

DE RHAM THEORY AND LOCALLY ANALYTIC VECTORS

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ABSTRACT. Let K_∞/K be a p -adic Lie extension of a p -adic field K . We study the subring of pro-analytic vectors in the de Rham period ring $\mathbf{B}_{\text{dR}}^+(K_\infty)$. We show that the pro-analytic subring admits a Galois-equivariant isomorphism with a formal power series ring $\widehat{K}_\infty^{\text{la}}[[t_{K_\infty}]]$ if and only if K_∞ satisfies a certain orientability condition, which says that the $\widehat{K}_\infty$ -level Sen operator admits a Galois-equivariant $\mathbf{B}_{\text{dR}}^+$ -lift. A key input is the vanishing of higher locally analytic vectors of $\widehat{K}_\infty$ -representations. As an application, we show that the lifted Sen operator induces regular connections on pro-analytic vectors of $\mathbf{B}_{\text{dR}}^+$ -representations, and can be used to compute Galois cohomology.

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1. INTRODUCTION

This article is concerned with the locally analytic structure of de Rham period rings attached to p -adic Lie extensions. Let K be a mixed characteristic complete discrete valuation field with perfect residue field of characteristic p and let $G_K = \text{Gal}(\bar{K}/K)$ be the absolute Galois group. Let K_∞/K be an infinitely ramified Galois extension such that $\Gamma(K_\infty) := \text{Gal}(K_\infty/K)$ is a p -adic Lie group. Write $H(K_\infty) = \text{Gal}(\bar{K}/K_\infty)$. Our aim is to understand the pro-analytic vectors of the action of $\Gamma(K_\infty)$ on $\mathbf{B}_{\text{dR}}^+(K_\infty) := (\mathbf{B}_{\text{dR}}^+)^{H(K_\infty)}$. A key observation is that this problem is controlled by the de Rham theory of the Lie algebra $\text{Lie } \Gamma(K_\infty)$ with the adjoint action of $\Gamma(K_\infty)$, viewed as a G_K -representation by precomposition with $G_K \rightarrow \Gamma(K_\infty)$. Before stating the main result, we explain how this problem arises, why the familiar cyclotomic answer does not extend formally to an arbitrary tower, and how the adjoint representation $\text{Lie } \Gamma(K_\infty)$ enters the picture.

1.1. p -adic Lie towers and analytic period rings. The choice of an infinitely ramified extension of K is a recurring feature of p -adic Hodge theory, Iwasawa theory, and the p -adic Langlands program. Different choices of K_∞ lead to rather different theories. We return to the principal examples after stating the main theorem.

The completion $\hat{K}_\infty$ is the most accessible period ring attached to the tower, and in some sense the computation $\hat{K}_\infty^{\text{la}} = K_\infty$ in the cyclotomic setting via Tate's trace maps ([Tat67]) can be considered as a starting point of p -adic Hodge theory. One of the main results of Berger and Colmez's article [BC16] is the computation of $\hat{K}_\infty^{\text{la}}$ for general K_∞ .

However, one would like to understand locally analytic vectors in much larger period rings. Perhaps the most interesting such period ring is the large Robba ring $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{H(K_\infty)}$. In the cyclotomic case, Berger ([Ber16]) proves that its pro-analytic vectors recover, up to iterates of Frobenius, the usual Robba ring. Further calculations for other towers and other rings of periods were carried out by L. P. ([Poy25], [Poy26]). These Robba rings are important because they form a bridge from Galois representations to p -adic analysis. In the cyclotomic theory they lead to connections with p -adic differential equations ([Ber02], [CC98], [Fon07], [Ked05]), to explicit reciprocity laws and Perrin–Riou's big exponential in Iwasawa theory ([Ber03]), and to the description of locally analytic representations in the p -adic local Langlands correspondence ([Col16], [CDP14]). Thus, from the point of view of applications, the locally analytic vectors in Robba-type period rings are the objects one would ultimately like to compute.

For general towers K_∞/K , such a computation appears to be much more difficult. Results of L. P. ([Poy25]) and Steingart ([Ste24]) suggest $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{H(K_\infty)}$ has nonzero higher locally analytic vectors whenever K_∞ does not contain the cyclotomic extension. This indicates that the correct locally analytic Robba ring for applications for general K_∞/K is likely to be the derived version $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{H(K_\infty), \text{RF}(K_\infty)\text{-la}}$. Our understanding is that strong theorems in this direction have been proven by Anschütz, Le Bras, Rodríguez Camargo and Scholze in their upcoming work on analytic prismatization, and they will be applied to investigate Iwasawa theory for general K_∞ in upcoming work of Ponsinet, Rodrigues Jacinto and Steingart.

The ring $\mathbf{B}_{\text{dR}}^+(K_\infty)$ has an intermediate level of complexity between $\hat{K}_\infty$ and $(\tilde{\mathbf{B}}_{\text{rig}}^\dagger)^{H(K_\infty)}$. It is larger than $\hat{K}_\infty$ and remembers the entire de Rham filtration, but its reduction modulo its maximal ideal I_θ is $\hat{K}_\infty$, and its successive graded pieces are rank-one semilinear $\hat{K}_\infty$ -representations. It is therefore natural to ask whether the calculation of Berger–Colmez can be lifted through the I_θ -adic filtration. This is the problem studied in the present article.

1.2. From Sen theory to de Rham theory. One can study $\hat{K}_\infty$ -semilinear representations of $\Gamma(K_\infty)$ using Sen's theory. The original theory uses vectors which lie in finite-dimensional K -subspaces stable under the Galois action. This works especially well when $\Gamma(K_\infty)$ is one-dimensional, but finite vectors become too small in higher dimension. Berger and Colmez ([BC16]) observed that locally analytic vectors are the correct replacement. The most difficult part of their article is the computation of $\hat{K}_\infty^{\text{la}}$ itself. They show that $\hat{K}_\infty^{\text{la}}$ is a field of convergent power series in $\dim \Gamma(K_\infty) - 1$ variables over K_∞ .

It is natural to ask for a de Rham version of this theorem. Reduction modulo the maximal ideal I_θ identifies $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta \cong \widehat{K}_\infty$. The quotients $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta^n$ are Banach representations of $\Gamma(K_\infty)$, and $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$ is their inverse limit. The natural object to consider in this setting is then the pro-analytic vectors $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}} = \varprojlim_n [\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta^n]^{\mathrm{la}}$.

In the cyclotomic case (i.e. $K_\infty = K_{\mathrm{cyc}}$), G. P. ([Por22, Proposition 2.6]) computed $\mathbf{B}_{\mathrm{dR}}^+(K_{\mathrm{cyc}})^{\mathrm{pa}} = K_{\mathrm{cyc}}[[t]]$, where t is Fontaine's element, where a key point is that there is a natural inclusion $K_{\mathrm{cyc}} \subset \overline{K} \subset \mathbf{B}_{\mathrm{dR}}^+$. In general, however, $\widehat{K}_\infty^{\mathrm{la}}/K$ may not be an algebraic extension, so the structure of $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$ becomes more complicated. An argument using Cohen's structure theorem shows that $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$ is non-canonically isomorphic to a power series ring $\widehat{K}_\infty^{\mathrm{la}}[[t_{K_\infty}]]$, where t_{K_∞} is a variable, see Theorem 5.3. This can be regarded as an analogue of the non-canonical ring isomorphism $\mathbf{B}_{\mathrm{dR}}^+ \cong \mathbf{C}[[t]]$, and thus prompts the following natural canonicity question, which will be the core of our investigation.

Question 1.1. When is there a $\Gamma(K_\infty)$ -equivariant¹ isomorphism of the form

$$\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}} \cong \widehat{K}_\infty^{\mathrm{la}}[[t_{K_\infty}]]?$$

In particular, when is there a $\Gamma(K_\infty)$ -equivariant section

$$\widehat{K}_\infty^{\mathrm{la}} \rightarrow \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}?$$

It turns out this question has a deformation-theoretic flavour, and its answer lies in the de Rham theory of the adjoint representation. Indeed, consider $\mathrm{Lie} \Gamma(K_\infty)$ with the adjoint action of $\Gamma(K_\infty)$, viewed as a G_K -representation by precomposition with $G_K \twoheadrightarrow \Gamma(K_\infty)$. A theorem of Sen ([Sen80, Theorem 12]) gives a canonical nonzero element, the Sen operator

$$\Theta(K_\infty) \in (\mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie} \Gamma(K_\infty))^{G_K} = \mathbf{D}_{\mathbf{C}}(\mathrm{Lie} \Gamma(K_\infty)).$$

Berger and Colmez prove that, when this element acts as a differential operator on $\widehat{K}_\infty^{\mathrm{la}}$, it acts by zero ([BC16, §6]). Our idea is to consider it as a $\mathbf{C}$ -period of $\mathrm{Lie} \Gamma(K_\infty)$ and try to lift it to $\mathbf{B}_{\mathrm{dR}}^+$. Thus we are led to the following key definition.

Definition 1.2 (Definition 4.1). An *orientation* of K_∞/K is a lift of $\Theta(K_\infty)$ to an element

$$\nabla(K_\infty) \in (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} \mathrm{Lie} \Gamma(K_\infty))^{G_K} = \mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie} \Gamma(K_\infty)).$$

We say that K_∞/K is *orientable* if an orientation exists. We say K_∞/K is *canonically orientable* if a unique orientation exists.

It is not difficult to show that orientability is a necessary condition for the existence of a $\Gamma(K_\infty)$ -equivariant isomorphism $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}} \cong \widehat{K}_\infty^{\mathrm{la}}[[t_{K_\infty}]]$ in Question 1.1 to hold. Indeed, such an isomorphism leads to a $\Gamma(K_\infty)$ -equivariant section map $\widehat{K}_\infty^{\mathrm{la}} \rightarrow \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$, along which one can Galois-equivariantly lift $\Theta(K_\infty) \in \widehat{K}_\infty^{\mathrm{la}} \otimes_{\mathbf{Q}_p} \mathrm{Lie} \Gamma(K_\infty)$ to $\nabla(K_\infty)$. In fact, we shall quickly see that orientability is also a sufficient condition in Theorem 1.4 below.

Example 1.3. (1) (Theorem 4.7). K_∞ is canonically orientable if and only if the representation $\mathrm{Lie} \Gamma(K_\infty)$ has no generalized Hodge-Tate weights in $\mathbf{Z}_{\leq -1}$. This includes familiar examples:

- (a) (Examples 4.8 and 4.9). Abelian extensions (and nilpotent extensions in general) K_∞/K are canonically orientable. In particular, cyclotomic, Lubin-Tate and 1-dimensional towers K_∞/K are canonically orientable. This observation matches with previous study of $\mathbf{B}_{\mathrm{dR}}^+$ -representations over these towers (see [Fon04], [Ber16], [Por22] etc.)

¹For simplicity, we are being imprecise in the introduction: one needs to explain how to equip $\widehat{K}_\infty^{\mathrm{la}}[[t_{K_\infty}]]$ with a natural $\Gamma(K_\infty)$ -action. It is in fact slightly more natural to consider instead the completed symmetric algebra $\widehat{\mathrm{Sym}}_{\widehat{K}_\infty^{\mathrm{la}}}(\mathrm{gr}^1 \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{la}})$, see Theorem 5.5; the $\Gamma(K_\infty)$ -action on $\widehat{K}_\infty^{\mathrm{la}}[[t_{K_\infty}]]$ then depends on a trivialization $\mathrm{gr}^1 \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{la}} \cong \widehat{K}_\infty^{\mathrm{la}} \cdot t_{K_\infty}$.

- (b) (Example 4.10). The false Tate (curve) extension $K(\zeta_{p^\infty}, \pi^{1/p^\infty})/K$ is canonically orientable (where ζ_{p^∞} resp. π^{1/p^∞} signifies taking a compatible system of p^n -th roots of 1 and a fixed uniformizer π). This is the tower used in the study of (φ, τ) -modules ([Car13], [GL20], [GP21] etc.), see also the $\mathbf{B}_{\text{dR}}^+$ -theory developed in [GMW].
- (2) (Proposition 4.4 and Theorem 6.2). If $\text{Lie}\Gamma(K_\infty)$ is a de Rham representation, then K_∞/K is orientable (although not necessarily canonically so). More generally, the tower K_∞ is orientable if and only if $\dim_K \mathbf{D}_{\text{dR}}(\text{Lie}\Gamma(K_\infty)) = \dim_K \mathbf{D}_{\text{HT}}(\text{Lie}\Gamma(K_\infty))$.
- (3) (Example 4.11). The well-known non-split extension $0 \rightarrow \mathbf{Q}_p \rightarrow V \rightarrow \mathbf{Q}_p(1) \rightarrow 0$ induces a p -adic Lie extension which is non-orientable (in particular, the representation $\text{Lie}\Gamma(K_\infty)$ is Hodge–Tate but not de Rham).
- (4) (Example 4.12). Suppose $\Gamma(K_\infty) \cong \text{SL}_2(\mathbf{Z}_p)$. Berger and Colmez studied these extensions in [BC16, §5] and described $\widehat{K}_\infty^{\text{la}}$. The standard two-dimensional representation V_{std} may be considered as a G_K -representation by precomposing it with $G_K \rightarrow \Gamma(K_\infty)$. It has generalized Hodge–Tate weights $\pm s$ for some $s \in \overline{K}^\times$. Depending on the value of s and properties of V_{std} , it turns out K_∞/K could be either canonically orientable, non-canonically orientable or non-orientable.

Our main theorem shows that for every orientation we get a $\Gamma(K_\infty)$ -equivariant isomorphism of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ with a 1-variable power series ring over $\widehat{K}_\infty^{\text{la}}$.

Theorem 1.4 (Theorem 5.5). *Let K_∞/K be orientable with a chosen orientation $\nabla = \nabla(K_\infty)$. Then the reduction map $\mathbf{B}_{\text{dR}}^+(K_\infty) \rightarrow \widehat{K}_\infty$ induces a $\Gamma(K_\infty)$ -equivariant isomorphism*

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0} \cong \widehat{K}_\infty^{\text{la}}.$$

Furthermore, there exists an element $t_{K_\infty} \in (\text{Fil}^1 \mathbf{B}_{\text{dR}}^+(K_\infty))^{\text{pa}}$ such that $\nabla(t_{K_\infty}) = t_{K_\infty}$, and for any such element, the natural inclusion map

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0} \llbracket t_{K_\infty} \rrbracket \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$$

is an isomorphism.

The following theorem then completely answers Question 1.1: an orientation is exactly the data which determines a $\Gamma(K_\infty)$ -equivariant power series description of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$.

Theorem 1.5 (Theorem 6.3). *The orientations of K_∞ are naturally in bijection with the set of $\Gamma(K_\infty)$ -equivariant isomorphisms between $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ and 1-variable power series rings over $\widehat{K}_\infty^{\text{la}}$ inducing the identity on graded pieces.*

Remark 1.6. In particular, if a tower K_∞/K is canonically orientable, then there is a canonical isomorphism

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cong \widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket.$$

1.3. A cohomological application. The key fact $\nabla(t_{K_\infty}) = t_{K_\infty}$ in Theorem 1.4 makes it possible to connect $\mathbf{B}_{\text{dR}}^+$ -representations with *regular connections* by using any orientable tower. Let U be a finite free $\mathbf{B}_{\text{dR}}^+$ -module equipped with a semilinear G_K -action; using a standard dévissage argument which makes use of our vanishing theorem for higher locally analytic vectors (Theorem 2.1), one can show that

$$D := (U^{H(K_\infty)})^{\Gamma(K_\infty)-\text{pa}}$$

is a finite free $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ -module with full rank, which then admits a $\nabla = \nabla(K_\infty)$ -action. Using the identification $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cong \widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket$, and using the fact $\nabla(t_{K_\infty}) = t_{K_\infty}$, it is clear that ∇ induces a *regular connection* on D in the sense that it satisfies the Leibniz rule with respect to the (regular) derivation $t_{K_\infty} \cdot \frac{d}{dt_{K_\infty}}$, that is:

$$\nabla(ax) = a\nabla(x) + t_{K_\infty} \cdot \frac{d}{dt_{K_\infty}}(a)x$$

for all $a \in \widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket$ and $x \in D$. We have the following cohomological consequence.

