} \circ d_{\varphi,h}$ with $\varphi(x, y) = \frac{1}{2}y^2 + V(x)$ and

$$G(x, h) = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 2\,\mathrm{Id} \end{pmatrix}.$$

In particular, G is invertible and $G^{-1} = \mathcal{O}(1)$. In [8], this permitted the authors to define the whole associated Witten complex and to obtain precise information on the spectrum of $K(h)$. This situation can be easily generalized. Let us go back to the above situation where $E = \ker A(x, h)$ is independent of x and suppose that $B(E) \subset F$. Denote $\Pi : X \to E$ as the orthogonal projection and $\hat{\Pi} = 1 - \Pi$. Then, $\Pi B \Pi = 0$ and

$$G = \hat{\Pi} A \hat{\Pi} + \Pi B \hat{\Pi} + \hat{\Pi} B \Pi + \hat{\Pi} B \hat{\Pi}.$$

Therefore, the equation $G\xi = 0$ is equivalent to

$$\begin{cases} \hat{\Pi} A \hat{\Pi}\xi + \hat{\Pi} B \Pi\xi + \hat{\Pi} B \hat{\Pi}\xi = 0 \\ \Pi B \hat{\Pi}\xi = 0. \end{cases}$$

Taking the scalar product with ξ, we get $\langle \hat{\Pi} B \Pi \xi, \xi \rangle = \langle \Pi \xi, B^t \hat{\Pi} \xi \rangle = -\langle \xi, \Pi B \hat{\Pi} \xi \rangle = 0$ and hence

$$\langle A \hat{\Pi} \xi, \hat{\Pi} \xi \rangle + \langle B \hat{\Pi} \xi, \hat{\Pi} \xi \rangle = 0.$$

Since A is definite positive on F and B is antisymmetric, it follows immediately that $\hat{\Pi} \xi = 0$. Hence G is injective if and only if $B_{|E} : E \to F$ is injective. In particular it is necessary that $\dim F \geq \dim E$. In order to estimate G^{-1} one starts from $G\xi = \eta$. Working as above we show easily that $\|\hat{\Pi} \xi\| \leq 2C_0^{-1} \|\eta\|$, where C_0 is the smallest eigenvalue of A on F. One also has

$$\hat{\Pi} B \Pi \xi = \hat{\Pi} \eta - \hat{\Pi} A \hat{\Pi} \xi - \hat{\Pi} B \hat{\Pi} \xi.$$

Hence, assuming additionally that $B : E \to \mathrm{Im}(B)$ has an inverse which is uniformly bounded with respect to h, G^{-1} is automatically uniformly bounded with respect to h.

3. About assumption (1.11)

The aim of this section is to discuss the necessity of assumption (1.11) in order to get temperate supersymmetry. Throughout this section, we assume that $\varphi = \psi$. Then, the question of the existence of a supersymmetric structure can be reduced to the same question for operators of order 1. Indeed, we can write $P = P_1 + P_2$ with $P_1 = \frac{1}{2}(P + P^*)$. Observe that P_1 is formally selfadjoint and P_2 is formally antiadjoint. Moreover, since $\varphi = \psi$, we have $P_1(e^{-\varphi/h}) = 0$, $P_2(e^{-\varphi/h}) = 0$ and we claim that P_1 automatically admits a temperate supersymmetric structure with phase functions $\varphi = \psi$. Indeed, we have

$$P_1 = h\delta \circ A \circ hd + \frac{h}{2} \delta(U) + v$$

and we know from the proof of Theorem 1.4 that P_1 admits a temperate supersymmetric structure if and only if there exists B antisymmetric, such that

$$d(\varphi - \psi)^\lrcorner \circ A = -h\delta(B) - d(\varphi + \psi)^\lrcorner \circ B.$$

Since $\varphi = \psi$, $B = 0$ solves this equation. We are then reduced to investigate the condition that ensures that

$$P_2 = U \circ hd - \frac{h}{2} \delta(U) \tag{3.1}$$

admits a temperate supersymmetric structure with phase functions $\varphi = \psi$.

