EYRING-KRAMERS TYPE FORMULAS FOR SOME PIECEWISE DETERMINISTIC MARKOV PROCESSES

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Abstract. In this work, we give sharp asymptotic equivalents in the small temperature regime of the smallest eigenvalues of the generator of some piecewise deterministic Markov processes (including the Zig-Zag process and the Bouncy Particle Sampler process) with refreshment rate α on the one-dimensional torus $\mathbb{T}$. These asymptotic equivalents are usually called Eyring-Kramers type formulas in the literature. The case when the refreshment rate α vanishes on $\mathbb{T}$ is also considered.

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1. Introduction

1.1. **Purpose and motivation.** Piecewise Deterministic Markov Processes [9] (PDMP hereafter) have recently attracted a lot of attention for their use within the Markov Chain Monte Carlo methodology. Given a potential $U : \mathsf{M} \to \mathbb{R}$ on a d-dimensional manifold M without boundary, such processes are indeed ergodic with respect to the Gibbs measure

$$\pi(dx) = \frac{e^{-\frac{2}{h}U(x)}}{\int_{\mathsf{M}} e^{-\frac{2}{h}U}}dx,$$

dx being the Lebesgue measure on the position space M. Here, the parameter $h > 0$ is proportional to the Boltzmann constant k_B through the relation $h = k_B T$, T being the temperature of the underlying system. When $h > 0$ is fixed, the ergodic properties and the rate of convergence of such processes with or without a refreshment rate α, have for instance been studied in [4, 10, 14, 12, 1, 2, 22, 23] (see also references therein). We also refer to [5] for a spectral analysis of the generator of the one-dimensional zigzag process and its corresponding semigroup when h is fixed.

In many applications in statistical physics where one needs to sample from π, the constant h is very small compared to the energetic barriers of U. Recently, the long time behavior of the semigroups generated by the generators L_h of these processes on L^2 have been investigated in [16] when $h \ll 1$, where it has also been proved that in the set $\{\mathrm{Re}(z) \geq -\epsilon_0 h\}$, L_h has exactly n_0 eigenvalues (n_0 being the number of local minima of U), which are non positive, real, and exponentially small as $h \to 0$. Such results exhibit a metastable behavior of the PDMP when $h \ll 1$, as it is the case for diffusion processes [11]. In this work, we want to push the analysis of the metastability of the PDMP further by proving that each of these n_0 eigenvalues satisfies a so-called Eyring-Kramers type formula when

$h \to 0$. Such sharp formulas describe completely the successive timescales involved in the convergence of the semigroup generated by $-\mathsf{L}_h$ to π, as it has been done in [8] (see also [17]) for elliptic reversible diffusions.

Let us be more precise on our results. We work in the following setting: $d = 1$ (i.e. the dimension of the position space is equal to 1) and $\mathsf{M} = \mathbb{T}$, the one-dimensional torus. In addition, we work with the operator P_h defined below in (1.1) which is (up to a multiplication by $-h$) unitary equivalent to L_h, see (1.2). Thus, all our results are easily translated in terms of L_h. The purpose of this work is to compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h (or equivalently, those of L_h) in the limit $h \to 0$, see Theorems 2 and 3. In particular these asymptotic equivalents allow us to provide a sharp exponential decay rate of the semigroup associated to $-\mathsf{P}_h$ in L^2, as $h \ll 1$, see Corollary 1.4. These results hold when (1.4) is satisfied, which implies that the refreshment rate α does not vanish at critical points of U (i.e. when $\partial_x U = 0$), see (1.5). The case $\alpha = 0$ when $\partial_x U = 0$ is investigated in Section 1.4, where we show that the smallest eigenvalues of P_h satisfy different asymptotic equivalents as $h \to 0$.

To compute sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h we proceed as follows. We first introduce a suitable change of variables to turn the eigenvalue problem $(\mathsf{P}_h - \lambda)u = 0$ into a nonlinear eigenvalue problem: $(\Delta_{V,h} - \lambda W + \lambda^2)g = 0$, where $V = -U$, $\Delta_{V,h}$ is the Witten Laplacian associated with V, and $W = 2|\partial_x U| + \alpha$ (see (2.6) and Lemma 2.1). Using a Grushin problem and known results on the low-lying spectrum of $\Delta_{V,h}$, we prove that if λ is small enough, the kernel of $\Delta_{V,h} + \lambda W + \lambda^2$ is composed of the singularities of a holomorphic function $\lambda \mapsto E_{-+}(\lambda) \in \mathbb{C}^{n_0 \times n_0}$ (see (2.21)). We finally investigate the localization of the singularities of $\lambda \mapsto E_{-+}^{-1}(\lambda)$ to deduce sharp asymptotic equivalents of the n_0 smallest eigenvalues of P_h.

1.2. **Setting.** Let $\mathsf{E} = \{(x, v) \in \mathbb{T} \times \{\pm 1\}\}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus. Consider the following unbounded operator P_h on $L^2(\mathsf{E})$ associated with a smooth function $U : \mathbb{T} \to \mathbb{R}$ defined by

$$\mathsf{P}_h = -v\, \mathsf{d}_{U,h} + 2(v\, \partial_x U)_+ (\mathsf{I} - \mathsf{B}) + \alpha(\mathsf{I} - \pi_v) \tag{1.1}$$

where $\mathsf{d}_{U,h} = h\partial_x + \partial_x U$, $\alpha : \mathbb{T} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ non-negative function (the refreshment rate), $h > 0$ is a parameter proportional to the temperature of the underlying statistical system, I is the identity operator, and B and π_v are defined by

$$\forall (x, v) \in \mathsf{E},\ \mathsf{B} f(x, v) = f(x, -v)$$

and

$$\pi_v f(x, v) = \frac{1}{2}\big(f(x, +1) + f(x, -1)\big)$$

for all $f \in L^2(\mathsf{E})$. Here and in the following, for $u \in L^2(\mathsf{E})$, we denote by $u_\pm = \max(0, \pm u)$. The operator P_h is linked to the Zig-Zag (or the Bouncy Particle Sampler generator) process generator L_h where $\mathsf{L}_h = v\partial_x - (\frac{2}{h} v \partial_x U)_+ (\mathsf{I} - \mathsf{B}) - \frac{1}{h}\alpha(\mathsf{I} - \pi_v)$ through the relation

$$\mathsf{P}_h = -h\, e^{-\frac{1}{h}U}\, \mathsf{L}_h e^{\frac{1}{h}U}. \tag{1.2}$$

We refer to [3, 24, 7] and references therein for more details on these two processes (see also [15] and [13, Section 3.1]). The space $E = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product

$$\langle f, g \rangle_{L^2(\mathsf{E})} = \frac{1}{2} \sum_{v=\pm 1} \int_{\mathbb{T}} f(x,v)\overline{g(x,v)}dx. \tag{1.3}$$

Throughout this work, we will assume that

$$U : \mathbb{T} \to \mathbb{R} \text{ is a smooth Morse function.}$$

By definition, this means that $\partial_x^2 U(x) \neq 0$ when $\partial_x U(x) = 0$, $x \in \mathbb{T}$. In particular, U has a finite number of critical points on $\mathbb{T}$. It is proved in [16] that P_h with domain $D(\mathsf{P}_h) = \{f \in L^2(\mathsf{E}),\ v\partial_x f \in L^2(\mathsf{E})\}$ is maximal accretive. In [16], the authors prove the following spectral result on P_h in the limit $h \to 0$.

Theorem 1. *([16, Theorem 1 and Proposition 13]) Assume that U is a Morse function with n_0 local minimum points. Assume also that*

$$\min_{\mathbb{T}} \big(2|\partial_x U| + \alpha \big) > 0. \tag{1.4}$$

Then, there exist $\epsilon_0 > 0$ and $h_0 > 0$ such that for all $h \in]0, h_0]$, $\sigma(\mathsf{P}_h) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h^2\}$ is made of n_0 real nonnegative eigenvalues $\lambda_{1,h} \leq \ldots \leq \lambda_{n_0,h}$ (counted with algebraic multiplicity). Moreover, their algebraic multiplicities equal their geometric multiplicities and there exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0]$, $\lambda_{n_0,h} \leq e^{-C/h}$. Finally, $\lambda_{1,h} = 0$ and has algebraic multiplicity 1.

Let us mention that $\dim \mathrm{Ker}(\mathsf{P}_h) = 1$ holds for all $h > 0$ (see [16]). In [16], it is assumed that $\min_{\mathbb{T}} \alpha > 0$, but all the results of [16] still hold if the less stringent assumption (1.4) is satisfied. Indeed, if (1.4) holds and $u \in D(\mathsf{P}_h)$, $\mathrm{Re}\, \langle \mathsf{P}_h u, u \rangle_{L^2(\mathsf{E})} \geq r_m \|(\mathsf{I} - \pi_v)u\|^2$, where $r_m = \min_{\mathsf{E}} 2|\partial_x U| + \alpha$. This follows from the following computations. By [16, Lemma 5], since $(\mathsf{I} - \mathsf{B}) = 2(\mathsf{I} - \pi_v)$, for all $u \in D(\mathsf{P}_h)$, one has, denoting by $w = (\mathsf{I} - \pi_v)u$ (notice that $|w(\cdot, 1)| = |w(\cdot, -1)|$):

$$\begin{aligned}
\mathrm{Re}\, \langle \mathsf{P}_h u, u \rangle_{L^2(\mathsf{E})} &= \frac{1}{2} \int_{\mathsf{E}} (2v\, \partial_x U)_+ |(\mathsf{I} - \mathsf{B})u|^2 + \int_{\mathsf{E}} \alpha |(\mathsf{I} - \pi_v)u|^2 \\
&= \int_{\mathsf{E}} 4(v\, \partial_x U)_+ |w|^2 + \int_{\mathsf{E}} \alpha |w|^2 \\
&= \frac{1}{2} \int_{\mathbb{T}} 4(\partial_x U)_+ |w|^2 + \frac{1}{2} \int_{\mathbb{T}} 4(\partial_x U)_- |w|^2 + \int_{\mathsf{E}} \alpha |w|^2 = \int_{\mathsf{E}} (2|\partial_x U| + \alpha)|w|^2.
\end{aligned}$$

Notice that (1.4) implies that α can vanish on $\mathbb{T}$ but not everywhere since (1.4) is equivalent to:

$$\text{for any } x \in \mathbb{T}, \quad \partial_x U(x) = 0 \ \Rightarrow\ \alpha(x) > 0. \tag{1.5}$$

The case when there exists $x \in \mathbb{T}$ such that $\partial_x U(x) = \alpha(x) = 0$ is be considered in Section 1.4.

For our analysis, it will be convenient to introduce the function

$$V = -U,$$

which is also a Morse function on $\mathbb{T}$. We shall denote by $\mathsf{U}^{(0)}$ the set of local minima of V and by $\mathsf{U}^{(1)}$ the set of its local maxima. Since we are on the torus, the set $\mathsf{U}^{(0)}$ and $\mathsf{U}^{(1)}$ have the same cardinality n_0.

Throughout the paper, we will say that a family of complex numbers $(a_h)_{h>0}$ admits a classical expansion in power of h^β (where $\beta > 0$) if there exists a sequence $(a^k)_{k\geq 0}$ such that, for all $K \geq 0$, one has $a_h = \sum_{k=0}^K a^k h^{\beta k} + O(h^{\beta(K+1)})$ when $h \to 0$. In that case, we will denote $a_h \sim \sum_{k\geq 0} a^k h^{\beta k}$.