Theorem 1.7 (Theorem 7.5). *Suppose K_∞ is orientable with a chosen orientation ∇ . Then there is a $\Gamma(K_\infty)$ -equivariant quasi-isomorphism*

$$[D \xrightarrow{\nabla} D] \simeq \mathrm{R}\Gamma(G_K, U) \otimes_K \widehat{K}_\infty^{\mathrm{la}}.$$

In addition, taking $\Gamma(K_\infty)$ -invariants, we have

$$[D \xrightarrow{\nabla} D]^{\Gamma(K_\infty)=1} \simeq \mathrm{R}\Gamma(G_K, U).$$

1.4. Structure of the article. In §2 we prove the vanishing of the higher locally analytic vectors of a finite free semilinear $\widehat{K}_\infty$ -representation. This gives the exactness needed to lift locally analytic vectors through the quotients of $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$. In §3 we recall the Sen operator of a p -adic Lie tower and introduce the adjoint representation $\mathrm{Lie}\, \Gamma(K_\infty)$. In §4 we define orientations, give criteria for their existence and uniqueness, and study their behavior under morphisms of towers. In §5 we compute the pro-analytic vectors in $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$. In §6 we deduce consequences of the main theorem for orientations. In §7 we give cohomological applications in the theory of Sen modules and $\mathbf{B}_{\mathrm{dR}}^+$ -representations.

1.5. Notation and conventions.

1.5.1. p -adic Hodge theory. Let K be a mixed characteristic complete discrete valuation field with perfect residue field of characteristic p . Let K_{cyc}/K be the cyclotomic extension by adjoining all p -power roots of unity, and let χ_{cyc} be the cyclotomic character inducing $\mathrm{Gal}(K_{\mathrm{cyc}}/K) \rightarrow \mathbb{Z}_p^\times$. Our Hodge-Tate weights are normalized so that χ_{cyc} is of weight 1. Let $\mathbf{C}$ be the completion of the algebraic closure $\overline{K}$. Let $\mathbf{B}_{\mathrm{dR}}^+$ be Fontaine's usual DVR with uniformizer t . Fontaine's G_K -equivariant map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \mathbf{C}$ gives an identification $\mathbf{B}_{\mathrm{dR}}^+/t\mathbf{B}_{\mathrm{dR}}^+ \cong \mathbf{C}$.

Whenever we write K_∞/K , we always mean an infinitely ramified extension such that $\Gamma(K_\infty/K) := \mathrm{Gal}(K_\infty/K)$ is a p -adic Lie group. The completion $\widehat{K}_\infty$ is a perfectoid field (see [CG96, Theorem 2.13] and [Sch12, Remark 3.3]). Most of the properties discussed in this article only depend on $\Gamma(K_\infty/K)$ up to an open subgroup, and so only depend on K up to a finite extension; for this reason we sometimes omit K from the notation and simply write $\Gamma(K_\infty)$ for $\Gamma(K_\infty/K)$. We let $\mathbf{B}_{\mathrm{dR}}^+(K_\infty) := \mathbf{B}_{\mathrm{dR}}^{+, \mathrm{Gal}(\overline{K}/K_\infty)}$. Then $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$ is a DVR (see [Sch13, Corollary 6.4]) whose maximal ideal we denote by I_θ , and by pro-étale descent one shows that θ induces an isomorphism $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta \cong \widehat{K}_\infty$. Each quotient $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta^n$ is a $\mathbf{Q}_p$ -Banach space with a continuous action of $\Gamma(K_\infty)$. The expression $\mathbf{B}_{\mathrm{dR}}^+(K_\infty) = \varprojlim_n (\mathbf{B}_{\mathrm{dR}}^+(K_\infty)/I_\theta^n)$ exhibits $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$ as a $\mathbf{Q}_p$ -Fréchet space with a continuous action of $\Gamma(K_\infty)$. We let $L(K_\infty) = \mathrm{gr}^1 \mathbf{B}_{\mathrm{dR}}^+(K_\infty)$. It is a 1-dimensional $\widehat{K}_\infty$ -vector space endowed with a $\widehat{K}_\infty$ -semilinear action of $\Gamma(K_\infty)$.

1.5.2. Locally analytic vectors. Given a compact p -adic Lie group G acting continuously on a $\mathbf{Q}_p$ -Banach space M , we let M^{la} denote the G -locally analytic vectors in M (the group G is omitted from the notation, since it will always be clear from the context). If M is a $\mathbf{Q}_p$ -Fréchet space with a G -action, so that $M = \varprojlim_i M_i$ with the M_i forming a compatible system of $\mathbf{Q}_p$ -Banach representations of G , we define the pro-analytic vectors in M as $M^{\mathrm{pa}} := \varprojlim_i M_i^{\mathrm{la}}$.

The following lemma summarizes two well-known results which are repeatedly used in this paper.

Lemma 1.8. *Let R be a $\mathbf{Q}_p$ -Fréchet algebra with G -action.*

- (1) $(R^{\mathrm{pa}})^\times = R^{\mathrm{pa}} \cap R^\times$. *In particular, if R is a field, then so is R^{pa} .*
- (2) *Let M be a finite free R -module equipped with an R -semilinear G -action. Suppose $e_1, \dots, e_d$ is a basis of M such that each e_i lies in M^{pa} . Then M^{pa} is a finite free R^{pa} -module with $e_1, \dots, e_d$ as a basis.*

Proof. For item (1), see [BC16, Lemma 2.5] and [GP21, Lemma 3.1.2]. For item (2), see [Ber16, Proposition 2.4]. $\square$

Given a compact p -adic Lie group G , we denote by $R_{G\text{-la}}^i$ for $i \geq 0$ the right derived functors of the functor of G -locally analytic vectors; concretely, given a G -representation on a $\mathbf{Q}_p$ -Banach space M , we have

$$R_{G\text{-la}}^i(M) = \varinjlim_{G' \leq G} H^i(G', \mathcal{C}^{\text{la}}(G', M)),$$

where the direct limit ranges over open normal uniform subgroups of G . Alternatively, one may fix an open normal uniform $G_0 \leq G$ and set $G_n = G_0^{p^n}$, and then it is also the case that

$$R_{G\text{-la}}^i(M) = \varinjlim_n H^i(G_n, \mathcal{C}^{\text{an}}(G_n, M)).$$

Here $\mathcal{C}^{\text{an}}(G_n, M)$ is endowed with the action $g(f)(x) = g(f(g^{-1}x))$. We refer the reader to [RJRC22] for more details on the definitions of these derived functors. In the particular case $i = 0$, we have the subspaces of analytic vectors $M^{G_n\text{-an}} = H^0(G_n, \mathcal{C}^{\text{an}}(G_n, M))$. These can also be defined in an alternative way, which we now explain. Since G_0 is uniform, we can define a homeomorphism $c : G_0 \cong \mathbf{Z}_p^d$, so that there are induced homeomorphisms $c|_{G_n} : G_n \cong (p^n \mathbf{Z}_p)^d$. Then $M^{G_n\text{-an}}$ can also be defined as the space of $m \in M$ such that there exists an expansion $g(m) = \sum_{\mathbf{k} \in \mathbf{N}^d} c(g)^{\mathbf{k}} m_{\mathbf{k}}$ with $p^{n|\mathbf{k}|} m_{\mathbf{k}} \rightarrow 0$. Using this perspective one sees that $M^{G_n\text{-an}}$ is a Banach space with valuation given by $\text{val}_{G_n}(m) = \inf_{\mathbf{k}} (n|\mathbf{k}| + \text{val}_M(m_{\mathbf{k}}))$.

1.5.3. Derivations. We often consider a linear extension of an action by derivations. These are defined in the following way. Let R be a commutative $\mathbf{Q}_p$ -algebra, and let M be an R -module. Suppose M has an action of $\text{Lie}\Gamma$ by $\mathbf{Q}_p$ -linear derivations. Given $\Theta \in R \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma$, we can act with $\Theta = \sum b_i \otimes \partial_i$ on M by setting $\Theta(x) := \sum_i b_i \partial_i(x)$. This action is well defined (independent of the specific writing $\Theta = \sum b_i \otimes \partial_i$).

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2. VANISHING OF HIGHER LOCALLY ANALYTIC VECTORS

2.1. The vanishing theorem. Recall our notation and conventions §1.5.1 and §1.5.2. In this section we prove the following theorem. It is well known to the experts (see [Por24, Remark 5.7] which follows the idea of [Pan22, Theorem 3.6.1]), but we include a proof here due to a lack of a suitable reference.

Theorem 2.1. *Let W be a finite free $\widehat{K}_\infty$ -semilinear representation of $\Gamma(K_\infty)$. Then $R_{\Gamma(K_\infty)\text{-la}}^i(W) = 0$ for $i \geq 1$.*

The following corollary will be used often in §5. It follows from the Theorem by dévissage.

Corollary 2.2. *Let W be a finite length $\mathbf{B}_{\text{dR}}^+(K_\infty)$ -semilinear representation of $\Gamma(K_\infty)$. Then $R_{\Gamma(K_\infty)\text{-la}}^i(W) = 0$ for $i \geq 1$.*

2.2. Reductions. We first show that we are allowed to replace K by a finite extension.

Lemma 2.3. *Let K' be a finite extension of K and let $K'_\infty = K'K_\infty$. Then Theorem 2.1 is true for K'_∞/K' and $W' := \widehat{K}'_\infty \otimes_{\widehat{K}_\infty} W$ if and only if it is true for K_∞/K and W .*

Proof. Assume first that $K'_\infty = K_\infty$, so that $W' = W$. Since $\Gamma(K_\infty/K')$ is an open subgroup of $\Gamma(K_\infty/K)$, we have $R_{\Gamma(K_\infty/K')\text{-la}}^i(W) = R_{\Gamma(K_\infty/K)\text{-la}}^i(W)$ and in this case the vanishing for K'_∞/K' or K_∞/K is clearly equivalent.

In the general case, by [Ser64, p. 97, Lemma 6], there exists a finite extension K'_0/K such that K'_0 is linearly disjoint from K_∞/K and $K'_0 K_\infty = K'_\infty$. Let $E = K'_0 K'$. Then by the case already proved, the theorem is equivalent for either of the towers K'_∞/K' , K'_∞/E or K'_∞/K'_0 . Hence we are free to replace K' with K'_0 .

Thus, we reduce to the case where K' is linearly disjoint from K_∞ . We have then a natural isomorphism $\Gamma(K'_\infty/K') \cong \Gamma(K_\infty/K)$, and so open subgroups in these two groups are identified. We may identify a system of open normal uniform subgroups $\Gamma_n(K_\infty/K)$ in $\Gamma(K_\infty/K)$ with a system of open normal uniform subgroups $\Gamma_n(K'_\infty/K')$ in $\Gamma(K'_\infty/K')$. Furthermore, we have

$$W' = \widehat{K}'_\infty \otimes_{\widehat{K}_\infty} W = K' \otimes_K W,$$

So there is a $\Gamma_n(K'_\infty/K') \cong \Gamma_n(K_\infty/K)$ -equivariant isomorphism

$$\begin{aligned} \mathcal{C}^{\text{an}}(\Gamma_n(K'_\infty/K'), W') &= \mathcal{C}^{\text{an}}(\Gamma_n(K'_\infty/K'), W \otimes_K K') \\ &= \mathcal{C}^{\text{an}}(\Gamma_n(K_\infty/K), W) \otimes_K K' \\ &\cong \mathcal{C}^{\text{an}}(\Gamma_n(K_\infty/K), W)^{\oplus [K':K]}. \end{aligned}$$

Hence, for every $i \geq 0$ we have

$$R_{\Gamma(K'_\infty/K')\text{-la}}^i(W') \cong R_{\Gamma(K_\infty/K)\text{-la}}^i(W)^{\oplus [K':K]}$$

and the result follows. $\square$

In what follows, we let $L_\infty = K_\infty K_{\text{cyc}}$. The tower L_∞/K still fits into our framework, in the sense that $\Gamma(L_\infty/K)$ is an infinitely ramified p -adic Lie group.

Proposition 2.4. *If L_∞/K_∞ is a finite extension then Theorem 2.1 holds for K_∞/K .*

Proof. By the previous lemma, it suffices to prove it for the tower L_∞/K , which contains K_{cyc} . The tower L_∞/K satisfies the Tate-Sen axioms (TS1)-(TS3) by [BC08, Proposition 4.1.1], [BC09, §14.1], and (TS4) by [Por24, Example 5.5]. Hence we conclude by [Por24, Theorem C]. $\square$

Lemma 2.5. *Suppose L_∞/K_∞ is infinite. Then for the purpose of proving Theorem 2.1, we may assume $\text{Gal}(L_\infty/K_\infty) \cong \mathbf{Z}_p$. Writing γ for a topological generator of $\text{Gal}(L_\infty/K_\infty)$, we may also assume there exists an element $z \in \widehat{L}_\infty^{\Gamma(L_\infty/K)\text{-la}}$ with $\gamma(z) = z + 1$.*

Proof. The cyclotomic character gives an isomorphism of $\Delta = \text{Gal}(L_\infty/K_\infty)$ with an open subgroup of $\mathbf{Z}_p^\times$. Replacing K by a finite extension, as we may, forces $\text{Gal}(K_{\text{cyc}}/K) \cong \mathbf{Z}_p$, and so we may assume $\Delta \cong \mathbf{Z}_p$. By [BC16, Lemme 3.6], after replacing K by a finite extension (as we may) there exists an element $z \in \widehat{L}_\infty^{\Gamma(L_\infty/K)\text{-la}}$ with $g(z) = z + \log \chi_{\text{cyc}}(g)$ for $g \in \Delta$. Let γ be a generator of Δ . Replacing z with $z/\log \chi_{\text{cyc}}(\gamma)$, we may assume $\gamma(z) = z + 1$. $\square$

2.3. Proof of the theorem. We now prove Theorem 2.1. In this subsection, we use some condensed mathematics to simplify cohomology computations; these techniques will not be used elsewhere in the article. All the objects in what follows will be considered in the sense explained in [RJRC22], [RJRC23], using the fully faithful inclusion of Banach representations of compact p -adic Lie groups into the appropriate categories of solid representations in $\mathbf{Q}_p$ -modules, as explained in detail in loc. cit.

We need the following lemma.

Lemma 2.6. *Let $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$ be an exact sequence of compact p -adic Lie groups. Then the inflation functor $\text{Inf}_{G,N}$ from complexes of $\mathbf{Q}_p[G/N]$ -modules to complexes of $\mathbf{Q}_p[G]$ -modules (respectively the locally analytic inflation functor $\text{Inf}_{G,N}^{\text{la}}$ from complexes of locally analytic G/N -representations to complexes of locally analytic G -representations) is left adjoint to the functor of N -fixed points $C \mapsto \text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C)$.*

Proof. Consider first the continuous setting. The inflation functor $\text{Inf}_{G,N}$ from $\mathbf{Q}_p[G/N]$ -complexes to $\mathbf{Q}_p[G]$ -complexes is given by restricting from $\mathbf{Q}_p[G/N]$ -complexes to $\mathbf{Q}_p[G]$ -complexes. We have the series of identities

$$\begin{aligned} \text{RHom}_{\mathbf{Q}_p[G]}(\text{Inf}_N^G(C_1), C_2) &= \text{RHom}_{\mathbf{Q}_p[G/N]}(C_1, \text{RHom}_{\mathbf{Q}_p[G]}(\mathbf{Q}_p[G/N], C_2)) \\ &= \text{RHom}_{\mathbf{Q}_p[G/N]}(C_1, \text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C_2)), \end{aligned}$$

where the last equality follows from Shapiro's lemma ([RJC22, Proposition 5.15]). This proves the lemma in the continuous setting. In the locally analytic setting, one can argue similarly by restricting from $D^{h^+}(G/N, \mathbf{Q}_p)$ -complexes to $D^{h^+}(G, \mathbf{Q}_p)$ -complexes and using the h^+ -analytic version of Shapiro's lemma of loc. cit. $\square$

Proposition 2.7. *Let $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$ be an exact sequence of compact p -adic Lie groups. Let C be a complex of solid $\mathbf{Q}_p[G]$ -modules. Then there is a natural quasi-isomorphism of complexes*

$$\text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C^{\text{RG-la}}) = (\text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C))^{\text{R}(G/N)\text{-la}}.$$

Here RG-la and $\text{R}(G/N)\text{-la}$ denote derived locally analytic vectors as in [RJC22].