3.1. **The two-dimensional case.** In this section we assume that $X = \mathbb{R}^2$, the Euclidean plane, and we consider operators P of the form (3.1). We make the additional assumption that $\delta(U) = 0$. We denote by $\mathscr{C}$ the set of critical points of φ and by $\omega = \partial_{x_1} \wedge \partial_{x_2}$ the canonical element of $\mathcal{C}^\infty(X, \Lambda^2 TX)$. Using the Euclidean structure we can write $P = U(x) \cdot h\nabla$. The following lemma shows that away from critical points U necessarily has the form (1.11).

Lemma 3.1. *For any $x_0 \in \mathbb{R}^2 \setminus \mathscr{C}$, there exists a neighborhood V of x_0 and a smooth function $f_{x_0} : \mathbb{R} \to \mathbb{R}$ such that $U = \delta((f_{x_0} \circ \varphi)\omega)$.*

Proof. Under the above assumptions, the eikonal equation reads

$$U(d\varphi) = 0,$$

and since $\delta(U) = 0$, there exists $\alpha \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $U = \delta(\alpha\omega) = \partial_2 \alpha \partial_{x_1} - \partial_1 \alpha \partial_{x_2}$. Going back to the eikonal equation we obtain $\partial_2 \alpha \partial_1 \varphi - \partial_1 \alpha \partial_2 \varphi = 0$, which can be interpreted as $\det(\nabla\alpha, \nabla\varphi) = 0$. From this equation we deduce that near any point $x_0 \in \mathbb{R}^2$ there exists a smooth function f_{x_0} such that $\alpha = f_{x_0} \circ \varphi$. To see this, recall that x_0 is noncritical, hence there exists V a neighborhood of x_0 such that $V \cap \mathscr{C} = \emptyset$. Shrinking V if necessary and changing coordinates, we can assume that there exists $\nu \in \mathbb{R}^2$ such that

$$\varphi(z) = \varphi(x_0) + \langle \nu, z - x_0 \rangle \text{ for all } z \in V. \tag{3.2}$$

Suppose that $x, y \in V$ are such that $\varphi(x) = \varphi(y)$. Then, we deduce from (3.2) that there exists a smooth path $\gamma : [0,1] \to V$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\varphi \circ \gamma$ is constant (take $\gamma(t) = x + t(y - x)$). Since

$$\alpha(x) - \alpha(y) = \int_0^1 \frac{d}{dt}\alpha \circ \gamma(t)dt = \int_0^1 \nabla\alpha \circ \gamma \cdot \dot{\gamma}(t)dt = 0$$

for that $\dot{\gamma}$ orthogonal to $\nabla\varphi$, this shows that α depends only on φ on V. Hence there exists a function f_{x_0} such that $\alpha = f_{x_0} \circ \varphi$. Moreover, using the fact that $\nabla\varphi$ doesn't vanish on V one can easily show that f_{x_0} is smooth. $\square$

Without additional assumption on φ it seems difficult to globalize the above result. However, in the case where φ is a Morse function one can get further information on the structure of U. In the following, we assume that φ is a Morse function. For $k = 0, 1, 2$, let $\mathscr{C}^{(k)}$ denote the set of critical points of index k. The following is an improvement of Lemma 3.1.

Lemma 3.2. *Assume that φ is a Morse function. Then for any $x_0 \in \mathbb{R}^2 \setminus \mathscr{C}^{(1)}$ there exists a neighborhood V of x_0 and a smooth function $f_{x_0} : \mathbb{R} \to \mathbb{R}$ such that $U = \delta((f_{x_0} \circ \varphi)\omega)$.*