1.3. **Main results.** In this section we give sharp asymptotics of the exponentially small eigenvalues $\lambda_{j,h}$, $j = 1, \dots, n_0$. Observe that if $n_0 = 1$, there is only one small eigenvalue by Theorem 1 which is 0, and there is thus nothing to compute. We then consider the case when $n_0 \geq 2$. We start with the following theorem which gives the result in the simplified setting of a non-symmetric double well potential V (see Theorem 3 below for the general case $n_0 \geq 2$).

Theorem 2. *Let U be a Morse function. Assume that (1.4) holds and that $\mathsf{U}^{(0)}$ is made of two elements $\mathbf{m}_1$ and $\mathbf{m}_2$ such that $V(\mathbf{m}_1) < V(\mathbf{m}_2)$. Assume also that the two elements $\mathbf{s}_1, \mathbf{s}_2$ of $\mathsf{U}^{(1)}$ satisfy $V(\mathbf{s}_1) > V(\mathbf{s}_2)$. Let $\epsilon_0 > 0$ be as in Theorem 1. Then, the second smallest eigenvalue $\lambda_{2,h}$ of P_h satisfies as $h \to 0$:*

$$\lambda_{2,h} = \zeta_h\, h\, e^{-\frac{2}{h}(V(\mathbf{s}_2)-V(\mathbf{m}_2))}, \text{ where } \zeta_h \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k \text{ and } \zeta_0 = \frac{1}{2\pi}\frac{\sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}}{\alpha(\mathbf{m}_2)}.$$

The situation where $V(\mathbf{m}_1) = V(\mathbf{m}_2)$ and/or $V(\mathbf{s}_1) = V(\mathbf{s}_2)$ could be handled easily by constructing adapted quasimodes in the spirit of [21]. Here, we decided to state our result in the above simplified setting in order to lighten the formulas.

Let us now state our result in the general setting $n_0 \geq 2$. To this end, we need to label the local minima and maxima of V in a suitable way. The following construction is inspired from [19] (see also [21] and [20]). In order to simplify, we assume from now that V uniquely attains its maximum at the point $\mathbf{s}_{max} \in \mathsf{U}^{(1)}$, i.e. that

$$\operatorname{argmax}_{\mathbb{T}} V = \{\mathbf{s}_{max}\}. \tag{1.6}$$

Then, set $\underline{\mathsf{U}}^{(1)} = \mathsf{U}^{(1)} \setminus \{\mathbf{s}_{max}\}$. Since $n_0 \geq 2$, $\underline{\mathsf{U}}^{(1)} \neq \emptyset$. We denote the elements of $V(\underline{\mathsf{U}}^{(1)})$ by $\sigma_2 > \sigma_3 > \dots > \sigma_N$, where $N \geq 2$. For convenience, we also introduce a fictive infinite saddle value $\sigma_1 = +\infty$ and we denote $\Sigma = \{\sigma_1, \dots, \sigma_N\}$. Starting from σ_1, we will recursively associate to each σ_i a finite family of local minima $(\mathbf{m}_{i,j})_j$ and a finite family $(\mathsf{C}_{i,j})_j$ of connected components of $\{V < \sigma_i\}$ in the following way

- ⋆ Let $X_{\sigma_1} = \{x \in \mathbb{T};\ V(x) < \sigma_1 = +\infty\} = \mathbb{T}$. We let $\mathbf{m}_{1,1}$ be any global minimum of V (not necessarily unique) and $\mathsf{C}_{1,1} = \mathbb{T}$. In the following, we will denote $\underline{\mathbf{m}} = \mathbf{m}_{1,1}$.
- ⋆ Next we consider $X_{\sigma_2} = \{x \in \mathbb{T};\ V(x) < \sigma_2\}$. This is the union of its finitely many connected components. Exactly one of these components contains $\mathbf{m}_{1,1}$ and the other components are denoted by $\mathsf{C}_{2,1}, \dots, \mathsf{C}_{2,N_2}$. In each component $\mathsf{C}_{2,j}$, we pick up a point $\mathbf{m}_{2,j}$ which is a global minimum of $V_{|\mathsf{C}_{2,j}}$.
- ⋆ Suppose now that the families $(\mathbf{m}_{k,j})_j$ and $(\mathsf{C}_{k,j})_j$ have been constructed until rank $k = i - 1$. The set $X_{\sigma_i} = \{x \in \mathbb{T};\ V(x) < \sigma_i\}$ has again finitely many connected

components and we label $\mathsf{C}_{i,j}$, $j = 1, \dots, N_i$ those of these components which do not contain any $\mathbf{m}_{k,\ell}$ with $k < i$. In each $\mathsf{C}_{i,j}$ we pick up a point $\mathbf{m}_{i,j}$ which is a global minimum of $V_{|\mathsf{C}_{i,j}}$.

We run the procedure until all the minima have been labeled.

Remark 1.1. *Since we work on $\mathbb{T}$, using the terminology of [19], every maximum point $\mathbf{s} \in \underline{\mathsf{U}}^{(1)}$ is a separating saddle points (ssp) and $\mathbf{s}_{max}$ is not a ssp. In the case where V attains its global maximum at several distinct points $\mathbf{s}_{max,1}, \dots, \mathbf{s}_{max,k}$, every maximum point is separating and the situation can be handled easily by a modification of the construction above. However, this would lead to a slightly more complicated presentation of the results that we prefer to avoid in this work.*

We now recall some constructions of [21] and [20] that will be useful in the sequel. Throughout we denote $\underline{\mathsf{U}}^{(0)} = \mathsf{U}^{(0)} \setminus \{\underline{\mathbf{m}}\}$, $\mathbf{s}_1$ is a fictive saddle point such that $V(\mathbf{s}_1) = \sigma_1 = +\infty$. For any set A, $\mathcal{P}(A)$ denotes the power set of A. From the above labelling we define two mappings

$$\mathsf{C} : \mathsf{U}^{(0)} \to \mathcal{P}(\mathbb{R}^d) \qquad \text{and} \qquad \mathbf{j} : \mathsf{U}^{(0)} \to \mathcal{P}(\underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_1\}),$$

as follows: for every $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N_i\}$,

$$\mathsf{C}(\mathbf{m}_{i,j}) := \mathsf{C}_{i,j}, \tag{1.7}$$

and

$$\mathbf{j}(\underline{\mathbf{m}}) := \{\mathbf{s}_1\} \qquad \text{and} \qquad \mathbf{j}(\mathbf{m}_{i,j}) := \partial \mathsf{C}_{i,j} \cap \underline{\mathsf{U}}^{(1)} \text{ for } i \geq 2. \tag{1.8}$$

In particular, we have $\mathsf{C}(\underline{\mathbf{m}}) = \mathbb{T}$ and for all $i, j \in \{1, \dots, N\}$, one has $\emptyset \neq \mathbf{j}(\mathbf{m}_{i,j}) \subset \{V = \sigma_i\}$. We then define the mappings

$$\boldsymbol{\sigma} : \mathsf{U}^{(0)} \to \Sigma \qquad \text{and} \qquad S : \mathsf{U}^{(0)} \to (0, +\infty],$$

by

$$\forall \mathbf{m} \in \mathsf{U}^{(0)}, \qquad \boldsymbol{\sigma}(\mathbf{m}) := V(\mathbf{j}(\mathbf{m})) \qquad \text{and} \qquad S(\mathbf{m}) := \boldsymbol{\sigma}(\mathbf{m}) - V(\mathbf{m}), \tag{1.9}$$

where, with a slight abuse of notation, we have identified the set $V(\mathbf{j}(\mathbf{m}))$ with its unique element. Note that $S(\mathbf{m}) = +\infty$ if and only if $\mathbf{m} = \underline{\mathbf{m}}$. With the above notations, our last assumption is the following:

(1.10)

- ⋆ Equation (1.6) is satisfied.
- ⋆ For any $\mathbf{m} \in \mathsf{U}^{(0)}$, $\mathbf{m}$ is the unique global minimum of $V_{|\mathsf{C}(\mathbf{m})}$.
- ⋆ For all $\mathbf{m}' \in \mathsf{U}^{(0)} \setminus \{\mathbf{m}\}$, $\mathbf{j}(\mathbf{m}) \cap \mathbf{j}(\mathbf{m}') = \emptyset$.
- ⋆ The map S: $\mathsf{U}^{(0)} \to (0, +\infty]$ is injective.

In particular, (1.10) implies that V uniquely attains its global minimum on $\mathbb{T}$ at $\underline{\mathbf{m}} \in \mathsf{U}^{(0)}$. In the following, when (1.10) holds, we label the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V such that $(S(\mathbf{m}_j))_{j \in \{1,\dots,n_0\}}$ is decreasing (see (1.9)), that is

$$\text{for all } j \in \{1, \dots, n_0 - 1\},\ S(\mathbf{m}_{j+1}) < S(\mathbf{m}_j) \text{ and } S(\mathbf{m}_1) = +\infty \text{ (i.e. } \mathbf{m}_1 = \underline{\mathbf{m}}). \tag{1.11}$$

Remark 1.2. *Notice that, in the geometrical setting of Theorem 2, one has by construction of S (see (1.9)) and $\mathbf{j}$ (see (1.8)),*

$$S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2) \ \text{and}\ \mathbf{j}(\mathbf{m_2}) = \{\mathbf{s_2}\}.$$

The main result of this work is the following.

Theorem 3. *Let $V = -U$ be a Morse function. Assume that (1.4) and (1.10) are satisfied. Let $\epsilon_0 > 0$ be given by Theorem 1. Then, there exists $h_0 > 0$ such that, for all $h \in]0, h_0]$, the n_0 eigenvalues $\lambda_{1,h} \leq \ldots \leq \lambda_{n_0,h}$ of P_h in $\{\mathrm{Re}(z) \leq \epsilon_0 h^2\}$ (counted with algebraic multiplicity) satisfy: $\lambda_{1,h} = 0$ and, for all $j \in \{2, \ldots, n_0\}$,*

$$\lambda_{j,h} = \zeta_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h}S(\mathbf{m}_j)}, \tag{1.12}$$

where $S : \mathsf{U}^{(0)} \to (0, +\infty]$ is defined in (1.9) (see also (1.11)) and ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$\zeta_0(\mathbf{m}_j) = \frac{1}{2\pi} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_j)} \frac{\sqrt{|V''(\mathbf{m}_j)V''(\mathbf{s})|}}{\alpha(\mathbf{m}_j)}, \tag{1.13}$$

where $\mathbf{j} : \mathsf{U}^{(0)} \to \mathcal{P}(\underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_1\})$ is defined in (1.8).

Notice that, according to Theorem 3, $\lambda_{j,h}$ is a simple eigenvalue of P_h for all $j \in \{1, \ldots, n_0\}$ and h small enough (that is $\dim\big(\mathrm{Ker}(\mathsf{P}_h - \lambda_{j,h})^m\big) = 1$ for every $m \in \mathbb{N}^*$).