Proof. For $\bullet \in \{G, G/N\}$, let $D(\mathbf{Q}_p[\bullet])$ (respectively $\text{Rep}_{\mathbf{Q}_p}^{\text{la}}(\bullet)$) be the derived category of solid $\mathbf{Q}_p[\bullet]$ -modules (respectively the subcategory of locally analytic $\bullet$ -representations). Let

$$\iota_{\bullet} : \text{Rep}_{\mathbf{Q}_p}^{\text{la}}(\bullet) \rightarrow D(\mathbf{Q}_p[\bullet])$$

be the fully faithful inclusion functor (see [RJC23, Remark 1.2.1(2)]). We have the following commutative diagram:

$$\begin{array}{ccc} \text{Rep}^{\text{la}}(G/N) & \xrightarrow{\text{Inf}_{G,N}^{\text{la}}} & \text{Rep}^{\text{la}}(G) \\ \downarrow \iota_{G/N} & & \downarrow \iota_G \\ D(\mathbf{Q}_p[G/N]) & \xrightarrow{\text{Inf}_{G,N}} & D(\mathbf{Q}_p[G]). \end{array}$$

By the previous lemma, the functors $\text{Inf}_{G,N}, \text{Inf}_{G,N}^{\text{la}}$ have as right adjoint the N -fixed points functor $C \mapsto \text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C)$. By [RJC23, Theorem A, Remark 1.2.1(2)], for $\bullet \in \{G, G/N\}$ the functor $\iota_{\bullet}$ has as right adjoint the functor $\text{R}(\bullet)\text{-la}$. It follows that $\iota_G \circ \text{Inf}_{G,N}^{\text{la}}$ has right adjoint the functor

$$C \mapsto \text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C^{\text{RG-la}})$$

while $\text{Inf}_{G,N} \circ \iota_{G/N}$ has as right adjoint the functor

$$C \mapsto (\text{RHom}_{\mathbf{Q}_p[N]}(\mathbf{Q}_p, C))^{\text{R}(G/N)\text{-la}}.$$

Since $\iota_G \circ \text{Inf}_{G,N}^{\text{la}} = \text{Inf}_{G,N} \circ \iota_{G/N}$, these two right adjoints are naturally identified, which concludes the proof. $\square$

Using the reductions of the previous subsection (Proposition 2.4 and Lemma 2.5), we may assume L_{∞}/K_{∞} is an infinite extension, with $\Delta = \text{Gal}(L_{\infty}/K_{\infty}) = \gamma^{\mathbf{Z}_p}$ and that there exists $z \in \widehat{L}_{\infty}^{\text{la}}$ with $\gamma(z) = z + 1$. Let $\nabla = \nabla_{\gamma}$ be the generator of $\text{Lie}\Delta$ corresponding to γ , we have $\nabla(z) = 1$. The map $\Gamma(L_{\infty}/K) \rightarrow \Gamma(K_{\infty}/K)$ has image $\text{Gal}(K_{\infty}/K)$. We may and do assume $K_{\text{cyc}} \cap K_{\infty} = K$ again by performing a finite extension of K . For the rest of the proof, write $\Gamma(L_{\infty}) := \Gamma(L_{\infty}/K)$ and $\Gamma(K_{\infty}) := \Gamma(K_{\infty}/K)$ to lighten the notation. We let $W(K_{\infty}) := W$ and $W(L_{\infty}) := \widehat{L}_{\infty} \otimes_{\widehat{K}_{\infty}} W(K_{\infty})$, which is a $\Gamma(L_{\infty})$ -semilinear representation. We can write the following chain of quasi-isomorphisms:

$$\begin{aligned} W(K_{\infty})^{\text{R}\Gamma(K_{\infty})\text{-la}} &= \text{RHom}_{\mathbf{Q}_p[\Delta]}(\mathbf{Q}_p, W(L_{\infty}))^{\text{R}\Gamma(K_{\infty})\text{-la}} \\ &= \text{RHom}_{\mathbf{Q}_p[\Delta]}(\mathbf{Q}_p, W(L_{\infty}))^{\text{R}\Gamma(L_{\infty})\text{-la}} \\ &= \text{RHom}_{\mathbf{Q}_p[\Delta]}(\mathbf{Q}_p, W(L_{\infty}))^{\Gamma(L_{\infty})\text{-la}}. \end{aligned}$$

We explain the equalities above. Since the extension $\widehat{L}_\infty/\widehat{K}_\infty$ is pro-étale ($\widehat{K}_\infty$ is perfectoid), we know by the almost purity theorem that $H_{\text{cont}}^i(\Delta, W(L_\infty)) = 0$ for $i > 0$ and is equal to $W(K_\infty)$ for $i = 0$, which gives the first equality. The second equality follows from Proposition 2.7. For the third equality, use that as in the proof of Proposition 2.4, we know that $R_{\Gamma(L_\infty)\text{-la}}^i(W(L_\infty)) = 0$ for $i > 0$. Hence, the proof of the theorem reduces to the following key computation.

Lemma 2.8. *We have $H_{\text{cont}}^i(\Delta, W(L_\infty)^{\Gamma(L_\infty)\text{-la}}) = 0$ for $i > 0$.*

Proof. In the proof of [BC16, Théorème 3.4] it is shown that

$$W(L_\infty)^{\Gamma(L_\infty)\text{-la}} = \widehat{L}_\infty^{\text{la}} \otimes_{\widehat{K}_\infty^{\text{la}}} W(K_\infty)^{\Gamma(K_\infty)\text{-la}}.$$

As Δ acts trivially on $W(K_\infty)^{\Gamma(K_\infty)\text{-la}}$, we reduce to the case $W(L_\infty) = \widehat{L}_\infty$. Since $\Delta \cong \mathbf{Z}_p$, the only nontrivial case is $i = 1$. Let Γ_n be the usual sequence of open normal uniform subgroups for $\Gamma(L_\infty)$. Then we have $\widehat{L}_\infty^{\text{la}} = \varinjlim_n \widehat{L}_\infty^{\Gamma_n\text{-an}}$. The complex computing continuous cohomology of Δ is $\text{RHom}_{\mathbf{Q}_p[\Delta]}(\mathbf{Q}_p, \widehat{L}_\infty^{\text{la}})$, and since $\mathbf{Q}_p$ is a compact object in the category of solid $\mathbf{Q}_p[\Delta]$ -modules (it is the cokernel of a map $\mathbf{Q}_p[\Delta] \rightarrow \mathbf{Q}_p[\Delta]$), we learn that $H_{\text{cont}}^1(\Delta, \widehat{L}_\infty^{\text{la}}) = \varinjlim_n H_{\text{cont}}^1(\Delta, \widehat{L}_\infty^{\Gamma_n\text{-an}})$. Hence, we can use Lazard's theorem ([RJRC22, Theorem 1.7]) to get $H_{\text{cont}}^1(\Delta, \widehat{L}_\infty^{\Gamma_n\text{-an}}) = H_{\text{Lie}\Delta}^1(\widehat{L}_\infty^{\Gamma_n\text{-an}})^{\Delta=1}$.

We therefore reduce to showing that the direct limit $\varinjlim_n H_{\text{Lie}\Delta}^1(\widehat{L}_\infty^{\Gamma_n\text{-an}})$ is zero. We are going to do this by showing that for every n there exists $N \geq n$ such that the map $H_{\text{Lie}\Delta}^1(\widehat{L}_\infty^{\Gamma_n\text{-an}}) \rightarrow H_{\text{Lie}\Delta}^1(\widehat{L}_\infty^{\Gamma_N\text{-an}})$ is zero. Equivalently, we show that for every $x \in \widehat{L}_\infty^{\Gamma_n\text{-an}}$ there exists $y \in \widehat{L}_\infty^{\Gamma_N\text{-an}}$ such that $\nabla(y) = x$.

We are going to define y as a power series. For $m \geq 1$, let val_{Γ_m} denote the valuation of the Banach space $\widehat{L}_\infty^{\Gamma_m\text{-an}}$. By [BC16, Lemme 2.6], we have $\nabla(\widehat{L}_\infty^{\Gamma_n\text{-an}}) \subset \widehat{L}_\infty^{\Gamma_n\text{-an}}$ and there exists some constant C_n (possibly negative) such that for $x \in \widehat{L}_\infty^{\Gamma_n\text{-an}}$ we have

$$\text{val}_{\Gamma_n}(\nabla(x)) \geq C_n + \text{val}_{\Gamma_n}(x).$$

Let $z_0 \in L_\infty$ such that the p -adic valuation $a = \text{val}_{\widehat{L}_\infty}(z - z_0)$ satisfies

$$a > 1/(p-1) + \max(0, -C_n + 1).$$

Since z_0 is smooth, we still have $\nabla(z - z_0) = 1$, so we may replace $z - z_0$ with z . We may choose $N \geq n$ sufficiently large such that $z \in \widehat{L}_\infty^{\Gamma_N\text{-an}}$ and $\text{val}_{\Gamma_N\text{-an}}(z) = \text{val}_{\widehat{L}_\infty}(z) = a$.

For N , we thus have

$$\begin{aligned} \text{val}_{\Gamma_N\text{-an}}(\nabla^i(x) \cdot \frac{z^{i+1}}{(i+1)!}) &\geq \text{val}_{\Gamma_N\text{-an}}(\nabla^i(x)) + \text{val}_{\Gamma_N\text{-an}}(z^{i+1}/(i+1)!) \\ &\geq \text{val}_{\Gamma_n\text{-an}}(\nabla^i(x)) + \text{val}_{\widehat{L}_\infty}(z^{i+1}/(i+1)!) \\ &\geq \text{val}_{\Gamma_n\text{-an}}(x) + iC_n + (i+1)a - (i+1)/(p-1) \\ &\geq \text{val}_{\Gamma_n\text{-an}}(x) + i, \end{aligned}$$

where the last step is because of the condition on a .

Hence, we have $\text{val}_{\Gamma_N\text{-an}}(\nabla^i(x) \cdot z^{i+1}/(i+1)!) \rightarrow \infty$ as $i \rightarrow \infty$. Thus, the power series

$$y = \sum_{i=0}^{\infty} (-1)^i \nabla^i(x) \frac{z^{i+1}}{(i+1)!}$$

converges in $\widehat{L}_\infty^{\Gamma_N\text{-an}}$. A straightforward check shows that $\nabla(y) = x$, which concludes the proof. $\square$

3. SEN OPERATORS

Recall our notation from §1.5. In §3.1 and §3.2, we recall the Sen operator for the extension K_∞/K , following [Sen80] and [BC16]. In §3.3, we study the field of rationality of the Sen operator.

We start by recording an interesting fact (well known to the experts) which shows that p -adic Lie extensions K_∞/K are abundant. This fact serves as a justification of our setup, and will not be used elsewhere in the article.

Lemma 3.1. *Let G be a topologically finitely generated pro- p group. Then there exist a finite extension $K/\mathbf{Q}_p$ and a Galois extension L/K such that $\text{Gal}(L/K) \cong G$ as topological groups.*

Proof. By Theorems 7.5.11-7.5.12 of [NSW08], one can choose K such that G_K surjects onto a free pro- p group with arbitrarily large rank, which then surjects onto G . $\square$

3.1. The Sen operator of the tower K_∞/K . In this subsection we recall the construction of the Sen operator for V a finite-dimensional $\mathbf{Q}_p$ -linear representation of G_K .

Let $V_{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{Q}_p} V$, considered as a *semilinear* representation of G_K . We have the K_{cyc} -vector space

$$\mathbf{D}_{\text{Sen}}(V) := (V_{\mathbf{C}})^{\text{Gal}(\overline{K}/K_{\text{cyc}}), \Gamma(K_{\text{cyc}})\text{-fin}},$$

that is, the K_{cyc} -subspace of elements in $V_{\mathbf{C}}$ which are fixed under H_{cyc} , and which lie in a $\Gamma(K_{\text{cyc}})$ -stable finite-dimensional K -vector space. It is $\Gamma(K_{\text{cyc}})$ -stable, and hence G_K -stable. In [Sen80], Sen proves that the natural map

$$\mathbf{D}_{\text{Sen}}(V)_{\mathbf{C}} := \mathbf{C} \otimes_{K_{\text{cyc}}} \mathbf{D}_{\text{Sen}}(V) \rightarrow V_{\mathbf{C}}$$

is an isomorphism of semilinear G_K -representations.

With this given, one defines the Sen operator $\Theta(V) \in \text{End}_{K_{\text{cyc}}}(\mathbf{D}_{\text{Sen}}(V))$ (a linear operator!) as follows. Choose $\gamma \in \Gamma(K_{\text{cyc}})$ which is sufficiently close to 1. Then

$$\Theta(V) := \log \gamma / \log \chi_{\text{cyc}}(\gamma).$$

This is independent of any choice. There is an inclusion

$$\text{End}_{K_{\text{cyc}}}(\mathbf{D}_{\text{Sen}}(V)) \hookrightarrow \text{End}_{\mathbf{C}}(\mathbf{D}_{\text{Sen}}(V)_{\mathbf{C}}) \cong \text{End}_{\mathbf{C}}(V_{\mathbf{C}}) = \mathbf{C} \otimes_{\mathbf{Q}_p} \text{End}_{\mathbf{Q}_p}(V)$$

which allows us to consider $\Theta(V)$ as an element of $\text{End}_{\mathbf{C}}(V_{\mathbf{C}})$. In fact, we have the following theorem of Sen ([Sen80, Theorem 11]), which shows $\Theta(V)$ belongs to a smaller subspace.

Theorem 3.2. *Let $\rho : G_K \rightarrow \text{GL}_{\mathbf{Q}_p}(V)$ denote the linear representation defining V . Then we have*

$$\Theta(V) \in \mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}(\rho(G_K)) \subset \mathbf{C} \otimes_{\mathbf{Q}_p} \text{End}_{\mathbf{Q}_p}(V).$$

Given a faithful representation V of $\Gamma(K_\infty)$ (which exists by Ado's theorem [Bou75, §7.3]), we set (implicitly using Theorem 3.2)

$$\Theta(K_\infty/K, V) := \Theta(V) \in \mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty).$$

Proposition 3.3. *The element $\Theta(K_\infty/K, V)$ is independent of V and depends on K only up to a finite extension. It is nonzero.*

Proof. See [Sen80, Theorem 12, (a)-(b)]. To show it is nonzero, one can use for example the Corollary to [Sen80, Theorem 11]. $\square$

This allows us to give the following definition.

Definition 3.4 (The Sen operator of K_∞). The Sen operator of K_∞ is the unique element

$$\Theta(K_\infty) \in \mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$$

satisfying $\Theta(K_\infty) = \Theta(K_\infty/K, V)$ for any faithful representation V of $\Gamma(K_\infty)$.

The following compatibility will be useful.

Lemma 3.5. *Let $K_\infty \subset L_\infty$ be two p -adic Lie extensions of K . Then the natural morphism*

$$\mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(L_\infty) \rightarrow \mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty),$$

which is induced by the quotient map $\Gamma(L_\infty) \rightarrow \Gamma(K_\infty)$, maps $\Theta(L_\infty)$ to $\Theta(K_\infty)$.

Proof. Let $V(K_\infty)$ be a faithful representation of $\Gamma(K_\infty)$, and let $V(L_\infty)$ be a faithful representation of $\Gamma(L_\infty)$. By assumption, there is a morphism $\Gamma(L_\infty) \rightarrow \Gamma(K_\infty)$. Precomposing this morphism, we may consider $V(K_\infty)$ as a representation of $\Gamma(L_\infty)$. The representation $V(L_\infty) \oplus V(K_\infty)$ is then a faithful representation of $\Gamma(L_\infty)$. We have a commutative diagram

$$\begin{array}{ccc} \Gamma(L_\infty) & \longrightarrow & \Gamma(K_\infty) \\ \downarrow & & \downarrow \\ \mathrm{GL}_{\mathbf{Q}_p}(V(L_\infty)) \times \mathrm{GL}_{\mathbf{Q}_p}(V(K_\infty)) & \longrightarrow & \mathrm{GL}_{\mathbf{Q}_p}(V(K_\infty)), \end{array}$$

where the bottom horizontal map is induced by the projection $V(L_\infty) \oplus V(K_\infty) \rightarrow V(K_\infty)$. Differentiating and tensoring with $\mathbf{C}$, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(L_\infty) & \longrightarrow & \mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty) \\ \downarrow & & \downarrow \\ \mathrm{End}_{\mathbf{C}}(V(L_\infty)_{\mathbf{C}}) \oplus \mathrm{End}_{\mathbf{C}}(V(K_\infty)_{\mathbf{C}}) & \longrightarrow & \mathrm{End}_{\mathbf{C}}(V(K_\infty)_{\mathbf{C}}). \end{array}$$

It therefore suffices to show that the bottom horizontal map sends the Sen operator of $V(L_\infty) \oplus V(K_\infty)$ to the Sen operator of $V(K_\infty)$. This follows from the equality $\Theta(V(L_\infty) \oplus V(K_\infty)) = \Theta(V(L_\infty)) \oplus \Theta(V(K_\infty))$ (see Remark (4) after [Sen80, Theorem 4]). $\square$

3.2. The adjoint p -adic Galois representation of K_∞ . Continue with the notation of the previous subsection. We now introduce one of the key objects of this paper: the p -adic representation $\mathrm{Lie}\Gamma(K_\infty)$, also called the adjoint representation of K_∞ . We give it the structure of a p -adic representation by precomposing the adjoint representation of $\Gamma(K_\infty)$ with the projection $G_K \rightarrow \Gamma(K_\infty)$. We may consider p -adic Hodge theoretic invariants of $\mathrm{Lie}\Gamma(K_\infty)$ as in classical p -adic Hodge theory. Two key players for us will be the 0-Hodge Tate weight space

$$\mathbf{D}_{\mathbf{C}}(\mathrm{Lie}\Gamma(K_\infty)) = (\mathbf{C} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty))^{G_K}$$

and the positive de Rham space

$$\mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie}\Gamma(K_\infty)) = (\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty))^{G_K}.$$

Because of the definition of the G_K action on $\mathrm{Lie}\Gamma(K_\infty)$, item (1) of the following theorem can be interpreted as saying that as a $\mathbf{C}$ -linear operator, $\Theta(K_\infty)$ is G_K -equivariant.