Proof. It suffices to check the conclusion in the case where x_0 is either a minimum or a maximum of φ. Assume that x_0 is a minimum of φ (the maximum case can be treated in the same way). As in the previous lemma, we first choose a small neighborhood V of x_0 and new coordinates such that

$$\varphi(z) = \varphi(x_0) + |z - x_0|^2 \text{ for all } z \in V. \tag{3.3}$$

Without loss of generality, we can also assume that $x_0 = 0$ and $\varphi(0) = 0$. Let $x, y \in V$ be such that $\varphi(x) = \varphi(y)$, that is, $|x| = |y|$, and denote by α the angle between x and y. Let $\gamma : [0,1] \to V$ be the path defined by $\gamma(t) = r_{t\alpha}(x)$, where r_θ denotes the rotation of angle θ. Then $\gamma(0) = x$, $\gamma(1) = y$ and $\varphi \circ \gamma$ is constant. The same argument as in Lemma 3.1 shows that $\alpha(x) = \alpha(y)$. Hence α depends only on φ on V and there exists a function f_{x_0} such that $\alpha = f_{x_0} \circ \varphi$. It is clear that f_{x_0} is smooth away from $x_0 = 0$ since the gradient of φ doesn't vanish. In order to show that f_{x_0} is smooth in x_0 we write $f_{x_0}(t) = \alpha(\sqrt{t}, 0)$. Let us write the Taylor expansion of α near the origin

$$\alpha(x_1, x_2) \simeq \sum_{j,k} \alpha_{j,k} x_1^j x_2^k.$$

The equation $\det(\nabla\alpha, \nabla\varphi) = 0$ yields $x_2\partial_1\alpha = x_1\partial_2\alpha$, and it follows that for all j, k, $j\alpha_{j,k-2} = k\alpha_{j-2,k}$ with the convention $\alpha_{p,q} = 0$ for $p < 0$ or $q < 0$. Using this relation, we immediately get $\alpha_{j,k} = 0$ for any j, k such that j or k is odd. In particular $\alpha_{j,0} = 0$ for any odd j, which shows that f_{x_0} is smooth at the origin. $\square$

From now, we will assume additionally that the set $\mathscr{C}$ of critical points of φ is finite. Let $\Sigma = \varphi(\mathscr{C}^{(1)}) \subset \mathbb{R}$ denote the saddle values of φ. Then $\mathbb{R}^2 \setminus \varphi^{-1}(\Sigma)$ has a finite number of connected components $\Omega_1, \dots, \Omega_J$ and only one (say Ω_J) is unbounded.

Lemma 3.3. *Assume that φ is a Morse function with a finite critical set. Then there exists some functions $f_1, \dots, f_J \in \mathcal{C}^\infty(\mathbb{R})$ such that*

$$U = \delta((f_j \circ \varphi)\omega) \text{ on } \Omega_j.$$

Proof. Let $j \in \{1, \dots, J\}$ be fixed and let $x, y \in \Omega_j$ such that $\varphi(x) = \varphi(y) =: \sigma$. We shall prove that the set $F_\sigma := \varphi^{-1}(\sigma)$ is arcwise connected. We first assume that Ω_j is bounded. Hence there exists a covering of $\overline{\Omega}_j$ by a finite collection of convex open set $(\omega_k)_{k=1,\dots,K}$ such that on each ω_k, there exists some change of coordinates $\theta_k : \mathcal{O}_k \to \omega_k$ such that $\varphi_k = \varphi \circ \theta_k$ takes one of the forms

$$\varphi_k(z) = \langle z, \nu \rangle,\ \nu \in \mathbb{R}^2 \setminus 0 \text{ or } \varphi_k(z) = |z|^2 \text{ or } \varphi_k(z) = -|z|^2$$

for any $z \in \mathcal{O}_k$ neighborhood of $0 \in \mathbb{R}^2$. Let $M = 2\pi K \sup_{k=1,\dots,K}(\|D\theta_k\|_\infty \operatorname{diam}(\mathcal{O}_k))$ and

$$\Gamma_j = \{M\text{-Lipschitz path } \gamma \text{ contained in } \Omega_j \text{ and joining } x \text{ to } y\}. \tag{3.4}$$

