LOCALLY ANALYTIC VECTORS AND RINGS OF PERIODS

by

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Abstract. — Let K be a finite extension of $\mathbf{Q}_p$. In this paper, we try to extend Berger's and Colmez's point of view, using locally analytic vectors in order to generalize classical cyclotomic theory, in higher rings of periods. We also provide a construction of analogs of the ring $\mathbf{B}_{Sen}$ of Colmez, one of which computes Sen theory in the de Rham case, and one which computes classical (φ, Γ)-modules theory. We explain what happens when we try to generalize constructions of (φ, Γ)-modules to arbitrary infinitely ramified p-adic Lie extensions, and provide a conjecture on the structure of the locally analytic vectors in the corresponding rings. In particular, we highlight the fact that the situation should be very different, depending on wether the p-adic Lie extension contains a cyclotomic extension or not. Finally, we explain how some of these constructions may be related to the construction of a potential ring of trianguline periods.

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Introduction

Let p be a prime, and let K be a finite extension of $\mathbf{Q}_p$. We fix $\overline{\mathbf{Q}}_p = \overline{K}$ an algebraic closure of K, and we let $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ be its absolute Galois group.

A classical idea in p-adic Hodge theory in order to study p-adic representations of $\mathcal{G}_K$ is to use an intermediate extension K_∞/K such that K_∞/K is nice enough but such that it contains "most of the ramification" of $\overline{\mathbf{Q}}_p/K$, so that $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale in the sense of Faltings (which is the same as saying that the p-adic completion of K_∞ is perfectoid). The main example of such an extension is the cyclotomic extension $K(\mu_{p^\infty})$ of K, which has been thoroughly used in p-adic Hodge theory, notably in Sen theory and (φ, Γ)-modules theory.

In some sense, Kummer extensions are simpler than the cyclotomic extension, and work from Breuil [**Bre98**] and Kisin [**Kis06**] show that Kummer extensions are very useful in order to study semistable representations. However, Kummer extensions are never Galois and this implies that we usually have to replace them by their Galois closure which increases the difficulty of the situation. Lubin-Tate extensions attached to uniformizers of K, of which the cyclotomic extension when $K = \mathbf{Q}_p$ is a particular case, trivialize local class field theory and thus seem particularly useful in order to extend the p-adic Langlands correspondence to $\mathrm{GL}_2(K)$ (see for example [**KR09, FX14, Ber16b**] for work in this direction). More generally, the interesting framework should be the one of infinitely ramified Galois extensions whose Galois group is a p-adic Lie group, with potential applications in Iwasawa theory [**Ven03**].

Let V be a p-adic Lie extension of $\mathcal{G}_K$, and let $K_\infty = K(\mu_{p^\infty})$, $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Recall that the cyclotomic character $\chi_{\mathrm{cycl}} : \Gamma_K \to \mathbf{Z}_p^\times$ identifies Γ_K with an open subgroup of $\mathbf{Z}_p^\times$. Since $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale, $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K} \otimes_{\widehat{K_\infty}} \mathbf{C}_p \simeq V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$, so that the study of the $\mathbf{C}_p$-representation $V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$ is reduced to the one of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$. The idea of Sen to study such a representation [**Sen80**] is to consider the subspace $D_{\mathrm{Sen}}(V)$ of K-finite vectors, which are elements of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$ which belong to finite dimensional sub-K-vector spaces stable by Γ_K. This is a sub-K_∞-vector space of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$, and Sen proved that $D_{\mathrm{Sen}}(V) \otimes_{K_\infty} \widehat{K_\infty} \simeq V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K}$.

If K_∞ is any infinitely ramified p-adic Lie extension K_∞/K, and if V is a $\mathbf{Q}_p$-representation of $\mathcal{G}_K$, then since $\overline{\mathbf{Q}}_p/K_\infty$ is almost étale, we still have an isomorphism $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{H_K} \otimes_{\widehat{K_\infty}} \mathbf{C}_p \simeq V \otimes_{\mathbf{Q}_p} \mathbf{C}_p$, but if the dimension of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ as a p-adic Lie group is greater or equal to 2, then the space of K-finite vectors of this semilinear $\widehat{K_\infty}$-representation of Γ_K is no longer suitable, as shown by [**BC16**, Prop. 1.5].

In order to generalize Sen theory to any infinitely ramified p-adic Lie extension K_∞/K, Berger and Colmez suggest to replace the space of K-finite vectors and the use of normalized Tate's traces maps (which no longer exist in general [**Fou09**]) by the space of locally analytic vectors, which are elements x such that the orbit map $g \mapsto g(x)$ is a locally analytic function on Γ_K. This gives a decompletion of $(V \otimes_{\mathbf{Q}_p} \mathbf{C}_p)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)}$ into a $\widehat{K_\infty}^{\mathrm{la}}$-vector space of dimension $\dim_{\mathbf{Q}_p} V$, but in general $\widehat{K_\infty}^{\mathrm{la}}$ strictly contains K_∞.

Recall that the strategy developped by Fontaine (see [**Fon94b**]) to study p-adic representations of $\mathcal{G}_K$ is to construct some p-adic rings of periods B, which are topological $\mathbf{Q}_p$-algebras endowed with an action of $\mathcal{G}_K$ and additional structures such that if V is a p-adic representation of $\mathcal{G}_K$, then the $B^{\mathcal{G}_K}$-module $D_B(V) := (B \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is endowed with the structures coming from those on B, and such that the functor $B \mapsto D_B(V)$ gives some interesting invariants attached to V. For Fontaine's strategy to work, one requires that these rings of periods B are $\mathcal{G}_K$-regular in the sense of [**Fon94b**, 1.4.1] (this implies in particular that $B^{\mathcal{G}_K}$ is a field). We then say that a p-adic representation V of $\mathcal{G}_K$ of dimension d is B-admissible if $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as B-representations. The strategy of Fontaine then consists of classifying p-adic representations according to the rings of periods for which they are admissible. In the case where V is admissible, $D_B(V)$ can usually be used to recover V, or at least $V_{|\mathcal{G}_L}$ for some finite extension L of K.

Fontaine has constructed several p-adic rings of periods, and in particular the rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}_{\mathrm{dR}}$. Recall that $\mathbf{B}_{\mathrm{crys}}$ is endowed with a Frobenius φ, $\mathbf{B}_{\mathrm{st}}$ contains $\mathbf{B}_{\mathrm{crys}}$, is endowed with a Frobenius φ and a monodromy operator N such that $\mathbf{B}_{\mathrm{crys}} = \mathbf{B}_{\mathrm{st}}^{N=0}$, and $\mathbf{B}_{\mathrm{dR}}$ is a field endowed with a filtration $\{\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}\}_{i\in\mathbf{Z}}$ and such that there is an injective map $\mathbf{B}_{\mathrm{st}} \to \mathbf{B}_{\mathrm{dR}}$. Moreover, these rings all contain an element t which is "Fontaine's p-adic $2i\pi$", and there exists rings $\mathbf{B}^+_{\mathrm{crys}}$, $\mathbf{B}^+_{\mathrm{st}}$ and $\mathbf{B}^+_{\mathrm{dR}}$ such that $\mathbf{B}_{\mathrm{crys}} = \mathbf{B}^+_{\mathrm{crys}}[1/t]$, $\mathbf{B}_{\mathrm{st}} = \mathbf{B}^+_{\mathrm{st}}[1/t]$ and $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}^+_{\mathrm{dR}}[1/t]$. Representations that are $\mathbf{B}_{\mathrm{crys}}$-admissible, $\mathbf{B}_{\mathrm{st}}$-admissible and $\mathbf{B}_{\mathrm{dR}}$-admissible are respectively called crystalline, semi-stable and de Rham. The relation between those rings imply that crystalline representations are semi-stable and that semi-stable representations are de Rham.

Colmez has constructed in [**Col94**] a ring of periods $\mathbf{B}_{\mathrm{Sen}}$ which recovers Sen's theory in the cyclotomic setting. Precisely, he defines $\mathbf{B}^n_{\mathrm{Sen}}$ as the set of power series in the variable u over $\mathbf{C}_p$, with radius of convergence $\geq p^{-n}$, and endows it with an action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K(\mu_{p^n}))$ by $g(u) = u + \log\chi_{\mathrm{cycl}}(g)$ (this makes sense since $\log\chi_{\mathrm{cycl}}(g) \in p^n\mathcal{O}_K$ if $g \in \mathcal{G}_{K_n}$). He then shows that $(\mathbf{B}^n_{\mathrm{Sen}})^{\mathcal{G}_{K(\mu_{p^n})}} = K(\mu_{p^n})$ and that $K_\infty \otimes_{K_n} (\mathbf{B}^n_{\mathrm{Sen}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K(\mu_{p^n})}}$ is isomorphic to $\mathbf{D}_{\mathrm{Sen}}(V)$.

One other key ingredient in the study of p-adic representations of $\mathcal{G}_K$ is the theory of (φ, Γ_K)-modules, which provides an equivalence of categories $V \mapsto D(V)$ between the category of all p-adic representations of $\mathcal{G}_K$ and the category of étale (φ, Γ_K)-modules. In Fontaine's theory, (φ, Γ_K)-modules are finite dimensional vector spaces, defined over a dimension 2 local ring $\mathbf{B}_K$ and endowed with semilinear actions of a Frobenius φ and of Γ_K which commutes one to another. The ring $\mathbf{B}_K$ is isomorphic to the ring of power series $\sum_{k\in\mathbf{Z}} a_k T^k$ where the sequence (a_k) is a bounded sequence of elements of $K_0 = K_\infty \cap \mathbf{Q}_p^{\mathrm{unr}}$ such that $a_{-k} \to 0$ when $k \to +\infty$, and the actions of φ and Γ_K on T are constructed through the theory of the field of norms [**Win83**].

One variant of the theory, which has been used with many useful applications, is the theory of (φ, Γ_K)-modules over the Robba ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$, which consists of the power series $\sum_{k\in\mathbf{Z}} a_k T^k$ where $a_k \in K_0$ and for which there exists ρ such that the series converges on the p-adic annulus $\rho \leq |T|_p < 1$. The theorem of Cherbonnier-Colmez [**CC98**] shows that the category of étale (φ,Γ_K)-modules over $\mathbf{B}_K$ actually embedds into the category of (φ,Γ_K)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ of slope 0, and the slope filtration theorem of Kedlaya [**Ked05**] shows that this is an equivalence of categories.

One interesting feature of the Robba ring is that it can be used as a bridge between the classical theory of (φ,Γ_K)-modules and p-adic Hodge theory, as its elements can be embedded inside $\mathbf{B}^+_{\mathrm{dR}}$. In particular, Berger has shown [**Ber02**] how to recover the invariants attached to a p-adic representation V in p-adic Hodge theory from its (φ,Γ_K)-module on the Robba ring.

In particular, the overconvergence of cyclotomic (φ,Γ_K)-modules is a really important component for their application. Kisin and Ren have defined Lubin-Tate (φ_q,Γ_K)-modules [**KR09**] and proved that the category of Lubin-Tate étale (φ_q,Γ_K)-modules is equivalent to the one of $\mathbf{Q}_p$-representations, but unfortunately a result from Fourquaux and Xie [**FX14**] shows that those (φ_q,Γ_K)-modules are usually not overconvergent. Results from Berger [**Ber13**] [**Ber16b**] suggest that the right objects to consider are once again the locally analytic vectors inside some higher rings of periods.

In this paper, we try to understand what happens if we use the point of view of Berger-Colmez of locally analytic vectors in "higher rings of periods".

Our first result, which should be well known to the experts, is that Colmez's construction of $\mathbf{B}_{\mathrm{Sen}}$ can be generalized to construct rings of periods which "compute the cyclotomic theory". More precisely, if B is a $\mathbf{Q}_p$-Banach (or Fréchet) ring endowed with an action of $\mathcal{G}_K$, such that the functor $V \mapsto D_B^{\mathrm{la}}(V) := (B \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/K(\mu_{p^\infty}),\mathrm{la})}$ gives interesting invariants of V, then there exists a ring $\mathbf{B}\{\{u\}\}$ that "computes" the functor $V \mapsto D_B^{\mathrm{la}}(V)$: Let u be a variable and B be a $\mathbf{Q}_p$-algebra endowed with a topology for which it is complete, and equipped with an action of $\mathcal{G}_K$. We denote by $B\{\{u\}\}_n$ the set of power series $\sum_{k\geq 0} a_k u^k$ with coefficients in B such that the series $\sum_{k \geq 0}(p^n)^k a_k$ converges in B and we equip it with the natural topology and with an action of $\mathcal{G}_{K_n}$ by setting

$$g(\sum_{k\geq 0} a_k u^k) = \sum_{k \geq 0} g(a_k)(u+\log \chi_{\mathrm{cycl}}(g))^k.$$

We let $B\{\{u\}\} = \bigcup_{n\geq 0} B\{\{u\}\}_n$, endowed with the inductive limit topology.

Theorem 0.1. — *Let B be a p-adic Banach or Fréchet ring, endowed with an action of $\mathcal{G}_K$. Let V be a p-adic representation of $\mathcal{G}_K$. Then we have an isomorphism*

$$(B\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \simeq ((B \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n-\mathrm{an}}.$$

In particular, this allows us to provide constructions recovering cyclotomic (φ,Γ)-modules and cyclotomic Sen theory for $\mathbf{B}^+_{\mathrm{dR}}$-representations in this spirit. We also define an analogue of these constructions in the F-analytic Lubin-Tate case.

In order to generalize (φ,Γ_K)-modules theory to any infinitely ramified p-adic Lie extension, one would like to understand the structure of the rings $(\widetilde{\mathbf{B}}^I)^{H_K,\mathrm{la}}$, where the rings $\widetilde{\mathbf{B}}^I$ are some higher rings of periods which are properly defined in §1. For the theory to behave well and indeed generalize, we should expect that $(\widetilde{\mathbf{B}}^I)^{H_K,\mathrm{la}}$ can be interpreted as a ring of power series in d variables, where d is the dimension of Γ_K as a p-adic Lie group. We expect that if K_∞ contains a twist by an unramified character of the cyclotomic extension, then the theory does generalize and the rings $(\widetilde{\mathbf{B}}^I)^{H_K,\mathrm{la}}$ can be interpreted as rings of power series in d variables:

Conjecture 0.2. — *If K_∞/K contains a cyclotomic extension, then the rings $(\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ are the completion for the locally analytic topology of rings of power series in d variables.*

We are able to prove this conjecture in a particular case:

Theorem 0.3. — *Let K_∞/K be an infinitely ramified p-adic Lie extension which is a successive extension of $\mathbf{Z}_p$-extensions and contains a cyclotomic extension. Then the conjecture above is true for K_∞/K.*

The fact that we expect the need to contain a cyclotomic extension follows from the following, which shows that for p-adic Lie extensions which do not contain a cyclotomic extension, the situation looks different:

Theorem 0.4. — *Let $K_\infty/\mathbf{Q}_{p^2}$ be the anticyclotomic extension, where $\mathbf{Q}_{p^2}$ is the unramified extension of $\mathbf{Q}_p$ of degree 2. Then the rings $(\widetilde{\mathbf{B}}^I)^{H_K,\mathrm{la}}$ are equal to $\mathbf{Q}_{p^2}$ if $0 \in I$.*

If W is a Fréchet representation of a p-adic lie group, the space of locally analytic vectors W^{la} can be defined but is too small in general to be able to recover W from W^{la}. We provide in this paper computations of locally analytic vectors for Robba rings in the F-analytic Lubin-Tate case, which provides such an example. We also show that taking locally analytic vectors in the (φ_q,Γ_K)-modules on Robba rings recovers modules defined by Colmez in [**Col14**] through different methods:

Theorem 0.5. — *Let V be an F-analytic representation of $\mathcal{G}_K$, and let $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ be its attached (φ_q,Γ_K)-module on the Robba ring $\mathbf{B}^\dagger_{\mathrm{rig},K}$. We have the following:*

- $\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\mathrm{la}} = \cap_{n\geq 0}\varphi^n(\mathbf{D}^\dagger_{\mathrm{rig}}(V))$ *and is a free* $(\mathbf{B}^\dagger_{\mathrm{rig},K})^{\mathrm{la}}$*-module of rank* $\leq \dim_{\mathbf{Q}_p} V$ *;*
- $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K})^{F-\mathrm{la}} = (\mathbf{B}^\dagger_{\mathrm{rig},K})^{\mathrm{la}} = K\langle\langle t_\pi\rangle\rangle$*, where t_π is the “Lubin-Tate analog of t” and $K\langle\langle T\rangle\rangle$ denote the set of power series in T with coefficients in K and infinite radius of convergence.*

This theorem alongside theorem 3.23 of [**Col14**] show that in general the rank of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\mathrm{la}}$ as a $K\langle\langle t_\pi\rangle\rangle$-module is strictly smaller that $\dim_{\mathbf{Q}_p} V$ and is thus too small to recover $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$.

Finally, we highlight the fact that the constructions of the rings of periods $\mathbf{B}\{\{u\}\}$ could have applications in order to define rings of periods for trianguline representations: A trianguline representation is a representation such that its attached (φ,Γ_K)-module on the Robba ring is a successive extension of rank 1 (φ,Γ_K)-modules, but as stated above, that does not mean that the corresponding representation itself is a successive extension of rank 1 representations, because the (φ,Γ_K)-modules of rank 1 that appear in the decomposition do not need to be étale. Trianguline representations are assumed to be related to representations coming from global geometric objects (see for example [**Eme09**] and [**Kis03**]) and for example the representations attached to overconvergent modular forms of finite slope are trianguline.

In order to better understand and parametrize trianguline representations, it would make sense to construct a ring which would be to trianguline representations what $\mathbf{B}_{\mathrm{crys}}$ is to crystalline representations, and we try to offer candidate rings for that purpose. The reason why one would have to define several rings is the following:

Proposition 0.6. — *There is no reasonable ring of periods B such that, for any finite extension K of $\mathbf{Q}_p$, B is a trianguline periods ring for $\mathcal{G}_K$.*

Therefore, our ring of trianguline periods of $\mathcal{G}_K$ has to be dependent on K. In the case $K = \mathbf{Q}_p$, since every rank 1 representation is trianguline, our ring has to contain every $\exp(\alpha \log t)$ with $\alpha \in E$, a field of coefficients. In particular, the ring $\mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{\mathrm{an}}$ we define is to $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}$ what the ring $\mathbf{B}_{Sen}$ introduced in [**Col94**] is to $\mathbf{C}_p$: $\mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{\mathrm{an}}$ is the increasing union of the rings $\mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{n}$, where $\mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{n} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}\{\{u\}\}_n$ is the ring of power series in a variable u with coefficients in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}$ and "radius of convergence $\geq p^{-n}$".

If we let $x = e^{-u}t \in \mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{1}$, then proposition 7.2 shows that $(\mathbf{B}_{\mathrm{tri},\mathbf{Q}_p}^{n})^{\mathcal{G}_{K_n}} = \mathbf{Q}_p\langle\langle x\rangle\rangle$, the set of power series in x with infinite radius of convergence. The module $\mathbf{D}_{\mathrm{tri},\mathbf{Q}_p}^{\mathrm{an}}(V)$ is therefore a module over $\mathbf{Q}_p\langle\langle x\rangle\rangle$ and is also endowed with a Frobenius φ coming from the one on $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}$ and an operator ∇_u coming from the operator $-\frac{d}{du}$ which commutes with the action of φ. We then extend these constructions to the F-analytic case, constructing a ring $\mathbf{B}_{\mathrm{tri},K}^{\mathrm{an}}$ in the same fashion but using a different variable u_K, and we extend Fontaine's classical formalism of admissibility to take this setting into account.