Theorem 3.6.

- (1) *We have $\Theta(K_\infty) \in \mathbf{D}_{\mathbf{C}}(\mathrm{Lie}\Gamma(K_\infty))$.*
- (2) *We have $\Theta(K_\infty) \in \widehat{K}_\infty^{\mathrm{la}} \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty)$.*

Proof. Item (1) is [Sen80, Theorem 12 (d)]. For item (2), as $\Theta(K_\infty)$ is a $\Gamma(K_\infty)$ -fixed point of $\widehat{K}_\infty \otimes_{\mathbf{Q}_p} \mathrm{Lie}\Gamma(K_\infty)$, it is in particular a locally analytic vector; we can conclude using Lemma 1.8, as $\mathrm{Lie}\Gamma(K_\infty)$ is automatically locally analytic. $\square$

Remark 3.7. Theorem 3.6(1) gives a nice way to see the functoriality of the association $K_\infty \mapsto \Theta(K_\infty)$. Let LieExt_K be the category whose objects are p -adic Lie extensions K_∞/K , with morphisms $K_\infty/K \rightarrow K'_\infty/K$ being the same as a morphism $K_\infty \rightarrow K'_\infty$ which is the identity on K . Then the associations $K_\infty/K \mapsto \mathrm{Lie}\Gamma(K_\infty)$ and $K_\infty/K \mapsto \mathbf{D}_{\mathbf{C}}(\mathrm{Lie}(\Gamma(K_\infty)))$ are contravariant functors, and $\mathbf{D}_{\mathbf{C}}(\mathrm{Lie}(\Gamma(K'_\infty))) \rightarrow \mathbf{D}_{\mathbf{C}}(\mathrm{Lie}(\Gamma(K_\infty)))$ maps $\Theta(K'_\infty)$ to $\Theta(K_\infty)$, by Lemma 3.5.

We shall need the following important result due to Berger and Colmez. Recall §1.5.3 for our definitions for the action of $\widehat{K}_\infty^{\text{la}} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$ on $\widehat{K}_\infty^{\text{la}}$.

Proposition 3.8. *The Sen operator $\Theta(K_\infty) \in \widehat{K}_\infty^{\text{la}} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$ acts by zero on $\widehat{K}_\infty^{\text{la}}$.*

Proof. To show this, consider the proof of [BC16, Théorème 6.1]. It is shown there that there is an isomorphism

$$\widehat{K}_\infty(\mu_{p^m}) \otimes_{K_n} \widehat{K}_\infty^{\Gamma_n\text{-an}} \xrightarrow{\cong} \mathcal{C}^{\text{an}}(\Gamma_n, \widehat{K}_\infty)^{D=0},$$

where for $\phi \in \mathcal{C}^{\text{an}}(\Gamma_n, \widehat{K}_\infty)$, we have $D(\phi)(g) = \lim_{t \rightarrow 0} \frac{\phi(e^{t\mathfrak{a}}g) - \phi(g)}{t}$, for some specific $\mathfrak{a} \in \widehat{K}_\infty(\mu_{p^m}) \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$. In [BC16, Proposition 6.3] it is proved that $\mathfrak{a}$ is the Sen operator of the representation V_1 of $\Gamma(K_\infty)$ which is given as follows: V_1 is the space of functions $\Gamma(K_\infty) \rightarrow \mathbf{Q}_p$ which are restrictions of linear functions from $\text{Mat}_d(\mathbf{Q}_p)$ to $\Gamma(K_\infty)$, and where $\Gamma(K_\infty)$ acts on V_1 by the formula $(g\phi)(x) = \phi(g^{-1}(x))$. The representation V_1 of $\Gamma(K_\infty)$ is faithful, and so $\mathfrak{a} = \Theta(K_\infty)$. It thus suffices to explain why $\mathfrak{a}$ kills $\widehat{K}_\infty^{\Gamma_n\text{-an}}$. In the isomorphism mentioned above, an element $x \in \widehat{K}_\infty^{\Gamma_n\text{-an}}$ is mapped to the function ϕ which is given by $\phi(g) = g(x)$. In particular $\phi(1) = x$. Hence as $D(\phi)(1) = 0$, we get

$$0 = D(\phi)(1) = \lim_{t \rightarrow 0} \frac{\phi(e^{t\mathfrak{a}}) - \phi(1)}{t} = \lim_{t \rightarrow 0} \frac{e^{t\mathfrak{a}}(x) - x}{t} = \mathfrak{a}(x),$$

as required. $\square$

3.3. The field of rationality of $\Theta(K_\infty)$. In this subsection, we discuss the smallest field of $\mathbf{C}$ over which $\Theta(K_\infty)$ is defined.

Definition 3.9 (The field of rationality $E(K_\infty)$). We let $E(K_\infty)$ be the smallest subfield of $\mathbf{C}$ containing K such that $\Theta(K_\infty) \in E(K_\infty) \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$. Equivalently, let $\{\partial_i\}_i$ be a $\mathbf{Q}_p$ -basis of $\text{Lie}\Gamma(K_\infty)$. Writing $\Theta(K_\infty) = \sum a_i \otimes \partial_i \in \mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$, one sees that the field $E(K_\infty)$ generated over K by the a_i is independent of the choice of basis $\{\partial_i\}_i$.

In the following theorem, write $\text{ad}(K_\infty) : \Gamma(K_\infty) \rightarrow \text{GL}(\text{Lie}(\Gamma(K_\infty)))$ for the adjoint representation. Then $\widehat{K}_\infty^{\ker(\text{ad}(K_\infty))} \subset \widehat{K}_\infty$ is a closed subfield.

Theorem 3.10. *The field $E(K_\infty)$ is contained in $\widehat{K}_\infty^{\ker(\text{ad}(K_\infty)), \Gamma(K_\infty)\text{-fin}}$, the subring of elements in $\widehat{K}_\infty^{\ker(\text{ad}(K_\infty))}$ which have $\Gamma(K_\infty)$ -orbit of finite $\mathbf{Q}_p$ -dimension.*

Proof. Let $\pi : G_K \rightarrow \Gamma(K_\infty)$ be the projection. The subgroup $\ker(\text{ad}(K_\infty) \circ \pi)$ fixes $\text{Lie}\Gamma(K_\infty)$ and $\Theta(K_\infty)$ (the latter being fixed by all of G_K). Hence all of the a_i in Definition 3.9 are fixed by this subgroup also. It follows that the field $E(K_\infty)$ is contained in the corresponding fixed field $\widehat{K}_\infty^{\ker \text{ad}(K_\infty)}$. Now express $\Theta(K_\infty) = \sum a_i \otimes \partial_i$ with $a_i \in \widehat{K}_\infty^{\ker \text{ad}(K_\infty)}$ such that the ∂_i form a basis of $\text{Lie}\Gamma(K_\infty)$. Then for every $g \in \Gamma(K_\infty)$,

$$(3.1) \quad \sum a_i \otimes \partial_i = \sum g(a_i) \otimes \text{ad}(g)(\partial_i).$$

Write $h_{ij}(g)$ for the coefficients of the matrix of $\text{ad}(g^{-1})^t$ in the basis $\{\partial_i\}_{i=1,\dots,d}$. They lie in $\mathbf{Q}_p$. The equation (3.1) implies that we have $g(a_i) = \sum_j h_{ij}(g)a_j$, which shows that the $\Gamma(K_\infty)$ -orbit of the a_i is finite-dimensional over $\mathbf{Q}_p$. $\square$

Corollary 3.11. *If $\Gamma(K_\infty)$ is abelian, then $\Theta(K_\infty) \in K \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$.*

When $\text{Lie}\Gamma(K_\infty)$ is Hodge-Tate we can get finer results for $E(K_\infty)$. We record these for the interest of the reader although they will not be used in the rest of the article.

Proposition 3.12. *Suppose $\text{Lie}\Gamma(K_\infty)$ is a Hodge-Tate representation. Then the following three numbers are equal:*

$$(1) \text{ trdeg}_K E(K_\infty).$$

(2) $\dim \Gamma(K_\infty) - \dim Z_{\text{Lie}\Gamma(K_\infty)}(\Theta(K_\infty))$, where $Z_{\text{Lie}\Gamma(K_\infty)}(\Theta(K_\infty))$ is the centralizer of $\Theta(K_\infty)$ in $\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$.

(3) $\dim \Gamma(K_\infty) - \dim_K \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$.

Proof. The equality between (1) and (2) is [Ser79, §1.5, Theorem 3]. The equality between (2) and (3) follows from the fact that $\mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$ is the space where $\Theta(\text{Lie}\Gamma(K_\infty))$ acts by zero, and the observation that as an element of $\text{End}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$, the Sen operator $\Theta(\text{Lie}\Gamma(K_\infty))$ is none other than $\text{ad}(\Theta(K_\infty))$. $\square$

Corollary 3.13. *Suppose $\text{Lie}\Gamma(K_\infty)$ is a Hodge-Tate representation. Then the Sen operator is defined over an algebraic extension of K if and only if $\text{Lie}\Gamma(K_\infty)$ is abelian.*

Proof. The previous proposition shows that $E(K_\infty) \subset \bar{K}$ if and only if $\Theta(K_\infty)$ is central in $\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$. It therefore suffices to show that this condition implies that $\text{Lie}\Gamma(K_\infty)$ is abelian. Let $Z(K_\infty)$ be the center of $\Gamma(K_\infty)$. Possibly replacing $\Gamma(K_\infty)$ with an open subgroup, as we may do, $Z(K_\infty)$ becomes a p -adic Lie group with Lie algebra $Z(\text{Lie}\Gamma(K_\infty))$. Let $\text{P}\Gamma(K_\infty) = \Gamma(K_\infty)/Z(K_\infty)$. Assume by contradiction that $\text{Lie}\Gamma(K_\infty)$ is not abelian. Then we have $\text{P}\Gamma(K_\infty) = \Gamma(K'_\infty)$ for some p -adic Lie extension K'_∞ of K . We have an exact sequence

$$0 \rightarrow Z(\text{Lie}\Gamma(K_\infty)) \rightarrow \text{Lie}\Gamma(K_\infty) \rightarrow \text{Lie}\text{P}\Gamma(K_\infty) \rightarrow 0.$$

By the second paragraph of [Bou75, Chapter I, §3, no. 8], we have

$$Z(\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)) = \mathbf{C} \otimes_{\mathbf{Q}_p} Z(\text{Lie}\Gamma(K_\infty)).$$

Hence by assumption we have

$$\Theta(K_\infty) \in \mathbf{C} \otimes_{\mathbf{Q}_p} Z(\text{Lie}\Gamma(K_\infty)),$$

so the map

$$\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty) \rightarrow \mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\text{P}\Gamma(K_\infty)$$

sends $\Theta(K_\infty)$ to 0. But by Lemma 3.5, this is also equal to $\Theta(K'_\infty)$, which is nonzero by Proposition 3.3. $\square$

4. ORIENTATIONS

In this section we introduce the concept of an orientation, which is a certain Galois-equivariant lift of a Sen operator. As explained in the introduction, this lift will lead to canonical descriptions of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ in §5.

4.1. Definition of orientations.

Definition 4.1. Recall by Theorem 3.6, we have

$$\Theta(K_\infty) \in \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty)) = (\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty))^{G_K}$$

By an orientation of K_∞ , we mean a lift of $\Theta(K_\infty)$ to an element

$$\nabla(K_\infty) \in \mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty)) = (\mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty))^{G_K}$$

We say K_∞ is orientable if there exists an orientation of K_∞ . We say K_∞ is canonically orientable if a unique orientation exists.

Lemma 4.2. *If an orientation $\nabla(K_\infty)$ exists, then necessarily*

$$\nabla(K_\infty) \in (\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty))^{\Gamma(K_\infty)}$$

Proof. This is similar to Theorem 3.6: a $\Gamma(K_\infty)$ -fixed point is necessarily pro-analytic. $\square$

Notation 4.3. We set up some notation which will be convenient in this section. Let $V_{\text{Sen}}(K_\infty)$ be the K -span of $\Theta(K_\infty)$ in $\mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$. Let $V_{\text{dR}}^+(K_\infty)$ be the inverse image of $V_{\text{Sen}}(K_\infty)$ under the natural map $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$. Thus K_∞ is orientable if and only if the map $V_{\text{dR}}^+(K_\infty) \rightarrow V_{\text{Sen}}(K_\infty)$ is non-zero, and canonically orientable if and only if the map $V_{\text{dR}}^+(K_\infty) \rightarrow V_{\text{Sen}}(K_\infty)$ is an isomorphism.

4.2. General criteria. We start with an easy sufficient condition for orientability.

Proposition 4.4 (de Rham extensions are orientable). *Suppose $\mathrm{Lie}\Gamma(K_\infty)$ is de Rham as a representation. Then K_∞ is orientable. It is canonically orientable if and only if $\mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie}\Gamma(K_\infty)(1)) = 0$.*

Proof. The map $\mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\mathbf{C}}(\mathrm{Lie}\Gamma(K_\infty))$ identifies with the surjection

$$\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(\mathrm{Lie}\Gamma(K_\infty)) \twoheadrightarrow \mathrm{gr}^0 \mathbf{D}_{\mathrm{dR}}(\mathrm{Lie}\Gamma(K_\infty))$$

and has kernel $\mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie}\Gamma(K_\infty)(1))$. $\square$

Remark 4.5. Write $V = \mathrm{Lie}\Gamma(K_\infty)$. We have an exact sequence

$$(4.1) \quad 0 \rightarrow \mathbf{D}_{\mathrm{dR}}^+(V(1)) \rightarrow \mathbf{D}_{\mathrm{dR}}^+(V) \rightarrow \mathbf{D}_{\mathbf{C}}(V) \xrightarrow{\delta} H^1(G_K, \mathbf{B}_{\mathrm{dR}}^+ \otimes V(1)).$$

The tower K_∞ is orientable if and only if $\delta(\Theta(K_\infty)) = 0$. Thus the image $\delta(\Theta(K_\infty))$ is the precise obstruction for orientability. However, this obstruction is in general difficult to compute directly; we shall see in Theorem 6.2 below an (equivalent) characterization of orientability (the proof will use results in §5).

To give a useful characterization for K_∞ being canonically orientable, we recall a classical lemma.

Lemma 4.6. *Let V be a p -adic Galois representation of G_K . The following are equivalent.*

- (1) *The representation V has no generalized Hodge-Tate weights in $\mathbf{Z}_{\leq 0}$.*
- (2) *We have $H^i(G_K, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V) = 0$ for $i = 0, 1$.*
- (3) *We have $\mathbf{D}_{\mathrm{dR}}^+(V) = 0$.*

Proof. First we show that item (1) implies item (2). For $n \geq 0$, we have exact sequences

$$0 \rightarrow t^{n+1} \mathbf{B}_{\mathrm{dR}}^+ \rightarrow t^n \mathbf{B}_{\mathrm{dR}}^+ \rightarrow \mathbf{C}(n) \rightarrow 0.$$

Tensoring with V and taking G_K -cohomology, we get that the natural maps

$$H^i(G_K, t^{n+1} \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V) \rightarrow H^i(G_K, t^n \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)$$

for $i = 0, 1$ are isomorphisms. Hence we conclude by t -adic continuity.

Clearly item (2) implies item (3). To conclude, we show that the negation of item (1) implies the negation of item (3). Suppose then that V has a generalized Hodge-Tate weight in $\mathbf{Z}_{\leq 0}$, and let n be the minimal integer. Observe that $V(-n+1)$ has no generalized Hodge-Tate weights in $\mathbf{Z}_{\leq 0}$, so by the direction already established, we have $H^i(G_K, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V(-n+1)) = 0$ for $i = 0, 1$. Hence $\mathbf{D}_{\mathrm{dR}}^+(V(-n)) \cong \mathbf{D}_{\mathbf{C}}(V(-n))$, so $\mathbf{D}_{\mathrm{dR}}^+(V(-n))$ is nonzero. But $\mathbf{D}_{\mathrm{dR}}^+(V(-n)) = \mathrm{Fil}^{-n} \mathbf{D}_{\mathrm{dR}}^+(V)$ is contained in $\mathbf{D}_{\mathrm{dR}}^+(V)$, so $\mathbf{D}_{\mathrm{dR}}^+(V)$ itself is nonzero. $\square$

We deduce the following characterization of canonically orientable extensions:

Theorem 4.7 (Characterization of canonically orientable K_∞). *The following are equivalent.*

- (1) *The extension K_∞ is canonically orientable.*
- (2) *The representation $\mathrm{Lie}\Gamma(K_\infty)$ has no generalized Hodge-Tate weights in $\mathbf{Z}_{\leq -1}$.*

Proof. Write $V = \mathrm{Lie}\Gamma(K_\infty)$.