Since Ω_j is arcwise connected, Γ_j is nonempty. Indeed there exists a smooth path $\gamma_0 : [0,1] \to \Omega_j$ joining x to y, and up to reparametrization we can also assume that $|\gamma_0'|$ is constant. Moreover, using the specific form of φ on each ω_k we can modify γ_0 into a piecewise $\mathcal{C}^1$ path so that $I_k := \{t \in [0,1],\ \gamma_0(t) \in \omega_k\}$ is an interval for all $k = 1, \dots, K$. It follows easily that

$$|\gamma_0'(t)| \le 2\pi \sum_{k,\ I_k \neq \emptyset} \operatorname{diam}(\mathcal{O}_k)\|D\theta_k\|_\infty \le M,$$

except for a finite number of values of t. Therefore $\gamma_0 \in \Gamma_j$.

Introduce next the set $\mathcal{M} = \{\sup_{[0,1]} \varphi \circ \gamma,\ \gamma \in \Gamma_j\} \subset [\sigma, +\infty[$ since $\varphi(x) = \sigma$, and let $m = \inf \mathcal{M} \ge \sigma$. We claim that $m = \sigma$. Indeed, it follows from the Ascoli theorem that Γ_j is relatively compact in $\mathcal{C}([0,1], \overline{\Omega}_j)$. Hence there exists a path γ_1 contained in Ω_j and joining x and y such that $m = \sup \varphi \circ \gamma_1 = \varphi \circ \gamma_1(t_1)$. Suppose by contradiction that $m > \sigma$ and let $x_1 = \gamma_1(t_1)$. By definition of Ω_j, x_1 cannot be a saddle point of φ, and since $m = \sup \varphi \circ \gamma_1$ it is not a minimum. Hence x_1 is either a local maximum or a noncritical point of φ. In both cases, it is easy to locally modify the path γ_1 in order to decrease m. This gives a contradiction.

Hence $m = \sigma$ and there exists a continuous path $\tilde{\gamma}_1 \subset \Omega_j$ joining x and y and such that $\sup_{[0,1]} \varphi \circ \tilde{\gamma} = \sigma$. Moreover, by construction $\tilde{\gamma}_1$ is M-Lipschitz. Therefore, the set

$$\tilde{\Gamma}_j = \{M\text{-Lipschitz path } \gamma \text{ contained in } \Omega_j \cap \{\varphi \le \sigma\} \text{ and joining } x \text{ to } y\} \tag{3.5}$$

is nonempty. Let $\mathcal{L} = \{\inf_{[0,1]} \varphi \circ \tilde{\gamma},\ \gamma \in \tilde{\Gamma}_j\}$ and $\ell = \sup \mathcal{L}$. As before, there exists a Lipschitz path γ_2 such that $\ell = \sup \varphi \circ \gamma_2$ and we can show easily that $\varphi \circ \gamma_2$ is a constant equal to σ. Using this path γ_2 and the fact that $\varphi^{-1}(\sigma)$ is locally connected, we construct a path $\gamma_3 \subset \varphi^{-1}(\sigma)$ from x to y which is piecewise $\mathcal{C}^1$.

Using this path γ_3 and repeating the argument of the proof of Lemma 3.2, it follows easily that $\alpha(x) = \alpha(y)$ and hence α depends only on φ. This permits us to construct a function f_j such that $\alpha = f_j \circ \varphi$ on Ω_j. The smoothness of f_j is a local property and then follows from Lemma 3.2.

Let us now prove the result for the unbounded component Ω_J. Let $\sigma \in \mathbb{R}$ be fixed and let $x, y \in \Omega_J \cap \varphi^{-1}(\sigma)$. By definition, there exists a path γ contained in Ω_J joining x and y. Let $R > 0$ be such that $\gamma \subset B(0, R)$. Since $\Omega_J \cap B(0, R)$ is relatively compact, one can follow the same strategy as for the bounded component with Ω_j replaced by $\Omega_J \cap B(0, R)$. □