Generalizing the notion of refinements of p-adic representations [**Maz00**] [**BC09a**] to our setting, we prove the following:

Theorem 0.7. — *Let V be an F-analytic representation of $\mathcal{G}_K$ which is $\mathbf{B}_{\mathrm{tri},K}^{\mathrm{an}}$-admissible. Then V is trianguline.*

While the ring $\mathbf{B}_{\mathrm{tri},K}^{\mathrm{an}}$ is too small to contain the periods of all F-analytic trianguline representations of $\mathcal{G}_K$, we could adapt our constructions to "add a log to our ring", which would cover the semistable periods, but we would still be missing the "nongeometric" periods of trianguline representations, which appear in item (ii) of theorem 3.23 of [**Col14**]. It is not yet clear how many periods one would have to add to $\mathbf{B}_{\mathrm{tri},K}^{\mathrm{an}}$ to get a ring of trianguline periods.

Structure of the paper

The first section of the paper recalls the theory of classical rings of periods and the theory of (φ,Γ)-modules and the rings it involves. The second section recalls the theory of locally and pro-analytic vectors. In §3, we recall the main results from [**BC16**] and develop the framework of the rings $\mathbf{B}\{\{u\}\}$, proving theorem 0.1. We explain in §4 how this framework recovers classical Sen theory for $\mathbf{B}_{\mathrm{dR}}^{+}$-representations, and we compute what $(\mathbf{B}_{\mathrm{dR}}^{+})^{H_K,\mathrm{la}}$ looks like in some particular cases mainly the Lubin-Tate one). Section 5 is dedicated to how we recover (φ,Γ)-modules theory in our framework, and we explain what happens in the anticyclotomic case. In §6, we explain what we expect to happen in general when trying to generalize (φ,Γ)-modules theory by using locally analytic vectors, and prove the particular case of the conjecture. The computations of locally analytic vectors in the Robba rings is done in §7. Finally, §8 is devoted to the applications to trianguline representations and towards a construction of rings of trianguline periods.

1. Classical p-adic rings of periods and (φ,Γ)-modules

1.1. Fontaine's strategy and some rings of periods. — Let p be a prime, let K be a finite extension of $\mathbf{Q}_p$ and let $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ be its absolute Galois group. Let k be the residual field ok K and let $F = W(k)[1/p]$ be the maximal unramified extension of $\mathbf{Q}_p$ inside K. Let $\mathbf{C}_p$ be the p-adic completion of $\overline{K}$. Let $F_\infty = \mathbf{Q}_p(\mu_{p^\infty})$ be the cyclotomic extension of $\mathbf{Q}_p$. For $n \geq 1$ let $K_n = K(\mu_{p^n})$ be the extension of K generated by the p^n-th roots of unity, and let $K_\infty = \bigcup_{n\geq 1} K(\mu_{p^n}) = K \cdot F_\infty$ be the cyclotomic extension of K. Let $H_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$ and $\Gamma_{\mathbf{Q}_p} = \mathrm{Gal}(F_\infty/\mathbf{Q}_p)$. Let $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Recall that the cyclotomic character $\chi_{\mathrm{cycl}} : \mathcal{G}_K \to \mathbf{Z}_p^\times$ factors through Γ_K and identifies it with an open subset of $\mathbf{Z}_p^\times$. We also let K_0 denote the maximal unramified extension of $\mathbf{Q}_p$ inside K_∞.

Recall that the strategy developped by Fontaine (see [**Fon94b**]) to study p-adic representations of $\mathcal{G}_K$ is to construct some p-adic rings of periods B, which are topological $\mathbf{Q}_p$-algebras endowed with an action of $\mathcal{G}_K$ and additional structures such that if V is a p-adic representation of $\mathcal{G}_K$, then the $B^{\mathcal{G}_K}$-module $D_B(V) := (B \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is endowed with the structures coming from those on B, and such that the functor $B \mapsto D_B(V)$ gives some interesting invariants attached to V. For Fontaine's strategy to work, one requires that these rings of periods B are $\mathcal{G}_K$-regular in the sense of [**Fon94b**, 1.4.1] (this implies in particular that $B^{\mathcal{G}_K}$ is a field). We then say that a p-adic representation V of $\mathcal{G}_K$ of dimension d is B-admissible if $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as B-representations. The strategy of Fontaine then consists of classifying p-adic representations according to the rings of periods for which they are admissible. In the case where V is admissible, $D_B(V)$ can usually be used to recover V, or at least $V_{|\mathcal{G}_L}$ for some finite extension L of K.

We now recall the construction of some rings of periods.

Let $\widetilde{\mathbf{E}}^+ = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}_p} = \left\{(x^{(0)},\dots) \in \mathcal{O}_{\mathbf{C}_p}^{\mathbf{N}} : (x^{(n+1)})^p = x^{(n)}\right\}$ and recall [**Win83**, Thm. 4.1.2] that this ring is naturally endowed with a ring structure which makes it a perfect ring of characteristic p which is complete for the valuation $v_{\mathbf{E}}$ defined by $v_{\mathbf{E}}(x) = v_p(x^{(0)})$. Let $\widetilde{\mathbf{E}}$ be its field of fractions and note that it is algebraically closed. We denote by φ the absolute Frobenius $x \mapsto x^p$ on $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{E}}$. The action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\mathcal{O}_{\mathbf{C}_p}$ induces a continuous action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{E}}$.

Choose a sequence $\varepsilon \in \widetilde{\mathbf{E}}^+$ of compatible p^n-th roots of unity (with $\varepsilon^{(1)} \neq 1$). Let $\overline{v} = \varepsilon - 1 \in \widetilde{\mathbf{E}}^+$ and let $\mathbf{E}_{\mathbf{Q}_p} := \mathbf{F}_p(\!(\overline{v})\!) \subset \widetilde{\mathbf{E}}$. Let $\mathbf{E} = \mathbf{E}_{\mathbf{Q}_p}^{sep}$ be the separable closure of $\mathbf{E}_{\mathbf{Q}_p}$ inside $\widetilde{\mathbf{E}}$. The field $\mathbf{E}_{\mathbf{Q}_p}$ is left invariant by the action of $H_{\mathbf{Q}_p}$ so that we have a morphism $H_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{E}/\mathbf{E}_{\mathbf{Q}_p})$. By [**Win83**, Thm. 3.2.2], it is actually an isomorphism.

We also let $\mathbf{E}_K = \mathbf{E}^{H_K}$. Note that Γ_K acts on $\mathbf{E}_K$, and that the action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\overline{v}$ is given by $g(\overline{v}) = (1+\overline{v})^{\chi_{\mathrm{cycl}}(g)} - 1$.

Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ and let $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$. We also let $\widetilde{\mathbf{B}} = \mathrm{Frac}(\widetilde{\mathbf{A}}) = \widetilde{\mathbf{A}}[1/p]$ and $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p]$. By functoriality of Witt vectors, the action of $\mathcal{G}_{\mathbf{Q}_p}$ extends to an action on $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ that commutes with the Frobenius φ.

Note that any element x of $\widetilde{\mathbf{A}}^+$ can be written as $x = \sum_{k\geq 0} p^k[x_k]$ where the x_k belong to $\widetilde{\mathbf{E}}^+$ and $[\cdot]$ denotes the Teichmüller lift. Recall [**Fon94a**, 1.5.1] that we have a surjective morphism of rings $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$ given by $\theta(x) = \sum_{k\geq 0} p^k x_k^{(0)}$ and whose kernel is a principal maximal ideal of $\widetilde{\mathbf{A}}^+$. This morphism θ naturally extends to $\widetilde{\mathbf{B}}^+$ to a surjective morphism that we still denote by $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}_p$. For $m \in \mathbf{N}$, we let $\mathbf{B}_m$ be the ring $\widetilde{\mathbf{B}}^+/\ker(\theta)^m\widetilde{\mathbf{B}}^+$ and we endow it with the structure of a p-adic Banach ring by taking the image of $\widetilde{\mathbf{A}}^+$ as its ring of integers. We let $\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_{m\in\mathbf{N}} \mathbf{B}_m$ be the completion of $\widetilde{\mathbf{B}}^+$ for the $\ker(\theta)$-adic topology and we endow it with the Fréchet topology of the projective limit. By construction, θ extends to a continuous morphism $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$ and the action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{B}}^+$ extends by continuity to a continuous action on $\mathbf{B}_{\mathrm{dR}}^+$. We let $\mathbf{B}_{\mathrm{dR}}$ be the fraction field of $\mathbf{B}_{\mathrm{dR}}^+$. The power series defining $\log[\varepsilon]$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element t that generates the maximal ideal $\ker(\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p)$ of $\mathbf{B}_{\mathrm{dR}}^+$, so that $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$. Note that the action of $\mathcal{G}_{\mathbf{Q}_p}$ on t is given by $g(t) = \chi_{\mathrm{cycl}}(g)\cdot t$. We endow $\mathbf{B}_{\mathrm{dR}}$ with a filtration by setting $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} = t^i\mathbf{B}_{\mathrm{dR}}^+$. We call representations that are $\mathbf{B}_{\mathrm{dR}}$-admissible "de Rham representations".

Fontaine has also defined several other rings of periods, among which $\mathbf{B}_{\mathrm{crys}}$ and $\mathbf{B}_{\mathrm{st}}$, in order to study p-adic representations. Recall that $\mathbf{B}_{\mathrm{crys}}$ is endowed with a Frobenius φ, $\mathbf{B}_{\mathrm{st}}$ contains $\mathbf{B}_{\mathrm{crys}}$, is endowed with a Frobenius φ and a monodromy operator N such that $\mathbf{B}_{\mathrm{crys}} = \mathbf{B}_{\mathrm{st}}^{N=0}$, and $\mathbf{B}_{\mathrm{dR}}$ is a field endowed with a filtration $\{\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}\}_{i\in\mathbf{Z}}$ and such that there is an injective map $\mathbf{B}_{\mathrm{st}} \to \mathbf{B}_{\mathrm{dR}}$. Moreover, these rings all contain the element t, and there exist rings $\mathbf{B}_{\mathrm{crys}}^+$ and $\mathbf{B}_{\mathrm{st}}^+$ such that $\mathbf{B}_{\mathrm{crys}} = \mathbf{B}_{\mathrm{crys}}^+[1/t]$ and $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{st}}^+[1/t]$. Representations that are $\mathbf{B}_{\mathrm{crys}}$-admissible and $\mathbf{B}_{\mathrm{st}}$-admissible are respectively called crystalline and semi-stable representations. The relations between those rings imply that crystalline representations are semi-stable and that semi-stable representations are de Rham. We do not recall the proper definitions of $\mathbf{B}_{\mathrm{crys}}$ and $\mathbf{B}_{\mathrm{st}}$ as they are not needed in this note.

1.2. Cyclotomic (φ,Γ)-modules. — Let us now recall briefly the theory of (φ,Γ)-modules and some of the rings involved in the theory. Let $v = [\varepsilon]-1$. Let $\mathbf{A}_{\mathbf{Q}_p}$ be the p-adic completion of $\mathbf{Z}_p(\!(v)\!)$ inside $\widetilde{\mathbf{A}}$. This is a discrete valuation ring with residue field $\mathbf{E}_{\mathbf{Q}_p}$. Since

$$\varphi(v) = (1+v)^p - 1 \quad \text{and} \quad g(v) = (1+v)^{\chi_{\mathrm{cycl}}(g)} - 1 \text{ if } g \in \mathcal{G}_{\mathbf{Q}_p},$$

the ring $\mathbf{A}_{\mathbf{Q}_p}$ and its field of fractions $\mathbf{B}_{\mathbf{Q}_p} = \mathbf{A}_{\mathbf{Q}_p}[1/p]$ are both stable by φ and $\mathcal{G}_{\mathbf{Q}_p}$. If K is a finite extension of $\mathbf{Q}_p$, we let $\widetilde{\mathbf{B}}_K = \widetilde{\mathbf{B}}^{H_K}$ and $\widetilde{\mathbf{A}}_K = \widetilde{\mathbf{A}}^{H_K}$.

For $r > 0$, we define $\widetilde{\mathbf{B}}^{\dagger,r}$ the subset of overconvergent elements of "radius" r of $\widetilde{\mathbf{B}}$, by

$$\left\{x = \sum_{n \ll -\infty} p^n[x_n] \text{ such that } \lim_{k \to +\infty} v_{\mathbf{E}}(x_k) + \frac{pr}{p-1}k = +\infty\right\}$$

and we let $\widetilde{\mathbf{B}}^\dagger = \bigcup_{r>0} \widetilde{\mathbf{B}}^{\dagger,r}$ be the subset of all overconvergent elements of $\widetilde{\mathbf{B}}$.

Let $\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r}$ be the subset of $\mathbf{B}_{\mathbf{Q}_p}$ given by

$$\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r} = \left\{\sum_{i \in \mathbf{Z}} a_i v^i, a_i \in \mathbf{Q}_p \text{ such that the } a_i \text{ are bounded and } \lim_{i \to -\infty} v_p(a_i) + i\frac{pr}{p-1} = +\infty\right\},$$

and note that $\mathbf{B}_{\mathbf{Q}_p}^{\dagger,r} = \mathbf{B}_{\mathbf{Q}_p} \cap \widetilde{\mathbf{B}}^{\dagger,r}$.

Let $\mathbf{B}_{\mathbf{Q}_p}^{\dagger} = \bigcup_{r>0} \mathbf{B}_{\mathbf{Q}_p}^{\dagger,r}$. By §2 of [**M$^+$95**], this is a Henselian field, and its residue ring is still $\mathbf{E}_{\mathbf{Q}_p}$. Since $\mathbf{B}_{\mathbf{Q}_p}^{\dagger}$ is Henselian, there exists a finite unramified extension $\mathbf{B}_K^\dagger/\mathbf{B}_{\mathbf{Q}_p}^\dagger$ inside $\widetilde{\mathbf{B}}$, of degree f and whose residue field is $\mathbf{E}_K$. Therefore, there exists $r(K) > 0$ and elements $x_1, \ldots, x_f$ in $\mathbf{B}_K^{\dagger,r(K)}$ such that $\mathbf{B}_K^{\dagger,s} = \bigoplus_{i=1}^f \mathbf{B}_{\mathbf{Q}_p}^{\dagger,s} \cdot x_i$ for all $s \geq r(K)$. We let $\mathbf{B}_K$ be the p-adic completion of $\mathbf{B}_K^\dagger$ and we let $\mathbf{A}_K$ be its ring of integers for the p-adic valuation. One can show that $\mathbf{B}_K$ is a subfield of $\widetilde{\mathbf{B}}$ stable under the action of φ and Γ_K (see for example [**Col08a**, Prop. 6.1]). Let $\mathbf{A}$ be the p-adic completion of $\bigcup_{K/\mathbf{Q}_p} \mathbf{A}_K$, taken over all the finite extensions $K/\mathbf{Q}_p$. Let $\mathbf{B} = \mathbf{A}[1/p]$. Note that $\mathbf{A}$ is a complete discrete valuation ring whose field of fractions is $\mathbf{B}$ and with residue field $\mathbf{E}$. Once again, both $\mathbf{A}$ and $\mathbf{B}$ are stable by φ and $\mathcal{G}_{\mathbf{Q}_p}$. Moreover, we have $\mathbf{A}^{H_K} = \mathbf{A}_K$ and $\mathbf{B}_K = \mathbf{B}^{H_K}$, so that $\mathbf{A}_K$ is a complete discrete valuation ring with residue field $\mathbf{E}_K$ and fraction field $\mathbf{B}_K = \mathbf{A}_K[1/p]$. If L is a finite extension of K, then $\mathbf{B}_L/\mathbf{B}_K$ is an unramified extension of degree $[L_\infty : K_\infty]$ and if L/K is Galois then so is $\mathbf{B}_L/\mathbf{B}_K$, and we have the following isomorphisms: $\mathrm{Gal}(\widetilde{\mathbf{B}}_L/\widetilde{\mathbf{B}}_K) = \mathrm{Gal}(\mathbf{B}_L/\mathbf{B}_K) = \mathrm{Gal}(\mathbf{E}_L/\mathbf{E}_K) = \mathrm{Gal}(L_\infty/K_\infty) = H_K/H_L$.

Definition 1.1. — If K is a finite extension of $\mathbf{Q}_p$, a (φ, Γ_K)-module D on $\mathbf{A}_K$ (resp. $\mathbf{B}_K$) is an $\mathbf{A}_K$-module of finite rank (resp. a finite dimensional $\mathbf{B}_K$-vector space) endowed with semilinear actions of Γ_K and φ that commute one to another.

It is said to be étale if $1 \otimes \varphi : \varphi^* D \to D$ is an isomorphism (resp. if there exists a basis of D such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathbf{A}_K)$).

If K is a finite extension of $\mathbf{Q}_p$ and if V is a p-adic representation of $\mathcal{G}_K$, we set

$$D(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}.$$

Note that $D(V)$ is a (φ, Γ_K)-module. Moreover, if V is a p-adic representation of $\mathcal{G}_K$, then $D(V)$ is étale and $(\mathbf{B} \otimes_{\mathbf{B}_K} D(V))^{\varphi=1}$ is canonically isomorphic to V (see [**Fon90**, Prop. 1.2.6]). The functors $V \mapsto D(V)$ and $D \mapsto (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$ then induce an equivalence of tannakian categories between p-adic representations of $\mathcal{G}_K$ and étale (φ, Γ_K)-modules.

For $r \geq 0$, we define a valuation $V(\cdot, r)$ on $\widetilde{\mathbf{B}}^+[\frac{1}{v}]$ by setting

$$V(x, r) = \inf_{k \in \mathbf{Z}} (k + \frac{p-1}{pr} v_{\mathbf{E}}(x_k))$$

for $x = \sum_{k \gg -\infty} p^k [x_k]$. If I is a closed subinterval of $[0; +\infty[$, we let $V(x, I) = \inf_{r \in I} V(x, r)$. We then define the ring $\widetilde{\mathbf{B}}^I$ as the completion of $\widetilde{\mathbf{B}}^+[1/v]$ for the valuation

$V(\cdot, I)$ if $0 \notin I$, and as the completion of $\widetilde{\mathbf{B}}^+$ for $V(\cdot, I)$ if $I = [0; r]$. We will write $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ for $\widetilde{\mathbf{B}}^{[r,+\infty[}$ and $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ for $\widetilde{\mathbf{B}}^{[0,+\infty[}$. We also define $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} = \bigcup_{r \geq 0} \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$. We also let $\widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}$ be the ring of integers of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ for the valuation $V(\cdot, [r; r])$.

The ring $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ is actually equal to $\bigcap_{n \geq 0} \varphi^n(\mathbf{B}^+_{\mathrm{crys}})$ (see for example [**Ber02**, §1.2]). The fact that periods of crystalline representations of $\mathcal{G}_K$ live inside finite dimensional F-vector spaces that are φ-stable implies that crystalline representations are exactly the representations that are $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$-admissible.

Let I be a subinterval of $]1, +\infty[$ or such that $0 \in I$. Let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $v_p(a_k) + k/\rho \to +\infty$ when $|k| \to +\infty$ for all $\rho \in I$. The series $f(v)$ converges in $\widetilde{\mathbf{B}}^I$ and we let $\mathbf{B}^I_{\mathbf{Q}_p}$ denote the set of all $f(\pi)$ with f as above. It is a subring of $\widetilde{\mathbf{B}}^I_{\mathbf{Q}_p}$.

We also write $\mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbf{Q}_p}$ for $\mathbf{B}^{[r;+\infty[}_{\mathbf{Q}_p}$. It is a subring of $\mathbf{B}^{[r;s]}_{\mathbf{Q}_p}$ for all $s \geq r$ and note that the set of all $f(v) \in \mathbf{B}^{\dagger,r}_{\mathrm{rig},\mathbf{Q}_p}$ such that the sequence $(a_k)_{k \in \mathbf{Z}}$ is bounded is exactly the ring $\mathbf{B}^{\dagger,r}_{\mathbf{Q}_p}$. Let $\mathbf{B}^{\dagger}_{\mathbf{Q}_p} = \cup_{r \gg 0} \mathbf{B}^{\dagger,r}_{\mathbf{Q}_p}$.