Remark 1.3. *The assumptions (1.6) and $\big(S\colon \mathsf{U}^{(0)} \to (0, +\infty]$ is injective$\big)$ in (1.10) are generic. They can be relaxed following the procedure of [20]. The whole assumption (1.10) could also be relaxed by following the strategy of [21].*

Let us recall that by the Hille-Yosida Theorem, $-\mathsf{P}_h$ generates a strongly continuous contraction semigroup $(e^{-t\mathsf{P}_h})_{t\geq 0}$ on $L^2(\mathsf{E})$. Let us recall that under the assumptions of Theorem 3, according to [16, Theorem 2], we have, for some $c > 0$ and every $h > 0$ small enough:

$$e^{-t\mathsf{P}_h} = \sum_{j=1}^{n_0} e^{-t\lambda_{j,h}} \Pi_{j,h} + O(e^{-cth^2}) \quad \text{in } \mathcal{L}(L^2(\mathsf{E})),$$

where, for $j = 1, \ldots, n_0$, $\Pi_{j,h}$ is the spectral projector associated with the eigenvalue $\lambda_{j,h}$ of P_h, and $\Pi_{j,h} = O(1)$ in $\mathcal{L}(L^2(\mathsf{E}))$. Using in addition Theorem 3, we get sharp asymptotic equivalents of the different timescales $1/\lambda_{j,h}$ involved in the return to equilibrium. This leads in particular to the following accurate exponential decay rate in $L^2(\mathsf{E})$ of the semigroup $(e^{-t\mathsf{P}_h})_{t\geq 0}$ as $h \ll 1$.

Corollary 1.4. *Let U be a Morse function. Assume that (1.4) and (1.10) are satisfied. Then, there exist $C > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$, it holds for all $t \geq 0$:*

$$\big\| e^{-t\mathsf{P}_h} - \Pi_{1,h} \big\|_{\mathcal{L}(L^2(\mathsf{E}))} \leq C e^{-t\lambda_{2,h}},$$

where, as $h \to 0$, $\lambda_{2,h} = \zeta_h(\mathbf{m}_2)\, h\, e^{-\frac{2S(\mathbf{m}_2)}{h}}$ with $\zeta_h(\mathbf{m}_2) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2)$ given by (1.13), and where $\Pi_{1,h}$ is the $L^2(\mathsf{E})$ orthogonal projection on $\mathrm{Span}(e^{-U/h} 1_{\{\pm 1\}})$, where $1_{\{\pm 1\}}$ is the constant function on $\{\pm 1\}$ which equals 1.

1.4. Extension of the results to the case when (1.4) is not satisfied.

In this section, we assume that

$$\text{for any } x \in \mathbb{T}, \quad \partial_x U(x) = 0 \;\Rightarrow\; \alpha(x) = 0, \tag{1.14}$$

and we give asymptotic equivalents of the n_0 first eigenvalues of P_h when (1.14) holds (and then, in particular, when there is no refreshment at all, i.e. when $\alpha = 0$).

Theorem 4. *Let U be a Morse function. Assume that (1.14) and (1.10) are satisfied. Then, for any $c > 0$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0]$, $\sigma(\mathsf{P}_h) \cap D(0, ch^2)$ is made of n_0 real eigenvalues $\lambda_{1,h} \leq \ldots \leq \lambda_{n_0,h}$ (counted with geometric multiplicity), which all have geometric multiplicity one. In addition, $\lambda_{1,h} = 0$ and, for all $j \in \{2, \ldots, n_0\}$,*

$$\lambda_{j,h} = \zeta_h(\mathbf{m}_j)\sqrt{h}\, e^{-\frac{2}{h}S(\mathbf{m}_j)}, \tag{1.15}$$

where ζ_h admits a classical expansion $\zeta_h(\mathbf{m}_j) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ with

$$\zeta_0(\mathbf{m}_j) = \frac{1}{4} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_j)} \sqrt{\frac{|V''(\mathbf{s})|}{\pi}}. \tag{1.16}$$

Compared with Theorem 3, we cannot exclude the existence of other eigenvalues in the strip $0 < \mathrm{Re}(z) < ch^2$ with large imaginary part. The reason for this is that Theorem 1 does not apply when α and $\partial_x U$ vanish simultaneously.

The prefactor (1.16) is consistent with the one obtained in [23, Theorem 1.1] for the expected hitting time of a 1-dimensional PDMP with no refreshment. Indeed, when $\alpha = 0$ and V, and thus $U = -V$, is a non-symmetric double well potential (i.e. when $\mathsf{U}^{(0)} = \{\mathbf{m}_1, \mathbf{m}_2\}$ with $V(\mathbf{m}_1) < V(\mathbf{m}_2)$ and $\mathsf{U}^{(1)} = \{\mathbf{s}_1, \mathbf{s}_2\}$ with $V(\mathbf{s}_1) > V(\mathbf{s}_2)$), Theorem 4 and [23, Theorem 1.1] imply that $h\,\lambda_{2,h}\,\mathbb{E}[\tau] = 1 + o(1)$ as $h \to 0$, where τ is the first time the process (X_t, Y_t) with generator L_h on $\mathbb{T} \times \{\pm 1\}$ hits $\{\mathbf{m}_2\}$ when it starts at $(\mathbf{s}_2, -1)$. Let us also recall here that $h\,\lambda_{2,h}$ is the first nonzero eigenvalue of $-\mathsf{L}_h$.

Of course, the situation where the refreshing function α vanishes at some critical points $x \in \mathbb{T}$ of U but not at all could easily be handled and would lead, for each eigenvalue $\lambda_{j,h}$, either to the formula given in Theorem 3 or to the formula given in Theorem 4, depending on whether $\alpha(\mathbf{m}_j) > 0$ or $\alpha(\mathbf{m}_j) = 0$.

2. Reduction to a finite dimensional problem

In this section, we prove that, if λ is small enough, we can reduce the infinite dimensional problem $(\mathsf{P}_h - \lambda_h)u = 0$ into a finite dimensional nonlinear eigenvalue problem.

2.1. A suitable change of variables.

Let us introduce some notation. For $\lambda \in \mathbb{C}$ and $r > 0$ we denote $D(\lambda, r) = \{z \in \mathbb{C}, |z - \lambda| < r\}$. For two families of numbers $a = (a_h)_{h>0}$ and $b = (b_h)_{h>0}$, we say that $a \in \mathcal{E}_{cl}(b)$ if there exists a family $c = (c_h)_{h>0}$ such that $a_h = b_h c_h$ and c admits a classical expansion $c_h \sim \sum_{j\geq 0} c_j h^j$ with $c_0 = 1$ as $h \to 0$. In all this work, $C > 0$ and $c > 0$ are constant which are independent of h and which can change from one occurence to another.

Recall that $\mathsf{E} = \mathbb{T} \times \{\pm 1\}$ is endowed with the natural scalar product (1.3). Let $\mathsf{F} = L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and let $\langle , \rangle_{\mathsf{F}}$ be the Hilbertian structure induced by the isomorphism

$$\begin{aligned} \Omega_1 : L^2(\mathbb{T}\times\{\pm1\}) &\to L^2(\mathbb{T}) \times L^2(\mathbb{T}) \\ f &\longmapsto \begin{pmatrix} f(.,+1) \\ f(.,-1) \end{pmatrix}. \end{aligned} \tag{2.1}$$

Then, Ω_1 is unitary from $(\mathsf{E}, \langle , \rangle_{L^2(\mathsf{E})})$ onto $(\mathsf{F}, \langle , \rangle_{\mathsf{F}})$, and

$$\Omega_1 D(\mathsf{P}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T}).$$

Direct computations show that

$$\Omega_1\, v\, \mathsf{d}_{U,h}\, \Omega_1^{-1} = \begin{pmatrix} \mathsf{d}_{U,h} & 0 \\ 0 & -\mathsf{d}_{U,h} \end{pmatrix}, \quad \Omega_1\, (v\, \partial_x U)_+\, \Omega_1^{-1} = \begin{pmatrix} (\partial_x U)_+ & 0 \\ 0 & (\partial_x U)_- \end{pmatrix}$$

$$\Omega_1 \mathsf{B} \Omega_1^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \Omega_1 \pi_v \Omega_1^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Combining these identities with (1.1), we get

$$\Omega_1 \mathsf{P}_h \Omega_1^{-1} = \begin{pmatrix} -\mathsf{d}_{U,h} & 0 \\ 0 & \mathsf{d}_{U,h} \end{pmatrix} + 2\begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix} + \frac{\alpha}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.2}$$

We now change the variables in F and consider the unitary transformation $\Omega_2 : \mathsf{F} \to \mathsf{F}$ defined by

$$\Omega_2(f,g) = \frac{1}{\sqrt{2}}(f+g, g-f).$$

Consider the two vectors of $\mathbb{R}^2$ given by $e_1 = \binom{1}{1}$ and $e_2 = \binom{-1}{1}$. Then, one has:

$$\Omega_1 \mathsf{P}_h \Omega_1^{-1}(f e_1) = \mathsf{d}_{U,h} f e_2. \tag{2.3}$$

On the other hand,

$$\begin{aligned} 2\begin{pmatrix} (\partial_x U)_+ & -(\partial_x U)_+ \\ -(\partial_x U)_- & (\partial_x U)_- \end{pmatrix}\begin{pmatrix} -f \\ f \end{pmatrix} &= 4f\begin{pmatrix} -(\partial_x U)_+ \\ (\partial_x U)_- \end{pmatrix} \\ &= 2((\partial_x U)_- - (\partial_x U)_+)\begin{pmatrix} f \\ f \end{pmatrix} + 2((\partial_x U)_- + (\partial_x U)_+)\begin{pmatrix} -f \\ f \end{pmatrix} \\ &= -2\partial_x U f e_1 + 2|\partial_x U| f e_2. \end{aligned}$$

It follows that:

$$\begin{aligned} \Omega_1 \mathsf{P}_h \Omega_1^{-1}(f e_2) &= \mathsf{d}_{U,h} f e_1 - 2\partial_x U f e_1 + (\alpha + 2|\partial_x U|) f e_2 \\ &= \mathsf{d}_{-U,h} f e_1 + (\alpha + 2|\partial_x U|) f e_2 \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\Omega_2 \Omega_1 \mathsf{P}_h \Omega_1^{-1} \Omega_2^{-1} = \begin{pmatrix} 0 & \mathsf{d}_{-U,h} \\ \mathsf{d}_{U,h} & \alpha + 2|\partial_x U| \end{pmatrix}.$$

Set $\mathsf{Q}_h := \Omega_2 \Omega_1 \mathsf{P}_h \Omega_1^{-1} \Omega_2^{-1}$ with domain $D(\mathsf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ on $L^2(\mathbb{T})^2$, i.e.

$$\mathsf{Q}_h = \begin{pmatrix} 0 & \mathsf{d}_{-U,h} \\ \mathsf{d}_{U,h} & W \end{pmatrix} = \begin{pmatrix} 0 & \mathsf{d}_{V,h} \\ -\mathsf{d}_{V,h}^* & W \end{pmatrix} \tag{2.5}$$

where we recall that $V = -U$ and where we denote

$$W(x) := \alpha(x) + 2|\partial_x U(x)| = \alpha(x) + 2|\partial_x V(x)|,\ x \in \mathbb{T}.$$

Since the transformation $\Omega_2\Omega_1$ is unitary from E onto F,

$$\sigma(\mathsf{P}_h) = \sigma(\mathsf{Q}_h).$$

We will in the following study the spectrum of Q_h to prove Theorems 2 and 3. Introduce the semiclassical Witten Laplacian $\Delta_{V,h}$ associated with V on $\mathbb{T}$, that is

$$\Delta_{V,h} = \mathsf{d}^*_{V,h}\mathsf{d}_{V,h} = -h^2\Delta + |\partial_x V(x)|^2 - h\partial_x^2 V(x), \quad \text{with domain } D(\Delta_{V,h}) = H^2(\mathbb{T}).$$

For $\lambda \in \mathbb{C}$, we finally define

$$\mathsf{T}_h(\lambda) = \Delta_{V,h} - \lambda W + \lambda^2 \text{ with domain } D(\Delta_{V,h}). \tag{2.6}$$

The following result is the key point of our analysis: it establishes an equivalence between the spectrum of Q_h and the kernels of the operators $\mathsf{T}_h(\lambda)$, $\lambda \in \mathbb{C}$.