As a consequence of the above lemma we get the following

Theorem 3.4. *Let $P(h) = U \circ hd$ with $\delta(U) = 0$. Assume that φ is a Morse function with a finite number of critical points and such that $U \cdot \nabla\varphi = 0$. Assume additionally that for all $i, j = 1, \dots, J$, $i \neq j$, and all $x \in \Omega_i, y \in \Omega_j$, such that $\varphi(x) = \varphi(y)$, there exists a smooth path γ from x to y such that*

$$\int_\gamma \star U = 0, \tag{3.6}$$

where $\star$ denotes the Hodge star operator. Then P satisfies (1.11) *and hence admits a temperate supersymmetric structure.*

Proof. Let I denote the image of φ which is a (bounded or unbounded) interval. From (3.6), one knows that for all $i \neq j$ and all $x \in \Omega_i$, $y \in \Omega_j$ such that $\varphi(x) = \varphi(y)$, one has $f_i \circ \varphi(x) = f_j \circ \varphi(y)$. Hence, the function $f : I \to \mathbb{R}$ given by $f \circ \varphi(x) = \alpha(x)$ is well defined. One has to show that f is smooth and the only point which has not already been examined is the smoothness near saddle points. Let s_0 be a saddle point of φ. Without loss of generality, we assume $s_0 = 0$ and $\varphi(x) = x_1^2 - x_2^2$ near the origin. As in the proof of Lemma 3.2, we write $U = \delta(\alpha\omega)$ with $\alpha \simeq \sum_{j,k} \alpha_{j,k} x_1^j x_2^k$, and it follows from the equation $U(d\varphi) = 0$ that

$$j\alpha_{j,k-2} = -k\alpha_{j-2,k} \tag{3.7}$$

for all j, k (with the convention $\alpha_{j,k} = 0$ for negative j or k). As before, we get $\alpha_{j,k} = 0$ for j or k odd and one has

$$f(t) = \begin{cases} \alpha(\sqrt{t}, 0) & \text{if } t > 0 \\ \alpha(0, \sqrt{-t}) & \text{if } t < 0. \end{cases}$$

From the Taylor expansion of α we see that f is smooth if and only if $\alpha_{2j,0} = (-1)^j \alpha_{0,2j}$, which is a consequence of (3.7). □

In consideration of the above theorem, one could think that operators admitting temperate supersymmetric structure are more or less of the form (1.11). In fact, when the dimension is greater than 2, this is not the case. One way to see this is to notice that supersymmetric structure can be easily tensorized.

Let X_j, $j = 1, 2$, be either a Euclidean space or a smooth connected compact manifold. Let $P_j(x_j, hD_{x_j})$ denote a second order semiclassical differential operator on X_j, and let $\varphi_j, \psi_j \in \mathcal{C}^\infty(X_j, \mathbb{R})$.

Theorem 3.5. *Assume that the P_j admit a supersymmetric structure $G_j(x_j; h)$ associated to the phase φ_j, ψ_j. Then the operator $P(x, hD_x) = P_1(x_1, hD_{x_1}) + P_2(x_2, hD_{x_2})$ acting on $X = X_1 \times X_2$ admits a supersymmetric structure*

$$P = d^{G,*}_{\psi,h} d_{\varphi,h}$$

with $\varphi(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2)$, $\psi(x_1, x_2) = \psi_1(x_1) + \psi_2(x_2)$ and $G(x; h) = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$.

Proof. This is immediate. □

Using this result, we can easily construct some examples where U does not necessarily have the form (1.11), even locally. For instance, let $X = \mathbb{R}^3$, $\varphi(x) = x_1^2 + x_2^2 + x_3^2$ and

$$P(x, hD_x) = x_1 \cos(x_1^2 + x_2^2)h\partial_{x_2} - x_2 \cos(x_1^2 + x_2^2)h\partial_{x_1} = U \circ hd$$

with $U(x) = -x_2 \cos(x_1^2 + x_2^2)\partial_{x_1} + x_1 \cos(x_1^2 + x_2^2)\partial_{x_2}$. Then P admits a supersymmetric structure with phases $\varphi = \psi$, and one has $U = \delta(\sin(x_1^2 + x_2^2)\partial_{x_1} \wedge \partial_{x_2})$ but U cannot be written under the form (1.11).