Recall that, for K a finite extension of $\mathbf{Q}_p$, there exists a separable extension $\mathbf{E}_K/\mathbf{E}_{\mathbf{Q}_p}$ of degree $f = [K_\infty : F_\infty]$ and an attached unramified extension $\mathbf{B}^\dagger_K/\mathbf{B}^\dagger_{\mathbf{Q}_p}$ of degree f with residue field $\mathbf{E}_K$, so that there exists $r(K) > 0$ and elements $x_1, \cdots x_f \in \mathbf{B}^{\dagger,r(K)}_K$ such that $\mathbf{B}^{\dagger,s}_K = \bigoplus_{i=1}^f \mathbf{B}^{\dagger,s}_{\mathbf{Q}_p} \cdot x_i$ for all $s \geq r(K)$. If $r(K) \leq \min(I)$, we let $\mathbf{B}^I_K$ be the completion of $\mathbf{B}^{\dagger,r(K)}_K$ for $V(\cdot, I)$, so that $\mathbf{B}^I_K = \oplus_{i=1}^f \mathbf{B}^I_{\mathbf{Q}_p} \cdot x_i$.

We actually have a better description of the rings $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ in general:

Proposition 1.2. — *Let K be a finite extension of $\mathbf{Q}_p$.*

1. *There exists $v_K \in \mathbf{A}^{\dagger,r(K)}_K$ whose image modulo p is a uniformizer of $\mathbf{E}_K$ and such that, for $r \geq r(K)$, every element $x \in \mathbf{B}^{\dagger,r}_K$ can be written as $x = \sum_{k \in \mathbf{Z}} a_k v_K^k$, where $a_k \in F' = \mathbf{Q}_p^{\mathrm{unr}} \cap K_\infty$, and the power series $\sum_{k \in \mathbf{Z}} a_k T^k$ is holomorphic and bounded on $\left\{p^{-1/e_K r} \leq |T| < 1\right\}$.*
2. *Let $\mathcal{H}^\alpha_{F'}(T)$ be the set of power series $\sum_{k \in \mathbf{Z}} a_k T^k$ where $a_k \in F'$ and such that, for all $\rho \in [\alpha; 1[$, $\lim\limits_{k \to \pm\infty} |a_k| \rho^k = 0$ and let $\alpha_K^r = p^{-1/e_K r}$. Then the map $\mathcal{H}^\alpha_{F'}(T) \to \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ sending f to $f(v_K)$ is an isomorphism.*

Proof. — The first item is proved in [**Col08a**, Prop. 7.5] and the second one in [**Col08a**, Prop. 7.6]. Be careful that the notations for the rings and the normalizations of the valuations used in Colmez's paper are a bit different than ours. □

The following theorem is the main result of [**CC98**] and shows that every étale (φ, Γ_K)-module is the base change to $\mathbf{B}_K$ of an overconvergent module:

Theorem 1.3. — *If D is an étale (φ, Γ_K)-module, then the set of free sub-$\mathbf{B}^\dagger_K$-modules of finite type stable by φ and Γ_K admits a bigger element $D^\dagger$ and one has $D = \mathbf{B}_K \otimes_{\mathbf{B}^\dagger_K} D^\dagger$.*

In particular, if V is a p-adic representation of $\mathcal{G}_K$, then there exists an étale (φ, Γ_K)-module over $\mathbf{B}^\dagger_K$ which we will denote by $\mathbf{D}^\dagger(V)$ and such that $D(V) = \mathbf{B}_K \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V)$. We let $\mathbf{D}^\dagger_{\mathrm{rig}}(V) = \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V)$.

If E if a finite extension of $\mathbf{Q}_p$, we can make the following definition:

Definition 1.4. — A (φ, Γ_K)-module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},K}$ is a finite module D over $E \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},K}$, equipped with a semi-linear Frobenius φ_D and a continuous semi-linear action of Γ_K such that D is free as a $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module, $\mathrm{id} \otimes \varphi_D : \mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\varphi, \mathbf{B}^\dagger_{\mathrm{rig},K}} D \to D$ is an isomorphism and that the actions of φ_D and Γ_K commute.

By [**Nak09**, Lemm. 1.30], a (φ, Γ_K)-module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},K}$ is free as an $E \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},K}$-module. We say that a (φ, Γ_K)-module over $E \otimes_{\mathbf{Q}_p} \mathbf{B}^\dagger_{\mathrm{rig},K}$ is étale if its underlying φ-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is étale.

1.3. Lubin-Tate (φ, Γ)-modules. — We now recall the theory of (φ, Γ)-modules in the Lubin-Tate setting. We let F be a finite extension of $\mathbf{Q}_p$, π a uniformizer of $\mathcal{O}_F$ and LT be a Lubin-Tate formal $\mathcal{O}_F$-module attached to the uniformizer π of $\mathcal{O}_F$. Let q be the cardinal of the residue field of F. Let $F_0 = F \cap \mathbf{Q}_p^{\mathrm{unr}}$. We let F_n denote the extension of F generated by the points of π^n-torsion of LT for $n \geq 1$, and $F_\infty = \bigcup_{n \geq 1} F_n$. We let $\Gamma_F = \mathrm{Gal}(F_\infty/F)$ and $H_F = \mathrm{Gal}(F_\infty/F)$. By Lubin-Tate's theory [**LT65**, Thm. 2], the Lubin-Tate character $\chi_\pi : \mathcal{G}_F \to \mathcal{O}_F^\times$ induces an isomorphism $\Gamma_F \simeq \mathcal{O}_F^\times$. For $a \in \mathcal{O}_F$, we let $[a](T)$ denote the power series that corresponds to the the multiplication by a map on LT. Let $v_0 = 0$ and for each $n \geq 1$, let $v_n \in \overline{\mathbf{Q}}_p$ be such that $[\pi](v_n) = v_{n-1}$, with $v_1 \neq 0$.

Recall that we defined rings $\widetilde{\mathbf{A}}, \widetilde{\mathbf{A}}^+, \widetilde{\mathbf{A}}^I$ and $\widetilde{\mathbf{B}}, \widetilde{\mathbf{B}}^+, \widetilde{\mathbf{B}}^I$ previously, and in what follows we will keep the same notations for those rings tensored over F_0 (resp. $\mathcal{O}_{F_0}$ in the case of $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{A}}^I$), by F (resp. $\mathcal{O}_F$). We let $\varphi_q = \varphi^{\circ k}$ where k is such that $p^k = q$.

Recall that by [**Col02**, §9.2], there exists $v \in \widetilde{\mathbf{A}}^+$ whose image in $\widetilde{\mathbf{E}}^+$ is $(v_0, v_1, \cdots)$, where $\widetilde{\mathbf{E}}^+ = \varprojlim_{x \mapsto x^q} \mathcal{O}_{\mathbf{C}_p}/\pi$ (by [**BC09b**, Prop. 4.3.1], this is the same ring $\widetilde{\mathbf{E}}^+$ as before) and such that $g(v) = [\chi_\pi(g)](v)$ and $\varphi_q(v) = [\pi](v)$. We also let $t_\pi = \log_{\mathrm{LT}}(v) \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ so that $g(t_\pi) = \chi_\pi(g) \cdot t_\pi$ and $\varphi_q(t_\pi) = \pi t_\pi$.

For $\rho > 0$, let $\rho' = \rho \cdot e \cdot p/(p-1) \cdot (q-1)/q$, where e is the ramification index of $F/\mathbf{Q}_p$. Let I be a subinterval of $]1, +\infty[$ or such that $0 \in I$. Let $f(Y) = \sum_{k \in \mathbf{Z}} a_k Y^k$ be a power series with $a_k \in F$ and such that $v_p(a_k) + k/\rho' \to +\infty$ when $|k| \to +\infty$ for all $\rho \in I$. The series $f(v)$ converges in $\widetilde{\mathbf{B}}^I$ and we let $\mathbf{B}^I_F$ denote the set of all $f(v)$ with f as above. It is a subring of $\widetilde{\mathbf{B}}^I_F$. We also write $\mathbf{B}^{\dagger,r}_{\mathrm{rig},F}$ for $\mathbf{B}_F^{[r;+\infty[}$.

We let $\mathbf{A}_F$ denote the p-adic completion of $\mathcal{O}_F(\!(v)\!)$ inside $\widetilde{\mathbf{A}}$, and we let $\mathbf{B}_F = \mathbf{A}_F[1/p]$. As in the cyclotomic case, to any extension K/F finite, there corresponds extensions $\mathbf{A}_K/\mathbf{A}_F$ and $\mathbf{B}_K/\mathbf{B}_F$, of degree $[K_\infty : F_\infty]$ where $K_\infty = K \cdot F_\infty$, equipped with actions of φ_q and $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, and there is a theory of (φ_q, Γ_K)-modules over $\mathbf{B}_K$, which are finite dimensional $\mathbf{B}_K$ vector spaces endowed with commuting semilinear actions of Γ_K and φ_q, Once again, such a (φ_q, Γ_K)-module is said to be étale if there exists a basis in which $\mathrm{Mat}(\varphi_q)$ belongs to $\mathrm{GL}_d(\mathbf{A}_K)$. By specializing Fontaine's constructions [**Fon90**, A.1.2.6 and A.3.4.3], Kisin and Ren prove the following, which is [**KR09**, Thm. 1.6]:

Theorem 1.5. — *There is a tannakian equivalence of categories between F-linear representations of $\mathcal{G}_F$ and étale (φ_q, Γ_K)-modules over $\mathbf{B}_K$.*

However, unlike in the cyclotomic case, these (φ_q, Γ_K) modules are rarely overconvergent. Berger showed in [**Ber16b**] that the right subcategory of representations corresponding to overconvergent (φ_q, Γ_K)-modules was the one of F-analytic representations (note however that there are representations which are not F-analytic but whose attached (φ_q, Γ_K)-module is overconvergent). An E-representation V of $\mathcal{G}_K$, where $E \supset F^{\mathrm{Gal}}$, is said to be F-analytic if for any $\tau \in \mathrm{Emb}(E, \overline{\mathbf{Q}}_p)$, $\tau \neq \mathrm{id}$, the semilinear $\mathbf{C}_p$-representation $\mathbf{C}_p \otimes^\tau V$ is trivial. In that case, theorem 10.4 of [**Ber16b**] shows that one can attach to V an étale F-analytic (φ_q, Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ on $\mathbf{B}^\dagger_{\mathrm{rig},K}$, which means that the operator $\frac{\log g}{\log \chi_\pi(g)}$ is F-linear on $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Note that, when $F = \mathbf{Q}_p$, every representation of $\mathcal{G}_K$ is $\mathbf{Q}_p$-analytic.

For $\delta \in \widehat{\mathfrak{I}}_K(E)$ which is F-analytic, we let $w(\delta)$ denote its weight, which is defined by $w(\delta) = \delta'(1)$.

Lemma 1.6. — *Let $\mathbf{D}$ be a rank 1 F-analytic (φ_q, Γ_K)-module over $E \otimes_K \mathbf{B}^\dagger_{\mathrm{rig},K}$. Then there exists an F-analytic character $\delta : K^\times \to E^\times$ and a basis e of $\mathbf{D}$ in which $g(e) = \delta(\chi_\pi) \cdot e$ and $\varphi_q(e) = \delta(\pi) \cdot e$.*

Proof. — This is the same as in [**Col08b**, Prop. 3.1], using [**Ber16b**, Thm. 10.4]. □

2. Locally and pro-analytic vectors

Here, we recall some of the theory of locally- and pro-analytic vectors, following the presentation of Emerton in [**Eme17**] and of Berger in [**Ber16b**].

Let G be a p-adic Lie group, and let W be a $\mathbf{Q}_p$-Banach representation of G. Let H be an open subgroup of G such that there exists coordinates $c_1, \cdots, c_d : H \to \mathbf{Z}_p$ giving rise to an analytic bijection $\mathbf{c} : H \to \mathbf{Z}_p^d$. We say that $w \in W$ is an H-analytic vector if there exists a sequence $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}^d}$ such that $w_{\mathbf{k}} \to 0$ in W and such that $g(w) = \sum_{\mathbf{k} \in \mathbf{N}^d} \mathbf{c}(g)^{\mathbf{k}} w_{\mathbf{k}}$ for all $g \in H$. We let $W^{H-\mathrm{an}}$ be the space of H-analytic vectors. This space injects into $\mathcal{C}^{\mathrm{an}}(H, W)$, the space of all analytic functions $f : H \to W$. Note that $\mathcal{C}^{\mathrm{an}}(H, W)$ is a Banach space equipped with its usual Banach norm, so that we can endow $W^{H-\mathrm{an}}$ with the induced norm, that we will denote by $|| \cdot ||_H$. With this definition, we have $||w||_H = \sup_{\mathbf{k} \in \mathbf{N}^d} ||w_{\mathbf{k}}||$ and $(W^{H-\mathrm{an}}, || \cdot ||_H)$ is a Banach space.

The space $\mathcal{C}^{\mathrm{an}}(H, W)$ is endowed by an action of $H \times H \times H$, given by

$$((g_1, g_2, g_3) \cdot f)(g) = g_1 \cdot f(g_2^{-1} g g_3)$$

and one can recover $W^{H-\mathrm{an}}$ as the closed subspace of $\mathcal{C}^{\mathrm{an}}(H, W)$ of its $\Delta_{1,2}(H)$-invariants, where $\Delta_{1,2} : H \to H \times H \times H$ denotes the map $g \mapsto (g, g, 1)$ (we refer the reader to [**Eme17**, §3.3] for more details).

We say that a vector w of W is locally analytic if there exists an open subgroup H as above such that $w \in W^{H-\mathrm{an}}$. Let W^{la} be the space of such vectors, so that $W^{\mathrm{la}} = \bigcup_H W^{H-\mathrm{an}}$, where H runs through a sequence of open subgroups of G. The space W^{la} is naturally endowed with the inductive limit topology, so that it is an LB space.

Lemma 2.1. — *If W is a ring such that $||xy|| \leq ||x|| \cdot ||y||$ for $x, y \in W$, then*

1. $W^{H-\mathrm{an}}$ *is a ring, and* $||xy||_H \leq ||x||_H \cdot ||y||_H$ *if* $x, y \in W^{H-\mathrm{an}}$*;*

2. if $w \in W^\times \cap W^{\mathrm{la}}$*, then* $1/w \in W^{\mathrm{la}}$*. In particular, if* W *is a field, then* W^{la} *is also a field.*

Proof. — See [**BC16**, Lemm. 2.5]. □

Let W be a Fréchet space whose topology is defined by a sequence $\{p_i\}_{i \geq 1}$ of seminorms. Let W_i be the Hausdorff completion of W at p_i, so that $W = \varprojlim_{i \geq 1} W_i$. The space W^{la} can be defined but as stated in [**Ber16b**] and as will be explained in §7, this space is too small in general for what we are interested in, and so we make the following definition, following [**Ber16b**, Def. 2.3]:

Definition 2.2. — If $W = \varprojlim_{i \geq 1} W_i$ is a Fréchet representation of G, then we say that a vector $w \in W$ is pro-analytic if its image $\pi_i(w)$ in W_i is locally analytic for all i. We let W^{pa} denote the set of all pro-analytic vectors of W.

We extend the definition of W^{la} and W^{pa} for LB and LF spaces respectively.

Proposition 2.3. — *Let G be a p-adic Lie group, let B be a Banach G-ring and let W be a free B-module of finite rank, equipped with a compatible G-action. If the B-module W has a basis $w_1, \ldots, w_d$ in which $g \mapsto \mathrm{Mat}(g)$ is a globally analytic function $G \to \mathrm{GL}_d(B) \subset M_d(B)$, then*

1. $W^{H-\mathrm{an}} = \bigoplus_{j=1}^d B^{H-\mathrm{an}} \cdot w_j$ *if H is a subgroup of G;*

2. $W^{\mathrm{la}} = \bigoplus_{j=1}^d B^{\mathrm{la}} \cdot w_j$*.*

Let G be a p-adic Lie group, let B be a Fréchet G-ring and let W be a free B-module of finite rank, equipped with a compatible G-action. If the B-module W has a basis $w_1, \ldots, w_d$ in which $g \mapsto \mathrm{Mat}(g)$ is a pro-analytic function $G \to \mathrm{GL}_d(B) \subset M_d(B)$, then

$$W^{\mathrm{pa}} = \bigoplus_{j=1}^d B^{\mathrm{pa}} \cdot w_j.$$

Proof. — The part for Banach ring is proven in [**BC16**, Prop. 2.3] and the one for Fréchet rings is proven in [**Ber16b**, Prop. 2.4]. □

If W a K-linear Banach representation of $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, and if $n \geq 1$, we say that $w \in W$ is F-analytic on $\Gamma_n = \mathrm{Gal}(K_\infty/K_n)$ if there exists a sequence $\{w_k\}_{k \geq 0}$ of elements of W such that $\pi^{nk} w_k \to 0$ such that $g(w) = \sum_{k \geq 0} \log \chi_\pi(g)^k w_k$ for all $g \in \Gamma_n$. We let $W^{\Gamma_n-\mathrm{an}, K-\mathrm{la}}$ denote the set of such elements and $W^{K-\mathrm{la}} = \bigcup_{n \geq 1} W^{\Gamma_n-\mathrm{an}, K-\mathrm{la}}$.

Lemma 2.4. — *We have* $W^{\Gamma_n-\mathrm{an}, K-\mathrm{la}} = W^{\Gamma_n-\mathrm{an}} \cap W^{K-\mathrm{la}}$*.*

Proof. — This is [**Ber16b**, Lemm. 2.5]. □

On locally analytic representations of Γ_K, we can define operators ∇_τ in the following way, as in [**Ber16b**, §2].

Definition 2.5. — Let L be a field that contains F^{Gal}. If $\tau \in \Sigma_F$, then we have the derivative in the direction τ, which is an element $\nabla_\tau \in L \otimes_{\mathbf{Q}_p} \mathrm{Lie}(\Gamma_F)$. The L-vector space $\mathrm{Hom}_{\mathbf{Q}_p}(F, L)$ is generated by the elements of Σ_F. If W is an L-linear Banach representation of Γ_F and if $w \in W^{\mathrm{la}}$ and $g \in \Gamma_F$, then there exists elements $\{\nabla_\tau\}_{\tau \in \Sigma_F}$ of $F^{\mathrm{Gal}} \otimes_{\mathbf{Q}_p} \mathrm{Lie}(\Gamma_F)$ such that we can write

$$\log g(w) = \sum_{\tau \in \Sigma_F} \tau(\log \chi_\pi(g)) \cdot \nabla_\tau(w).$$

In particular, there exist $m \gg 0$ and elements $\{w_{\mathbf{k}}\}_{\mathbf{k} \in \mathbf{N}^{\Sigma_F}}$ such that if $g \in \Gamma_m$, then $g(w) = \sum_{\mathbf{k} \in \mathbf{N}^{\Sigma_F}} \log \chi_\pi(g)^{\mathbf{k}} w_{\mathbf{k}}$, where $\log \chi_\pi(g)^{\mathbf{k}} = \prod_{\tau \in \Sigma_F} \tau \circ \log \chi_\pi(g)^{k_\tau}$. We have $\nabla_\tau(w) = w_{\mathbf{1}_\tau}$ where $\mathbf{1}_\tau$ is the Σ_F-tuple whose entries are 0 except the τ-th one which is 1. If $\mathbf{k} \in \mathbf{N}^{\Sigma_K}$, and if we set $\nabla^{\mathbf{k}}(w) = \prod_{\tau \in \Sigma_F} \nabla_\tau^{k_\tau}(w)$, then $w_{\mathbf{k}} = \nabla^{\mathbf{k}}(w)/\mathbf{k}!$.

Remark 2.6. — If $w \in W^{\mathrm{la}}$, then $w \in W^{F-\mathrm{la}}$ if and only if $\nabla_\tau(w) = 0$ for all $\tau \in \Sigma_F \setminus \{\mathrm{id}\}$.

3. Sen theory by Berger-Colmez

Recall that to a p-adic representation V of $\mathcal{G}_K$, one can attach to it the K_∞-vector space $D_{\mathrm{Sen}}(V)$ which is the set of elements of $W = (\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$ which belong to some finite dimensional K-vector subspace of W which is stable by Γ_K. The K_∞-vector space D_{Sen} comes equipped with an action of the Lie algebra of Γ_K and admits a canonical generator $\nabla = \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi_{\mathrm{cycl}}(\gamma) - 1}$ which is the operator of Sen, usually denoted by Θ_{Sen} and whose eigenvalues are called the generalized Hodge-Tate weights of the representation V.