Suppose item (1) holds. Then $\mathbf{D}_{\mathrm{dR}}^+(V(1))$, which is the kernel of $\mathbf{D}_{\mathrm{dR}}^+(\mathrm{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\mathbf{C}}(\mathrm{Lie}\Gamma(K_\infty))$, must be zero. So the previous lemma implies item (2).

Suppose item (2) holds. We have an exact sequence

$$(4.2) \quad 0 \rightarrow \mathbf{D}_{\mathrm{dR}}^+(V(1)) \rightarrow \mathbf{D}_{\mathrm{dR}}^+(V) \rightarrow \mathbf{D}_{\mathbf{C}}(V) \rightarrow H^1(G_K, \mathbf{B}_{\mathrm{dR}}^+ \otimes V(1)).$$

Since V has no Hodge-Tate weights in $\mathbf{Z}_{\leq -1}$, the twist $V(1)$ has no Hodge-Tate weights in $\mathbf{Z}_{\leq 0}$. By the previous lemma applied to $V(1)$, $\mathbf{D}_{\mathrm{dR}}^+(V(1))$ and $H^1(G_K, \mathbf{B}_{\mathrm{dR}}^+ \otimes V(1))$ both vanish. Hence the map $\mathbf{D}_{\mathrm{dR}}^+(V) \rightarrow \mathbf{D}_{\mathbf{C}}(V)$ is an isomorphism and K_∞ is canonically orientable. $\square$

4.3. Examples. We now discuss specific examples of the concept of orientability.

Example 4.8 (Abelian extensions are canonically orientable). Suppose $\Gamma(K_\infty)$ is abelian. The Hodge-Tate weights of $\text{Lie}\Gamma(K_\infty)$ are 0, so by Theorem 4.7 we get that K_∞ is canonically orientable. We also know this independently of Theorem 4.7, as we have already established in Corollary 3.11 that the Sen operator lies in $K \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)$, and this clearly has a canonical lift to $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty))$.

Example 4.9 (Virtually nilpotent extensions are canonically orientable). Suppose more generally $\Gamma(K_\infty)$ is virtually nilpotent (i.e. a finite index subgroup of $\Gamma(K_\infty)$ is nilpotent). Then $\text{Lie}\Gamma(K_\infty)$ is nilpotent. The Sen operator of $\text{Lie}\Gamma(K_\infty)$, as an element of

$$\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty) \subset \text{End}_{\mathbf{C}}(\mathbf{C} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)),$$

is equal to $\text{ad}(\Theta(K_\infty))$. In particular, it is a nilpotent operator, which means the generalized Hodge-Tate weights of $\text{Lie}\Gamma(K_\infty)$ are all zero. It follows from Theorem 4.7 that K_∞ is canonically orientable.

Example 4.10 (The false Tate extension is canonically orientable). Let $\pi_0 = \pi$ be a uniformizer of K , inductively for each $n \geq 1$, choose some $\pi_n \in \bar{K}$ such that $\pi_n^p = \pi_{n-1}$. Let $K_\infty = K_{\text{fT}} = \cup_n K_{\text{cyc}}(\pi_n)$ be the false Tate extension. We claim that $\text{Lie}\Gamma(K_{\text{fT}})$ has Hodge-Tate weights 0 and 1, and so K_∞ is canonically orientable by Theorem 4.7. To see this, take the faithful representation V of $\Gamma(K_{\text{fT}})$ given by

$$g \mapsto \begin{bmatrix} \chi_{\text{cyc}}(g) & c(g) \\ 0 & 1 \end{bmatrix},$$

where $c(g)$ is the usual Kummer cocycle defined by the equation $g(\pi^{1/p^\infty}) = \zeta_{p^\infty}^{c(g)} \pi^{1/p^\infty}$. Then we have an identification

$$\text{Lie}\Gamma(K_{\text{fT}}) = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} \subset \text{End}_{\mathbf{Q}_p}(V).$$

Using the above description of $\text{Lie}\Gamma(K_{\text{fT}})$, one computes it is the 2-dimensional representation given by

$$g \mapsto \begin{bmatrix} 1 & 0 \\ -c(g) & \chi_{\text{cyc}}(g) \end{bmatrix},$$

whose Hodge-Tate weights are 0, 1. Let us calculate the Sen operator and its lift in this case. We claim there is a unique element $\tilde{\alpha} \in \mathbf{B}_{\text{dR}}^+(K_{\text{fT}})^{\text{pa}}$ (respectively $\alpha \in \hat{K}_{\text{fT}}^{\text{la}}$) satisfying the identity $g(\tilde{\alpha})\chi_{\text{cyc}}(g) = \tilde{\alpha} + c(g)$ (respectively $g(\alpha)\chi_{\text{cyc}}(g) = \alpha + c(g)$). To show existence, take $\tilde{\alpha} = \log([\pi^b]/\pi)/t$, respectively $\theta(\tilde{\alpha})$. We explain uniqueness in the case of $\tilde{\alpha}$; the case of α is similar but easier. Suppose $\tilde{\alpha}_1, \tilde{\alpha}_2$ are two such elements, and let $z = \tilde{\alpha}_1 - \tilde{\alpha}_2$. Then $g(\tilde{\alpha}_1 - \tilde{\alpha}_2)\chi_{\text{cyc}}(g) = \tilde{\alpha}_1 - \tilde{\alpha}_2$. In particular, z is fixed by $\text{Gal}(\bar{K}/K_{\text{cyc}})$, hence is contained in $\mathbf{B}_{\text{dR}}^+(K_{\text{cyc}})^{\text{pa}}$, which is equal to $K_{\text{cyc}}[[t]]$ by [Por22, Proposition 2.6(2)]. Writing $z = \sum a_i t^i$ with $a_i \in K_{\text{cyc}}$, and observing the identity $\nabla(K_{\text{cyc}})(z) = -z$, we get $a_i = 0$ for all i , and so $z = 0$, which shows $\tilde{\alpha}_1 = \tilde{\alpha}_2$. One then verifies that the Sen operator and its canonical lift are given by

$$\Theta(K_{\text{fT}}) = \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix} \text{ and } \nabla(K_{\text{fT}}) = \begin{bmatrix} 1 & \tilde{\alpha} \\ 0 & 0 \end{bmatrix}.$$

In terms of more traditional notation (see [GMW23, Notation 6.5]) this is

$$\Theta(K_{\text{fT}}) = \nabla_\gamma + \alpha \nabla_\tau \text{ and } \nabla(K_{\text{fT}}) = \nabla_\gamma + \tilde{\alpha} \nabla_\tau.$$

Example 4.11 (A non-orientable example). By [BC09, Example 6.3.5], we may choose a nonsplit extension

$$0 \rightarrow \mathbf{Q}_p \rightarrow V \rightarrow \mathbf{Q}_p(1) \rightarrow 0.$$

²This element appeared in [BC16, §4.5].

Let K_∞ correspond to the kernel of this representation. In matrix notation, we may write this representation as an extension

$$g \mapsto \begin{bmatrix} 1 & c(g) \\ 0 & \chi_{\text{cyc}}(g) \end{bmatrix}$$

for some cocycle $c(g)$. As a subrepresentation of $\text{End}(V)$, we have

$$\text{Lie}\Gamma(K_\infty) \cong \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \subset \begin{bmatrix} * & * \\ * & * \end{bmatrix} = \text{End}(V).$$

From this, it is straightforward to calculate $\text{Lie}\Gamma(K_\infty)$ is the representation given by

$$g \mapsto \begin{bmatrix} \chi_{\text{cyc}}(g)^{-1} & c(g)\chi_{\text{cyc}}(g)^{-1} \\ 0 & 1 \end{bmatrix}.$$

In particular, it is Hodge-Tate with weights $-1, 0$, but is not de Rham (see [BC09, Example 6.3.5]). As it is 2-dimensional, the map $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$ must be zero. So the element $\Theta(K_\infty) \in \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$ cannot be lifted to $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty))$, which means K_∞ is not orientable. We remark that this non-orientability also follows from a more general criterion, see Theorem 6.2.

Example 4.12 ($\text{SL}_2(\mathbf{Z}_p)$ -examples). Suppose that $\Gamma(K_\infty) \cong \text{SL}_2(\mathbf{Z}_p)$. This case is discussed at length in [BC16, §5]. By considering the standard representation of $\text{SL}_2(\mathbf{Z}_p)$, we get a representation of G_K with two generalized Hodge-Tate weights $\pm s$ for some $s \neq 0$ in $\mathbf{Q}_p$. The representation $\text{Lie}\Gamma(K_\infty)$ then has generalized Hodge-Tate weights $0, \pm 2s$.

There are two cases:

- (1) $s \notin \frac{1}{2}(\mathbf{Z} \setminus \{0\})$. In this case, we have $\pm 2s \notin \mathbf{Z}_{\leq -1}$, so Theorem 4.7 applies, and K_∞ is canonically orientable.
- (2) $s \in \frac{1}{2}(\mathbf{Z} \setminus \{0\})$. In this case, either K_∞ is non-orientable, or K_∞ is orientable, but necessarily not canonically so, because $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty)(1))$ is nonzero. This can happen, for example, if one takes a modular form f of weight 2, which is non-CM and non-ordinary at p , and twists the restrictions to $G_{\mathbf{Q}_p(\mu_{2p})}$ of the Galois representation of V_f by $\det(V_f)^{-1/2}$. Later in Corollary 6.1, we will see that for $\text{Lie}\Gamma(K_\infty)$ Hodge-Tate, being orientable is equivalent to being de Rham.

4.4. Compatibilities. In the next section we shall need functoriality properties of orientations. The following result shows that orientations act compatibly along towers; this can be regarded as a compatibility complementing Lemma 3.5.

Lemma 4.13. *Suppose that $K_\infty \subset L_\infty$ are p -adic Lie extensions of K with compatible orientations $\nabla(L_\infty)$, $\nabla(K_\infty)$, i.e. $\nabla(L_\infty)$ is mapped to $\nabla(K_\infty)$ under the map $V_{\text{dR}}^+(L_\infty) \rightarrow V_{\text{dR}}^+(K_\infty)$. Let $x \in \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. Then $\nabla(L_\infty)(x) = \nabla(K_\infty)(x)$.*

Proof. Let $\pi : \Gamma(L_\infty) \rightarrow \Gamma(K_\infty)$ be the projection with the induced projection $\text{Lie}\pi : \text{Lie}\Gamma(L_\infty) \rightarrow \text{Lie}\Gamma(K_\infty)$. The element x lies in the subring $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, hence if $\sum b_i \otimes \partial_i \in \mathbf{B}_{\text{dR}}^+(L_\infty)^{\text{pa}} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(L_\infty)$ then $(\sum b_i \otimes \partial_i)(x) = (\sum b_i \otimes (\text{Lie}\pi)(\partial_i))(x)$. On the other hand, $(1 \otimes \text{Lie}\pi)$ induces the natural map from $V_{\text{dR}}^+(L_\infty) \rightarrow V_{\text{dR}}^+(K_\infty)$, so by assumption $(1 \otimes \text{Lie}\pi)(\nabla(L_\infty)) = \nabla(K_\infty)$. Thus we get

$$\nabla(L_\infty)(x) = (1 \otimes \text{Lie}\pi)(\nabla(L_\infty))(x) = \nabla(K_\infty)(x),$$

as required. $\square$

We will conclude the section by proving a result that will help us shift orientations along central extensions.

Lemma 4.14. *Let L_∞/K be p -adic Lie Galois extension with $K_\infty \subset L_\infty$ and $\Delta = \text{Gal}(L_\infty/K_\infty)$ abelian. Suppose Δ is central in $\Gamma(L_\infty) = \text{Gal}(L_\infty/K)$ up to a finite extension of K ; in other words, that the conjugation map $\Gamma(K_\infty) = \text{Gal}(K_\infty/K) \rightarrow \text{Aut}(\Delta)$ has finite image. Then the natural map $V_{\text{dR}}^+(L_\infty) \rightarrow V_{\text{dR}}^+(K_\infty)$ is an isomorphism and induces a bijection between orientations of L_∞ and orientations of K_∞ .*

Proof. To make the notation lighter, we write for this proof only $\mathbf{D}_{\text{dR}}^+(K_\infty)$ for $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty))$; similarly with $\mathbf{D}_C$ instead of $\mathbf{D}_{\text{dR}}^+$ and L_∞ instead of K_∞ . We are free to replace K by a finite extension contained in K_∞ , and so we may assume Δ is central in $\Gamma(L_\infty)$. We have an exact sequence

$$1 \rightarrow \Delta \rightarrow \Gamma(L_\infty) \rightarrow \Gamma(K_\infty) \rightarrow 1$$

of p -adic Lie groups. We may consider $\text{Lie}(\Gamma(L_\infty))$ as a G_K -representation by letting G_K first project onto $\Gamma(L_\infty)$ and then act via the adjoint action. We have an exact sequence of G_K -representations with $\mathbf{Q}_p$ -coefficients:

$$0 \rightarrow \text{Lie}\Delta \rightarrow \text{Lie}\Gamma(L_\infty) \rightarrow \text{Lie}\Gamma(K_\infty) \rightarrow 0,$$

and $\text{Lie}\Delta$ is the trivial representation, since Δ is central.

We now argue in steps:

Step 1: Consider the exact sequence obtained by tensoring with $\mathbf{C}$ and taking G_K -fixed points:

$$0 \rightarrow \mathbf{D}_C(\text{Lie}\Delta) \rightarrow \mathbf{D}_C(L_\infty) \rightarrow \mathbf{D}_C(K_\infty) \rightarrow H_C^1(\text{Lie}\Delta) \rightarrow \dots$$

Then we claim that $V_{\text{Sen}}(K_\infty)$ is in the kernel of the connecting map from $\mathbf{D}_C(K_\infty)$ to $H_C^1(\text{Lie}\Delta)$. Indeed, by Lemma 3.5, $\Theta(K_\infty)$ is the image of $\Theta(L_\infty)$ under the map from $\mathbf{D}_C(L_\infty)$ to $\mathbf{D}_C(K_\infty)$, hence it is mapped to zero in the complex.

Step 2: Let ∂ be the boundary map

$$\partial : \mathbf{D}_C(K_\infty) \rightarrow H_C^1(\text{Lie}\Delta).$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{D}_C(\text{Lie}\Delta) & \longrightarrow & \mathbf{D}_C(L_\infty) & \longrightarrow & \ker(\partial) \longrightarrow 0 \\ & & \downarrow g & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker(f) & \longrightarrow & \mathbf{D}_C(L_\infty)/V_{\text{Sen}}(L_\infty) & \xrightarrow{f} & \mathbf{D}_C(K_\infty)/V_{\text{Sen}}(K_\infty) \end{array}$$

Both rows of the diagram are exact. We claim that the map g is an isomorphism, so that $\ker(f)$ is identified with $\mathbf{D}_C(\text{Lie}\Delta)$. Indeed, we have by the snake lemma an exact sequence

$$0 \rightarrow \ker(g) \rightarrow V_{\text{Sen}}(L_\infty) \rightarrow V_{\text{Sen}}(K_\infty) \rightarrow \text{coker}(g) \rightarrow 0,$$

where we use the previous step to identify the third term with $V_{\text{Sen}}(K_\infty)$. Since $V_{\text{Sen}}(L_\infty) \rightarrow V_{\text{Sen}}(K_\infty)$ is an isomorphism, we conclude the proof of the second step.