3.2. **Perturbation of a supersymmetric structure.** In this section we go back to the general situation where X is either a compact manifold without boundary or $\mathbb{R}^n$. We assume that $\varphi : X \to \mathbb{R}$ is a smooth function such that the set $\mathcal{V} = \varphi(\mathscr{C})$ of critical values of φ is finite. In the case where $X = \mathbb{R}^n$ we assume additionaly that $\lim_{|x|\to\infty} \varphi(x) = \infty$. For any $\sigma \in \mathbb{R}$ we denote $X_\sigma = \{x \in X, \varphi(x) < \sigma\}$ and we consider a fixed connected component ω_σ of X_σ. For any $\epsilon > 0$ we denote $\omega_\sigma^\epsilon = \omega_\sigma \cap X_{\sigma-\epsilon}$. Since $\mathcal{V}$ is finite, there exists $\epsilon_0 > 0$ small enough such that $]\sigma - \epsilon_0, \sigma[\cap \mathcal{V} = \emptyset$. Therefore, the set ω_σ^ϵ has smooth boundary for all $0 < \epsilon < \epsilon_0$ and ω_σ^ϵ is relatively compact in ω_σ (in the case $X = \mathbb{R}^n$ this is true since φ goes to infinity at infinity). Hence we can construct a smooth cut-off function χ_ϵ such that $\chi = 1$ on ω_σ^ϵ and $\operatorname{supp}(\chi_\epsilon) \subset \omega_\sigma$. Let $\alpha \in \mathcal{C}^\infty(\mathbb{R})$ such that $\operatorname{supp}(\alpha) \subset] - \infty, \sigma - \epsilon[$ and let θ be a 2-form such that $\delta\theta = 0$. Consider

$$U_\epsilon = \delta((\chi_\epsilon\, \alpha \circ \varphi)\theta).$$

Then U_ϵ is a smooth 1-form such that $\delta U_\epsilon = 0$. Moreover, by construction $\operatorname{supp}(d\chi_\epsilon) \subset \{\sigma - \epsilon < \varphi < \sigma\}$ and hence $\alpha \circ \varphi d\chi_\epsilon = 0$. Therefore, we have in fact

$$U_\epsilon = \chi_\epsilon \delta((\alpha \circ \varphi)\theta) = -\chi_\epsilon \alpha' \circ \varphi d\varphi \lrcorner \theta,$$

and hence $U_\epsilon(d\varphi) = 0$. Then, the operator $P_\epsilon = U_\epsilon \circ hd$ is formally self-adjoint and we have $P_\epsilon(e^{-\varphi/h}) = 0$.

Proposition 3.6. *Assume that the above assumptions are fulfilled; then P_ϵ admits a temperate supersymmetric structure.*

Proof. Set $\phi = 2\varphi$. Let

$$B^0(x) := \Big(\frac{1}{2h} e^{\phi/h} \int_{\phi(x)}^{2(\sigma-\epsilon)} \alpha'(\frac{s}{2}) e^{-s/h} ds\Big)\theta(x),$$

which is temperate since α is supported in $\{s < \sigma - \epsilon\}$. It follows from the proof of Theorem 1.4 that

$$h\delta B^0 + d\phi \lrcorner B^0 = (\alpha' \circ \varphi) d\varphi \lrcorner \theta = -\delta(\alpha \circ \varphi\theta).$$

Set $B = \chi_\epsilon B^0$. Thanks to the support properties of χ_ϵ and α, one has $\delta B = \chi\delta B^0$. Therefore,

$$h\delta B + d\phi \lrcorner B = -U_\epsilon,$$

which is exactly the eikonal equation we have to solve. Moreover, the same proof as in Theorem 1.4, with m_ϵ instead of m_∞, shows that B is temperate. □

Remark 3.7. Assume X_σ has two distinct connected components. Then U_ϵ has a temperate supersymmetric structure and doesn't satisfy (1.11).

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