Colmez has constructed in [**Col94**] a ring $\mathbf{B}_{\mathrm{Sen}}$ as follows:

Definition 3.1. — Let u be a variable and $\mathbf{B}^n_{Sen} = \mathbf{C}_p\{\{u\}\}_n$ be the set of power series $\sum_{k \geq 0} a_k u^k$ with coefficients in $\mathbf{C}_p$ such that the series $\sum_{k \geq 0} (p^n)^k a_k$ converges in $\mathbf{C}_p$ and equip it with the natural topology and with an action of $\mathrm{Gal}(\overline{K}/K(\mu_{p^n}))$ by setting

$$g(\sum_{k \geq 0} a_k u^k) = \sum_{k \geq 0} g(a_k)(u + \log \chi_{\mathrm{cycl}}(g))^k.$$

Note that this makes sense since $\log \chi_{\mathrm{cycl}}(g) \in p^n \mathbf{Z}_p$ if $g \in \mathrm{Gal}(\overline{K}/K(\mu_{p^n}))$. Let $\mathbf{B}_{Sen} = \bigcup_{n \geq 0} \mathbf{B}^n_{Sen}$, endowed with the inductive limit topology.

We now recall the following properties (for more details, see [**Col94**] and [**BC16**, §2.2]):

Proposition 3.2. —

1. *We have* $(\mathbf{B}^n_{\mathrm{Sen}})^{\mathcal{G}_{K_n}} = K_n$;
2. *if V is a p-adic representation of $\mathcal{G}_K$ and if n is an integer, let* $D'_{\mathrm{Sen},n}(V) := (\mathbf{B}^n_{\mathrm{Sen}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ *equipped with the operator ∇_u induced by the operator ∇_u on $\mathbf{B}^n_{\mathrm{Sen}}$ (meaning that $(\nabla_u)_{D'_{\mathrm{Sen},n}(V)}$ acts by $\nabla_u \otimes 1$ on $\mathbf{B}^n_{\mathrm{Sen}} \otimes_{\mathbf{Q}_p} V$) and let $D'_{\mathrm{Sen}}(V) := \bigcup_{n \geq 0} D'_{\mathrm{Sen},n}(V)$. Every element δ of $D'_{\mathrm{Sen},n}(V)$ can be written as $\delta^{(0)} + \delta^{(1)} u + \cdots$ where the $\delta^{(i)}$ belong to $\mathbf{C}_p \otimes_{\mathbf{Q}_p} V$. Then the map $\delta \mapsto \delta^{(0)}$ induces an isomorphism of K_∞-vector spaces between $D'_{\mathrm{Sen}}(V)$ and $D_{\mathrm{Sen}}(V)$, and of K_n-vector spaces between $D'_{\mathrm{Sen},n}(V)$ and $D_{\mathrm{Sen},n}(V)$ for $n \gg 0$. Moreover, the image of ∇_u by this isomorphism is Θ_{Sen}.*

Proof. — Item (i) is [**Col94**, Thm. 2 (i)]. For item (ii), see [**Col94**, Thm. 2 (ii)] and [**BC16**, Prop. 2.8]. □

When K_∞/K is any p-adic Lie extension with Galois group Γ_K (such that $\dim \Gamma_K \geq 2$ or such that K_∞/K is almost totally ramified), Berger and Colmez offer to replace classical Sen theory with the theory of locally analytic vectors, by considering the locally analytic vectors of semilinear $\widehat{K_\infty}$-representations of Γ_K:

Theorem 3.3. — *If W is a $\widehat{K_\infty}$-semilinear representation of Γ_K, then the map*

$$\widehat{K_\infty} \otimes_{\widehat{K_\infty}^{\mathrm{la}}} W^{\mathrm{la}} \to W$$

is an isomorphism. Moreover, if K_∞/K is the cyclotomic extension of K, and if $W = (\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K}$ then $W^{\Gamma_n-\mathrm{an}} = \mathbf{D}_{\mathrm{Sen},n}(V)$.

Proof. — The main claim is theorem 3.4 of [**BC16**], and the particular case for the cyclotomic extension follows from remark 3.3 of ibid. □

We also have in general a nice description of the structure of $\widehat{K_\infty}^{\mathrm{la}}$: if K_∞/K is a p-adic Lie extension with Galois group $\Gamma_K = \mathrm{Gal}(K_\infty/K)$, then by [**BC16**, Thm. 6.1], $\widehat{K_\infty}^{\mathrm{la}}$ can be thought of as the completion for the locally analytic topology of a ring of series in $d-1$ variables, where d is the dimension of Γ_K as a p-adic Lie group.

Using the rings $\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)$, we can recover Sen theory (and its generalization by Berger Colmez):

Proposition 3.4. — *Let K_∞/K be a p-adic Lie extension with Galois group $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and let V be a p-adic representation of $\mathcal{G}_K$. Then we have*

$$((\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K-\mathrm{la}} = \bigcup_{n\geq 1} ((\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}\mathbf{C}_p) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}},$$

where $\mathcal{G}_{K_n}$ acts on $\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}\mathbf{C}_p$ through the $\Delta_{1,2}$ map defined in §2.

This proposition is a direct consequence of the following more general proposition:

Proposition 3.5. — *Let K_∞/K be a p-adic Lie extension with Galois group $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ and let V be a p-adic representation of $\mathcal{G}_K$. Let B be a p-adic Banach ring, endowed with an action of $\mathcal{G}_K$. Then we have*

$$((B \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K-\mathrm{la}} = \bigcup_{n\geq 1} ((\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}B) \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}.$$

Proof. — This is tautological, as the set of Γ_n-locally analytic vectors of $W := B \otimes_{\mathbf{Q}_p} V)^{H_K}$ is by definition the subset of $\mathcal{C}^{\mathrm{an}}(\Gamma_n, B) = \mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}B$ which are invariant by the action given by $\Delta_{1,2}$ following the notations of §2. □

In particular, if B is a p-adic Banach ring, endowed with an action of $\mathcal{G}_K$ such that for V a p-adic representation of $\mathcal{G}_K$, $((B \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_K-\mathrm{la}}$ is related to some module attached to V which appears in p-adic Hodge theory (e.g. its (φ, Γ)-modules), then $\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}B$ can be thought of as a ring of periods that computes those modules. This is the main idea behind most of the constructions of the following sections.

In the cyclotomic case, we have seen that one could replace the rings $\mathcal{C}^{\mathrm{an}}(\Gamma_n, \mathbf{C}_p)$ by the rings $\mathbf{B}_{\mathrm{Sen}}^n$. This point of view can also be generalized when we replace $\mathbf{C}_p$ with other p-adic Banach rings endowed with an action of $\mathcal{G}_K$, and we make the following definition:

Definition 3.6. — Let u be a variable and B be a $\mathbf{Q}_p$-algebra endowed with a topology for which it is complete, and equipped with an action of $\mathcal{G}_K$. We denote by $B\{\{u\}\}_n$ the set of power series $\sum_{k\geq 0} a_k u^k$ with coefficients in B such that the series $\sum_{k\geq 0}(p^n)^k a_k$ converges in B and we equip it with the natural topology and with an action of $\mathcal{G}_{K_n}$ by setting

$$g(\sum_{k\geq 0} a_k u^k) = \sum_{k\geq 0} g(a_k)(u+\log\chi_{\mathrm{cycl}}(g))^k.$$

This makes sense since $\log\chi_{\mathrm{cycl}}(g) \in p^n\mathcal{O}_K$ if $g \in \mathcal{G}_{K_n}$. We let $B\{\{u\}\} = \bigcup_{n\geq 0} B\{\{u\}\}_n$, endowed with the inductive limit topology.

Proposition 3.7. — *Let B be a p-adic Banach ring, endowed with an action of $\mathcal{G}_K$. We have an isomorphism*

$$(B\{\{u\}\}_n)^{\mathcal{G}_{K_n}} \simeq (B^{H_K})^{\Gamma_n-\mathrm{an}}.$$

Proof. — Let $x \in (B\{\{u\}\}_n)^{\mathcal{G}_{K_n}}$. We can write $x = \sum_{i\geq 0} x_i u^i$. The fact that $x \in (B\{\{u\}\}_n)^{\mathcal{G}_{K_n}}$ shows that $x \in ((B\{\{u\}\}_n)^{H_K})^{\Gamma_n} = ((B^{H_K})\{\{u\}\}_n)^{\Gamma_n}$. Moreover, it implies that for all $g \in \mathcal{G}_{K_n}$, we have $g(x_i) = \sum_{j\geq 0} x_{i+j}\binom{i+j}{j}(-\log\chi_{\mathrm{cycl}}(g))^j$. Therefore, the x_i are all locally analytic vectors of B^{H_K}. We can define an operator ∇ on $(B^{H_K})^{\mathrm{la}}\{\{u\}\}_n$ by $\nabla = \frac{\log g}{\log\chi_\pi(g)}$ for g close enough to 1 and which does not depend on such a choice of g. A quick computation shows that

$$\nabla(x) = \sum_{i\geq 0}\nabla(x_i)u^i + \sum_{i\geq 0} i x_i u^{i-1}.$$

Since $x \in ((B\{\{u\}\}_n)^{H_K})^{\Gamma_n}$, it is killed by ∇, and thus we get that for all $i \geq 0$, we have $x_{i+1} = -\frac{\nabla(x_i)}{i+1}$. Therefore, we have $x_i = (-1)^i\frac{\nabla^i(x_0)}{i!}$ for all $i \geq 0$.

The fact that $x \in B\{\{u\}\}_n$ then implies that the sequence $\sum_{i\geq 0}(-1)^i\frac{\nabla^i(x_0)}{i!}p^{ni}$ converges in B, so that by definition x_0 is a Γ_n-analytic vector of B. It follows that the map $x \mapsto x_0$ is an injective morphism of rings $(B\{\{u\}\}_n)^{\mathcal{G}_{K_n}} \to (B^{H_K})^{\Gamma_n-\mathrm{an}}$, whose inverse is given by $z \mapsto \sum_{i\geq 0}(-1)^i\frac{\nabla^i(z)}{i!}u^i$, so that it is an isomorphism. □

Proposition 3.8. — *Let B be a p-adic Banach ring, endowed with an action of $\mathcal{G}_K$. Let V be a p-adic representation of $\mathcal{G}_K$. Then we have an isomorphism*

$$(B\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \simeq ((B\otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n-\mathrm{an}}.$$

Proof. — The proof is exactly the same as above, replacing B by $B\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V$. □

We can also define an F-analytic generalization of these constructions. In the following definitions, the cyclotomic setting is replaced by the Lubin-Tate one.

Definition 3.9. — Let u_K be a variable and B be a $\mathbf{Q}_p$-algebra endowed with a topology for which it is complete, and equipped with an action of $\mathcal{G}_K$. We denote by $B\{\{u_K\}\}_n$ the set of power series $\sum_{k\geq 0} a_k u_K^k$ with coefficients in B such that the series $\sum_{k\geq 0}(\pi^n)^k a_k$ converges in B and we equip it with the natural topology and with an action of $\mathcal{G}_{K_n}$ by

setting
$$g(\sum_{k\geq 0} a_k u_K^k) = \sum_{k\geq 0} g(a_k)(u_K + \log\chi_\pi(g))^k.$$
This once again makes sense since $\log\chi_\pi(g) \in \pi^n\mathcal{O}_K$ if $g \in \mathcal{G}_{K_n}$. We let $B\{\{u_K\}\} = \bigcup_{n\geq 0} B\{\{u_K\}\}_n$, endowed with the inductive limit topology.

Definition 3.10. — We endow the rings $B\{\{u_K\}\}$ and $B\{\{u_K\}\}_n$ with a continuous operator $\nabla_u = -\frac{d}{du_K}$ given by
$$\nabla_u(\sum_{k\geq 0} a_k u_K^k) = -\sum_{k\geq 1} k a_k u_K^{k-1}.$$

Proposition 3.11. — *Let B be a p-adic Banach ring, endowed with an action of $\mathcal{G}_K$. Let V be a p-adic representation of $\mathcal{G}_K$. We have isomorphisms*
$$(B\{\{u\}\}_n)^{\mathcal{G}_{K_n}} \simeq (B^{H_K})^{\Gamma_n-F-\mathrm{an}}$$
and
$$(B\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \simeq ((B\otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n-F-\mathrm{an}}.$$

Proof. — The proof is the same as in the cyclotomic case. □

4. de Rham computations

In this section we compute locally analytic vectors and pro-analytic vectors in $\mathbf{B}_{\mathrm{dR}}^+$, both in the cyclotomic case and in the Lubin-Tate case, and we explain how to recover the module $\mathbf{D}_{\mathrm{Dif}}^+(V)$ attached to a p-adic representation V thanks to the use of the locally analytic vectors. The fact that locally analytic vectors are able to recover $\mathbf{D}_{\mathrm{Dif}}^+(V)$ has already been proven in [**Por22**, §6.1] but here we will also use proposition 3.8 to produce a ring of periods which "computes" the functor $\mathbf{D}_{\mathrm{Dif}}^+$.

4.1. Computations in $\mathbf{B}_{\mathrm{dR}}^+$. — We let $\mathbf{B}_{\mathrm{dR},F}^+$ and $\mathbf{B}_{\mathrm{dR},F}$ denote respectively $\mathbf{B}_{\mathrm{dR}}^{H_F}$ and $\mathbf{B}_{\mathrm{dR}}^{H_F}$. Recall that there is a natural injective, $\mathcal{G}_F$-equivariant map $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \to \mathbf{B}_{\mathrm{dR}}^+$, which sends t_π to a generator of $\ker(\theta)$ in $\mathbf{B}_{\mathrm{dR}}^+$ and we still denote the image of t_π through this map by t_π. The image of t_τ, $\tau \in \Sigma$ through this map is still denoted by t_τ, and note that $t_\tau \in (\mathbf{B}_{\mathrm{dR},F}^+)^\times$ if $\tau \neq \mathrm{id}$ (see for example item 2 of [**BDM19**, Prop. 3.4]). We let $\partial_{\mathrm{id}} = \frac{1}{t_\pi}\nabla_{\mathrm{id}}$.

Lemma 4.1. — *We have $\partial_{\mathrm{id}}((\mathbf{B}_{\mathrm{dR},F}^+)^{\mathrm{pa}}) \subset (\mathbf{B}_{\mathrm{dR},F}^+)^{\mathrm{pa}}$.*

Proof. — Let $x \in (\mathbf{B}_{\mathrm{dR},F}^+)^{\mathrm{pa}}$. Then $\theta(x) \in \widehat{F_\infty}^{\mathrm{la}}$. Since $\nabla_{\mathrm{id}} = 0$ on $\widehat{F_\infty}^{\mathrm{la}}$, we get that $\nabla_{\mathrm{id}} \circ \theta(x) = 0 = \theta \circ \nabla_{\mathrm{id}}(x)$ so that $\nabla_{\mathrm{id}}(x) \in t_\pi \mathbf{B}_{\mathrm{dR}}^+$. Therefore, $\partial_{\mathrm{id}}(x) \in \mathbf{B}_{\mathrm{dR}}^+$. Since t_π is a pro-analytic vector of $\mathbf{B}_{\mathrm{dR},F}$ and since $x \in (\mathbf{B}_{\mathrm{dR},F}^+)^{\mathrm{pa}}$, we obtain $\partial_{\mathrm{id}}(x) \in (\mathbf{B}_{\mathrm{dR},F}^+)^{\mathrm{pa}}$. □

Lemma 4.2. — *We have $(\mathbf{B}_{\mathrm{dR},F})^{F-\mathrm{pa}} = F_\infty(\!(t_\pi)\!)$ and $(\mathbf{B}_{\mathrm{dR},F}^+)^{F-\mathrm{pa}} = F_\infty[\![t_\pi]\!]$.*

Proof. — See [**Por20**, Prop. 2.6]. □

We let $(\mathbf{B}_{\mathrm{dR},F}^+)^{\Sigma_0-\mathrm{pa}}$ denote the set of pro-analytic vectors of $\mathbf{B}_{\mathrm{dR},F}^+$ which are killed by ∇_{id}.

Proposition 4.3. — *We have* $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}} = \left\{\sum_{k\geq 0} a_k t_\pi^k, a_k \in (\mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}}\right\}$.

Proof. — Let $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$. For $i \geq 0$, we let $x_i = \frac{1}{i!}\sum_{k\geq 0}(-1)^k \frac{\partial_{\mathrm{id}}^{i+k}(x)}{k!} t_\pi^k$. By lemma 4.1, we have that for any $i,k \geq 0$, $\partial_{\mathrm{id}}^{i+k}(x)$ belongs to $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ so that the sum $\frac{1}{i!}\sum_{k\geq 0}(-1)^k \frac{\partial_{\mathrm{id}}^{i+k}(x)}{k!} t_\pi^k$ converges in $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ to an element x_i such that $\partial_{\mathrm{id}}(x_i) = 0$.

The sum $\sum_{i\geq 0} x_i t_\pi^i$ converges in $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ and a simple computation shows that $x = \sum_{i\geq 0} x_i t_\pi^i$.

Conversely, it is easy to check that if $(a_k)_{k\geq 0}$ is a sequence of elements of $(\mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}}$, the sum $\sum_{k\geq 0} a_k t_\pi^k$ converges to an element of $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$. $\square$

Lemma 4.4. — *Let* $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}}$ *such that* $t_\pi | x$ *in* $\mathbf{B}^+_{\mathrm{dR}}$. *Then* $x = 0$.

Proof. — Let $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}}$ such that $t_\pi | x$, and assume that $x \neq 0$. We can therefore write $x = t_\pi^k \alpha$ with $k \geq 1$, $\alpha \in \mathbf{B}^+_{\mathrm{dR}}$ and t_π does not divide α in $\mathbf{B}^+_{\mathrm{dR}}$. Moreover, since t_π is pro-analytic for the action of Γ_F, we get that α is pro-analytic for the action of Γ_K.

By proposition 4.3, we can write $\alpha = \sum_{j\geq 0} a_j t_\pi^j$ where the a_j are elements of $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ killed by ∇_{id}. The fact that x is killed by ∇_{id} translates into

$$\sum_{j\geq 0}(k+j)a_j t_\pi^{k+j} = 0.$$

Applying ∂_{id}^k to this equality and reducing mod t_π, we obtain that $a_0 = 0 \mod t_\pi$ and thus $t_\pi | \alpha$, which is not possible. $\square$

Corollary 4.5. — *For any* $N \geq 1$, *the map* $\theta_N : (\mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}} \to (\mathbf{B}^+_{\mathrm{dR},F}/t_\pi^N \mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}}$ *is injective.*

Note that $(\mathbf{B}^+_{\mathrm{dR},F}/t_\pi \mathbf{B}^+_{\mathrm{dR},F})^{\Sigma_0-\mathrm{pa}} = \widehat{F_\infty}^{\Sigma_0-\mathrm{la}}$ and that $\widehat{F_\infty}^{\Sigma_0-\mathrm{la}} = \widehat{F_\infty}^{\mathrm{la}}$ by [**Ber16b**, Prop. 2.10]. Note that this also implies that for any $m \geq 0$, the natural map $\theta : (\mathbf{B}^+_{\mathrm{dR},F})^{\Gamma_m-\mathrm{an},\Sigma_0-\mathrm{pa}} \to \widehat{F_\infty}^{\Gamma_m-\mathrm{an}}$ is injective. By [**Por22**], the map $\theta : (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}} \to \widehat{F_\infty}^{\mathrm{la}}$ is surjective. In particular, using proposition 4.3, we get the following "description" of $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$:

Proposition 4.6. — *The natural map* $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}} \mapsto \sum_{i\geq 0}\theta(x_i)t_\pi^i$, *where* $x_i = \frac{1}{i!}\sum_{k\geq 0}(-1)^k \frac{\partial_{\mathrm{id}}^{i+k}(x)}{k!} t_\pi^k$, *induces a* Γ_F*-equivariant isomorphism from* $(\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ *to* $\widehat{F_\infty}^{\mathrm{la}}[\![t_\pi]\!]$.