Step 3: Using the previous step, we get another commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{D}_{\text{dR}}^+(\text{Lie}\Delta) & \longrightarrow & \mathbf{D}_{\text{dR}}^+(L_\infty) & \longrightarrow & \mathbf{D}_{\text{dR}}^+(K_\infty) \longrightarrow H_{\text{dR}}^{1,+}(\text{Lie}\Delta) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \mathbf{D}_C(\text{Lie}\Delta) & \rightarrow & \mathbf{D}_C(L_\infty)/V_{\text{Sen}}(L_\infty) & \rightarrow & \mathbf{D}_C(K_\infty)/V_{\text{Sen}}(K_\infty) \rightarrow H_C^1(\text{Lie}\Delta) \end{array}$$

The two outer vertical maps are isomorphisms, because $\text{Lie}\Delta$ is a trivial G_K -representation. By a diagram chase, we conclude that the natural map $V_{\text{dR}}^+(L_\infty) \rightarrow V_{\text{dR}}^+(K_\infty)$, which is the map between the kernels of the middle columns of the diagram, is an isomorphism. The claim follows. $\square$

Corollary 4.15. *Let L/K be an abelian 1-dimensional p -adic Lie extension and let $L_\infty = K_\infty L$. Then L_∞/K is a p -adic Lie extension, the natural map $V_{\text{dR}}^+(L_\infty) \rightarrow V_{\text{dR}}^+(K_\infty)$ is an isomorphism, and there is a bijection between orientations of L_∞ and orientations of K_∞ .*

Proof. The injection $\Gamma(L_\infty) \rightarrow \Gamma(K_\infty/K) \times \text{Gal}(L/K)$ identifies $\Gamma(L_\infty)$ with a closed subgroup of the target, so it is a p -adic Lie group. Furthermore, under this injection, the subgroup $\Delta = \text{Gal}(L_\infty/K_\infty)$ is identified with a subgroup of $\text{Gal}(L/K)$. In particular, it follows that the relevant conjugation action on Δ has finite image. Now apply the previous lemma. $\square$

5. PRO-ANALYTIC VECTORS IN $\mathbf{B}_{\text{dR}}^+$

In this section we study the structure of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. The main purpose of this section is to prove Theorem 5.5, which says that an orientation gives rise to a Galois-equivariant description of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ in terms of a 1-variable power series ring over $\widehat{K}_\infty^{\text{la}}$. In §5.3, we first treat the case $K_{\text{cyc}} \subset K_\infty$; we then treat the general case in §5.4. This canonical description makes it possible to relate $\mathbf{B}_{\text{dR}}^+$ -representations to *regular connections* in §7.3. Before doing that, we first prove in §5.1 that even without an orientation, $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is abstractly isomorphic to such a ring.

5.1. The algebraic structure of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. Recall our notation from §1.5.1. The goal of this subsection is to prove that $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ and $\widehat{K}_\infty^{\text{la}}[[s]]$ are abstractly isomorphic as rings (Theorem 5.3).

We shall first need the following lemma.

Lemma 5.1. *For $n \geq 0$, the natural map $(I_\theta^n)^{\text{pa}}/(I_\theta^{n+1})^{\text{pa}} \rightarrow (I_\theta^n/I_\theta^{n+1})^{\text{la}}$ is an isomorphism.*

Proof. The map is clearly injective. For each $m \geq 0$ we have $R_{\text{la}}^1(I_\theta^m/I_\theta^{m+1}) = 0$ by Theorem 2.1. It follows that the map $(I_\theta^n/I_\theta^{n+1})^{\text{la}} \rightarrow (I_\theta^n/I_\theta^m)^{\text{la}}$ is surjective for all $m \geq n$. This implies the surjectivity of $(I_\theta^n)^{\text{pa}} \rightarrow (I_\theta^n/I_\theta^{n+1})^{\text{la}}$. $\square$

Proposition 5.2. *The ring $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is a complete discrete valuation ring with maximal ideal I_θ^{pa} and residue field $\widehat{K}_\infty^{\text{la}}$.*

Proof. We know that $\widehat{K}_\infty^{\text{la}}$ is a field by Lemma 1.8. By the previous lemma, we have $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}/I_\theta^{\text{pa}} \cong \widehat{K}_\infty^{\text{la}}$, which shows that I_θ^{pa} is maximal with residue field $\widehat{K}_\infty^{\text{la}}$. By Lemma 1.8 we have

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \times} = \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cap \mathbf{B}_{\text{dR}}^+(K_\infty)^\times,$$

so any element not in I_θ^{pa} is invertible. Therefore $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is local.

Let $n \geq 1$. Since $I_\theta^n/I_\theta^{n+1}$ is 1-dimensional over $\widehat{K}_\infty^{\text{la}}$, its locally analytic vectors form a one-dimensional vector space over $\widehat{K}_\infty^{\text{la}}$ ([BC16, Theorem 3.4]). By the previous lemma, we may choose $s \in I_\theta^{\text{pa}} \setminus (I_\theta^2)^{\text{pa}}$. Then s generates I_θ and thus $s^n \in (I_\theta^n)^{\text{pa}} \setminus (I_\theta^{n+1})^{\text{pa}}$. Lemma 1.8 then implies that $(I_\theta^n)^{\text{pa}}$ is the ideal in $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ generated by s^n . In particular, I_θ^{pa} is principal, so Exercise 4 of [AM69, Chapter 9] implies $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is a discrete valuation ring. It remains to prove $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is $(I_\theta^{\text{pa}})^n = (I_\theta^n)^{\text{pa}}$ -adically complete. This comes down to showing $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}/(I_\theta^n)^{\text{pa}} = (\mathbf{B}_{\text{dR}}^+(K_\infty)/I_\theta^n)^{\text{la}}$, which follows by a dévissage argument from the previous lemma. $\square$

Cohen's structure theorem ([Sta, Tag 0C0S]) then implies the following result.

Theorem 5.3. *There is an algebraic isomorphism*

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cong \widehat{K}_\infty^{\text{la}}[[s]]$$

which maps s to a generator of I_θ^{pa} .

Corollary 5.4. *The map $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)$ is faithfully flat.*

5.2. Statement of the main theorem. In the presence of an orientation we can get a much stronger $\Gamma(K_\infty)$ -equivariant version of Theorem 5.3.

Theorem 5.5. *Let K_∞/K be an orientable p -adic Lie Galois extension with a chosen orientation $\nabla = \nabla(K_\infty)$. Then:*

(1) *The composition*

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0} \subset \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \rightarrow \widehat{K}_\infty^{\text{la}}$$

induces a natural $\Gamma(K_\infty)$ -equivariant isomorphism of K -algebras

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0} \cong \widehat{K}_\infty^{\text{la}}.$$

In particular, $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ acquires in this way a natural $\widehat{K}_\infty^{\text{la}}$ -algebra structure, and the structure morphism $\widehat{K}_\infty^{\text{la}} \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is $\Gamma(K_\infty)$ -equivariant.

(2) *For $n \geq 1$, the $\Gamma(K_\infty)$ -stable $\widehat{K}_\infty^{\text{la}}$ -subspace*

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=n} \subset \mathbf{B}_{\text{dR}}^+(K_\infty)$$

is contained in $\text{Fil}^n \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. The induced map

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=n} \rightarrow L(K_\infty)^{\otimes n, \text{la}} = \text{gr}^n \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$$

is a natural isomorphism of 1-dimensional $\widehat{K}_\infty^{\text{la}}$ -semilinear representations of $\Gamma(K_\infty)$.

(3) *The natural map from $\prod_{n \geq 0} \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=n}$ to $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ is an isomorphism, and so one gets a natural $\Gamma(K_\infty)$ -equivariant isomorphism of $\widehat{K}_\infty^{\text{la}}$ -algebras*

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cong \widehat{\text{Sym}}_{\widehat{K}_\infty^{\text{la}}}(L(K_\infty)^{\text{la}}).$$

Remark 5.6. In particular, choosing a generator t_{K_∞} of $L(K_\infty)^{\text{la}}$, we get a $\Gamma(K_\infty)$ -trivialization $L(K_\infty)^{\text{la}} \cong \widehat{K}_\infty^{\text{la}} \cdot t_{K_\infty}$. By considering t_{K_∞} as an element in $\text{Fil}^1 \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, we get that the natural inclusion map

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0} \llbracket t_{K_\infty} \rrbracket \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$$

is an isomorphism. This induces the $\Gamma(K_\infty)$ -equivariant isomorphism

$$\widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket \cong \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$$

in Question 1.1.

Example 5.7 (The False Tate extension). Let $K_{\text{fT}}, \tilde{\alpha}$ and α be as in Example 4.10. We describe explicitly the *unique* Galois-equivariant section map

$$s : \widehat{K}_{\text{fT}}^{\text{la}} \rightarrow \mathbf{B}_{\text{dR}}^+(K_{\text{fT}})^{\text{pa}}.$$

We shall see it is completely determined by one rule $s(\alpha) = \tilde{\alpha}$. By [BC16, Proposition 4.12], we have

$$\widehat{K}_{\text{fT}}^{\text{la}} = \bigcup_{n \geq 1} K(\zeta_{p^{r(n)}}, \pi_{r(n)}) \{ \{ \alpha - \alpha_n \} \}_n.$$

See [BC16, §4] for the notation. We just mention here that the $\alpha_n \in K_{\text{fT}}$ is a sequence of (algebraic) elements approximating α , and the expression above says that each $f \in \widehat{K}_{\text{fT}}^{\text{la}}$ can be expressed as a (converging) power series with coefficients in K_{fT} and with variable $(\alpha - \alpha_n)$ for some n . It suffices to determine $s(\alpha)$, as everything else in the expression of $\widehat{K}_{\text{fT}}^{\text{la}}$ are elements in K_{fT} which embed canonically into $\mathbf{B}_{\text{dR}}^+$. In fact, we claim $s(\alpha) = \tilde{\alpha}$. Indeed, as the section s comes from the isomorphism $\mathbf{B}_{\text{dR}}^+(K_{\text{fT}})^{\text{pa}, \nabla(K_{\text{fT}})=0} \cong \widehat{K}_{\text{fT}}^{\text{la}}$, we reduce to verifying $\nabla(\tilde{\alpha}) = 0$, which follows from the computation of Example 4.10.

5.3. The case $K_{\text{cyc}} \subset K_\infty$.

Lemma 5.8. *Suppose that $K_{\text{cyc}} \subset K_\infty$ with K_∞ orientable. Fix an orientation $\nabla(K_\infty)$, so that $\nabla(K_\infty)$ acts on $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. Then $\nabla(K_\infty)(t) = t$, where t is Fontaine's element. In particular, $\nabla(K_\infty)((I_\theta^{\text{pa}})^n) \subset (I_\theta^{\text{pa}})^n$ for $n \geq 0$.*

Proof. Since K_{cyc} is canonically orientable, the operator $\nabla(K_\infty)$ must map to $\nabla(K_{\text{cyc}})$ under the map $V_{\text{dR}}^+(K_\infty) \rightarrow V_{\text{dR}}^+(K_{\text{cyc}})$. Now use Lemma 4.13 and the relation $\nabla(K_{\text{cyc}})(t) = t$. The last statement of the lemma follows from the Leibniz rule. $\square$

Lemma 5.9. *For $n \geq 0$, the operator $\nabla(K_\infty)$ acts on $(I_\theta^{\text{pa}})^n / (I_\theta^{\text{pa}})^{n+1}$ multiplication by n .*

Proof. As $K_{\text{cyc}} \subset K_\infty$, we can take $s = t$ in proof of Proposition 5.2. Recall that by Lemma 5.1 and Proposition 5.2, we have $(I_\theta^{\text{pa}})^n / (I_\theta^{\text{pa}})^{n+1} \cong [I_\theta^n / I_\theta^{n+1}]^{\text{la}}$. Then $[I_\theta^n / I_\theta^{n+1}]^{\text{la}} \cong \widehat{K}_\infty^{\text{la}} \cdot t^n$ and we conclude by the previous lemma and Proposition 3.8. $\square$

We shall need the following key lemma.

Lemma 5.10. *Let A be a $\mathbf{Q}$ -algebra, complete and separated with respect to a decreasing filtration $A = \text{Fil}^0 A \supset \text{Fil}^1 A \supset \dots$, with $\text{gr}^0 A$ a field and each $\text{gr}^i A$ a finite-dimensional vector space over $\text{gr}^0 A$. Let T be a $\mathbf{Q}$ -linear operator on A which respects the filtration and acts on $\text{gr}^i A$ as multiplication by i . Then:*

- (1) *For all $i \geq 0$ we have $A^{T=i} \subset \text{Fil}^i A$ and the induced map $A^{T=i} \rightarrow \text{gr}^i A$ is an isomorphism.*
- (2) *The natural map $\prod_{i \geq 0} A^{T=i} \rightarrow A$ is an isomorphism.*

Proof. We start by proving $A^{T=i} \subset \text{Fil}^i A$. For $0 \leq j < i$, we have an exact sequence of $\mathbf{Q}[T]$ -modules

$$0 \rightarrow \text{Fil}^{j+1} A \rightarrow \text{Fil}^j A \rightarrow \text{gr}^j A \rightarrow 0.$$

Since $\text{Hom}_{\mathbf{Q}[T]}(A^{T=i}, \text{gr}^j A) = 0$, we get $\text{Hom}_{\mathbf{Q}[T]}(A^{T=i}, \text{Fil}^j A) = \text{Hom}_{\mathbf{Q}[T]}(A^{T=i}, \text{Fil}^{j+1} A)$. Applying this consecutively for $j = 0, \dots, i-1$, starting from the inclusion $A^{T=i} \subset \text{Fil}^0 A$, we get $A^{T=i} \subset \text{Fil}^i A$.

We now show $A^{T=i} \rightarrow \text{gr}^i A$ is an isomorphism. Since $A = \varprojlim_n A / \text{Fil}^n A$, it suffices to prove that $[A / \text{Fil}^n A]^{T=i} \rightarrow \text{gr}^i A$ is an isomorphism for $n \geq i+1$. The assumption that T acts as multiplication by j on gr^j implies by dévissage that

$$[\text{Fil}^i A / \text{Fil}^n A]^{T=i} = [A / \text{Fil}^n A]^{T=i},$$

so we reduce to proving $[\text{Fil}^i A / \text{Fil}^n A]^{T=i} \rightarrow \text{gr}^i A$ is an isomorphism. Argue by induction on $n \geq i+1$, the base case being obvious. Then for the induction step, it suffices to show $[\text{Fil}^i A / \text{Fil}^{n+1} A]^{T=i} \rightarrow [\text{Fil}^i A / \text{Fil}^n A]^{T=i}$ is an isomorphism. To see this, apply the functor $\text{Hom}_{\mathbf{Q}[T]}(\mathbf{Q}[T]/(T-i), *)$ to the sequence

$$0 \rightarrow \text{gr}^n A \rightarrow \text{Fil}^i A / \text{Fil}^{n+1} A \rightarrow \text{Fil}^i A / \text{Fil}^n A \rightarrow 0$$

and observe that $T-i$ acts as an isomorphism on $\text{gr}^n A$. This completes the proof of item (1).

For item (2), we write

$$A_m := \left(\prod_{i \geq 0} A^{T=i} \right) / \left(\prod_{i \geq m+1} A^{T=i} \right).$$

It suffices to show that for each m , the map $A_m \rightarrow A / \text{Fil}^{m+1} A$ is an isomorphism. We prove this claim by induction on m . For $m = 0$, the desired claim is the same as the isomorphism $A^{T=0} \cong \text{gr}^0 A$, which we have already shown. Then by induction we consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A^{T=m+1} & \longrightarrow & A_{m+1} & \longrightarrow & A_m \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{gr}^{m+1}A & \longrightarrow & A/\mathrm{Fil}^{m+2}A & \longrightarrow & A/\mathrm{Fil}^{m+1}A.
\end{array}$$

We know that the left vertical arrow is an isomorphism; the right vertical arrow is an isomorphism by induction. It follows the middle vertical map is injective, but it is also a map of $\mathrm{gr}^0 A$ -vector spaces of the same finite dimension, hence it must be an isomorphism. $\square$

The following proposition completes the proof of Theorem 5.5 in the case where $K_{\mathrm{cyc}} \subset K_\infty$. In this case, there is a trivialization $L(K_\infty)^{\mathrm{la}} \cong \widehat{K}_\infty^{\mathrm{la}} \cdot t$, and so the statement simplifies.

Proposition 5.11. *Suppose that $K_{\mathrm{cyc}} \subset K_\infty$ with K_∞ orientable. Fix an orientation $\nabla(K_\infty)$. Then, depending on the orientation $\nabla(K_\infty)$, the ring $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$ is naturally a $\widehat{K}_\infty^{\mathrm{la}}$ -algebra, and there exists a natural isomorphism of $\widehat{K}_\infty^{\mathrm{la}}$ -algebras*

$$\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}} \cong \widehat{K}_\infty^{\mathrm{la}}[[t]],$$

where t is Fontaine's element. In this isomorphism, $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}, \nabla=n}$ is identified with $\widehat{K}_\infty^{\mathrm{la}} \cdot t^n$.