Lemma 2.1. *The operator Q_h with domain $D(\mathsf{Q}_h) = H^1(\mathbb{T}) \times H^1(\mathbb{T})$ is closed and has compact resolvent. In particular, it has only discrete spectrum. Moreover, $\lambda = 0$ is a simple eigenvalue of Q_h and* $\dim(\operatorname{Ker}\mathsf{T}_h(0)) = \dim(\operatorname{Ker}\mathsf{Q}_h) = 1$*. Besides, for every $\lambda \in \mathbb{C}\setminus\{0\}$, the application*

$$\begin{array}{rccc} \Psi: & \operatorname{Ker}(\mathsf{T}_h(\lambda)) & \longrightarrow & \operatorname{Ker}(\mathsf{Q}_h - \lambda) \\ & g & \longmapsto & \begin{pmatrix} \frac{1}{\lambda}\mathsf{d}_{V,h}g \\ g \end{pmatrix} \end{array} \tag{2.7}$$

is a linear isomorphism. Eventually, any $\mu \in \mathbb{C}$ is a singularity of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ if and only if there exists $g \in H^2(\mathbb{T})$, $g \neq 0$, such that $\mathsf{T}_h(\mu)g = 0$.

Proof. The operator Q_h is closed on $L^2(\mathbb{T})^2$ since it is a bounded perturbation of the closed operator $(0, h\partial_x; h\partial_x, 0)$. The resolvent of Q_h is compact since the injection $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ is compact. Since Q_h is unitarily equivalent to P_h (see (2.5)), it follows from Theorem 1 that $\lambda = 0$ is a simple eigenvalue of Q_h. Moreover, since $\mathsf{T}_h(0) = \Delta_{V,h}$, one gets $\operatorname{Ker}\mathsf{T}_h(0) = \mathbb{C}e^{-\frac{V}{h}}$ and then $\dim(\operatorname{Ker}\mathsf{T}_h(0)) = 1$.

Let us now consider $\lambda \in \mathbb{C}\setminus\{0\}$. For any $g \in \operatorname{Ker}(\mathsf{T}_h(\lambda))$, one has $g \in H^2(\mathbb{T})$ and thus $(\frac{1}{\lambda}\mathsf{d}_{V,h}g, g) \in D(\mathsf{Q}_h)$. It follows moreover from $\mathsf{T}_h(\lambda)g = 0$ that

$$\mathsf{Q}_h\Psi(g) = \begin{pmatrix} 0 & \mathsf{d}_{V,h} \\ -\mathsf{d}^*_{V,h} & W \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda}\mathsf{d}_{V,h}g \\ g \end{pmatrix} = \begin{pmatrix} \mathsf{d}_{V,h}g \\ (-\frac{1}{\lambda}\Delta_{V,h} + W)g \end{pmatrix} = \lambda\Psi(g).$$

This proves that Ψ is well defined. The linearity and the injectivity of Ψ are obvious. To prove its surjectivity, consider $u = \binom{f}{g} \in D(\mathsf{Q}_h)$ such that $\mathsf{Q}_h u = \lambda u$, i.e. such that

$$\begin{cases} \mathsf{d}_{V,h}g = \lambda f, \\ -\mathsf{d}^*_{V,h}f + Wg = \lambda g. \end{cases}$$

It follows from the first equation that $g \in H^2(\mathbb{T})$ and, since $\lambda \neq 0$, that $u = \binom{\frac{1}{\lambda}\mathsf{d}_{V,h}g}{g}$. Moreover, applying $\mathsf{d}^*_{V,h}$ to the second equation leads to

$$\Delta_{V,h}g = \lambda\mathsf{d}^*_{V,h}f = \lambda W g - \lambda^2 g.$$

Consequently, $g \in D(\Delta_{V,h})$ and $\mathsf{T}_h(\lambda)g = 0$, which proves the surjectivity.

It remains to prove the last statement of Lemma 2.1. To this end, let us consider $\mu \in \mathbb{C}$. It is a singularity of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ if and only if $\mathsf{T}_h(\mu) : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is not invertible. Furthermore, since $\Delta_{V,h} + 1 : H^2(\mathbb{T}) \to L^2(\mathbb{T})$ is invertible,

$$\mathsf{T}_h(\lambda) = \left[1 - (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}\right](\Delta_{V,h} + 1)$$

is not invertible if and only if $1 - \mathsf{B}(\lambda) : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is not invertible, where $\mathsf{B}(\lambda) = (1 + \lambda W - \lambda^2)(\Delta_{V,h} + 1)^{-1}$. Since $(\Delta_{V,h} + 1)^{-1}$: $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is compact, so is $\mathsf{B}(\lambda)$. The Fredholm alternative then implies that $\mathsf{T}_h(\mu)$ is not invertible if and only if there exists $u \in L^2(\mathbb{T})$, $u \neq 0$ such that $(1 - \mathsf{B}(\lambda))u = 0$, that is if and only if there exists $g \in H^2(\mathbb{T})$, $g \neq 0$ such that $\mathsf{T}_h(\mu)g = 0$. This concludes the proof of Lemma 2.1. ◻

According to Lemma 2.1, for every $\mu \in \mathbb{C}$:

$$\mu \in \sigma_d(\mathsf{P}_h) = \sigma(\mathsf{P}_h) \text{ if and only if } \mu \text{ is a singularity of } \lambda \mapsto \mathsf{T}_h(\lambda)^{-1} \tag{2.8}$$

and, for such a μ, one has

$$\dim(\mathrm{Ker}(\mathsf{P}_h - \mu)) = \dim(\mathrm{Ker}(\mathsf{T}_h(\mu)). \tag{2.9}$$

To prove Theorems 2 and 3, we will thus investigate the singularities of $\lambda \mapsto \mathsf{T}_h(\lambda)^{-1}$ near 0.

2.2. **Finite dimensional reduction, a Grushin problem.** In this section, we show that the singularities of $\mathsf{T}_h(\lambda)$ near 0 are those of a matrix valued holomorphic function $E_{-+}(\lambda)$. To this end, we will construct a so-called Grushin problem.

To build the Grushin problem, we first need to recall known results on the low-lying spectrum of $\Delta_{V,h}$. From the early works of Witten [26] and Helffer-Sjöstrand [18], we know that there is a one-to-one correspondence between the local minima of V and the smallest eigenvalues of $\Delta_{V,h}$. This correspondence was further investigated by several authors and sharp asymptotic equivalents of these small eigenvalues were finally obtained in [8] and [17] under assumptions on the relative positions of the minima and saddle points, and in [21] in the general case. The following version gives the general form of these asymptotic equivalents in a non-degenerate setting.

Theorem 5. [8], [17]. *Let $U = -V$ be a Morse function. There exist $\epsilon_* > 0$, $C > 0$, and $h_0 > 0$ such that, for all $h \in]0, h_0]$, the nonnegative self-adjoint operator $(\Delta_{V,h}, H^2(\mathbb{T}))$ admits exactly n_0 eigenvalues (counted with algebraic multiplicity) in $[0, \epsilon_* h]$:*

$$\sigma(\Delta_{V,h}) \cap [0, \varepsilon_* h] = \{0, \mu^{\Delta}_{2,h}, \dots, \mu^{\Delta}_{n_0,h}\}, \tag{2.10}$$

where 0 is a simple eigenvalue of $\Delta_{V,h}$. Let us order $\{\mu^{\Delta}_{2,h}, \dots, \mu^{\Delta}_{n_0,h}\}$ such that $\mu^{\Delta}_{j,h} \leq \mu^{\Delta}_{j+1,h}$ for $j = 1, \dots, n_0 - 1$. Then, if (1.10) holds, it holds for all $j = 2, \dots, n_0$:

$$\mu^{\Delta}_{j,h} = a_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h} S(\mathbf{m}_j)}, \tag{2.11}$$

where $S : \mathsf{U}^{(0)} \to (0, +\infty]$ *is defined in* (1.9) *(see also* (1.11)*) and* $a_h(\mathbf{m})$ *admits a classical expansion* $a_h(\mathbf{m}_j) \sim \sum_{k\geq 0} h^k a_k(\mathbf{m}_j)$ *with*

$$a_0(\mathbf{m}_j) = \frac{1}{2\pi} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_j)} \sqrt{|V''(\mathbf{m}_j)V''(\mathbf{s})|}, \tag{2.12}$$

where $\mathbf{j} : \mathsf{U}^{(0)} \to \mathcal{P}(\underline{\mathsf{U}}^{(1)} \cup \{\mathbf{s}_1\})$ *is defined in* (1.8).

We are now in position to construct a Grushin problem. Set

$$\Pi = \pi_{[0,\varepsilon_* h]}(\Delta_{V,h}) \text{ and } \mathsf{E}_0 = \mathrm{Ran}(\Pi),$$

where $\pi_{[a,b]}(\Delta_{V,h})$ is the spectral projector associated with $\Delta_{V,h}$ and the interval $[a,b]$. According to Theorem 5, the space E_0 has dimension n_0 for all $h > 0$ small enough. Let $(\Psi_j)_{j=1,\dots n_0}$ be an orthonormal basis of the space E_0. We assume without loss of generality that Ψ_1 is proportional to $e^{-V/h}$. Introduce the operators

$$\begin{aligned} R_- : \mathbb{C}^{n_0} &\to L^2(\mathbb{T}) \\ u &\mapsto \sum_{j=1}^{n_0} u_j \Psi_j \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} R_+ : L^2(\mathbb{T}) &\to \mathbb{C}^{n_0} \\ u &\mapsto (\langle u, \Psi_j\rangle_{L^2})_j. \end{aligned} \tag{2.14}$$

We equip $\mathbb{C}^{n_0}$ with the ℓ_2 norm. Notice that $R_+R_- = \mathsf{I}_{\mathbb{C}^{n_0}}$ and $R_-R_+ = \Pi$. In addition, $\|R_+\| \leq 1$ and $\|R_-\| \leq 1$, for all $h > 0$. These inequalities will be used many times in what follows. From now on, we denote

$$\widehat{\Pi} = \mathsf{I} - \Pi.$$

Lemma 2.2. *Let* U *be a Morse function. For any* $\lambda \in \mathbb{C}$*, the operator* $\widehat{\mathsf{T}}_h(\lambda) := \widehat{\Pi}\mathsf{T}_h(\lambda)\widehat{\Pi}$ *acting on* $\widehat{\Pi}L^2 := \widehat{\Pi}(L^2(\mathbb{T}))$ *with domain* $D(\widehat{\mathsf{T}}_h(\lambda)) = \{u \in \widehat{\Pi}L^2,\ u \in H^2(\mathbb{T})\}$ *is closed. Moreover, there exist* $C, \epsilon_0, h_0 > 0$ *such that for all* $h \in]0, h_0]$*, and for all* $\lambda \in D(0, \epsilon_0 h)$*,* $\widehat{\mathsf{T}}_h(\lambda)$ *is invertible, holomorphic with respect to* λ*, and* $\|\widehat{\mathsf{T}}_h(\lambda)^{-1}\| \leq Ch^{-1}$.