Proof. — We already know from the above that the map $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}} \mapsto \sum_{i\geq 0}\theta(x_i)t_\pi^i$ is injective. To prove that it is surjective, recall that the map $\theta : (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}} \to \widehat{K_\infty}^{\mathrm{la}}$ is surjective. If $y \in \widehat{K_\infty}^{\mathrm{la}}$, let $x \in (\mathbf{B}^+_{\mathrm{dR},F})^{\mathrm{pa}}$ such that $\theta(x) = y$. One can write $x = \sum_{i\geq 0} x_i t_\pi^i$ with $\partial_{\mathrm{id}}(x_i) = 0$ for all i, and thus x_0 satisfies $\theta(x_0) = \theta(x) = y$ and $\partial(x_0) = 0$, so that the map above is injective. $\square$

Remark 4.7. — We have $\mathbf{B}^+_{\mathrm{dR},F} \simeq \widehat{F_\infty}[\![t_\pi]\!]$ noncanonically but this isomorphism is not Γ_F-equivariant. However, taking only the pro-analytic vectors gives us a canonical isomorphism which is Γ_F-equivariant.

4.2. The modules $\mathbf{D}^+_{\mathrm{Dif}}(V)$. — When K_∞/K is the cyclotomic extension of K, Fontaine has proven in [**Fon04**] that the set of sub-$K_\infty[\![t]\!]$-modules free of finite type of $(\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K}$ and stable by the action of Γ_K admits a maximal element, usually denoted by $\mathbf{D}_{\mathrm{Dif}}^+(V)$, and which is such that $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{K_\infty[\![t]\!]} \mathbf{D}_{\mathrm{Dif}}^+(V) = \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V$.

If $\gamma \in \Gamma_K$ is close enough to 1, then the power series defining $\log(\gamma)$ converges as a power series of $\mathbf{Q}_p$-linear operators of $\mathbf{D}_{\mathrm{Dif}}^+(V)$, and the operator $\nabla_V = \frac{\log(\gamma)}{\log(\chi_{\mathrm{cycl}}(\gamma))}$ does not depend on the choice of γ and satisfies the Leibniz rule $\nabla_V(\lambda \cdot x) = \lambda \nabla_V(x) + \nabla(\lambda)x$. The map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$ induces an isomorphism of modules with connexions $(\mathbf{D}_{\mathrm{Dif}}^+(V), \nabla_V) \to (\mathbf{D}_{\mathrm{Sen}}(V), \Theta_V)$ (see for example [**Ber02**, §5.3]).

The map $\iota_n : \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger, r_n} \to \mathbf{B}_{\mathrm{dR}}^+$ sends $\mathbf{B}_K^{\dagger, r_n}$ into $K_n[\![t]\!] \subset \mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{D}^{\dagger, r_n}(V)$ in a sub-$K_n[\![t]\!]$-module of $\mathbf{D}_{\mathrm{Dif}}^+(V)$, and we let $\mathbf{D}_{\mathrm{Dif},n}^+(V) := K_n[\![t]\!] \otimes_{\iota_n(\mathbf{B}_K^{\dagger, r_n})} \iota_n(\mathbf{D}^{\dagger, r_n}(V))$. Proposition 5.7 of [**Ber02**] shows that $\mathbf{D}_{\mathrm{Dif}}^+(V) = K_\infty[\![t]\!] \otimes_{K_n[\![t]\!]} \mathbf{D}_{\mathrm{Dif},n}^+(V)$.

The fact that one could retrieve the modules $\mathbf{D}_{\mathrm{Dif}}^+(V)$ and $\mathbf{D}_{\mathrm{Dif},n}^+(V)$ using the theory of locally analytic vectors had already been noticed by Berger and Colmez [**BC16**, Rem. 3.3] and proven by Porat in [**Por20**, Prop. 3.3] and [**Por22**, Thm. 6.2] but we now explain how this incorporates into the setting laid out at the end of §3.

Note that $\mathbf{B}_{\mathrm{dR}}^+$, endowed with its natural topology, is not a Banach ring but a Fréchet ring, and as Berger points out in [**Ber16b**], locally analytic vectors in the setting of Fréchet spaces usually have to be replaced with the weaker notion of pro-analytic vectors, because the resulting objects are too small in general. However, in the setting of $\mathbf{B}_{\mathrm{dR}}^+$ and $\mathbf{D}_{\mathrm{Dif}}^+(V)$, locally analytic vectors are actually sufficient to recover the theory.

Lemma 4.8. — *We have* $(\mathbf{B}_{\mathrm{dR},K}^+)^{\Gamma_n-\mathrm{an}} = K_n[\![t]\!]$, $(\mathbf{B}_{\mathrm{dR},K}^+)^{\mathrm{la}} = \bigcup_n K_n[\![t]\!]$ *and* $(\mathbf{B}_{\mathrm{dR},K}^+)^{\mathrm{pa}} = K_\infty[\![t]\!]$.

Proof. — The second equality follows directly from the first one. For the first equality, take $x \in (\mathbf{B}_{\mathrm{dR},K}^+)^{\Gamma_n-\mathrm{an}}$. We have $\theta(x) \in \widehat{K_\infty}^{\Gamma_n-\mathrm{an}} = K_n$, so that we can write $x = x_0 + ty$, with $x_0 \in K_n$ and $y \in \mathbf{B}_{\mathrm{dR},K}^+$, and one checks that y is Γ_n-analytic because x, x_0 and t are. By induction, $x \in K_n[\![t]\!]$. Because $K_n \subset (\mathbf{B}_{\mathrm{dR},K}^+)^{\Gamma_n-\mathrm{an}}$ and because t is Γ_0-analytic, we have $K_n[\![t]\!] \subset (\mathbf{B}_{\mathrm{dR},K}^+)^{\Gamma_n-\mathrm{an}}$, which finishes the proof. □

Proposition 4.9. — *For* $n \gg 0$, *we have* $\mathbf{D}_{\mathrm{Dif},n}^+(V) = ((\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n-\mathrm{an}}$.

Proof. — Since $(\mathbf{B}_{\mathrm{dR},K}^+)^{\Gamma_n-\mathrm{an}} = K_n[\![t]\!]$, it suffices to prove that the elements of $\iota_n(\mathbf{D}^{\dagger, r_n}(V)$ are Γ_n-analytic for $n \gg 0$.

Let $m \geq 0$ be such that $\mathbf{D}^{\dagger, r_m}(V)$ has the right dimension, and let $e_1, \cdots, e_d$ be a basis of $\mathbf{D}^{\dagger, r_m}(V)$. We can see the elements of $\mathbf{D}^{\dagger, r_m}(V)$ as elements of $\mathbf{D}^{[r_m; r_m]}(V)$. By §2.1 of [**KR09**], these elements are Γ_n-analytic for $n \gg m$ big enough. A direct consequence of lemma 2.2 of [**BC16**] shows that if we let $u_i = \varphi^{n-m}(e_i)$, $1 \leq i \leq d$, then the u_i are Γ_m-analytic as elements of $\mathbf{D}^{[r_n; r_n]}(V)$, and we know that it is a basis of $\mathbf{D}^{\dagger, r_n}(V)$ (since $\varphi^*(\mathbf{D}^\dagger(V)) \simeq \mathbf{D}^\dagger(V)$) and thus of $\mathbf{D}^{[r_n; r_n]}(V)$. Therefore, $(\iota_n(u_1), \cdots, \iota_n(u_d))$ generates $\mathbf{D}_{\mathrm{Dif},n}^+(V)$, and forms a basis of Γ_n-analytic elements of $\mathbf{D}_{\mathrm{Dif},n}^+(V)$. □

Proposition 4.10. — *We have* $\mathbf{D}_{\mathrm{Dif}}^+(V) = ((\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K})^{\mathrm{pa}}$.

Proof. — This is proposition 3.3 of [**Por20**] and also follows from the previous proposition. □

In the Lubin-Tate case, one can also define a variant of $\mathbf{D}^+_{\mathrm{Dif}}(V)$, which we will denote by $\mathbf{D}^+_{\mathrm{Dif},F,\mathrm{LT}}(V)$, and which retrieves the K-Hodge Tate weights of V. We let $\mathbf{D}^+_{\mathrm{Dif},F,\mathrm{LT}}(V) := ((\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{H_K})^{F-\mathrm{pa}}$. Proposition 4.10 shows that this seems to be the good version for a generalization of $\mathbf{D}^+_{\mathrm{Dif}}(V)$ in the F-analytic Lubin-Tate case. We also let $\mathbf{D}^+_{\mathrm{Dif},F,\mathrm{LT},n}(V) := ((\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{H_K})^{\Gamma_n-F-\mathrm{an}}$.

We now study the case of definitions 3.6 and 3.9 applied to $B = \mathbf{B}^+_{\mathrm{dR}}$, endowed with its Fréchet topology.

Recall that by the discussion following lemma 3.4 of [**Ber16b**], there exists an element $t_\pi \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ such that $g(t_\pi) = \chi_\pi(g)t_\pi$ and $\varphi_q(t_\pi) = \pi t_\pi$ (in the cyclotomic case, we have $t_\pi = t$). Therefore, the element $x_K := t_\pi e^{-u_K}$ and all its powers are invariants under the action of $\mathcal{G}_{K_n}$. We have an inclusion $\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$ defined in [**Ber02**, §2] and we still denote by t_π the image of $t_\pi \in \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ through this map.

Proposition 4.11. — *We have* $(\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}} = K_n[\![x_K]\!]$ *and* $(\mathbf{B}_{\mathrm{dR}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}} = K_n(\!(x_K)\!)$.

Proof. — Since both K_n and x_K are invariant under the action of $\mathcal{G}_{K_n}$, it suffices to show the direct inclusions.

Recall that that there is a map $\theta : \mathbf{B}^+_{\mathrm{dR}} \to \mathbf{C}_p$ which is a surjective, $\mathcal{G}_K$-equivariant ring morphism and such that $\ker\theta$ is the maximal ideal of $\mathbf{B}^+_{\mathrm{dR}}$ and is generated by t_π. This induces a collection of maps $\theta : \mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n \to \mathbf{C}_p\{\{u_K\}\}_n$ given by

$$\theta(\sum_{k\geq 0} a_k u_K^k) = \sum_{k \geq 0} \theta(a_k) u_K^k.$$

and which are $\mathcal{G}_{K_n}$-equivariant surjective ring morphisms.

The same argument as in [**BC16**, Prop. 4.7] and theorem 2.7 of ibid. show that $\theta(y) \in (\mathbf{B}^n_{\mathrm{Sen}})^{\mathcal{G}_{K_n}} = K_n$ so that we can write $y = y_0 + ty_1$ with $y_0 \in K_n$ and $y_1 \in \mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n$. Using the fact that both y and $t_\pi e^{-u_K}$ are invariant under the action of $\mathcal{G}_{K_n}$ and that e^{-u_K} is invertible in $\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n$, we get that $y = y_0 + xy_1'$ with $y_0 \in K_n$ and $y_1 \in (\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$. Using the same procedure inductively, we obtain that $y \in K_n[\![x_K]\!]$.

For the case of $(\mathbf{B}_{\mathrm{dR}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$, it suffices to remark that if $y \in \mathbf{B}_{\mathrm{dR}}\{\{u_K\}\}_n$, then there exists $k \geq 0$ such that $x_K^k y \in (\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$ since x_K is invariant under the action of $\mathcal{G}_{K_n}$. □

Proposition 4.12. — *Let V be a p-adic representation of $\mathcal{G}_K$ of dimension d. For $n \gg 0$, the module $D_n^+(V) := (\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ is a free $K_n[\![x]\!]$-module of dimension d. Moreover, if we write an element δ of $\mathbf{B}^+_{\mathrm{dR}}\{\{u_K\}\}_n \otimes_{\mathbf{Q}_p} V$ as $\delta^{(0)} + \delta^{(1)}u + \cdots$, where the $\delta^{(i)}$ are elements of $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V$ then the map*

$$D_n^+(V) \to \mathbf{D}^+_{\mathrm{Dif},F,\mathrm{LT},n}(V)$$

given by $\delta \mapsto \delta^{(0)}$ is an isomorphism.

Proof. — This follows from proposition 4.9 in the cyclotomic case, and of the definition of $\mathbf{D}^+_{\mathrm{Dif},F,\mathrm{LT},n}(V)$ in the Lubin-Tate case, and of propositions 3.8 and 3.11. □

In particular, the rings $\mathbf{B}_{\mathrm{dR}}^+\{\{u_K\}\}_n$ allow us to compute the modules $\mathbf{D}_{\mathrm{Dif},F,\mathrm{LT},n}^+(V)$ in the spirit of Fontaine's strategy. Moreover, this shows that every p-adic representation of $\mathcal{G}_K$ is "$\mathbf{B}_{\mathrm{dR}}^+\{\{u_K\}\}$-admissible".

In general, when K_∞/K is any p-adic Lie extension, one could define a module $\mathbf{D}_{\mathrm{Dif},K}^+(V)$ in the same manner, taking the pro-analytic vectors of $(\mathbf{B}_{\mathrm{dR}}^+)^{H_K}$ for the action of Γ_K. The fact that this module has the same dimension as $\dim_{\mathbf{Q}_p} V$ follows from an unpublished result of Porat, and one could show in that case that the ring $\varprojlim_k \varinjlim_n (\mathcal{C}^{\mathrm{an}}(\Gamma_n,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+/t^k$ computes the said module.

5. (φ,Γ)-modules

Computations made by Berger in [**Ber16b**, §4, §8] show that classical cyclotomic (φ,Γ)-modules over the Robba ring $\mathbf{B}_{\mathrm{rig},K}^\dagger$ can be recovered by using pro-analytic vectors. We start by recalling Berger's results.

Given a p-adic representation V of $\mathcal{G}_K$, we let $\widetilde{\mathbf{D}}_{\mathrm{rig},K}^{\dagger,r}(V) = (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r}\otimes_{\mathbf{Q}_p} V)^{H_K}$. Recall that $\mathbf{D}_{\mathrm{rig},K}^{\dagger,r}(V)$ is the cyclotomic (φ,Γ)-module attached to V on $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$.

Proposition 5.1. — *We have*

1. $(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r_k})^{\mathrm{pa}} = \mathbf{B}_{\mathrm{rig},K,\infty}^{\dagger,r_k}$;
2. $\widetilde{\mathbf{D}}_{\mathrm{rig},K}^{\dagger}(V)^{\mathrm{pa}} = \mathbf{B}_{\mathrm{rig},K,\infty}^{\dagger}\otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger}\mathbf{D}_{\mathrm{rig},K}^{\dagger,r}(V)$;
3. *if D is a (φ,Γ)-module on $\mathbf{B}_{\mathrm{rig},K}^\dagger$ such that $\widetilde{\mathbf{D}}_{\mathrm{rig},K}^{\dagger}(V)^{\mathrm{pa}} = \mathbf{B}_{\mathrm{rig},K,\infty}^{\dagger}\otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} D$ then $D = \mathbf{D}_{\mathrm{rig},K}^{\dagger,r}(V)$.*

Proof. — The first item is item 3 of theorem 4.4 of [**Ber16b**], specialized in the cyclotomic case. The second item is item 2 of theorem 8.1 of [**Ber16b**]. For the last item, let M denote the base change matrix and P_1, P_2 denote the matrices of φ on D, $\mathbf{D}_{\mathrm{rig},K}^{\dagger,r}(V)$ respectively. There exists $n \gg 0$ such that $M \in \mathrm{GL}_d(\mathbf{B}_{\mathrm{rig},K,n}^\dagger)$, and the equation $M = P_2^{-1}\varphi(M)P_1$ implies that $M \in \mathrm{GL}_d(\mathbf{B}_{\mathrm{rig},K}^\dagger)$. □

In particular, taking the pro-analytic vectors of $\widetilde{\mathbf{D}}_{\mathrm{rig},K}^{\dagger,r}(V)$ allows us to recover the cyclotomic (φ,Γ)-module $\mathbf{D}_{\mathrm{rig},K}^{\dagger,r}(V)$.

5.1. Coadmissible modules over Fréchet-Stein algebras. — We recall the notion of a weak Fréchet-Stein structure on a locally convex K-algebra, following [**Eme17**, Def. 1.2.6]:

Definition 5.2. — For A a locally convex K-algebra, a weak Fréchet-Stein structure on A is the data of:

1. A sequence of locally convex topological K-algebras $\{A_n\}_{n\geq 1}$, such that each A_n is hereditarily complete.
2. For each $n \geq 1$, a continuous K-algebra homomorphism $A_{n+1} \longrightarrow A_n$ which is a BH-map (this is defined in [**Eme17**, Def. 1.1.13]) of convex vector spaces.

3. An isomorphism of locally convex topological K-algebras $A \simeq \varprojlim_n A_n$, where the inverse limit is taken with respect to the maps of item 2, such that each of the induced maps $A \longrightarrow A_n$ has dense image.

We can now define the notion of weak Fréchet-Stein and Fréchet-Stein algebras, as in [**Eme17**, Def 1.2.6 and 1.2.10]:

Definition 5.3. — We say that A is a weak Fréchet-Stein K-algebra if it admits a weak Fréchet-Stein structure, and we say that it is a Fréchet-Stein K-algebra if moreover for each $n \geq 1$, the K-algebra A_n is left Noetherian and the transition map $A_{n+1} \longrightarrow A_n$ is right flat.

Finally, we recall the notion of coadmissible modules over (weak) Fréchet-Stein algebras, following [**Eme17**, Def. 1.2.8].

Definition 5.4. — Given a weak Fréchet-Stein algebra A with weak-Fréchet-Stein structure $A \simeq \varprojlim_n A_n$, we say that a locally convex topological A-module is coadmissible if we may find the following data:

1. A sequence $\{M_n\}_{n\geq 1}$ such that for each $n \geq 1$, M_n is a finitely generated locally convex topological A_n-module.
2. An isomorphism of topological A_n-modules $A_n \widehat{\otimes}_{A_{n+1}} M_{n+1} \simeq M_n$ for each $n \geq 1$.
3. An isomorphism of topological modules $M \simeq \varprojlim_n M_n$, where the inverse limit is taken with respect to the transition maps induced by the isomorphisms of item 2.

The first two items of definition 5.4 say that the collection of modules (M_n) form a coherent sheaf for (A, A_n). Passing to global sections defines a functor H from the category of coherent sheaves for (A, A_n) to the category of coadmissible A-modules (relatively to the (A_n)).

Proposition 5.5. — *The functor H defined above is an equivalence of categories.*

Proof. — This is [**ST03**, Coro. 3.3]. □

For $n \geq 0$, let $r_n = p^{hn-1}(p-1)$, where h is such that the residue field of K is of cardinal $q = p^h$. We define compact intervals (I_n) by $I_0 = [0; r_0]$ and $I_n = [r_{n-1}; r_n]$ for $n \geq 1$. For any interval I, let $\widetilde{\mathbf{B}}^I$, $\widetilde{\mathbf{B}}_K^I$ and $\mathbf{B}_K^I$ be the rings defined in §1.3. The rings $\left(\widetilde{\mathbf{B}}^{I_n}\right)_{n\geq k}$ (resp. $\left(\widetilde{\mathbf{B}}_K^{I_n}\right)_{n\geq k}$) endow $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r_k}$ (resp. $\left(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r_k}\right)$) with a weak Fréchet-Stein structure, and the rings $\left(\mathbf{B}_K^{I_n}\right)_{n\geq k}$ endow $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_k}$ with a Fréchet-Stein structure. More generally, for any $r \geq 0$, the rings $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r}$ (resp. $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$) are naturally endowed with a weak Fréchet-Stein structure (resp. a Fréchet-Stein structure), by considering an increasing sequence $(p_n)_{n\geq 0}$ of real numbers such that $p_n \geq r$, $p_n \longrightarrow +\infty$ and by considering the corresponding rings for the intervals $[r; p_n]$.