Proof. This follows from Lemma 5.10 by taking $A = \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$ with its I_θ^{pa} -adic filtration and $T = \nabla$. To see the lemma applies, use Proposition 5.2, Lemma 5.8 and Lemma 5.9. $\square$

5.4. The general case. In this subsection, we complete the proof of Theorem 5.5. We let K_∞ be general in this section.

Lemma 5.12. *If Theorem 5.5 is true for K , and K'/K is a finite extension, then it is also true for K' .*

Proof. As usual let $K'_\infty = K_\infty K'$. If $K'_\infty = K_\infty$ the claim is clear. By the same argument as that appearing in the proof of Lemma 2.3, we may reduce to the case where K' is linearly disjoint from K_∞ . In this case we have a natural identification $\Gamma(K'_\infty) \cong \Gamma(K_\infty)$ as well as natural $\Gamma(K'_\infty)$ -equivariant isomorphisms $K'_\infty \cong K_\infty \otimes_K K'$, $L(K'_\infty) \cong L(K_\infty) \otimes_K K'$ and $\mathbf{B}_{\mathrm{dR}}^+(K'_\infty) \cong \mathbf{B}_{\mathrm{dR}}^+(K_\infty) \otimes_K K'$. The result follows from base change by Lemma 1.8. $\square$

Lemma 5.13. *If Theorem 5.5 is true after replacing K by a finite extension K'/K , then it is also true for K .*

Proof. By the previous lemma, we may assume K' is Galois over K . Set $K'_\infty = K_\infty K'$ once more and let $\Delta = \mathrm{Gal}(K'_\infty/K_\infty)$ be the finite Galois group of K'_∞ over K_∞ . Let $\tilde{\Gamma}(K'_\infty) = \mathrm{Gal}(K'_\infty/K)$. It is an extension of $\Gamma(K'_\infty)$ by Δ .

Let us first choose an orientation $\nabla(K'_\infty)$ for K'_∞ . Observing that the natural map $\mathrm{Lie}\Gamma(K'_\infty) \rightarrow \mathrm{Lie}\Gamma(K_\infty)$ is an isomorphism of $\mathbf{Q}_p$ -vector spaces, we may and do consider $\mathrm{Lie}\Gamma(K'_\infty)$ as a $\tilde{\Gamma}(K'_\infty)$ -representation (rather than just a $\Gamma(K'_\infty)$ -representation). We then have an identification

$$V_{\mathrm{dR}}^+(K'_\infty)^{\mathrm{Gal}(K'/K)} = V_{\mathrm{dR}}^+(K_\infty).$$

We choose the orientation $\nabla(K'_\infty)$ corresponding to $\nabla(K_\infty)$ under this isomorphism. It has the property of being $\tilde{\Gamma}(K'_\infty)$ -equivariant (rather than just being $\Gamma(K'_\infty)$ -equivariant).

The statements of Theorem 5.5 for K_∞ are now easily checked once one makes the easy observation that each of the maps for K'_∞ is actually $\tilde{\Gamma}(K'_\infty)$ -equivariant. For example, for item (1) of Theorem 5.5, we have by assumption that the canonical map from $\mathbf{B}_{\mathrm{dR}}^+(K'_\infty)^{\mathrm{pa}, \nabla(K'_\infty)=0}$ to $\widehat{K}_\infty^{\mathrm{la}}$ is a $\Gamma(K'_\infty)$ -equivariant isomorphism. But by construction this map is the composition of $\mathbf{B}_{\mathrm{dR}}^+(K'_\infty)^{\mathrm{pa}, \nabla(K'_\infty)=0} \subset \mathbf{B}_{\mathrm{dR}}^+(K'_\infty)^{\mathrm{pa}}$ and $\mathbf{B}_{\mathrm{dR}}^+(K'_\infty)^{\mathrm{pa}} \rightarrow \widehat{K}_\infty^{\mathrm{la}}$, both of which are $\tilde{\Gamma}(K'_\infty)$ -equivariant. Taking Δ -fixed points proves the natural map from $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}, \nabla(K_\infty)=0}$ to $\widehat{K}_\infty^{\mathrm{la}}$ is an isomorphism. The proofs of item (2) and item (3) are similar. $\square$

For the rest of this subsection, let $L_\infty = K_\infty K_{\text{cyc}}$ and $\Delta = \text{Gal}(L_\infty/K_\infty)$. If L_∞/K_∞ is a finite extension, then by Lemma 5.13, Theorem 5.5 holds. Thus as in Section 2, we may put ourselves in the situation where L_∞/K_∞ is 1-dimensional extension, $\Delta \cong \mathbf{Z}_p$ with generator γ , and there is an element $z \in \widehat{L}_\infty^{\Gamma(L_\infty)-\text{la}}$ satisfying $\gamma(z) = z + 1$. Furthermore, because of Corollary 4.15, the orientation $\nabla(K_\infty)$ of K_∞ induces a *unique* orientation $\nabla(L_\infty)$ of L_∞ which is compatible with the inclusion $K_\infty \subset L_\infty$.

The following key construction defines a “ K_∞ -analogue” of Fontaine’s element t .

Construction 5.14. Take $z_0 \in L_\infty$ close enough to z so that the series

$$u = \exp(-\log \chi_{\text{cyc}}(\gamma)(z - z_0))$$

converges inside $\widehat{L}_\infty^{\text{la}}$, which, via Proposition 5.11, can be regarded as an element in $\mathbf{B}_{\text{dR}}^+(L_\infty)^{\text{pa}, \nabla(L_\infty)=0}$. Then after possibly replacing K by a finite extension, we get that u has the property that $g(u) = \chi_{\text{cyc}}(g)^{-1}u$ for $g \in \Delta$. Thus $t_{K_\infty} = ut$ is fixed by Δ , and this means that

$$t_{K_\infty} \in \text{Fil}^1 \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}.$$

Note $\nabla(L_\infty)(t_{K_\infty}) = t_{K_\infty}$ by Lemma 5.8. Thus, by Lemma 4.13, we have

$$\nabla(K_\infty)(t_{K_\infty}) = t_{K_\infty}.$$

The construction implies:

Lemma 5.15. *We have $\nabla(K_\infty)((I_\theta^{\text{pa}})^n) \subset (I_\theta^{\text{pa}})^n$ and $\nabla(K_\infty)$ acts on $(I_\theta^{\text{pa}})^n/(I_\theta^{\text{pa}})^{n+1}$ as multiplication by n .*

Proof. The first claim holds because it is true for L_∞ , see Lemma 4.13. For the second claim argue as in the proof of Lemma 5.9, using t_{K_∞} instead of t . $\square$

The following Proposition then concludes the proof of Theorem 5.5.

Proposition 5.16. *The map*

$$\widehat{K}_\infty^{\text{la}}[[t_{K_\infty}]] \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$$

is a $\Gamma(K_\infty)$ -equivariant isomorphism.

Proof. This follows from the previous lemma and Lemma 5.10. $\square$

6. CONSEQUENCES FOR ORIENTATIONS

In the previous section we studied the structure of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, notably via Theorem 5.5. In this section, we show how the structural results there clarify further the concept of an orientation. In Theorem 6.2, we give an equivalent criterion of orientability (compare with Theorem 4.7 on criterion of canonical orientability). Then in Theorem 6.3, we classify equivariant power series descriptions of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, giving a complete answer to Question 1.1 from the introduction.

6.1. Representations of $\Gamma(K_\infty)$ and a criterion for orientability. We make the following observation: even though the condition of orientability asks for the lifting of a single element $\Theta(K_\infty)$ in $\mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$ to $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty))$, one automatically gets from this a splitting of the entire space $\mathbf{D}_{\mathbf{C}}(V)$ for an arbitrary p -adic representation V of $\Gamma(K_\infty)$. In what follows, recall that for such V we have identifications $\mathbf{D}_{\mathbf{C}}(V) = (\widehat{K}_\infty^{\text{la}} \otimes_{\mathbf{Q}_p} V)^{\Gamma(K_\infty)}$ and $\mathbf{D}_{\text{dR}}^+(V) = (\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \otimes_{\mathbf{Q}_p} V)^{\Gamma(K_\infty)}$.

Corollary 6.1. *Let K_∞ be orientable. Let V be a representation of $\Gamma(K_\infty)$, considered as a G_K -representation.*

- (1) *The natural map $\mathbf{D}_{\text{dR}}^+(V) \rightarrow \mathbf{D}_{\mathbf{C}}(V)$ has a splitting. In particular, $\text{gr}^0 \mathbf{D}_{\text{dR}}(V) \cong \mathbf{D}_{\mathbf{C}}(V)$.*

- (2) We have $\dim_K \mathbf{D}_{\text{HT}}(V) = \dim_K \mathbf{D}_{\text{dR}}(V)$. In particular, V is Hodge-Tate if and only if it is de Rham.

Proof. For item (1), use the existence of a G_K -equivariant section for the map $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \rightarrow \widehat{K}_\infty^{\text{la}}$ (Theorem 5.5).

For item (2), we may regard V as a representation of $\Gamma(K_\infty K_{\text{cyc}})$ via its surjection to $\Gamma(K_\infty)$. Corollary 4.15 implies $K_\infty K_{\text{cyc}}$ is still orientable. Thus, without loss of generality, we may assume K_∞ contains K_{cyc} , so that twists $V(i)$ of V are still representations of $\Gamma(K_\infty)$. Item (1) then implies that

$$\text{gr}^i \mathbf{D}_{\text{dR}}(V) \cong \text{gr}^0 \mathbf{D}_{\text{dR}}(V(i)) = \mathbf{D}_{\mathbf{C}}(V(i)),$$

hence we can sum over $i \in \mathbf{Z}$ to conclude. $\square$

Furthermore, we get the following simple characterization of orientable K_∞ :

Theorem 6.2. *The following are equivalent.*

- (1) The extension K_∞ is orientable.
- (2) $\dim_K \mathbf{D}_{\text{dR}}(\text{Lie}\Gamma(K_\infty)) = \dim_K \mathbf{D}_{\text{HT}}(\text{Lie}\Gamma(K_\infty))$.

Proof. Corollary 6.1 shows that item (1) implies item (2). Conversely, suppose item (2) holds. We have

$$\begin{aligned} \dim_K \mathbf{D}_{\text{dR}}(\text{Lie}\Gamma(K_\infty)) &= \sum_{i \in \mathbf{Z}} \dim_K \text{gr}^i \mathbf{D}_{\text{dR}}(\text{Lie}\Gamma(K_\infty)) \\ &\leq \sum_{i \in \mathbf{Z}} \dim_K \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty)(i)) \\ &= \dim_K \mathbf{D}_{\text{HT}}(\text{Lie}\Gamma(K_\infty)), \end{aligned}$$

so the assumption implies that for every $i \in \mathbf{Z}$, the inequality

$$\dim_K \text{gr}^i \mathbf{D}_{\text{dR}}(\text{Lie}\Gamma(K_\infty)) \leq \dim_K \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty)(i))$$

is an equality. Taking in particular $i = 0$ implies that $\mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty))$ is surjective. This shows $\Theta(K_\infty)$ lifts, hence K_∞ is orientable. $\square$

6.2. Orientations and power series descriptions of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. The following theorem shows orientations are in bijection with equivariant power series descriptions of $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, answering Question 1.1. It holds even if K_∞ is not assumed to be orientable.

Theorem 6.3. *The following are in bijection.*

- (1) The set of orientations of K_∞ .
- (2) The set of $\Gamma(K_\infty)$ -equivariant sections of the map $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \rightarrow \widehat{K}_\infty^{\text{la}}$.
- (3) The set of $\Gamma(K_\infty)$ -equivariant isomorphisms

$$\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \cong \widehat{\text{Sym}}_{\widehat{K}_\infty^{\text{la}}}(L(K_\infty)^{\text{la}})$$

inducing the identity on graded pieces.

Proof. The bijection between (2) and (3) follows from the proof of Theorem 5.5. Let us then construct the bijection between (1) and (2). In the proof of Theorem 5.5, we have given a map $\nabla \mapsto s(\nabla)$ from (1) to (2). It is given as follows: an orientation ∇_0 is mapped to the section

$$s(\nabla_0) : \widehat{K}_\infty^{\text{la}} \cong \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla_0=0} \subset \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}.$$

In the inverse direction, given a $\Gamma(K_\infty)$ -equivariant section $s_0 : \widehat{K}_\infty^{\text{la}} \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, we get an induced map $s_{0,*} : \mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty)) \rightarrow \mathbf{D}_{\text{dR}}^+(\text{Lie}\Gamma(K_\infty))$, and the corresponding orientation is

$$\nabla(s_0) := s_{0,*}(\Theta(K_\infty)).$$

It remains to show that the maps s and ∇ are inverses.

First we show that $s \circ \nabla = \text{Id}$. Given a section s_0 , we need to show that $s(\nabla(s_0)) = s_0$. We claim that s_0 , which is a priori only a section $\widehat{K}_\infty^{\text{la}} \rightarrow \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$, actually factors through $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla(s_0)=0}$. Indeed, write $\nabla(s_0) = \sum b_i \otimes \partial_i$ with the ∂_i forming a basis of $\text{Lie}\Gamma(K_\infty)$ and the $b_i \in \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. By definition of $\nabla(s_0)$, each b_i is of the form $s_0(a_i)$ for $a_i \in \widehat{K}_\infty^{\text{la}}$. For $x \in \widehat{K}_\infty^{\text{la}}$ we thus have

$$\begin{aligned} \nabla(s_0)(s_0(x)) &= \sum b_i \partial_i(s_0(x)) \\ &= s_0\left(\sum a_i \partial_i(x)\right) \\ &= s_0(\Theta(K_\infty)(x)) = 0, \end{aligned}$$

where the last equality holds by Proposition 3.8. This proves the claim. But now we see that both s_0 and $s(\nabla(s_0))$ are inverses of the natural map $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla(s_0)=0}$, so they must be equal.

Conversely, given an orientation ∇_0 , we need to show that $\nabla(s(\nabla_0)) = \nabla_0$. Write $\nabla_0 = \sum b_i \otimes \partial_i$ with the ∂_i forming a basis of $\text{Lie}\Gamma(K_\infty)$ and the $b_i \in \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. Then by Lemma 6.5, the elements b_i are killed by ∇_0 , hence are in $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla_0=0}$. But this means $\nabla_0 = s(\nabla_0)_*(\Theta(K_\infty))$, or $\nabla_0 = \nabla(s(\nabla_0))$, as required. $\square$

Remark 6.4. The tower K_∞ is orientable if and only if $\Theta(K_\infty)$ is mapped to zero under the boundary map

$$\mathbf{D}_{\mathbf{C}}(\text{Lie}\Gamma(K_\infty)) \rightarrow H^1(G_K, \mathbf{B}_{\text{dR}}^+ \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty)(1)).$$

Thus the above theorem is saying that the image of $\Theta(K_\infty)$ under this boundary map is the obstruction for $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$ to be $\Gamma(K_\infty)$ -equivariantly isomorphic to a power series ring over $\widehat{K}_\infty^{\text{la}}$.

The following lemma was used in proof of Theorem 6.3. It is the $\mathbf{B}_{\text{dR}}^+$ -analogue of Theorem 3.10.

Lemma 6.5. *Let $\nabla(K_\infty)$ be an orientation of K_∞ . Let ∂_i be a $\mathbf{Q}_p$ -basis of $\text{Lie}\Gamma(K_\infty)$, and write $\nabla(K_\infty) = \sum_i b_i \otimes \partial_i$ with the $b_i \in \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}}$. Then for every $i = 1, \dots, d$ we have $\nabla(K_\infty)(b_i) = 0$. More precisely,*

$$\nabla(K_\infty) \in \mathbf{B}_{\text{dR}}^+(K_\infty)^{\ker(\text{ad}), \Gamma(K_\infty)\text{-fin}, \nabla(K_\infty)=0} \otimes_{\mathbf{Q}_p} \text{Lie}\Gamma(K_\infty).$$

Proof. For $g \in \Gamma(K_\infty)$ we have the identity

$$\sum_i g(b_i) \otimes \text{ad}(g)(\partial_i) = \sum_i b_i \otimes \partial_i.$$

Differentiating the identity along $\partial \in \text{Lie}\Gamma(K_\infty)$, we get

$$\sum_i \partial(b_i) \otimes \partial_i + \sum_i b_i \otimes [\partial, \partial_i] = 0.$$

Now take $\partial = \partial_j$, multiply the equation by b_j , and sum over j . We get

$$\sum_{i,j} b_j \partial_j(b_i) \otimes \partial_i + \sum_{i,j} b_i b_j \otimes [\partial_j, \partial_i] = 0.$$

Since $[\partial_j, \partial_i] = -[\partial_i, \partial_j]$, the second summand is zero. Considering the coefficient of ∂_i , we get

$$\nabla(K_\infty)(b_i) = \sum_j b_j \partial_j(b_i) = 0,$$

which shows the first statement of the theorem. For the stronger statement, argue as in the proof of Theorem 3.10. $\square$

7. COHOMOLOGICAL APPLICATIONS

In this section we give cohomological applications of our methods. In §7.1 we give a version of Sen theory for general K_∞ . Then in §7.2 we extend the picture to $\mathbf{B}_{\text{dR}}^+$ -representations. In both of these subsections the main ingredient is Theorem 2.1. In §7.3 we establish the deeper link between $\mathbf{B}_{\text{dR}}^+$ -representations to regular connections when K_∞ is orientable; this uses the full force of Theorem 5.5.