Proof. The proof is very close to the proof of Lemma 2.1 in [20]. We sketch it for reader's convenience. We first observe that since W is bounded, the operator $\mathsf{T}_h(\lambda)$ with domain $D(\mathsf{T}_h(\lambda)) = D(\Delta_{V,h}) = H^2(\mathbb{T})$ is closed and densely defined. Its adjoint $\mathsf{T}_h(\lambda)^*$ satisfies $D(\mathsf{T}_h(\lambda)^*) = H^2(\mathbb{T})$. Suppose that $(u_n, \widehat{\mathsf{T}}_h(\lambda)u_n) \in D(\widehat{\mathsf{T}}_h(\lambda)) \times L^2(\mathbb{T})$ converges to $(u,v) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$. For any $j = 1, \dots, n_0$, one has $\Psi_j \in D(\Delta_{V,h}) = D(\mathsf{T}_h(\lambda)^*)$ and hence for all $n \in \mathbb{N}$, one has

$$\Pi\mathsf{T}_h(\lambda)u_n = \sum_{j=1}^{n_0} \langle \mathsf{T}_h(\lambda)u_n, \Psi_j\rangle \Psi_j = \sum_{j=1}^{n_0} \langle u_n, \mathsf{T}_h(\lambda)^*\Psi_j\rangle \Psi_j. \tag{2.15}$$

Consequently, the sequence $(\Pi\mathsf{T}_h(\lambda)u_n)_n$ converges and using the identity

$$\mathsf{T}_h(\lambda)u_n = \Pi\mathsf{T}_h(\lambda)u_n + \widehat{\mathsf{T}}_h(\lambda)u_n,$$

it follows that $(\mathsf{T}_h(\lambda)u_n)$ converges. Since $\mathsf{T}_h(\lambda)$ is closed as a bounded perturbation of a closed operator, and $u_n \in H^2(\mathbb{T}) = D(\mathsf{T}_h(\lambda))$, one deduces that $u \in D(\mathsf{T}_h(\lambda))$ and

$\mathsf{T}_h(\lambda)u = \lim \mathsf{T}_h(\lambda)u_n$. Since $\widehat{\Pi} = 1 - \Pi$ is bounded, this implies that $u \in D(\widehat{\mathsf{T}}_h(\lambda))$ and $\widehat{\mathsf{T}}_h(\lambda)u = v$ which proves that $\widehat{\mathsf{T}}_h(\lambda)$ is closed. In addition, the operator $\widehat{\mathsf{T}}_h(\lambda)$ is clearly densely defined on $\widehat{\Pi}L^2$. Let us now study the invertibility of $\widehat{\mathsf{T}}_h(\lambda)$. For any $u \in D(\widehat{\mathsf{T}}_h(\lambda))$, one has by definition

$$\mathrm{Re}\langle\widehat{\mathsf{T}}_h(\lambda)u, u\rangle = \mathrm{Re}\langle(\Delta_{V,h} - \lambda W + \lambda^2)\widehat{\Pi}u, \widehat{\Pi}u\rangle.$$

Using Theorem 5, this implies that for $|\lambda| < \epsilon_0 h$, one has

$$\begin{aligned}\mathrm{Re}\langle\widehat{\mathsf{T}}_h(\lambda)u, u\rangle &\geq \epsilon_* h\|\widehat{\Pi}u\|^2 - \mathrm{Re}(\lambda)\langle W\widehat{\Pi}u, \widehat{\Pi}u\rangle + \mathrm{Re}(\lambda^2)\|\widehat{\Pi}u\|^2 \\ &\geq (\epsilon_* - \epsilon_0\|W\|_{L^\infty})h\|\widehat{\Pi}u\|^2 - \epsilon_0^2 h^2\|\widehat{\Pi}u\|^2 \\ &\geq \frac{\epsilon_*}{2}h\|\widehat{\Pi}u\|^2\end{aligned}$$

for ϵ_0 small enough. This proves that $\widehat{\mathsf{T}}_h(\lambda)$ is injective when $|\lambda| < \epsilon_0 h$. We observe that the same proof shows that $\widehat{\mathsf{T}}_h(\lambda)^* = \widehat{\Pi}\mathsf{T}_h(\lambda)^*\widehat{\Pi}$ is injective. Moreover, Cauchy-Schwarz inequality implies that for all $u \in D(\widehat{\mathsf{T}}_h(\lambda))$,

$$\|\widehat{\mathsf{T}}_h(\lambda)u\| \geq \frac{\epsilon_*}{2}h\|\widehat{\Pi}u\|. \tag{2.16}$$

Let us now prove that $\widehat{\mathsf{T}}_h(\lambda)$ is surjective if $|\lambda| < \epsilon_0 h$. We first observe that $\mathrm{Ran}(\widehat{\mathsf{T}}_h(\lambda))$ is closed since $\widehat{\mathsf{T}}_h(\lambda)$ is closed and the convergence of any sequence $(\widehat{\mathsf{T}}_h(\lambda)u_n)$ implies the convergence of (u_n) thanks to (2.16). Hence it is sufficient to prove that $\mathrm{Ran}(\widehat{\mathsf{T}}_h(\lambda))^\perp = \{0\}$. Suppose that $v \in L^2(\mathbb{T})$ satisfies $\langle\widehat{\mathsf{T}}_h(\lambda)u, v\rangle = 0$ for all $u \in D(\widehat{\mathsf{T}}_h(\lambda))$. Then

$$\langle\mathsf{T}_h(\lambda)\widehat{\Pi}u, \widehat{\Pi}v\rangle = 0$$

which implies that $\widehat{\Pi}v \in D(\mathsf{T}_h(\lambda)^*)$ and $\langle u, \widehat{\Pi}\mathsf{T}_h(\lambda)^*\widehat{\Pi}v\rangle = 0$ for all $u \in D(\widehat{\mathsf{T}}_h(\lambda))$ which is dense in $\widehat{\Pi}L^2$. Hence $\widehat{\Pi}\mathsf{T}_h(\lambda)^*\widehat{\Pi}v = 0$ and since $\widehat{\mathsf{T}}_h(\lambda)^*$ is injective, this implies that $v = 0$. This ends the proof of the lemma. □

We now introduce the Grushin operator

$$\mathcal{K}_h(\lambda) = \begin{pmatrix} \mathsf{T}_h(\lambda) & R_- \\ R_+ & 0 \end{pmatrix}. \tag{2.17}$$

Proposition 2.3. *Let U be a Morse function. There exist $\epsilon_0 > 0$ and $h_0 > 0$ such that, for all $h \in]0, h_0]$ and all $\lambda \in D(0, \epsilon_0 h)$, the operator $\mathcal{K}_h(\lambda)$ is invertible. Moreover, its inverse $\mathcal{E}_h(\lambda)$ writes*

$$\mathcal{E}_h(\lambda) = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{-+}(\lambda) \end{pmatrix},$$

where E, E_-, E_+, E_{-+} are holomorphic in $D(0, \epsilon_0 h)$ and satisfy the following formulas:

$$E_+(\lambda) = R_- - \widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\mathsf{T}_h(\lambda)R_-, \quad E_-(\lambda) = R_+ - R_+\mathsf{T}_h(\lambda)\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi},$$

$$E_{-+}(\lambda) = -R_+\mathsf{T}_h(\lambda)R_- + R_+\mathsf{T}_h(\lambda)\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}\mathsf{T}_h(\lambda)R_- : \mathbb{C}^{n_0} \to \mathbb{C}^{n_0}, \tag{2.18}$$

and

$$E(\lambda) = \widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}. \tag{2.19}$$

Moreover, for $\lambda \in D(0,\epsilon_0 h)$, $\mathsf{T}_h(\lambda)$ is invertible if and only if the matrix $E_{-+}(\lambda)$ is invertible, in which case it holds

$$\mathsf{T}_h(\lambda)^{-1} \;=\; E(\lambda) - E_+(\lambda)E_{-+}^{-1}(\lambda)E_-(\lambda).$$

Proof. Thanks to Lemma 2.2, the proof is reduced to an algebraic computation which is completely analogous to the one used in the proof of [20, Lemma 2.2]. □

We now give some direct consequences of Proposition 2.3 and Lemma 2.2, which will be used in the following. By definition, one has $R_+\mathsf{T}_h(\lambda)\widehat{\Pi} = -\lambda R_+ W\widehat{\Pi}$. Hence (2.18) becomes

$$E_{-+}(\lambda) \;=\; -R_+\mathsf{T}_h(\lambda)R_- + \lambda^2 R_+ W\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}WR_-.$$

Introducing the matrices

$$\mathsf{M}_{V,h} = R_+\Delta_{V,h}R_-,\ \ \mathsf{W}_h = R_+WR_-,\ \text{ and } \mathsf{G}_h(\lambda) = \lambda^2(\mathsf{I}_{\mathbb{C}^{n_0}} - R_+W\widehat{\Pi}\widehat{\mathsf{T}}_h(\lambda)^{-1}\widehat{\Pi}WR_-), \tag{2.20}$$

this rewrites

$$-E_{-+}(\lambda) \;= \mathsf{M}_{V,h} - \lambda\mathsf{W}_h + \mathsf{G}_h(\lambda). \tag{2.21}$$

Moreover, it follows from Lemma 2.2 that $\lambda \in D(0,\epsilon_0 h) \mapsto \mathsf{G}_h(\lambda)$ is holomorphic and that there exists $C > 0$ such that for all h small enough and all $\lambda \in D(0,\epsilon_0 h)$,

$$|\mathsf{G}_h(\lambda)| \le C|\lambda|^2 h^{-1}. \tag{2.22}$$

According to (2.8) and to Proposition 2.3,

$$\lambda \in D(0,\epsilon_0 h)\cap\sigma(\mathsf{P}_h) \quad \text{if and only if} \quad \lambda \in D(0,\epsilon_0 h) \text{ is a singularity of } E_{-+}^{-1}(\lambda). \tag{2.23}$$

To prove Theorems 2 and 3, the strategy consists in studying the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0,\epsilon_0 h)$, which first requires to compute asymptotic equivalents of $\mathsf{M}_{V,h}$ and W_h as $h \to 0$ (see (2.21)).

3. The double well case

In this section we prove Theorem 2. To this end, we assume throughout this section that $\mathsf{U}^{(0)}$ has exactly two elements $\mathbf{m}_1$ and $\mathbf{m}_2$ such that $V(\mathbf{m}_1) < V(\mathbf{m}_2)$. Assume also that the two elements $\mathbf{s}_1, \mathbf{s}_2$ of $\mathsf{U}^{(1)}$ satisfy $V(\mathbf{s}_1) > V(\mathbf{s}_2)$. Recall that in this case, one has by construction of S (see (1.9)) and $\mathbf{j}$ (see (1.8)), $S(\mathbf{m}_2) = V(\mathbf{s}_2) - V(\mathbf{m}_2)$ and $\mathbf{j(m_2)} = \{\mathbf{s_2}\}$ (see Remark 1.2). Thus, by Theorem 5, it holds

$$\mu^{\Delta}_{2,h} = a_h(\mathbf{m}_2)\, h\, e^{-\frac{2(V(\mathbf{s}_2)-V(\mathbf{m}_2))}{h}}, \tag{3.1}$$

with $a_h(\mathbf{m}_2) \sim \sum_{k\ge 0} h^k a_k(\mathbf{m}_2)$ and $a_0(\mathbf{m}_2) = \frac{1}{2\pi}\sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}$.