This also allows us to endow the rings $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r}\{\{u_K\}\}_n$ with a weak Fréchet-Stein structure. Since the rings $\mathbf{B}_K^I\{\{u_K\}\}_n$ can be seen as rings of analytic functions defined over the product of an annulus (corresponding to I) by a disk (with center 0 and of radius

$|\pi|^{-n}$), they are noetherian and as above this allows us to endow the rings $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}\{\{u_K\}\}_n$ with a Fréchet-Stein structure.

The main theorem regarding coadmissible modules over Fréchet-Stein algebras is the following, which is theorem 1.2.11 of [**Eme17**] and recalls the main results from [**ST03**, §3]:

Theorem 5.6. — *Let A be a Fréchet-Stein algebra, and let $A \simeq \varprojlim_n A_n$ be a choice of weak Fréchet-Stein structure on A.*

1. *If M is a coadmissible A-module, and if $\{M_n\}_{n\geq 1}$ is an $\{A_n\}_{n\geq 1}$-sequence for which there is a topological isomorphism $M \simeq \varprojlim_n M_n$, then for each value of n, the natural map $A_n\widehat{\otimes}_A M \to M_n$ is an isomorphism. Consequently, the natural map $A \to \varprojlim_n A_n\widehat{\otimes}_A M$ is an isomorphism.*
2. *The full subcategory of the category of topological A-modules consisting of coadmissible A-modules is closed under passing to finite direct sums, closed submodules and Hausdorff quotient modules, and is abelian.*

5.2. Recovering the theory. — We now explain how to recover part of the usual (φ,Γ)-modules theory through the "admissibility for rings of periods" formalism. As in the constructions for $\mathbf{B}_{\mathrm{dR}}$ and $\mathbf{C}_p$, the rings $\varinjlim_n \mathcal{C}^{\mathrm{an}}(\Gamma_n,\mathbf{B})$ or equivalently the rings $\mathbf{B}\{\{u\}\}$, for B an LB space, are not endowed with an action of Γ_K but only with an action of its Lie algebra, so that if V is a p-adic representation of $\mathcal{G}_K$, the module $(\mathbf{B}\{\{u\}\}\otimes_{\mathbf{Q}_p} V)^{H_K}$ is only endowed with an operator ∇ coming from the infinitesimal action of Γ_K. In particular, the constructions laid out in this subsection can only allow us to recover the (φ,∇)-module attached to a representation V.

Lemma 5.7. — *Let V be a p-adic representation of $\mathcal{G}_K$, let I be a compact subinterval of $[0;+\infty[$ and let $\widetilde{D}^I(V) = (\tilde{\mathbf{B}}^I\otimes_{\mathbf{Q}_p} V)^{H_K}$. Then*

$$\widetilde{D}^I(V)^{\mathrm{la}} \simeq \bigcup_n (\tilde{\mathbf{B}}^I\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}.$$

as modules with connections, where the connection on the LHS comes from the action of the Lie algebra of Γ_K, and the connection on the RHS is given by $-\frac{d}{du}$.

Proof. — This is just proposition 3.8 applied to the ring $\tilde{\mathbf{B}}^I$. □

Proposition 5.8. — *Let V be a p-adic representation of $\mathcal{G}_K$ and let $r > 0$. The collection $(\bigcup_n(\tilde{\mathbf{B}}^I\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}})_{\min(I)\geq r}$ equipped with natural transition maps $\bigcup_n(\tilde{\mathbf{B}}^I\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \to \bigcup_n(\tilde{\mathbf{B}}^J\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ and Frobenius maps $\varphi : \bigcup_n(\tilde{\mathbf{B}}^I\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \to \bigcup_n(\tilde{\mathbf{B}}^{pI}\{\{u\}\}_n\otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ defines a (φ,∇)-module $\widetilde{\mathbf{D}}$ over $\varprojlim_I(\tilde{\mathbf{B}}^I\{\{u\}\}_n)^{\mathcal{G}_{K_n}} \simeq (\tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K})^{\mathrm{pa}}$, and we have $\widetilde{\mathbf{D}} \simeq \widetilde{D}^{\dagger,r}_{\mathrm{rig}}(V)$ as (φ,∇)-modules.*

Moreover, there exists a (φ,∇)-module $\mathbf{D}$ on $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ inside $\widetilde{\mathbf{D}}$ such that $\widetilde{\mathbf{D}} = \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}$, and if $\mathbf{D}'$ is a (φ,∇)-module on $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ such that $\widetilde{\mathbf{D}}^{\dagger}_{\mathrm{rig},K}(V)^{\mathrm{pa}} = \mathbf{B}^{\dagger}_{\mathrm{rig},K,\infty} \otimes_{\mathbf{B}^{\dagger}_{\mathrm{rig},K}} \mathbf{D}'$ then $\mathbf{D} = \mathbf{D}'$.

Proof. — Let $r > 0$. For $I \subset [r;+\infty[$ compact subinterval, we let $B_I^n = (\widetilde{\mathbf{B}}^I\{\{u\}\}_n)^{\mathcal{G}_{K_n}}$ and $B_I = \bigcup_n B_I^n$. We let $B_r = \varprojlim_I B_I$, where the inverse limit is taken over all compact subintervals of $[r;+\infty[$. We claim that $B_r = \varprojlim_I B_I$ is Fréchet-Stein with respect to the family (B_I).

Indeed, if $f \in B_r$ and if f_I denotes the image of f in $\varprojlim_I B_I$, then proposition 3.7 shows that $f_I(u) \mapsto f_I(0)$ gives an isomorphism between B_I and $(\widetilde{\mathbf{B}}^I_K)^{\mathrm{la}}$, so that the collection $(f_I(0))_I$ defines an element of $\varprojlim_I (\widetilde{\mathbf{B}}^I_K)^{\mathrm{la}} = (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K})^{\mathrm{pa}}$, so that the map $(f_I(u))_I \in B_r \mapsto (f_I(0))_I$ induces an isomorphism of Fréchet-Stein algebras between B_r and $(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K})^{\mathrm{pa}}$, relative to the structures given by $(B_I)_I$ and $((\widetilde{\mathbf{B}}^I_K)^{\mathrm{la}})_I$ respectively.

The fact that $\varphi : \bigcup_n(\widetilde{\mathbf{B}}^I\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}} \to \bigcup_n(\widetilde{\mathbf{B}}^{pI}\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}}$ is an isomorphism proves that the collection $(\bigcup_n(\widetilde{\mathbf{B}}^I\{\{u\}\}_n \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_{K_n}})_I$ is glued by Frobenius and therefore gives rise to a coadmissible module $\widetilde{\mathbf{D}}$ over $\varprojlim_I(\widetilde{\mathbf{B}}^I\{\{u\}\}_n)^{\mathcal{G}_{K_n}} \simeq (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K})^{\mathrm{pa}}$.

Since $(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K})^{\mathrm{pa}} = \mathbf{B}^{\dagger,r}_{\mathrm{rig},K,\infty}$, there exist elements $v_1,\cdots,v_d$ of $\widetilde{\mathbf{D}}$ and $n \gg 0$ such that $\mathbf{D} := \oplus_{i=1}^d \mathbf{B}^{\dagger,p^n r}_{\mathrm{rig},K} \cdot \varphi^n(v_i)$ generates $\widetilde{\mathbf{D}}$. The unicity of $\mathbf{D}$ follows from the same argument as in the proof of the last item of proposition 5.1. □

5.3. The anticyclotomic case. — Berger and Colmez have proven in [**BC16**] that the theory of locally analytic vectors is the right object to consider in order to generalize classical Sen theory to arbitrary p-adic Lie extensions. With that in mind, and considering the results above that show that in the cyclotomic (and in the F-analytic Lubin-Tate) case one recovers classical (φ,Γ)-modules theory, it seems reasonable to assume that the theory of locally analytic vectors is the right object to consider in order to generalize (φ,Γ)-modules to arbitrary p-adic Lie extensions.

It has already been noticed that, even in the Lubin-Tate case, "one dimensional (φ_q,Γ_K)-modules" do not behave well [**FX14**] and that the kind of objects one should consider are multivariable Lubin-Tate (φ_q,Γ_K)-modules [**Ber13**] which arise from locally analytic vectors [**Ber16b**].

Therefore, in general, one should expect to use that theory for arbitrary p-adic Lie extensions to get a theory of (φ_q,Γ_K)-modules over $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{\mathrm{pa}}$, and such that the functor $V \mapsto ((V \otimes_{\mathbf{Q}_p} \widetilde{\mathbf{B}}_{\mathrm{rig}})^{H_K})^{\mathrm{pa}}$ is a faithfully exact functor. We now give some insight as to why such a generalization does not seem to be true in general, using the anticyclotomic extension as a potential counterexample.

Let $F/\mathbf{Q}_p$ be the unramified extension of $\mathbf{Q}_p$ of degree 2. We take π to be equal to p in our Lubin-Tate setting. We let σ denote the Frobenius on F. Since $[p](T) \in \mathbf{Z}_p[T]$, the series $Q_k(T), \log_{\mathrm{LT}}(T)$ and $\exp_{\mathrm{LT}}(T)$ have all their coefficients in $\mathbf{Q}_p$, so that $t_\sigma = \varphi(t_p) = \log_{\mathrm{LT}}(u_\sigma)$.

Let $F_{\mathrm{cycl}} = F(\mu_{p^\infty})$ denote the cyclotomic extension of F. We let F_{ac} be the anticyclotomic extension of F: it is the unique $\mathbf{Z}_p$ extension of F, Galois over $\mathbf{Q}_p$, which is pro-dihedral: the Frobenius σ of $\mathrm{Gal}(F/\mathbf{Q}_p)$ acts on $\mathrm{Gal}(F_{\mathrm{ac}}/F)$ by inversion. It is linearly disjoint from F_{cycl} over F, and the compositum $F_{\mathrm{cycl}} \cdot F_{\mathrm{ac}}$ is equal to the Lubin-Tate extension attached to p by local class field theory. The anticyclotomic extension is then the subfield of F_∞ fixed under $G_\sigma := \{g \in \mathrm{Gal}(F_\infty/F) \ : \chi_p(g) = \sigma(\chi_p(g))\}$, and the cyclotomic extension of F is the one fixed by $G := \{g \in \mathrm{Gal}(F_\infty/F) \ : \chi_p(g) = (\sigma(\chi_p(g)))^{-1}\}$. We let $H_{F,ac}$ denote the group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F_{\mathrm{ac}})$, and if B is a ring of periods we let $B_{F,\mathrm{ac}}$ denote $B^{H_{F,ac}}$. We write t_1 for t_p and t_2 for t_σ.

Proposition 5.9. — *We have* $(\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}} = F_{\mathrm{ac}}[\![\frac{t_1}{t_2}]\!]$.

Proof. — Clearly, if $z \in F_{\mathrm{ac}}[\![\frac{t_1}{t_2}]\!]$, then the corresponding power series converges to an element of $\mathbf{B}^+_{\mathrm{dR}}$ which is invariant by H_F and pro-analytic for the action of Γ_F.

Now if $z \in (\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}}$, we have $\theta(z) \in \widehat{F_{\mathrm{ac}}}^{\mathrm{la}} = F_{\mathrm{ac}}$ by [**BC16**, Thm. 3.2]. We can therefore write $z = \theta(z) + t_1 \cdot z'$ in $\mathbf{B}^+_{\mathrm{dR}}$. Since $\frac{t_1}{t_2}$ belongs to $(\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}} \cap \mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}$, we can write $z = \theta(z) + \frac{t_1}{t_2} z_2$ with z_2 in $(\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}}$. Now we can do the same thing for z_2, and doing this inductively gives us the result. □

If I is big enough, so that the corresponding annulus contains a zero of t_1 and t_2, then the localization map at the zero of t_1 gives an embedding $(\widetilde{\mathbf{B}}^I_{F,ac})^{\mathrm{la}} \to (\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}} = F_{\mathrm{ac}}[\![\frac{t_1}{t_2}]\!]$, and it seems difficult for an element in $\widetilde{\mathbf{B}}^I_{F,ac}$ to have an "essential singularity at a zero of t_2", even if it's after a localization at a zero of t_1. Moreover, it is easy to prove that the image of $(\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}}$ in $\mathbf{B}^+_{\mathrm{dR}}$ does not intersect $K_\infty[\frac{t_1}{t_2}] \setminus F$ as soon as I is such that the corresponding annulus contains a zero of t_2. It seems therefore reasonable to expect that $(\mathbf{B}^+_{\mathrm{dR},F,ac})^{\mathrm{pa}} = F$, even though we do not have a proof of that statement.

Remark 5.10. — If I is small enough, for example if $I = [r_s; r_s]$, then t_2 is invertible in $\widetilde{\mathbf{B}}^I_{F,ac}$ and thus $(\widetilde{\mathbf{B}}^I_{F,ac})^{\Gamma_m-\mathrm{an}} \neq F$ since it contains non trivial elements such as $\frac{t_1}{t_2}$.

Let V be a Banach representation of a p-adic Lie group G. The fact that the functor $V \mapsto V^{\mathrm{la}}$ is exact is equivalent to the vanishing of the higher locally analytic vectors cohomology groups $R^i_{G-\mathrm{la}}(V)$ for $i \geq 1$. If V is a finite free module over a Banach G-ring B, endowed with a semilinear action of G, then the vanishing of the $R^i_{G-\mathrm{la}}(V)$ is equivalent to the vanishing of the $R^i_{G-\mathrm{la}}(B)$. In what follows, we show that for the anticyclotomic extension, the $R^1_{G-\mathrm{la}}(\widetilde{\mathbf{B}}^I_{F,ac})$ are nonzero when $0 \in I$. It still does not prove that the $R^1_{G-\mathrm{la}}(\widetilde{\mathbf{B}}^I_{F,ac})$ do not vanish for arbitrary I, but it highlights the fact that the anticyclotomic extension's behaviour in regards to taking locally analytic vectors is quite strange.

We let $P(T) = [p](T) = T^q + pT$.

Lemma 5.11. — *We have* $P^{\circ k}(\varphi_q^{-k}(u^p)) \to y_\sigma$ *in* $\widetilde{\mathbf{A}}^+$ *for the* p*-adic topology.*

Proof. — Let $s_k := P^{\circ k}(\varphi_q^{-k}(u^p)) \in \widetilde{\mathbf{A}}^+$, for all $k \geq 0$. We therefore have $s_0 = u$, and $s_{k+1} = \varphi_q^{-1}(P(s_k))$.

Let us assume that $s_k - s_{k-1}$ belongs to $p^b\widetilde{\mathbf{A}}^+$, with $b \geq 1$.

Then we have $s_{k+1} = \varphi_q^{-1}(P(s_k))$, and we can write

$$P(s_k) = P(s_{k-1}) + \sum_{j=1}^{q} P^{(j)}(s_{k-1})\frac{(s_k - s_{k-1})^j}{j!}.$$

Since $b \geq 1$ and since $P^{(j)}(T) \in p\mathcal{O}_F[\![T]\!]$, this means that $P(s_k) = P(s_{k-1}) + (s_k - s_{k-1})h_k$, with $h_k \in p\widetilde{\mathbf{A}}^+$. But then this means that

$$s_{k+1} - s_k = \varphi_q^{-1}(s_k - s_{k-1})\varphi_q^{-1}(h_k)$$

and thus $s_{k+1} - s_k \in p^{b+1}\widetilde{\mathbf{A}}^+$.

We already know that $s_1 - s_0 \in p\widetilde{\mathbf{A}}^+$ (because $\overline{s_1} = \overline{s_0} = \overline{u}^p \bmod p$) so that the sequence (s_k) converges in $\widetilde{\mathbf{A}}^+$ to an element we will denote by s.

Because both φ and θ are continuous for the p-adic topology, we know that $\theta \circ \varphi_q^{-j}(s) = \lim_{k \to +\infty} P^{\circ k}(\theta \circ \varphi_q^{-k}(u^p)) = P^{\circ k}(u_{j+k}^p)$. Therefore by lemma 5.3 of [**Ber16a**], s is such that $\theta \circ \varphi_q^{-j}(s) = \theta \circ \varphi_q^{-j}(y_\sigma)$ for all $j \in \mathbf{N}$, so that $s = y_\sigma$. □

In particular, in lemma 5.3 of [**Ber16b**], we can actually take x_n to be equal to $P^{\circ k}(\varphi_q^{-k}(u^p))$ for some $k \gg 0$. In what follows, we let $h_\ell(u) := P^{\circ \ell}(\varphi_q^{-\ell}(u^p))$.

Let $I = [0, r_0]$, let $m \geq 0$ and let $x \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$. Then there exists $n \geq 0$ such that $\|\partial_\sigma(x)\|_{\Gamma_m} \leq p^{nk}\|x\|_{\Gamma_m}$. Moreover, by [**BC16**, Lemm. 2.4], there exists $k_0 \geq m$ such that $\|x\|_{\Gamma_k} = \|x\|$ for all $k \geq k_0$. There exists $\ell \geq k_0$ such that $h_\ell(u) - y_\sigma \in p^n\widetilde{\mathbf{A}}^I$, and there exists $m' \geq \ell$ such that $h_\ell(u), y_\sigma \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_{m'}-\mathrm{an}}$ and such that $\|h_\ell(u) - y_\sigma\|_{\Gamma_s} \leq p^{-n}$ for all $s \geq m'$.

Then for $s \geq m'$, the series $x_i := \frac{1}{i!}\sum_{k \geq 0}(-1)^k\partial_\sigma^{i+k}(x)\frac{(y_\sigma - h_\ell(u))^k}{k!}$ converges in $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_s-\mathrm{an}}$, and we have

$$x = \sum_{i \geq 0} x_i(y_\sigma - h_\ell(u))^i$$

in $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_s-\mathrm{an}}$ (this is the same as the proof of theorem 5.4 of [**Ber16b**]).

Now let

$$X_{\ell,s} := \left\{ x \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_\ell-\mathrm{an}}, x = \sum_{i \geq 0} x_i(y_\sigma - h_\ell(u))^i \text{ and } x_i \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_s-F-\mathrm{an}} \right\}.$$

The above shows that any $x \in (\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ belongs to some $X_{\ell,s}$, $s \geq \ell \geq 0$.

Proposition 5.12. — *There is a Galois-equivariant map $\iota_{\ell,s} : X_{\ell,s} \to F[\![T_1, T_2]\!]$.*

Proof. — Let $x \in X_\ell$. We can write $x = \sum_{i \geq 0} x_i(y_\sigma - h_\ell(u))^i$, where $x_i \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_s-F-\mathrm{an}}$. Note that $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_s-F-\mathrm{an}} \subset \mathbf{B}_{K,s}^I$ by [**Ber16b**, Thm. 4.4] so that we can write $x_i = f_i(\varphi_q^{-s}(u))$, with $f_i \in F[\![u]\!]$.

We can write $\varphi_q^s(x) = \sum_{i \geq 0} f_i(u)(P^{\circ s}(y_\sigma) - P^{\circ s}(u^p))^i$, so that

$$\varphi_q^s(x) = \sum_{i \geq 0} f_i(u)\sum_{k=0}^{i}\binom{i}{k}(P^{\circ s}(y_\sigma))^k(-P^{\circ s}(u^p))^{k-i}$$

and this is equal to (if everything converges)

$$\sum_{k\geq 0}(P^{\circ s}(y_\sigma))^k\sum_{j\geq 0}f_{j+k}(u)(-P^{\circ s}(u^p))^j.$$

Let $A_k := \sum_{j\geq 0} f_{j+k}(u)(-P^{\circ s}(u^p))^j \in F[\![u]\!]$. This is a well defined element of $F[\![u]\!]$ since $P^{\circ s}(u^p) \in u\cdot F[\![u]\!]$ and since the $f_{j+k}(u)$ belong to $F[\![u]\!]$. Since $P^{\circ s}(y_\sigma) \in y_\sigma \cdot F[\![y_\sigma]\!]$ (because $s\geq \ell$), the sum $\sum_{k\geq 0}(P^{\circ s}(y_\sigma))^k A_k$ defines an element of $F[\![y_\sigma, u]\!]$. Now because $t_\sigma \in y_\sigma\cdot F[\![y_\sigma]\!]$ and $t_p \in F[\![u]\!]$, this can be rewritten as an element of $F[\![T_1,T_2]\!]$. It remains to check that the map we have just constructed is well defined relative to the Galois action, which is straightforward (because φ_q^{-s} is Γ_K-equivariant and then the rest is just rewriting power series in $F[\![Y_1,Y_2]\!] = F[\![T_1,T_2]\!]$). □

Corollary 5.13. — *We have* $(\tilde{\mathbf{B}}_F^{[0,r_0]})^{\mathrm{ac,la}} = F$.