7.1. Sen modules and cohomology comparisons. The main result of this subsection is Theorem 7.2, where we construct a full analogue of Sen theory—namely, categorical (decompletion)-equivalence, cohomology comparison, etc.—for a general p -adic Lie extension. Many ingredients were known from the work of Sen [Sen80] and Berger–Colmez [BC16]; our new input here is the vanishing of higher locally analytic vectors which was proved in Theorem 2.1.

Definition 7.1. Suppose $\mathcal{G}$ is a topological group that acts continuously on a topological ring R . We use $\text{Rep}_{\mathcal{G}}(R)$ to denote the category where an object is a finite free R -module M (topologized via the topology on R) with a continuous and *semilinear* $\mathcal{G}$ -action in the usual sense that

$$g(rx) = g(r)g(x), \forall g \in \mathcal{G}, r \in R, x \in M.$$

The following is the main theorem of this section.

Theorem 7.2. *Let K_∞/K be an infinitely ramified p -adic Lie extension.*

(1) *There are equivalences of categories (see Definition 7.1)*

$$\text{Rep}_{G_K}(\mathbf{C}) \cong \text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty) \cong \text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty^{\text{la}})$$

Given $U \in \text{Rep}_{G_K}(\mathbf{C})$, the corresponding object in $\text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty)$ is $W = U^{H(K_\infty)}$, and the corresponding object in $\text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty^{\text{la}})$ is $D = W^{\Gamma(K_\infty)\text{-la}}$. The functors from right to left are base-change functors.

(2) *For $U \in \text{Rep}_{G_K}(\mathbf{C})$, we have cohomology comparisons*

$$\text{R}\Gamma(G_K, U) \simeq \text{R}\Gamma(\Gamma(K_\infty), W) \simeq \text{R}\Gamma(\Gamma(K_\infty), D) \simeq \text{R}\Gamma_{\text{la}}(\Gamma(K_\infty), D)$$

Here, the first three complexes are continuous group cohomologies, and the last complex denotes locally analytic cohomology.

(3) *We have*

$$\text{R}\Gamma(\Gamma(K_\infty), D) \otimes_K K_\infty \simeq \text{R}\Gamma(\text{Lie}\Gamma(K_\infty), D)$$

where the right hand side is Lie algebra cohomology. As a consequence,

$$\text{R}\Gamma(G_K, U) \simeq (\text{R}\Gamma(\text{Lie}\Gamma(K_\infty), D))^{\Gamma(K_\infty)}.$$

(4) *We have*

$$\text{R}\Gamma(\Gamma(K_\infty), D) \otimes_K \widehat{K}_\infty^{\text{la}} \simeq \text{R}\Gamma(\Theta(K_\infty), D) := [D \xrightarrow{\Theta(K_\infty)} D].$$

As a consequence,

$$\text{R}\Gamma(G_K, U) \simeq (\text{R}\Gamma(\Theta(K_\infty), D))^{\Gamma(K_\infty)}.$$

Proof. We first prove items (1) and (2). The equivalence $\text{Rep}_{G_K}(\mathbf{C}) \cong \text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty)$ and cohomological comparison $\text{R}\Gamma(G_K, U) \simeq \text{R}\Gamma(\Gamma(K_\infty), W)$ follows from almost purity (since $\widehat{K}_\infty$ is perfectoid). The equivalence $\text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty) \cong \text{Rep}_{\Gamma(K_\infty)}(\widehat{K}_\infty^{\text{la}})$ is proved by Berger–Colmez [BC16, Theorem 3.4]. The cohomological comparison

$$\text{R}\Gamma(\Gamma(K_\infty), W) \simeq \text{R}\Gamma(\Gamma(K_\infty), D) \simeq \text{R}\Gamma_{\text{la}}(\Gamma(K_\infty), D)$$

follows from the fact that W has no higher locally analytic vectors (Theorem 2.1) and the general comparison theorem [RJC22, Theorem 1.7].

Item (3) follows from the general fact (see [Tam15]) that

$$\mathrm{R}\Gamma(\mathrm{Lie}\Gamma(K_\infty), D) \simeq \mathrm{colim}_{K \subset_{\mathrm{fin}} M \subset K_\infty} \mathrm{R}\Gamma_{\mathrm{la}}(\mathrm{Gal}(K_\infty/M), D)$$

and Galois descent.

For item (4), first note that the $\Theta(K_\infty)$ -action on D is $\widehat{K}_\infty^{\mathrm{la}}$ -linear since $\Theta(K_\infty)$ kills $\widehat{K}_\infty^{\mathrm{la}}$ by Proposition 3.8. Let $L_\infty = K_\infty K_{\mathrm{cyc}}$. For convenience, use $D(K_\infty)$, $D(L_\infty)$ and $D(K_{\mathrm{cyc}})$ to denote the Sen modules for these towers; by Lemma 1.8, we have natural isomorphisms:

$$D(K_\infty) \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \widehat{L}_\infty^{\mathrm{la}} \simeq D(L_\infty) \simeq D(K_{\mathrm{cyc}}) \otimes_{K_{\mathrm{cyc}}} \widehat{L}_\infty^{\mathrm{la}}.$$

Recall Lemma 3.5 proved compatibilities between $\Theta(L_\infty)$, $\Theta(K_\infty)$ and $\Theta(K_{\mathrm{cyc}})$. We thus have

$$\begin{aligned} \mathrm{R}\Gamma(\Theta(K_\infty), D(K_\infty)) \otimes_{\widehat{K}_\infty^{\mathrm{la}}} \widehat{L}_\infty^{\mathrm{la}} &\simeq \mathrm{R}\Gamma(\Theta(L_\infty), D(L_\infty)) \\ &\simeq \mathrm{R}\Gamma(\Theta(K_{\mathrm{cyc}}), D(K_{\mathrm{cyc}})) \otimes_{K_{\mathrm{cyc}}} \widehat{L}_\infty^{\mathrm{la}}. \end{aligned}$$

Indeed, these complexes are computing the same endomorphism using different bases. Recall the classical (cyclotomic) fact

$$\mathrm{R}\Gamma(\Theta(K_{\mathrm{cyc}}), D(K_{\mathrm{cyc}})) \simeq \mathrm{R}\Gamma(G_K, U) \otimes_K K_{\mathrm{cyc}}.$$

We can thus take $\mathrm{Gal}(L_\infty/K_\infty)$ -invariants on the previous quasi-isomorphism to conclude. $\square$

7.2. Dévissage of $\mathbf{B}_{\mathrm{dR}}^+$ -representations. The goal of this subsection is to prove Theorem 7.3, which generalizes items (1)-(3) of Theorem 7.2 to the $\mathbf{B}_{\mathrm{dR}}^+$ -setting, for *any* p -adic Lie extension K_∞/K . As we see, a generalization of Theorem 7.2(4) (concerning cohomology of the Sen operator) is missing here: this will have to make full use of Theorem 5.5, under the *orientability* assumption; cf. Theorem 7.5.

The content of the following theorem is hinted above [BC16, Theorem 3.4].

Theorem 7.3. *Let K_∞/K be an infinitely ramified p -adic Lie extension.*

(1) *There are equivalences of categories*

$$\mathrm{Rep}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+) \simeq \mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR}}^+(K_\infty)) \simeq \mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}).$$

Given $U \in \mathrm{Rep}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+)$, the corresponding object in $\mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR}}^+(K_\infty))$ is $W = U^{H(K_\infty)}$, and the corresponding object in $\mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}})$ is $D = W^{\Gamma(K_\infty)-\mathrm{pa}}$. The functors from right to left are base-change functors.

(2) *For $U \in \mathrm{Rep}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+)$, we have cohomology comparisons*

$$\mathrm{R}\Gamma(G_K, U) \simeq \mathrm{R}\Gamma(\Gamma(K_\infty), W) \simeq \mathrm{R}\Gamma(\Gamma(K_\infty), D) \simeq \varprojlim_n \mathrm{R}\Gamma_{\mathrm{la}}(\Gamma(K_\infty), D/t_{K_\infty}^n D)$$

Here, the first three complexes are continuous group cohomologies, and the last complex denotes inverse limit of locally analytic cohomology.

(3) *We have*

$$\mathrm{R}\Gamma(\Gamma(K_\infty), D) \otimes_K K_\infty \simeq \mathrm{R}\Gamma(\mathrm{Lie}\Gamma(K_\infty), D)$$

where the right hand side is Lie algebra cohomology. As a consequence

$$\mathrm{R}\Gamma(G_K, U) \simeq (\mathrm{R}\Gamma(\mathrm{Lie}\Gamma(K_\infty), D))^{\Gamma(K_\infty)}$$

Proof. For each $m \in \mathbb{Z}_{\geq 1}$, let $\mathbf{B}_{\mathrm{dR},m}^+ = \mathbf{B}_{\mathrm{dR}}^+/I_\theta^m$; by a standard dévissage argument, it suffices to prove that we have equivalences of categories:

$$(7.1) \quad \mathrm{Rep}_{G_K}(\mathbf{B}_{\mathrm{dR},m}^+) \simeq \mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)) \simeq \mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)^{\mathrm{la}}).$$

via the analogous functors

$$U_m \mapsto W_m := (U_m)^{H(K_\infty)}, \quad W_m \mapsto D_m := (W_m)^{\Gamma(K_\infty)\text{-la}}.$$

Indeed, assuming these equivalences, the cohomological comparisons (for U_m, W_m, D_m) follow by obvious dévissage argument to the $m = 1$ case, which is already established in Theorem 7.2.

We first prove the first equivalence in (7.1) via induction on m with the $m = 1$ case known in Theorem 7.2. It is clear that we only need to prove that $W_m = (U_m)^{H(K_\infty)}$ is finite free over $\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)$ and the natural map

$$(7.2) \quad W_m \otimes_{\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)} \mathbf{B}_{\mathrm{dR},m}^+ \rightarrow U_m$$

is an isomorphism. For brevity, for any $k \leq m$, let $U_k := U_m \otimes_{\mathbf{B}_{\mathrm{dR},m}^+} \mathbf{B}_{\mathrm{dR},m}^+/I_\theta^k$ and similarly for W_k, D_k . Theorem 7.2 implies that higher (i.e., degree ≥ 1) $H(K_\infty)$ -cohomology of U_1 vanishes; thus so is that of U_k for any k . Thus the short exact sequence

$$0 \rightarrow U_{m-1} \xrightarrow{\times t} U_m \rightarrow U_1 \rightarrow 0$$

induces a short exact sequence

$$0 \rightarrow W_{m-1} \rightarrow W_m \rightarrow W_1 \rightarrow 0$$

This (via short five lemma, and induction hypothesis) implies that (7.2) is an isomorphism; this forces W_m to be finite free since $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)$ is a DVR.

We now prove the second equivalence in (7.1) via induction on m with $m = 1$ case known in Theorem 7.2. The strategy is quite similar to that in the preceding paragraph. It suffices to prove D_m is finite free over $\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)^{\mathrm{la}}$ and natural map

$$(7.3) \quad D_m \otimes_{\mathbf{B}_{\mathrm{dR},m}^+(K_\infty)^{\mathrm{la}}} \mathbf{B}_{\mathrm{dR}}^+(K_\infty) \rightarrow W_m$$

is an isomorphism. Corollary 2.2 implies W_k has no higher locally analytic vectors. Thus we have a short exact sequence

$$0 \rightarrow D_{m-1} \rightarrow D_m \rightarrow D_1 \rightarrow 0.$$

Thus (7.3) is an isomorphism by induction and short five lemma. Again, this forces D_m to be finite free since $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$ is a DVR by Theorem 5.3. $\square$

7.3. Orientations and regular connections. In this subsection, when K_∞ is orientable, we relate $\mathbf{B}_{\mathrm{dR}}^+$ -representations with regular connections.

Definition 7.4. Let E be a characteristic zero field. Let z be a variable. Equip $E[[z]]$ with the differentiation $z \frac{d}{dz}$. A regular connection over $E[[z]]$ is a finite free $E[[z]]$ -module M equipped with an E -linear morphism $\nabla : M \rightarrow M$ satisfying the Leibniz rule with respect to $z \frac{d}{dz}$. Given a regular connection, denote

$$\mathrm{R}\Gamma(\nabla, M) := [M \xrightarrow{\nabla} M].$$

Theorem 7.5. Use the same notation as in Theorem 7.3, and suppose furthermore K_∞ is orientable with a chosen orientation $\nabla(K_\infty)$. Let $U \in \mathrm{Rep}_{G_K}(\mathbf{B}_{\mathrm{dR}}^+)$ with corresponding $D \in \mathrm{Rep}_{\Gamma(K_\infty)}(\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}})$. Then the pair $(D, \nabla(K_\infty))$ is a regular connection over the ring $\mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}}$. In addition, we have

$$(7.4) \quad \mathrm{R}\Gamma(\Gamma(K_\infty), D) \otimes_K \mathbf{B}_{\mathrm{dR}}^+(K_\infty)^{\mathrm{pa}, \nabla=0} \simeq \mathrm{R}\Gamma(\nabla(K_\infty), D)$$

As a consequence

$$(7.5) \quad \mathrm{R}\Gamma(G_K, U) \simeq (\mathrm{R}\Gamma(\nabla(K_\infty), D))^{\Gamma(K_\infty)}.$$

Proof. We know $\nabla(t_{K_\infty}) = t_{K_\infty}$ from Theorem 5.5, thus clearly (D, ∇) is a regular connection. We focus on proving (7.4) as (7.5) is trivial consequence.

The idea of the proof is quite similar to that of Theorem 7.2(4) (i.e., using K_{cyc} and its composite with K_∞), except we need to upgrade the “linear endomorphism” base change comparison

$$R\Gamma(\Theta(K_\infty), D(K_\infty)) \otimes_{\widehat{K}_\infty^{\text{la}}} \widehat{L}_\infty^{\text{la}} \simeq R\Gamma(\Theta(K_{\text{cyc}}), D(K_{\text{cyc}})) \otimes_{K_{\text{cyc}}} \widehat{L}_\infty^{\text{la}}.$$

to a base change comparison of *regular connections*.

Let $L_\infty = K_\infty K_{\text{cyc}}$. Lemma 4.14 implies $\nabla(K_\infty)$ induces a unique compatible orientation $\nabla(L_\infty)$, which then necessarily specializes to the unique cyclotomic orientation $\nabla(K_{\text{cyc}})$ by Lemma 4.13. Use Theorem 5.5(1) to identify $\mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}, \nabla=0}$ with $\widehat{K}_\infty^{\text{la}}$; similarly for the towers K_{cyc} and L_∞ . The compatibilities among all these orientations imply that we have a commutative diagram

$$\begin{array}{ccc} \widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket & \xrightarrow{\simeq} & \mathbf{B}_{\text{dR}}^+(K_\infty)^{\text{pa}} \\ \downarrow & & \downarrow \\ \widehat{L}_\infty^{\text{la}} \llbracket t \rrbracket & \xrightarrow{\simeq} & \mathbf{B}_{\text{dR}}^+(L_\infty)^{\text{pa}} \\ \uparrow & & \uparrow \\ K_{\text{cyc}} \llbracket t \rrbracket & \xrightarrow{\simeq} & \mathbf{B}_{\text{dR}}^+(K_{\text{cyc}})^{\text{pa}} \end{array}$$

where all vertical arrows are inclusion maps. Use $D(K_\infty)$, $D(L_\infty)$ and $D(K_{\text{cyc}})$ to denote the regular connection for these towers, then Lemma 1.8 implies:

$$D(K_\infty) \otimes_{\widehat{K}_\infty^{\text{la}} \llbracket t_{K_\infty} \rrbracket} \widehat{L}_\infty^{\text{la}} \llbracket t \rrbracket \simeq D(L_\infty) \simeq D(K_{\text{cyc}}) \otimes_{K_{\text{cyc}} \llbracket t \rrbracket} \widehat{L}_\infty^{\text{la}} \llbracket t \rrbracket.$$

Thus base change property for cohomology of regular connections (e.g., see [GW, Proposition 8.7] which is elementary) imply

$$\begin{aligned} R\Gamma(\nabla(K_\infty), D(K_\infty)) \otimes_{\widehat{K}_\infty^{\text{la}}} \widehat{L}_\infty^{\text{la}} &\simeq R\Gamma(\nabla(L_\infty), D(L_\infty)) \\ &\simeq R\Gamma(\nabla(K_{\text{cyc}}), D(K_{\text{cyc}})) \otimes_{K_{\text{cyc}}} \widehat{L}_\infty^{\text{la}}. \end{aligned}$$

Using the classical cyclotomic fact (see e.g. [GMW, Theorem 10.12])

$$R\Gamma(\nabla(K_{\text{cyc}}), D(K_{\text{cyc}})) \simeq R\Gamma(G_K, U) \otimes_K K_{\text{cyc}},$$

we can thus take $\text{Gal}(L_\infty/K_\infty)$ -invariants on the previous quasi-isomorphism to conclude. $\square$

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