3.1. **Asymptotic equivalent of W_h, as $h \to 0$.** To compute an asymptotic equivalent of W_h in the limit $h \to 0$, we first need to define (Ψ_1, Ψ_2) with the help of so-called quasi-modes, where we recall that (Ψ_1, Ψ_2) is an orthonormal basis of eigenvectors of $\Delta_{V,h}$ associated respectively with the eigenvalues 0 and $\mu_h(\mathbf{m}_2)$. First of all, we choose for $h > 0$,

$$\Psi_1 = \frac{1}{Z_{1,h}} e^{-(V - V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})},$$

where by Laplace's method: $Z_{1,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4})$ (since $\mathbf{m}_1$ is the unique global minimum of V on $\mathbb{T}$). Because $\Delta_{V,h}\Psi_1 = 0$, $\Psi_1 \in \mathsf{E}_0$. We then construct Ψ_2 as follows. Define

$$\varphi_2 = \frac{1}{Z_{2,h}} \chi_2 e^{-(V-V(\mathbf{m}_2))/h} \text{ with } Z_{2,h} = \|e^{-(V-V(\mathbf{m}_2))/h}\|_{L^2(\mathbb{T})}, \tag{3.2}$$

and where $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{T}, [0,1])$ satisfies $\mathbb{1}_{B(\mathbf{m}_2,r)} \le \chi_2 \le \mathbb{1}_{B(\mathbf{m}_2,2r)}$, and where $r > 0$. If $r > 0$ is small enough, $\mathbf{m}_2$ is the unique global minimum of V on the closure of $B(\mathbf{m}_2, 2r)$ (because V is a Morse function). Thus, by Laplace's method: $Z_{2,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_2))^{1/4})$. In addition, for such fixed $r > 0$, $V > V(\mathbf{m}_2)$ on $\operatorname{supp}(\nabla \chi_2)$, and therefore, there exists $C > 0$ such that for h small enough:

$$\Delta_{V,h}\varphi_2 = O(e^{-C/h}) \text{ in } L^2(\mathbb{T}). \tag{3.3}$$

On the other hand, since $\Delta_{V,h}$ is self-adjoint, it follows from the localisation of the spectrum in Theorem 5 (with $n_0 = 2$ there) that

$$\forall z \in \partial D(0, \epsilon_* h/2),\ (\Delta_{V,h} - z)^{-1} = O(h^{-1}). \tag{3.4}$$

By definition of Π and by Theorem 5,

$$\Pi = \frac{1}{2i\pi} \int_{\partial D(0, \frac{\epsilon_*}{2} h)} (z - \Delta_{V,h})^{-1} dz$$

and using (3.3) and (3.4), it follows that $\Pi\varphi_2$ satisfies

$$\Pi\varphi_2 - \varphi_2 = \frac{1}{2i\pi} \int_{\partial D(0, \frac{\epsilon_*}{2} h)} z^{-1} (z - \Delta_{V,h})^{-1} \Delta_{V,h}\varphi_2 dz = O(e^{-C/h}). \tag{3.5}$$

Moreover, since $V(\mathbf{m}_2) > V(\mathbf{m}_1)$ and $\Pi\Psi_1 = \Psi_1$, one has for h small enough:

$$\langle \Pi\varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} = \langle \varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} = O(e^{-C/h}). \tag{3.6}$$

We finally set

$$\Psi_2 = \frac{\Pi\varphi_2 - \langle \Pi\varphi_2, \Psi_1 \rangle_{L^2(\mathbb{T})} \Psi_1}{\|\Pi\varphi_2 - \langle \Pi\varphi_2, \Psi_1 \rangle \Psi_1\|_{L^2(\mathbb{T})}}.$$

The function Ψ_2 belongs to E_0, is orthogonal to Ψ_1, and $\|\Psi_2\|_{L^2(\mathbb{T})} = 1$. From now on, we consider (Ψ_1, Ψ_2) constructed as above, as a orthonormal basis of E_0. Notice that, using (3.5) and (3.6), one has

$$\Psi_2 = \varphi_2 + O(e^{-C/h}) \text{ in } L^2(\mathbb{T}). \tag{3.7}$$

Lemma 3.1. *With the above choice of* Ψ_1, Ψ_2*, there exists* $C > 0$ *such that for all* h *small enough,*

$$\mathsf{W}_h = \begin{pmatrix} \gamma_{1,h} & 0 \\ 0 & \gamma_{2,h} \end{pmatrix} + O(e^{-C/h}),$$

where we recall that W_h *is defined in* (2.20) *and where, for* $i \in \{1,2\}$, $\gamma_{i,h}$ *satisfies as* $h \to 0$: $\gamma_{i,h} \sim \sum_{k\geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_i) + \sqrt{h}\sum_{k\geq 0} h^k \gamma_{V,k}(\mathbf{m}_i)$*, with*

$$\gamma_{\alpha,0}(\mathbf{m}_i) = \alpha(\mathbf{m}_i) \text{ and } \gamma_{V,0}(\mathbf{m}_i) = 2\sqrt{\frac{V''(\mathbf{m}_i)}{\pi}}.$$

Proof. Since (Ψ_1, Ψ_2) is an orthonormal family, one has $\mathsf{W}_h = (\langle W\Psi_i, \Psi_j\rangle_{L^2(\mathbb{T})})_{i,j=1,2}$. In addition, $\Psi_1 = O(e^{-C/h})$ in $L^2(\mathrm{supp}(\varphi_2))$. Hence, using (3.7), for all $i \neq j$, $\langle W\Psi_i, \Psi_j\rangle_{L^2(\mathbb{T})} = O(e^{-C/h})$. Suppose now that $i \in \{1,2\}$ is fixed. By definition of W, one has $\langle W\Psi_i, \Psi_i\rangle_{L^2(\mathbb{T})} = \langle \alpha\Psi_i, \Psi_i\rangle_{L^2(\mathbb{T})} + 2\langle |\partial_x V|\Psi_i, \Psi_i\rangle_{L^2(\mathbb{T})}$. By definition of Ψ_1 and Ψ_2 above, using Laplace's method, one has as $h \to 0$: $\langle \alpha\Psi_i, \Psi_i\rangle_{L^2(\mathbb{T})} \sim \sum_{k\geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_i)$ with $\gamma_{\alpha,0}(\mathbf{m}_i) = \alpha(\mathbf{m}_i)$. On the other hand, for $i \in \{1,2\}$, one has for $\delta > 0$ small enough,

$$\begin{aligned}\langle |\partial_x V|\Psi_i, \Psi_i\rangle_{L^2(\mathbb{T})} &= Z_{i,h}^{-2}\int_{|x-\mathbf{m}_i|<\delta} |\partial_x V(x)| e^{-2(V(x)-V(\mathbf{m}_i))/h}dx + O(e^{-c/h}) \\ &= Z_{i,h}^{-2}\Big(\int_{\mathbf{m}_i}^{\mathbf{m}_i+\delta} \partial_x V(x) e^{-2(V(x)-V(\mathbf{m}_i))/h}dx \\ &\qquad - \int_{\mathbf{m}_i-\delta}^{\mathbf{m}_i} \partial_x V(x) e^{-2(V(x)-V(\mathbf{m}_i))/h}dx\Big) + O(e^{-c/h}) \\ &= hZ_{i,h}^{-2} + O(e^{-c/h}).\end{aligned}$$

Using the fact that $Z_{i,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$, this ends the proof of the lemma. ☐

We are now in position to prove Theorem 2.

3.2. **End of the proof of Theorem 2.** Recall that the strategy consists in localizing the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0,\epsilon_0 h)$ (see (2.23)). First of all, $\lambda = 0$ is always a singularity of $\lambda \in D(0,\epsilon_0 h) \mapsto E_{-+}^{-1}(\lambda)$, since $\mathsf{T}_h(0)\Psi_1 = 0$ (see Proposition 2.3). Let us now look for the other singularities of $\lambda \in D(0,\epsilon_0 h) \mapsto E_{-+}^{-1}(\lambda)$. Since $\dim \mathsf{E}_0 = 2$, $\Delta_{V,h}\Psi_1 = 0$, and $\Psi_2 \in \mathsf{E}_0$ is orthogonal to Ψ_1, it holds: $\Delta_{V,h}\Psi_2 = \mu_{2,h}^{\Delta}\Psi_2$ and therefore, $\mathsf{M}_{V,h} = \mathrm{diag}(0, \mu_{2,h}^{\Delta})$. By Lemma 3.1, (2.21), and (2.22), one then has

$$-E_{-+}(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \mu_{2,h}^{\Delta} \end{pmatrix} - \lambda\begin{pmatrix} \gamma_{1,h} & 0 \\ 0 & \gamma_{2,h} \end{pmatrix} + \mathsf{r}_h(\lambda),$$

where $\lambda \in D(0,\epsilon_0 h) \mapsto \mathsf{r}_h(\lambda)$ is holomorphic and $\mathsf{r}_h(\lambda) = O(\lambda^2 h^{-1} + \lambda e^{-C/h})$ for all $\lambda \in D(0,\epsilon_0 h)$ and h small enough. Set

$$\Gamma_h = \mathrm{diag}(\gamma_{1,h}, \gamma_{2,h}).$$

Then, by Lemma 3.1:

$$\Gamma_h^{-1} = O(h^{-1/2}). \tag{3.8}$$

Therefore, one deduces that

$$-E_{-+}(\lambda) = \Gamma_h\big(F_h(\lambda) + \mathsf{R}_h(\lambda)\big) \text{ with } F_h(\lambda) = \begin{pmatrix} -\lambda & 0 \\ 0 & \mu^{\Delta}_{2,h}/\gamma_{2,h} - \lambda \end{pmatrix}, \tag{3.9}$$

where $\mathsf{R}_h(\lambda)$ is holomorphic with respect to $\lambda \in D(0, \epsilon_0 h)$ and $\mathsf{R}_h(\lambda) = O(\lambda^2 h^{-\frac{3}{2}} + \lambda h^{-\frac{1}{2}} e^{-C/h})$ for all $\lambda \in D(0, \epsilon_0 h)$ and h small enough. Set

$$\eta_{2,h} := \mu^{\Delta}_{2,h}/\gamma_{2,h}.$$

According to Lemma 3.1, to (3.1), and to the relation $\alpha(\mathbf{m}_2) > 0$ implied by (1.4) (see indeed (1.5)), one has in the limit $h \to 0$:

$$\eta_{2,h} = \zeta_h(\mathbf{m}_2)\, h\, e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}, \tag{3.10}$$

where $\zeta_h(\mathbf{m}_2) \sim \sum_{k\geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_2)$ and $\zeta_0(\mathbf{m}_2) = \frac{1}{2\pi}\sqrt{|V''(\mathbf{m}_2)V''(\mathbf{s}_2)|}/\alpha(\mathbf{m}_2)$. Let $K \geq 2$ be fixed in what follows. Set

$$\mathcal{D}_K = \big\{\lambda \in \mathbb{C},\ |\lambda - \eta_{2,h}| < h^K e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}\big\},$$

whose closure is by (3.10) included in $D(0, \epsilon_0 h)$ for h small enough. Hence, for any $\lambda \in \overline{\mathcal{D}_K}$, since $K \geq 2$, one has, for all h small enough,

$$C^{-1} h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))} \leq |\lambda| \leq C h e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}. \tag{3.11}$$

Consequently, for any $\lambda \in \partial\mathcal{D}_K$, the matrix $F_h(\lambda)$ is invertible and

$$F_h(\lambda)^{-1} = O(h^{-K} e^{\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}) \text{ on } \partial\mathcal{D}_K. \tag{3.12}$$

On the other hand, it follows from the estimate on R_h below (3.9) that for any $\lambda \in \partial\mathcal{D}_K$,

$$\mathsf{R}_h(\lambda) = O(\sqrt{h}\, e^{-4(V(\mathbf{s}_2) - V(\mathbf{m}_2))/h} + \sqrt{h}\, e^{-(2(V(\mathbf{s}_2) - V(\mathbf{m}_2)) + C)/h}) = O(e^{-(2(V(\mathbf{s}_2) - V(\mathbf{m}_2)) + C)/h}).$$