Proof. — By the previous proposition, it suffices to prove that $F[\![T_1,T_2]\!]^{\nabla_1+\nabla_2=0} = K$, which is straightforward because

$$(\nabla_1+\nabla_2)(\sum_{i,j}a_{ij}T_1^iT_2^j) = \sum_{i,j}(i+j)a_{ij}T_1^iT_2^j$$

which is equal to 0 if and only if $a_{ij}=0$ for all $i,j\neq 0$. □

Corollary 5.14. — *We have* $R^1_{G-\mathrm{la}}(\tilde{\mathbf{B}}^I_{F,ac}) \neq 0$ *if* $0\in I$.

Proof. — Assume that $R^1_{G-\mathrm{la}}(\tilde{\mathbf{B}}^I_{F,ac}) = 0$. Then taking the locally analytic vectors in the exact sequence

$$0\to \ker(\theta : \tilde{\mathbf{B}}^I_{K,ac}\to \mathbf{C}_p)\to \tilde{\mathbf{B}}^I_{F,ac}\to \widehat{K_\infty}\to 0$$

gives us an exact sequence

$$0\to \ker(\theta : (\tilde{\mathbf{B}}^I_{K,ac})^{\mathrm{la}}\to \mathbf{C}_p)\to (\tilde{\mathbf{B}}^I_{F,ac})^{\mathrm{la}}\to \widehat{K_\infty}^{\mathrm{la}}\to R^1_{G-\mathrm{la}}(\tilde{\mathbf{B}}^I_{F,ac})$$

and thus the map $\theta : (\tilde{\mathbf{B}}^I_{F,ac})^{\mathrm{la}}\to\widehat{K_\infty}^{\mathrm{la}}$ is surjective, but $(\tilde{\mathbf{B}}^I_{F,ac})^{\mathrm{la}} = F$ by the above so this can't be true. □

6. Generalization to other p-adic Lie extensions

6.1. General results when K_∞ contains a cyclotomic extension. — The results from §5.3 highlight that in general, the rings $(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}}$ could be really small, even if we restrict ourselves to the case of p-adic abelian extensions. In this section, we show that if we assume that K_∞/K contains a cyclotomic extension, then most of those problems should disappear. Note that the case we considered of the anticyclotomic extension in which we proved that the rings $(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}}$ were trivial when $0\in I$ is precisely a case where we removed the cyclotomic extensions contained inside the Lubin-Tate extension. In particular, the author does not know what the answer to the following question is:

Question 1. — *Are there Galois p-adic Lie extensions K_∞/K almost totally ramified, not containing any cyclotomic extension, such that for all compact subinterval I of* $[0;+\infty[$, $(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \neq K$?

In what follows, K is a finite extension of $\mathbf{Q}_p$, K_∞/K is an finite Galois p-adic Lie extension, with $\dim \Gamma_K \geq 2$, and such that K_∞ contains a cyclotomic extension, in the sense that there exists an unramified character $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ such that $K_\infty \cap \overline{\mathbf{Q}}_p^{\eta\chi_{\mathrm{cycl}}}$ is infinitely ramified. We let K_∞^η denote the extension $K_\infty \cap \overline{\mathbf{Q}}_p^{\eta\chi_{\mathrm{cycl}}}$.

Recall that K_∞^η/K is the extension of K attached to $\eta\chi_{\mathrm{cycl}}$. Let $\Gamma_K' = \mathrm{Gal}(K_\infty^\eta/K)$. Let $\mathbf{B}_{K,\eta}^\dagger$, $\mathbf{B}_{K,\eta}^I$ and $\mathbf{B}_{\mathrm{rig},K,\eta}^\dagger$ be as in [**Ber16b**, §8]. By the same arguments as in [**Ber16b**, §8], there is an equivalence of categories between étale (φ, Γ_K')-modules over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{rig},K,\eta}^\dagger$ (it is also true over $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{K,\eta}^\dagger$) and E-representations of $\mathcal{G}_K$.

If V is a p-adic representation of $\mathcal{G}_K$, we let $\mathbf{D}_\eta^\dagger(V) := \bigcup_{r \gg 0} \mathbf{D}_\eta^{\dagger,r}(V)$, where $\mathbf{D}_\eta^{\dagger,r}(V) := (\mathbf{B}_\eta^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$. Let $\mathbf{D}_\eta^{[r;s]}$ and $\mathbf{D}_{\mathrm{rig},\eta}^\dagger(V)$ denote the various completions of $\mathbf{D}_\eta^{\dagger,r}(V)$. We let $\widetilde{\mathbf{D}}_\eta^{[r;s]}(V) = (\widetilde{\mathbf{B}}^{[r;s]} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$ and $\widetilde{\mathbf{D}}_{\mathrm{rig},\eta}^{\dagger,r}(V) = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{H_{K,\eta}}$. By the variant of the Cherbonnier-Colmez theorem for twisted cyclotomic extensions, we have that $\widetilde{\mathbf{D}}_\eta^{[r;s]}(V) = \widetilde{\mathbf{B}}_{K,\eta}^{[r;s]} \otimes_{\mathbf{B}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]}$ and $\widetilde{\mathbf{D}}_{\mathrm{rig},\eta}^{\dagger,r}(V) = \widetilde{\mathbf{B}}_{\mathrm{rig},K,\eta}^{\dagger,r} \otimes_{\mathbf{B}_{\mathrm{rig},K,\eta}^{\dagger,r}} \mathbf{D}_{\mathrm{rig},K,\eta}^{\dagger,r}(V)$.

Lemma 6.1. — *Let $r \geq 0$ be such that $\mathbf{D}_\eta^{\dagger,r}(V)$ has the right dimension, and let $s \geq r$. Then the elements of $\mathbf{D}_\eta^{[r;s]}(V)$ are locally analytic for the action of $\mathrm{Gal}(K_\infty^\eta/K)$.*

Proof. — See the proof of [**Ber16b**, Thm. 8.1]. □

Corollary 6.2. — *If V is a p-adic representation of $\mathcal{G}_K$ which factors through Γ_K, then the coefficients of the base change matrix in $\mathrm{GL}_d(\mathbf{B}^{\dagger,r})$ belong to $(\mathbf{B}_K^{[r;s]})^{\mathrm{la}}$ for any $s \geq r$.*

Proof. — Let V be such a p-adic representation of $\mathcal{G}_K$. Since V factors through Γ_K, the elements of $V = V^{H_K}$ are locally analytic vectors for the action of Γ_K. Now, we have

$$\widetilde{\mathbf{D}}_K^{[r;s]}(V)^{\mathrm{la}} = (\mathbf{B}_K^{[r;s]} \otimes_{\mathbf{Q}_p} V)^{\mathrm{la}}$$

since V factors through Γ_K, and thus

$$\widetilde{\mathbf{D}}_K^{[r;s]}(V)^{\mathrm{la}} = (\mathbf{B}_K^{[r;s]})^{\mathrm{la}} \otimes_{\mathbf{Q}_p} V$$

by proposition 2.3.

Since Γ_K contains $\Gamma_{K,\eta}$, lemma 6.1 and proposition 2.3 imply that

$$\widetilde{\mathbf{D}}_K^{[r;s]}(V)^{\mathrm{la}} = (\widetilde{\mathbf{B}}_K^{[r;s]})^{\mathrm{la}} \otimes_{\mathbf{B}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]},$$

so that with what we wrote above imply that we have the equality

$$(\widetilde{\mathbf{B}}_K^{[r;s]})^{\mathrm{la}} \otimes_{\mathbf{B}_{K,\eta}^{[r;s]}} \mathbf{D}_\eta^{[r;s]} = (\mathbf{B}_K^{[r;s]})^{\mathrm{la}} \otimes_{\mathbf{Q}_p} V$$

. In particular, this implies that the coefficients of the base change matrix in $\mathrm{GL}_d(\mathbf{B}^{\dagger,r})$ belong to $(\mathbf{B}_K^{[r;s]})^{\mathrm{la}}$. □

This corollary will prove very useful in order to produce locally analytic vectors for Γ_K in the rings $(\widetilde{\mathbf{B}}_K^I)$.

Remark 6.3. — Note that the fact that Γ_K contains K_∞^η is crucial for the proof of corollary 6.2 to work.

We now recall the following result, which is corollary 5.4 of [**Por22**]:

Proposition 6.4. — *If I is a compact subinterval of $[\frac{p}{p-1};+\infty[$, then the derived analytic cohomology groups $R^i_{\mathrm{la}}(\tilde{\mathbf{B}}^I_K)$ are zero for $i \geq 1$.*

Corollary 6.5. — *Let $I=[r_k;r_\ell]$. Then for any $m \in [k;\ell]$ integer, the map $\theta \circ \varphi_q^{-m}$: $(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \to \widehat{K_\infty}^{\mathrm{la}}$ is surjective.*

Proof. — We have an exact sequence

$$0 \to Q_k(u)\tilde{\mathbf{B}}^I_K \to \tilde{\mathbf{B}}^I_K \to K_\infty \to 0$$

which gives rise to the exact sequence

$$0 \to Q_k(u)(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \to (\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \to (K_\infty)^{\mathrm{la}} \to R^1_{\mathrm{la}}(\tilde{\mathbf{B}}^I_K) \to \cdots$$

because $(Q_k(u)(\tilde{\mathbf{B}}^I_K))^{\mathrm{la}} = Q_k(u)(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}}$. By the previous proposition, $R^1_{\mathrm{la}}(\tilde{\mathbf{B}}^I_K)=0$ so that we get the exact sequence

$$0 \to Q_k(u)(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \to (\tilde{\mathbf{B}}^I_K)^{\mathrm{la}} \to (K_\infty)^{\mathrm{la}} \to 0.$$

□

Recall that by theorem 6.2 of [**BC16**], $\widehat{K_\infty}^{\mathrm{la}}$ is a ring of power series in $d-1$ variables. Since in the case we consider K_∞ contains a cyclotomic extension, $\ker(\theta \circ \varphi_q^{-m})$ is a principal ideal generated by a locally analytic vector of $\tilde{\mathbf{B}}^I_K$. This (and the computations of the next section) makes us think that the following conjecture should hold:

Conjecture 6.6. — *If K_∞/K contains a cyclotomic extension, then the rings $(\tilde{\mathbf{B}}^I_K)^{\mathrm{la}}$ are the completion for the locally analytic topology of rings of power series in d variables.*

6.2. A particular case of the conjecture. — We now explain how to prove the conjecture in a very particular case, which is already nontrivial and is a generalization of the Kummer case.

In this section, we assume that K_∞/K is a p-adic Lie extension which is a successive extension of $\mathbf{Z}_p$-extensions: there exist $(K_{\infty,i})_{i\in\{0,\dots,d\}}$ such that for all i, $K_{\infty,i}/K$ is Galois, $K_\infty = K_{\infty,d}$, $K_{\infty,0}=K$, and $\mathrm{Gal}(K_{\infty,i+1}/K_{\infty,i}) \simeq \mathbf{Z}_p$. We also assume that there exists $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ an unramified character such that $K_{\infty,1} = K_\infty^\eta$. In particular, this implies that Γ_K is isomorphic to a semi-direct product $\mathbf{Z}_p \rtimes \cdot \rtimes \mathbf{Z}_p$. We write $g \mapsto (c_d(g),\dots,c_1(g))$ for the isomorphism $\Gamma_K \simeq \mathbf{Z}_p \rtimes \cdot \rtimes \mathbf{Z}_p$.

For any $i \in \{1,\cdots,d\}$, we let $g_i \in \mathrm{Gal}(K_\infty/K_{\infty,i-1})$ be such that $c_i(g)=1$, so that its image in the quotient $\mathrm{Gal}(K_{\infty,i}/K_{\infty,i-1})$ is a topological generator, and we let $\nabla_i \in \mathrm{Lie}(\Gamma_K)$ denote the operator corresponding to $\log g_i$. Since it is clear that the g_i generate Γ_K topologically, the operators ∇_i define a basis of the Lie algebra of Γ_K. We also let $\Gamma_i = \mathrm{Gal}(K_{\infty,i}/K)$.

Lemma 6.7. — *If x is a locally analytic vector of a p-adic Banach representation of Γ_K such that there exists $j \geq 2$, such that for all $k \geq j$, $\nabla_k(x)=0$, then for all $\ell < j$ and for all $k \geq j$, $\nabla_k \circ \nabla_\ell(x) = \nabla_\ell \circ \nabla_k(x) = 0$.*

Proof. — Let W be a p-adic Banach representation of Γ_K. Let x be a locally analytic vector of W which is killed by ∇_d. By definition of ∇_d, this implies that for some $n \gg 0$, we have $g_d^{p^k}(x) = x$, so that $x \in W^{\mathrm{Gal}(K_\infty/M)}$ for some finite extension M of $K_{\infty,d-1}$. By induction, if x is killed by ∇_k for all $k \geq j$, then $x \in W^{\mathrm{Gal}(K_\infty/M_j)}$ for some finite extension M_j of $K_{\infty,j-1}$, which we can assume to be Galois over K. But then $g_\ell(x) \in W^{\mathrm{Gal}(K_\infty/M_j)}$ for all $\ell < j$, so that $\nabla_\ell(x) = 0$. $\square$

Proposition 6.8. — *For any $i \in \{2, \cdots, d\}$, there exists $r_i \geq 0$ and $b_i \in \mathbf{B}_{K_{\infty,i}}^{\dagger,r_i}$ such that $(g_i - 1)(b_i) = 1$ and $b_i \in (\widetilde{\mathbf{B}}_{\mathrm{rig},K_{\infty,i}}^{\dagger,r_i})^{\mathrm{pa}}$.*

Proof. — We only prove it for $i = d$, the proof for $i < d$ is the same replacing Γ_K by Γ_i.

Let V denote the 2-dimensional p-adic representation of $\mathcal{G}_K$ given by

$$g \mapsto \begin{pmatrix} 1 & c_d(g) \\ 0 & 1 \end{pmatrix}$$

By the theorem of Cherbonnier-Colmez, the (φ, Γ)-module attached to V is overconvergent, so that it admits a basis on $(\mathbf{B}_K^\eta)^{\dagger,r}$. If (e_1, e_2) was the basis of V giving rise to the matrix representation above, we see that a basis of the attached (φ, Γ)-module on $(\mathbf{B}_K^\eta)^{\dagger,r}$ is given by $(e_1 \otimes 1, e_2 \otimes 1 - e_1 \otimes b)$ for some $b \in \mathbf{B}_K^{\dagger,r}$. The fact that this basis is invariant by the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\infty,1})$ means that it also is invariant by the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K_{\infty,d-1})$ and thus we get that $g_d(b) = b + c_d(g_b) = b + 1$ by our choice of g_d. $\square$

We let $r_b = \max(r_i)$ so that the (b_i) can all be seen as elements of $(\widetilde{\mathbf{B}}_K^{\dagger,r_b})$.

Recall that if M_∞^η is a finite extension of K_∞^η then there corresponds a finite unramified extension $\mathbf{B}_{M,\eta}^\dagger/\mathbf{B}_{K,\eta}^\dagger$ of degree $[M_\infty^\eta : K_\infty^\eta]$, and there exists $r(M) > 0$ and elements $x_1, \ldots, x_f$ in $\mathbf{A}_{M,\eta}^{\dagger,r(M)}$ such that $\mathbf{A}_{M,\eta}^{\dagger,s} = \oplus_{i=1}^f \mathbf{A}_{K,\eta}^\dagger \cdot x_i$ for all $s \geq r(M)$.

Lemma 6.9. — *Let $M_\infty^\eta \subset K_\infty$ be a finite extension of K_∞^η. If $r_\ell \geq r(M)$ then the x_i defined above belong to $(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger,r_\ell})^{\mathrm{pa}}$ and are killed by ∇_i for all $i > 1$.*

Proof. — The fact that the x_i are pro-analytic is a consequence of the proof of item 2 of [**Ber16b**, Thm. 4.4]. The second part is straightforward as $M_\infty^\eta/K_\infty^\eta$ is finite. $\square$

If K is a finite extension of F then by the theory of the field of norms (see [**Win83**]), there corresponds to K/F a separable extension $\mathbf{E}_K/\mathbf{E}_F$, of degree $[K_\infty : F_\infty]$. Since $\mathbf{B}_F^\dagger$ is a Henselian field, there exists a finite unramified extension $\mathbf{B}_K^\dagger/\mathbf{B}_F^\dagger$ of degree $f = [K_\infty : F_\infty]$ whose residue field is $\mathbf{E}_K$ (see §2 and §3 of [**M⁺95**]). There exist therefore $r(K) > 0$, where $\mathbf{A}_K^{\dagger,s}$ is the ring of integers of $\mathbf{B}_K^{\dagger,s}$ for $V(\cdot, s)$,

If M_∞^η is a finite extension of K_∞^η, and if I is a compact subinterval of $[0; +\infty[$ such that $\min(I) \geq r(M)$, we let $\mathbf{A}_{M,\eta}^I$ be the completion of $\mathbf{A}_{M,\eta}^{\dagger,r(M)}$ for $V(\cdot, I)$.

Lemma 6.10. — *If $x \in \mathbf{A}_K^{\dagger,r}$ and if $k, n \in \mathbf{N}$ then there exists M_∞^η a finite extension of K_∞^η, $m \geq 0$ and $y \in \varphi^{-m}(\mathbf{A}_{M,\eta}^{\dagger,p^m r})$ such that $x - y \in \pi^j \widetilde{\mathbf{A}}^{\dagger,r} + u^k \widetilde{\mathbf{A}}^+$.*

Proof. — By reducing mod π, we obtain that $\overline{x} \in \mathbf{E}_K$. But $\mathbf{E}_K = \bigcup \mathbf{E}_M$ where M goes through the set of finite extensions of $K_{\infty,\eta}$ contained in K_∞. In particular, there exists a finite extension M_0 of $K_{\infty,\eta}$, contained in K_∞, and $y_0 \in \mathbf{A}_{M_0}$ such that $x - y_0 \in p\mathbf{A}_K$, since $\mathbf{A}_{M_0,\eta} \subset \mathbf{A}_K$. Since $\frac{x-y_0}{p} \in \mathbf{A}_K$, the same arguments show that there exists a finite extension M_1 of $K_{\infty,\eta}$, contained in K_∞, and $y_1 \in \mathbf{A}_{M_1,\eta}$ such that $\frac{x-y_0}{p} - y_1 \in \pi\mathbf{A}_K$, so that $x - y_0 - py_1 \in \pi\mathbf{A}_K$, and we can without loss of generality assume that $M_0 \subset M_1$.

By induction, we find $y_0, y_1, \cdots, y_n$ in $\mathbf{A}_{M_n,\eta}$, with M_n finite extension of $K_{\infty,\eta}$ contained in K_∞, such that $x - y_0 - py_1 - \cdots - p^n y_n \in \pi^{n+1}\mathbf{A}_K$. Let $z_n = y_0 + \cdots + \pi^n y_n$. Let $\sum_{i\geq 0} p^i[x_i]$ be the way x is written in $\widetilde{\mathbf{A}}_K = W(\widetilde{\mathbf{E}}_K)$. Then $x^{(n)} := \sum_{i=0}^n p^i[x_i]$ is such that $x - x^{(n)} \in \pi^{n+1}\widetilde{\mathbf{A}}_K$, and thus $x^{(n)} - z_n \in \pi^{n+1}\widetilde{\mathbf{A}}_K$. In particular, since $z_n \in \widetilde{\mathbf{A}}_{M_n,\eta}$ by construction, we deduce that the x_i all belong to $\widetilde{\mathbf{E}}_{M_n,\eta}$ for $i \leq n$, and thus $x^{(n)} \in \widetilde{\mathbf{A}}_{M_n,\eta}$.