Hence, by (3.9), $E_{-+}(\lambda)$ is invertible on $\partial\mathcal{D}_K$ for all h small enough and

$$-E^{-1}_{-+}(\lambda) = \big(1 + O(e^{-\frac{C}{2h}})\big) F_h(\lambda)^{-1} \Gamma_h^{-1} \text{ on } \partial\mathcal{D}_K,$$

where we have used (3.12) and the invertibility of Γ_h. Using in addition $\|F_h(\lambda)^{-1}\|\, |\partial\mathcal{D}_K| = O(1)$ and (3.8), this implies that for all h small enough:

$$\begin{aligned} \frac{1}{2i\pi}\int_{\partial\mathcal{D}_K} E^{-1}_{-+}(\lambda) d\lambda &= -\Big[\frac{1}{2i\pi}\int_{\partial\mathcal{D}_K} \begin{pmatrix} -\lambda & 0 \\ 0 & \eta_{2,h} - \lambda \end{pmatrix}^{-1} d\lambda\Big] \Gamma_h^{-1} + \|\Gamma_h^{-1}\| O(e^{-\frac{C}{2h}}) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma_{2,h}^{-1} \end{pmatrix} + O(e^{-\frac{C}{4h}}) \text{ is non trivial.} \end{aligned} \tag{3.13}$$

Hence, for all $K \geq 2$, $\lambda \mapsto E^{-1}_{-+}(\lambda)$ admits at least a singularity $\alpha_{h,K}$ in the disk $\mathcal{D}_K$. In particular, one has for all h small enough,

$$\alpha_{h,K} = \eta_{2,h} + O(h^K e^{-\frac{2}{h}(V(\mathbf{s}_2) - V(\mathbf{m}_2))}), \text{ where } \eta_{2,h} \text{ satisfies (3.10)}.$$

But, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathsf{P}_h) \cap \{\mathrm{Re}(z) \leq \epsilon_0 h^2\}$ is composed of two elements 0 and $\lambda_{2,h}$. According to (2.23), the nonzero eigenvalue $\lambda_{2,h}$ then necessarily satisfies $\lambda_{2,h} = \alpha_{h,K}$ for all $K \geq 2$. This ends the proof of Theorem 2.

4. The multiple well case

In this section, we prove Theorem 3. We assume that U is a Morse function and that (1.10) is satisfied. Recall that the local minima $\mathbf{m}_1, \dots, \mathbf{m}_{n_0}$ of V are labeled such that $(S(\mathbf{m}_j))_{j\in\{1,\dots,n_0\}}$ is decreasing (see (1.11)). We then label $\{\Psi_1, \dots, \Psi_{n_0}\}$, the basis of E_0, accordingly.

4.1. An adapted basis of quasimodes.

In order to compute the matrices $\mathsf{M}_{V,h}$ and W_h accurately, we will build the basis $(\Psi_j)_{j=1,\dots,n_0}$ of E_0 from a family of quasi-modes $(\varphi_j)_{j=1,\dots,n_0}$ constructed in [6]. First, as in the previous section, set, for $h>0$,

$$\varphi_1 = \frac{1}{Z_{1,h}} e^{-(V-V(\mathbf{m}_1))/h} \text{ with } Z_{1,h} = \|e^{-(V-V(\mathbf{m}_1))/h}\|_{L^2(\mathbb{T})} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_1))^{1/4}),$$

and $\Psi_1 = \varphi_1$. For any $j = 2, \dots, n_0$, let φ_j be the $L^2(\mathbb{T})$-normalized quasi-mode associated with $\mathbf{m}_j$ given in [6, Definition 4.3] applied to the case of the Witten Laplacian $\Delta_{V,h}$. Since φ_i is by definition supported in a neighborhood of $\mathsf{C}(\mathbf{m}_i)$ (see [6, Section 4]), it follows from the second item of Assumption (1.10) that, for all $i \in \{1, \dots, n_0\}$ and $r > 0$ small enough, one has for all h small enough:

$$\varphi_i = \frac{1_{D(\mathbf{m}_i,r)}}{Z_{i,h}} e^{-(V-V(\mathbf{m}_i))/h} + O(e^{-c/h}) \text{ in } L^2(\mathbb{T}), \tag{4.1}$$

where $Z_{i,h} \in \mathcal{E}_{cl}((\pi h/V''(\mathbf{m}_i))^{1/4})$. For $j \in \{1, \dots, n_0\}$, we set

$$\kappa_{j,h} := \langle \Delta_{V,h}\varphi_j, \varphi_j\rangle.$$

We have obviously $\kappa_{1,h} = 0$ and, from [6, Proposition 5.1], we have for all $j \in \{2, \dots, n_0\}$,

$$\kappa_{j,h} \in \mathcal{E}_{cl}\Big(h e^{-2S(\mathbf{m}_j)/h} \sum_{\mathbf{s}\in\mathbf{j}(\mathbf{m}_j)} \frac{|V''(\mathbf{m}_j)V''(\mathbf{s})|^{\frac12}}{2\pi}\Big). \tag{4.2}$$

The basis $(\Psi_j)_{j=1,\dots,n_0}$ of $\mathsf{E}_0 = \operatorname{Ran}\Pi$ is then constructed from the family $(\varphi_j)_{j=1,\dots,n_0}$ by the following procedure. Set, for $j = 1, \dots, n_0$: $v_j = \Pi\varphi_j$. For h small enough, the family $(v_j)_{j=1,\dots,n_0}$ is then a basis of E_0 (since, according to [6, Proposition 5.3], $\langle v_j, v_i\rangle_{L^2(\mathbb{T})} = \delta_{i,j} + O(e^{-c/h})$, for all $i, j = 1, \dots, n_0$). We then consider the family $(\Psi_j)_{j=1,\dots,n_0}$ obtained from $(v_j)_{j=1,\dots,n_0}$ by a Gram-Schmidt procedure as in [6, page 30]. According to [6, Proposition 5.3 and Eq. (5.15)], this orthonormal basis of E_0 satisfies the following properties:

$$\text{for all } j,k = 1, \dots, n_0, \quad \langle \Delta_{V,h}\Psi_j, \Psi_k\rangle = \delta_{j,k}\kappa_{j,h} + O\Big(h^\infty \sqrt{\kappa_j(h)\kappa_k(h)}\Big) \tag{4.3}$$

and

$$\text{for all } j = 1, \dots, n_0, \quad \Psi_j = \varphi_j + O(e^{-c/h}) \text{ in } L^2(\mathbb{T}). \tag{4.4}$$

Working in the basis $(\Psi_j)_{j=1,\dots,n_0}$ of E_0, we obtain the following asymptotic equivalent of W_h (see (2.20)).

Lemma 4.1. *There exists $C > 0$ such that the matrix W_h satisfies, for h small enough,*

$$\mathsf{W}_h = \Gamma_h + O(e^{-C/h}),$$

where $\Gamma_h = \mathrm{diag}(\gamma_{1,h},\dots,\gamma_{n_0,h})$ *and, for* $j=1,\dots,n_0$*:*

$$\gamma_{j,h} \sim \sum_{k\geq 0} h^k \gamma_{\alpha,k}(\mathbf{m}_j) + \sqrt{h}\sum_{k\geq 0} h^k\gamma_{V,k}(\mathbf{m}_j)$$

with $\gamma_{\alpha,0}(\mathbf{m}_j)=\alpha(\mathbf{m}_j)$*, and* $\gamma_{V,0}(\mathbf{m}_j) = 2\sqrt{\frac{V''(\mathbf{m}_j)}{\pi}}$.

Proof. The asymptotic computation of $\langle W\Psi_i,\Psi_i\rangle$ as $h\to 0$ is exactly the same as in Lemma 3.1, using (4.1) and (4.4). The only point to be checked is that for any $i\neq j$, one has $\langle W\Psi_i,\Psi_j\rangle = O(e^{-C/h})$. Thanks to (4.4), this is equivalent to say that for all $i\neq j$, one has $\langle W\varphi_i,\varphi_j\rangle = O(e^{-C/h})$, which follows directly from (4.1). □

4.2. **Proof of Theorem 3.** Let Γ_h be defined by Lemma 4.1 and let $\mathsf{M}_{V,h}$ be given by (2.20). Notice that in the case $n_0\geq 3$, the matrix $\mathsf{M}_{V,h}$ has not to be diagonal and we vill use the following result.

Lemma 4.2. *Introduce the symmetric positive semi-definite matrix*

$$\mathsf{M}^{\Gamma}_{V,h} = \Gamma_h^{-\frac12}\mathsf{M}_{V,h}\Gamma_h^{-\frac12}.$$

Then, there exists $\epsilon_0>0$ *such that, for all* h *small enough, the* n_0 *eigenvalues* $\beta_{1,h}\leq\dots\leq\beta_{n_0,h}$ *of* $\mathsf{M}^{\Gamma}_{V,h}$ *satisfy:* $\beta_{1,h}=0$ *and, for all* $k=2,\dots,n_0$, $\beta_{k,h}\in\mathcal{E}_{cl}(\mu^{\Delta}_{k,h}/\gamma_{k,h})$*, where* $\mu^{\Delta}_{k,h}$ *is given by* (2.10)*.*

Proof. First, observe that $\mathsf{M}_{V,h}$ admits 0 as a simple eigenvalue (since 0 is a simple eigenvalue of $\Delta_{V,h}$), so it is also a simple eigenvalue of $\mathsf{M}^{\Gamma}_{V,h}$. For $A=(a_{i,j})_{1\leq i,j\leq n_0}$ a matrix, we define $\tilde A := (a_{i,j})_{2\leq i,j\leq n_0}$. We then have:

$$\tilde{\mathsf{M}}^{\Gamma}_{V,h} = (\tilde\Gamma_h)^{-\frac12}\tilde{\mathsf{M}}_{V,h}(\tilde\Gamma_h)^{-\frac12}.$$

Moreover, by (4.2) and (4.3), the matrix $\tilde{\mathsf{M}}_{V,h}$ writes

$$\tilde{\mathsf{M}}_{V,h} = \Omega_h\big(\mathsf{D}_{V,h}+O(h^\infty)\big)\Omega_h,$$

with $\Omega_h=\mathrm{diag}(e^{-S(\mathbf{m}_2)/h},\dots,e^{-S(\mathbf{m}_{n_0})/h})$ and $\mathsf{D}_{V,h}=\mathrm{diag}(a_h(\mathbf{m}_2),\dots,a_h(\mathbf{m}_{n_0}))$. Since $\tilde\Gamma_h$ is diagonal, one has $\Omega_h\tilde\Gamma_h=\tilde\Gamma_h\Omega_h$ and it follows that $\tilde{\mathsf{M}}^{\Gamma}_{V,h}$ writes

$$\tilde{\mathsf{M}}^{\Gamma}_{V,h} = \Omega_h\big(\mathsf{D}^{\Gamma}_{V,h}+O(h^\infty)\big)\Omega_h$$

with $\mathsf{D}^{\Gamma}_{V,h}=\mathrm{diag}(\mu^{\Delta}_{j,h}/\gamma_{j,h}, j=2,\dots,n_0)$. Hence $\tilde{\mathsf{M}}^{\Gamma}_{V,h}$ admits a *graded structure* in the sense of [20, Definition A.1]. We can thus apply [20, Theorem A.4] which yields the result. □

We are now in position to prove Theorem 3. Let U be a Morse function and assume that (1.10) is satisfied. Recall that we look for the singularities of $E^{-1}_{-+}(\lambda)$ and that $\lambda=0$ is always a singularity of $E^{-1}_{-+}(\lambda)$ (since $\dim\mathrm{Ker}(\mathsf{P}_h)=1$). Let us now look for the remaining singularities of $E^{-1}_{-+}(\lambda)$ in $D(0,\epsilon_0 h)$. From Lemma 4.2, there exists a unitary change of basis B_h such that $\mathsf{M}^{\Gamma}_{V,h} = B_h^*\,\mathrm{diag}(\beta_{1,h},\dots,\beta_{n_0,h})B_h$ (since $\mathsf{M}^{\Gamma}_{V,h}$ is symmetric and thus diagonalizable in an orthonormal basis). Combined with (2.21), (2.22), and Lemma 4.1, this yields

$$-E_{-+}(\lambda) = \Gamma_h^{1/2} B_h^* \Big[\operatorname{diag}(\beta_{1,h}, \dots, \beta_{n_0,h}) - \lambda \mathsf{I}_{\mathbb{C}^{n_0}} - \mathsf{R}_h(\lambda) \Big] B_h \Gamma_h^{1/2}, \tag{4.5}$$

where, for h small enough, $\mathsf{R}_h(\lambda)$ is holomorphic with respect to $\lambda \in D(0, \epsilon_0 h)$ and

$$\mathsf{R}_h(\lambda) = O(\lambda^2 h^{-\frac{3}{2}} + \lambda h^{-\frac{1}{2}} e^{-C/h}), \tag{4.6}$$

and where we have used that $\Gamma_h^{-1} = O(h^{-1/2})$ by Lemma 4.1. Hence, the singularities of $E_{-+}^{-1}(\lambda)$ in $D(0, \epsilon_0 h)$ are exactly those of L_h^{-1}, where

$$L_h(\lambda) = F_h(\lambda) - \mathsf{R}_h(\lambda), \text{ with } F_h(\lambda) = \operatorname{diag}(\beta_{1,h} - \lambda, \dots, \beta_{n_0,h} - \lambda). \tag{4.7}$$

Recall that $\beta_{1,h} = 0$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^{\Delta}/\gamma_{j,h})$ for all $j \geq 2$. For all $j = 2, \dots, n_0$, using the asymptotic equivalents of $\mu_{j,h}^{\Delta}$ and $\gamma_{j,h}$ given in Theorem 5 and in Lemma 4.1, one has when $h \to 0$, using (1.4) which implies $\alpha(\mathbf{m}_j) > 0$:

$$\beta_{j,h} = \zeta_h(\mathbf{m}_j)\, h\, e^{-\frac{2}{h} S(\mathbf{m}_j)}, \tag{4.8}$$

where $\zeta_h(\mathbf{m}_j) \sim \sum_{k \geq 0} h^{\frac{k}{2}} \zeta_k(\mathbf{m}_j)$ and $\zeta_0(\mathbf{m}_j)$ is given by (1.13). Let us now consider $j \in \{2, \dots, n_0\}$ and $K \geq 2$. Denote $\mathcal{D}_{j,K} = D(\beta_{j,h}, h^K e^{-\frac{2}{h} S(\mathbf{m}_j)})$ and let $\lambda \in \partial \mathcal{D}_{j,K}$. Since $S(\mathbf{m}_\ell) < S(\mathbf{m}_k)$ when $\ell \geq k$, for $h > 0$ small enough, the $\mathcal{D}_{j,K}$ are pairwise disjoint, their closures are included in $D(0, \epsilon_0 h)$, and for all $i \in \{2, \dots, n_0\}$:

$$\begin{aligned} &\forall i > j,\ |\lambda - \beta_{i,h}| \geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_i)} \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)} \\ &\forall i < j,\ |\lambda - \beta_{i,h}| \geq |\beta_{j,h} - \beta_{i,h}| - |\lambda - \beta_{j,h}| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)}. \end{aligned} \tag{4.9}$$

Moreover, for h small enough,

$$|\lambda| \geq C h e^{-\frac{2}{h} S(\mathbf{m}_j)}. \tag{4.10}$$

Consequently, for $\lambda \in \partial \mathcal{D}_{j,K}$, the matrix $F_h(\lambda)$ is invertible and

$$F_h(\lambda)^{-1} = O(h^{-K} e^{\frac{2}{h} S(\mathbf{m}_j)}) \text{ on } \mathcal{D}_{j,K}.$$

Combining this estimate with (4.6) and reasoning as around (3.13), we prove that for all $j = 2, \dots, n_0$, $K \geq 2$, and h small enough,

$$L_h^{-1}(\lambda) \text{ (and thus } E_{-+}^{-1}(\lambda)) \text{ admits a singularity } \alpha_{j,h,K} \text{ in } \mathcal{D}_{j,K}, \tag{4.11}$$

so in particular $\alpha_{j,h,K} \neq \alpha_{i,h,K}$ when $i \neq j \in \{2, \dots, n_0\}$ (the $\mathcal{D}_{j,K}$ being pairwise disjoint).

In addition, since (1.4) holds, Theorem 1 implies that for all h small enough, $\sigma(\mathsf{P}_h) \cap \{\operatorname{Re}(z) \leq \epsilon_0 h^2\} \setminus \{0\}$ is made of $n_0 - 1$ real eigenvalues $0 < \lambda_{2,h} \leq \ldots \leq \lambda_{n_0,h}$ (counted with algebraic multiplicity). It then follows from (2.23) that, for each $j = 2, \dots, n_0$, the eigenvalue $\lambda_{j,h}$ satisfies $\lambda_{j,h} = \alpha_{j,h,K}$ for all $K \geq 2$. Since $\alpha_{j,h,K} = \beta_{j,h} + O(h^K e^{-\frac{2}{h} S(\mathbf{m}_j)})$ for all $K \geq 2$ and $\beta_{j,h} \in \mathcal{E}_{cl}(\mu_{j,h}^{\Delta}/\gamma_{j,h})$, this completes the proof of Theorem 3, using the asymptotic equivalent of $\mu_{j,h}^{\Delta}/\gamma_{j,h}$ as $h \to 0$ (see Theorem 5 and Lemma 4.1).

4.3. **The case when (1.4) is not satisfied.** In this section, we prove Theorem 4, where, compared to Theorem 3, we do no longer assume that (1.4) holds but that (1.14) holds. We can thus no longer use Theorem 1 as we did at the very end of the proof of Theorem 3 to say that P_h admits n_0-1 nonzero eigenvalues (counted with multiplicity) in $\{\mathrm{Re}(z)\le \epsilon_0 h^2\}$. We can however make use of all the intermediate results in the proof of Theorem 3 until (4.11) (included), except that one has the following minor scaling changes:

(1) Equation (4.8) must be changed into

$$\beta_{j,h}=\zeta_h(\mathbf{m}_j)\sqrt{h}\,e^{-\frac{2}{h}S(\mathbf{m}_j)}, \tag{4.12}$$

where $\zeta_h(\mathbf{m}_j)\sim\sum_{k\ge0}h^{\frac{k}{2}}\zeta_k(\mathbf{m}_j)$ with $\zeta_0(\mathbf{m}_j)$ satisfying (1.16).

(2) In Equations (4.9) and (4.10), all the Ch must be replaced by $C\sqrt{h}$.

Let $j\in\{2,\dots,n_0\}$ and note that, for h small enough, the sequence $(\alpha_{j,h,K})_{K\ge2}$ defined in the proof of Theorem 3 is stationary by analyticity of the nontrivial map $\lambda\mapsto E_{-+}(\lambda)$. We denote by $\alpha_{j,h}$ its limit and recall that, according to (2.23), $\{0,\alpha_{2,h},\dots,\alpha_{n_0,h}\}\subset\sigma(\mathsf{P}_h)$. In addition, the $\alpha_{j,h}$'s are exponentially small and thus belong to $D(0,ch^2)$ for any $c>0$ and all h small enough. Let us now prove that, for h small enough, 0 and $\alpha_{j,h}$, $j=2,\dots,n_0$, are the only singularities of $\lambda\mapsto E_{-+}^{-1}(\lambda)$ in $D(0,ch^2)$, that they are all real, and have geometric multiplicity 1 as eigenvalues of P_h. Recall that by (4.5), $\lambda\in D(0,\epsilon_0 h)$ is a singularity of $E_{-+}^{-1}(\lambda)$ if and only if it is a singularity of $L_h(\lambda)^{-1}$ (see (4.7)). Let us denote, for $\lambda\in\mathbb{C}$,

$$d_h(\lambda)=\det L_h(\lambda)=\det(F_h(\lambda)-\mathsf{R}_h(\lambda)),$$

which is holomorphic on $D(0,\epsilon_0 h)$. For all $\lambda\in\partial D(0,ch^2)$, using (4.6), we get $\mathsf{R}_h(\lambda)=O(h^4h^{-3/2})=O(h^{5/2})$, and by (4.12),

$$\big|\det F_h(\lambda)\big|=\big|\lambda\times\Pi_{j=2}^{n_0}(\lambda-\beta_{j,h})\big|=c^{n_0}h^{2n_0}(1+o_h(1)).$$

Thus, for h small enough, it holds uniformly on $\partial D(0,ch^2)$:

$$d_h(\lambda)=\lambda\times\Pi_{j=2}^{n_0}(\lambda-\beta_{j,h})+O(h^{5/2}h^{2(n_0-1)})=\det F_h(\lambda)(1+o_h(1)). \tag{4.13}$$

In particular, for any $\lambda\in\partial D(0,ch^2)$, one has

$$|d_h(\lambda)-\det F_h(\lambda)|<|\det F_h(\lambda)|,$$

which implies by Rouché's theorem that $d_h(\lambda)$ and $\det F_h(\lambda)$ have the same number n_0 of zeros (counted with multiplicity) in $D(0,ch^2)$. These n_0 zeros are thus 0 and $\alpha_{j,h}$, $j=2,\dots,n_0$. Let us recall that the $\alpha_{j,h}$ are pairwise disctinct and satisfy $\alpha_{j,h}=\beta_{j,h}+O(h^Ke^{-\frac{2}{h}S(\mathbf{m}_j)})$ for all $K\ge2$ and $j=2,\dots,n_0$. They are then all simple zeros of $d_h(\lambda)$ (and thus $\dim\operatorname{Ker}E_{-+}(\alpha_{j,h})=1$ for $j=2,\dots,n_0$). Using also (2.9) and [25, Equation (2.7)], $\dim\operatorname{Ker}(\mathsf{P}_h-\alpha_{j,h})=\dim\operatorname{Ker}\mathsf{T}_h(\alpha_{j,h})=\dim\operatorname{Ker}E_{-+}(\alpha_{j,h})=1$, for all $j=\{2,\dots,n_0\}$. Lastly, since the operator P_h has real coefficients, its spectrum is stable by complex conjugation. Hence, since moreover 0 and the $\alpha_{j,h}$, $j=2,\dots,n_0$, have different asymptotic equivalents as $h\to0$, one has $\overline{\alpha_{j,h}}=\alpha_{j,h}$ for every $j=2,\dots,n_0$. To conclude the proof of Theorem 4, it then just remains to use that, for every $j=2,\dots,n_0$, $\beta_{j,h}\in\mathcal{E}_{cl}(\mu^{\Delta}_{j,h}/\gamma_{j,h})$, and the asymptotic equivalents of $\mu^{\Delta}_{j,h}/\gamma_{j,h}$ as $h\to0$ given by Theorem 5 and Lemma 4.1.

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