Since $x \in \widetilde{\mathbf{A}}_K^{\dagger,r}$, we have in particular that $x^{(n)} \in \widetilde{\mathbf{A}}_K^{\dagger,r}$. By corollary 8.11 of [**Col08a**], $\mathbf{A}^{\dagger,r}_{M_n,\infty,\eta}$ is dense in $\widetilde{\mathbf{A}}^{\dagger,r}_{M_n,\eta}$ for the topology induced by $V(\cdot,r)$, so that we can find $y \in \mathbf{A}^{\dagger,r}_{M_n,\infty,\eta}$ such that $x^{(n)} - y \in \pi^n\widetilde{\mathbf{A}}^{\dagger,r} + u^k\widetilde{\mathbf{A}}^+$. We thus have $x - y = (x - x^{(n)}) + (x^{(n)} - y) \in \pi^n\widetilde{\mathbf{A}}^{\dagger,r} + u^k\widetilde{\mathbf{A}}^+$. □

Lemma 6.10 shows that for any $I = [r;s]$ with $r \geq r_b$, and any integer n we can find elements b_n^ℓ such that $b_\ell - b_n^\ell \in p^n\widetilde{\mathbf{A}}^I$ for all $\ell \in \{2,\dots,d\}$, which by lemma 6.9 belong to $(\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ and are killed by ∇_j, for all $j \in \{2,\dots,d\}$. Since they are locally analytic vectors, we let $m = m(n,I)$ be such that all these elements belong to $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$.

Proposition 6.11. — *Let $I = [r;s]$ with $r \geq r_b$. Let $\ell \in \{2,\dots,d\}$ and let $x \in (\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ be such that for all $k > \ell$, $\nabla_k(x) = 0$. Then there exist $(x_j)_{j\in\mathbf{N}} \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$ such that $\|x_j p^{nj}\| \to 0$, for all $k \geq \ell, \nabla_k(x_j) = 0$ and $x = \sum_{j\geq 0} x_j(b_\ell - b_n^\ell)^j$.*

Proof. — Let $m \geq 1$ be such that $x \in (\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$. By [**BC16**, Lemm. 2.6], there exists $n \geq 1$ such that for all $j \geq 1$, $\|\nabla_\ell^j(x)\|_{\Gamma_m} \leq p^{nj}\|x\|$ for all $\ell \in \{2,\dots,d\}$. Up to increasing m, we can assume that $m \geq m(n,I)$. Let

$$x_j = \frac{1}{j!}\sum_{k\geq 0}(-1)^k\frac{(b_\ell - b_n^\ell)}{k!}\nabla_\ell^j(x).$$

Similarly to the proof of [**Ber16b**, Thm. 5.4], the series converges in $(\widetilde{\mathbf{B}}_K^I)^{\Gamma_m-\mathrm{an}}$ to an element x_j such that $\nabla_\ell(x_j) = 0$. Moreover, by construction of the b_ℓ and b_n^ℓ, we have $\nabla_k(b_\ell - b_n^\ell) = 0$ for all $k > \ell$, and thus using lemma 6.7, the x_j are killed by ∇_k, $k > \ell$. □

Using proposition 6.11 inductively, we obtain the following theorem for the structure of locally analytic vectors in our particular case:

Theorem 6.12. — *Let K_∞/K be a p-adic Lie extension which is a successive extension of $\mathbf{Z}_p$-extensions over a cyclotomic extension. Then the rings $(\widetilde{\mathbf{B}}_K^I)^{\mathrm{la}}$ are the completion for the locally analytic topology of rings of power series in d variables.*

7. Locally analytic vectors for Robba rings

In this section we give an example of why locally analytic vectors are usually not the right object to consider when working with Fréchet rings. We study the case of locally analytic vectors in some Robba rings and show that we surprisingly recover objects that were defined by Colmez using completely different methods in [**Col14**].

7.1. Locally analytic vectors in Robba rings and the corresponding rings $B\{\{u\}\}$. — We now study the ring $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\{\{u\}\}$.

If T is a variable and L is a finite extension of $\mathbf{Q}_p$, we let $L\langle\langle T\rangle\rangle$ denote the set of power series in T with coefficients in L and with infinite radius of convergence.

Proposition 7.1. — *We have* $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K})^{F-\mathrm{la}} = (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{F-\mathrm{la}} = K\langle\langle t_\pi\rangle\rangle$.

Proof. — Let $r \geq 0$ and let $z \in (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})^{F-\mathrm{la}}$. It is therefore Γ_n-analytic for some $n \geq 0$, so that for any $s \geq r$, z is a Γ_n-F-analytic vector of $\widetilde{\mathbf{B}}^{[r;s]}$. By item 1. of [**Ber16b**, Thm. 4.4], we have that the images of z in $\widetilde{\mathbf{B}}^{[r;s]}$ all belong to $\mathbf{B}^{[r;s]}_K$ as long as s is such that $r_n \leq s$. Taking the inverse limit, this implies that $z \in \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$.

Since φ commutes with the Galois action, the reasoning above also applies to $\varphi_q^{-1}(z)$. Therefore, for all $k \geq 0$, $\varphi_q^{-k}(z) \in \mathbf{B}^{\dagger}_{\mathrm{rig},K}$. This implies that z belongs to the ring $K\langle\langle t_\pi\rangle\rangle$. Indeed, in the cyclotomic case this is [**Col14**, Prop. 3.9], and for the general case this follows from the same arguments, using the dictionary developped by Colmez in the Lubin-Tate case in [**Col16**, §2].

To finish the proof, it suffices to notice that any element of $K\langle\langle t_\pi\rangle\rangle$ is indeed locally analytic (and is actually Γ_0-analytic). □

Proposition 7.1 already shows that the set of F-analytic vectors of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}$ is really small compared with the set of F-pro-analytic vectors of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}$.

As in the de Rham case, we let $x_K = e^{-u_K}t_\pi \in (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{F-\mathrm{la}}\{\{u_K\}\}$.

Proposition 7.2. — *For any* $r \geq 0$, *we have* $(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}} = K\langle\langle x_K\rangle\rangle$.

Proof. — This follows from the previous proposition and from the proof of proposition 3.7 which extends to locally analytic vectors for a Fréchet ring. □

In order to simplify the notations, we let $\mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K} := \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}\{\{u_K\}\}_n$ and $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K} := \bigcup_{n\geq 1}\mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K}$.

Proposition 7.3. — *The rings* $((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})\{\{u_K\}\}_n)^{\nabla=0}$ *are principal.*

Proof. — We first prove that $((\widetilde{\mathbf{B}}^{I}_{K})^{K-\mathrm{la}})\{\{u_K\}\}_n)^{\nabla=0} = \left\{\sum_{k=0}^{\infty}\frac{(-1)^k\nabla^k(z)}{k!}u_K^k, z \in \varphi_q^{-n}(\mathbf{B}^{q^nI}_K)\right\}$.

Let $y \in ((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})\{\{u_K\}\}_n)^{\nabla=0}$. Recall [**Ber02**, §2.2] that there is a $\mathcal{G}_K$-equivariant injective map $\iota_k : \widetilde{\mathbf{B}}^{I}_{K} \to \mathbf{B}^{+}_{\mathrm{dR}}$ which induces an injective map $\iota_k : ((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})\{\{u_K\}\}_n \to (\mathbf{B}^{+}_{\mathrm{dR}})^{H_K}\{\{u_K\}\}_n$.

The image of y in $(\mathbf{B}^{+}_{\mathrm{dR}})^{H_K}\{\{u_K\}\}_n$ is therefore pro-analytic and killed by $\nabla = \frac{\log g}{\log(\chi_\pi(g))}$ for g close enough to 1, so that it is invariant by Γ_N for $N \gg 0$. By proposition 4.11, this implies that the image of y in $\mathbf{B}^{+}_{\mathrm{dR}}\{\{u_K\}\}_n$ belongs to $K_N[\![x_K]\!]$, and we can even say that it belongs to $K_m[\![x_K]\!]$ since ι_k is Γ_m-equivariant. Let us write $\iota_k(y) = \sum_{k=0}^{\infty} b_k x_K^k$ with $b_k \in K_k$. We also know that $\iota_k(y) \in \mathbf{B}^{+}_{\mathrm{dR}}\{\{u_K\}\}_n$ can be written as $\iota_k(y) = \sum_{k\geq 0} a_k u_K^k$ with the $a_k \in \mathbf{B}^{+}_{\mathrm{dR}}$. The equality $\sum_{k=0}^{\infty} b_k x_K^k = \sum_{k=0}^{\infty} a_k u_K^k$ tells us that $a_k = \frac{(-1)^k}{k!}\sum_{j=0}^{\infty} t_\pi^j b_j j^k$.

Let $\alpha := \sum_{j=0}^{\infty} t_\pi^j b_j$. Since ∇ acts on $K_k[\![x_K]\!]$ by $t_\pi\frac{d}{dx_K}$, we can write $a_\ell = \frac{(-1)^\ell}{\ell!}\nabla^\ell(\alpha)$. Let us write $y = \sum_{\ell=0}^{\infty}\beta_\ell u_K^\ell$ in $((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})\{\{u_K\}\}_n$. By definition of ι_k, we get that β_0 is such that $\iota_k(\beta_0) = \alpha$, and since it is Γ_k-equivariant, we obtain that $\beta_j = \frac{(-1)^j}{j!}\nabla^j(\beta_0)$ in $((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})$. But by definition of $((\widetilde{\mathbf{B}}^{I}_{K})^{F-\mathrm{la}})\{\{u_K\}\}_n$, this implies that the series

$\sum_{j\geq 0}\frac{(-1)^j}{j!}\pi^{nj}\nabla^j(\beta_0)$ converges in $(\widetilde{\mathbf{B}}_K^I)^{F-\mathrm{la}}$, so that β_0 is a Γ_n-analytic vector. By [**Ber16b**, Thm. 4.4], this implies that $\beta_0 \in \varphi_q^{-n}(\mathbf{B}_K^{q^nI})$.

For the other direction, if z is in $\varphi_q^{-n}\mathbf{B}_K^{q^nI}$, then it is Γ_n-F-analytic by [**Ber16b**, Prop. 4.1], so that the series $\sum_{j\geq 0}\frac{(-1)^j}{j!}\pi^{nj}\nabla^j(z)$ converges in $\widetilde{\mathbf{B}}_K^I$ and thus $y := \sum_{j\geq 0}\frac{(-1)^j}{j!}\nabla^j(z)u_K^j$ belongs to $((\widetilde{\mathbf{B}}_K^I)^{F-\mathrm{la}})\{\{u_K\}\}_n)$, and a simple computation shows that $\nabla(y)=0$.

This implies that the map $f(u_K)\mapsto f(0)$, where $f(u_K)=\sum_{k=0}^{\infty}a_ku_K \in ((\widetilde{\mathbf{B}}_K^I)^{F-\mathrm{la}})\{\{u_K\}\}_n)^{\nabla=0}$, induces an isomorphism of K-algebras between $((\widetilde{\mathbf{B}}_K^I)^{F-\mathrm{la}})\{\{u_K\}\}_n)^{\nabla=0}$ and $\varphi_q^{-n}(\mathbf{B}_K^{q^nI})$, and thus concludes the proof. □

Lemma 7.4. — *The natural map $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}\to\mathbf{B}_K^I\{\{u_K\}\}_n$ is faithfully flat.*

Proof. — This map is clearly injective, and since $\mathbf{B}_K^I\{\{u_K\}\}_n$ is an integral domain, $\mathbf{B}_K^I\{\{u_K\}\}_n$ is torsion free over $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$ which is a PID by corollary 7.3. Therefore, the map $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}\to\mathbf{B}_K^I\{\{u_K\}\}_n$ is flat. To show that it is faithfully flat, it therefore suffices to prove that for any maximal ideal $\mathfrak{m}$ of $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$, $\mathfrak{m}\mathbf{B}_K^I\{\{u_K\}\}_n \neq \mathbf{B}_K^I\{\{u_K\}\}_n$. But if $\mathfrak{m}$ is a maximal ideal of $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$, then by the proof of proposition 7.3 it is generated by an element f of $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}\simeq \varphi_q^{-n}(\mathbf{B}_K^{q^nI})$ which corresponds through this isomorphism to an element of $\varphi_q^{-n}(\mathbf{B}_K^{q^nI})$ which vanishes on some point z of the corresponding annulus. But then every element of $\mathfrak{m}\mathbf{B}_K^I\{\{u_K\}\}_n$ will vanish on $(z,0)$ after identifying $\mathbf{B}_K^I\{\{u_K\}\}_n$ with the set of analytic functions on the product of the annulus corresponding to I with the disk $D(0,|\pi|^n)$, so that $\mathfrak{m}\mathbf{B}_K^I\{\{u_K\}\}_n \neq \mathbf{B}_K^I\{\{u_K\}\}_n$. □

Proposition 7.5. — *Let V be an F-analytic E-representation of K. Then*

$$(\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}\{\{u_K\}\}_n\otimes_E V)^{\mathcal{G}_{K_n}} = (\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)\{\{u_K\}\})^{\Gamma_{K_n}}$$

and

$$\bigcup_{n\geq 1}(\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}\{\{u_K\}\}_n\otimes_E V)^{\mathcal{G}_{K_n}} = (\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)\{\{u_K\}\})^{\nabla=0}.$$

Proof. — We have

$$(\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}\{\{u_K\}\}_n\otimes_E V)^{\mathcal{G}_{K_n}} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}\{\{u_K\}\}_n\otimes_E \mathbf{D}_{\mathrm{rig}}^{\dagger}(V))^{\mathcal{G}_{K_n}}$$

since $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ contains the periods of F-analytic representations used to define the (φ_q,Γ_K)-modules attached to such representations. Taking the invariants under H_K, we get that

$$(\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}\{\{u_K\}\}_n\otimes_E V)^{\mathcal{G}_{K_n}} = (\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger}\{\{u_K\}\}_n\otimes_E \mathbf{D}_{\mathrm{rig}}^{\dagger}(V))^{\Gamma_{K_n}}.$$

Since the elements of $(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^{\dagger}\{\{u_K\}\}_n\otimes_E \mathbf{D}_{\mathrm{rig}}^{\dagger}(V))^{\Gamma_{K_n}}$ are fixed by Γ_{K_n} they are in particular pro-analytic vectors of Γ_K. By proposition 2.3 and by the fact that $\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ is composed of pro-analytic vectors of Γ_K (this follows from example from [**Ber16b**, Thm.

10.4]), we get using [**Ber16b**, Thm. 4.4] that

$$(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}\{\{u_K\}\}_n \otimes_E \mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\Gamma_{K_n}} = \left(\bigcup_{m\geq 1} \varphi_q^{-m}(\mathbf{B}^{\dagger}_{\mathrm{rig},K}\{\{u_K\}\}_n) \otimes \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\right)^{\Gamma_{K_n}}.$$

Since the action of φ_q and Γ_{K_n} commute and since $\varphi_q^*\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \simeq \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, we obtain that

$$(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},K}\{\{u_K\}\}_n \otimes_E \mathbf{D}^{\dagger}_{\mathrm{rig}}(V))^{\Gamma_{K_n}} = \bigcup_{m\geq 0} (\varphi_q^{-m}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\Gamma_{K_n}}.$$

Using once again that $\varphi_q^*\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \simeq \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, we get that

$$\bigcup_{m\geq 0} (\varphi_q^{-m}(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\Gamma_{K_n}} = \bigcup_{m\geq 0} (\varphi_q^{-m}(\mathbf{B}^{\dagger}_{\mathrm{rig},K})^{\Gamma_{K_n}} \otimes_{K\langle\langle x_K\rangle\rangle} (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\Gamma_{K_n}}$$

and this is equal to

$$K\langle\langle x_K\rangle\rangle \otimes_{K\langle\langle x_K\rangle\rangle} (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\Gamma_{K_n}} = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\Gamma_{K_n}}.$$

For the second point of the proposition, it suffices to see that an element z in a pro-F-analytic representation of Γ_K is killed by ∇ if and only if there exists $m \gg 0$ such that z is invariant under Γ_m. □

In particular, the module $\bigcup_{n\geq 1}(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}}$ can be thought of as the module of solutions of the differential operator ∇ over a ring of power series in two variables, defined over a polyannulus (here a product between a disk and an annulus). In the cyclotomic case for example, the operator on the ring of power series in two variables X and Y is given by $(1+X)\cdot\log(1+X)\cdot\frac{d}{dX} + \frac{d}{dY}$.

If V is an F-analytic E-representation of $\mathcal{G}_K$, if $r \geq 0$ is such that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and all its structures are defined over $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and if I is a compact subinterval of $[r,+\infty[$, we let $\mathcal{D}^I_n(V) := (\mathbf{B}^I_K \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n)^{\nabla=0}$.

Proposition 7.6. — *Let V be an F-analytic E-representation of $\mathcal{G}_K$, let $r \geq 0$ be such that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and all its structures are defined over $\mathbf{B}^{\dagger,r}_{\mathrm{rig},K}$ and let I be a compact subinterval of $[r,+\infty[$. Then $\mathcal{D}^I_n(V)$ is a free $(\mathbf{B}^I_K\{\{u_K\}\}_n)^{\nabla=0}$-module of finite type.*

Proof. — Let $\mathcal{M}^I_n \subset \mathbf{B}^I_K \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n$ denote the image of the map

$$\alpha : \mathbf{B}^I_K\{\{u_K\}\}_n \otimes_{\mathbf{B}^I_K\{\{u_K\}\}_n^{\nabla=0}} \mathcal{D}^I_n(V) \to \mathbf{B}^I_K \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n$$

deduced from the inclusion $\mathcal{D}^I_n(V) \subset \mathbf{B}^I_K \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n$. Since $\mathbf{B}^I_K\{\{u_K\}\}_n$ is noetherian, and since $\mathbf{B}^I_K \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n$ is free of rank $\dim_E(V)$, we know that $\mathcal{M}^I_n$ is finitely generated.

In order to prove that α is injective, note that it suffices to prove that

$$\alpha' : \mathrm{Frac}(\mathbf{B}^I_K\{\{u_K\}\}_n) \otimes_{\mathbf{B}^I_K\{\{u_K\}\}_n^{\nabla=0}} \mathcal{D}^I_n(V) \to \mathrm{Frac}(\mathbf{B}^I_K) \otimes_{\mathbf{B}^{\dagger,r}_{\mathrm{rig}}} \mathbf{D}^{\dagger,r}_{\mathrm{rig}}(V))\{\{u_K\}\}_n$$

deduced from the inclusion $\mathcal{D}_n^I(V) \subset \mathrm{Frac}(\mathbf{B}_K^I) \otimes_{\mathbf{B}_{\mathrm{rig}}^{\dagger,r}} \mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V))\{\{u_K\}\}_n$ is injective, because the natural map

$$\mathbf{B}_K^I\{\{u_K\}\}_n \otimes_{\mathbf{B}_K^I\{\{u_K\}\}_n^{\nabla=0}} \mathcal{D}_n^I(V) \to \mathrm{Frac}(\mathbf{B}_K^I\{\{u_K\}\}_n) \otimes_{\mathbf{B}_K^I\{\{u_K\}\}_n^{\nabla=0}} \mathcal{D}_n^I(V)$$

is injective since $\mathcal{D}_n^I(V)$ is flat over $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$ (it is torsion free over $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$ which is a PID by corollary 7.3).

In order to prove that α' is injective, we are reduced to check that, given a family of elements $(u_1, \cdots, u_d)$ linearly independent over $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$, they are linearly independent over $\mathrm{Frac}(\mathbf{B}_K^I\{\{u_K\}\}_n)$. We prove the result by induction on d. The result is trivial when $d = 1$ because $(\mathrm{Frac}(\mathbf{B}_K^I) \otimes_{\mathbf{B}_{\mathrm{rig}}^{\dagger,r}} \mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V))\{\{u_K\}\}_n$ is free over $\mathrm{Frac}(\mathbf{B}_K^I\{\{u_K\}\}_n)$. Assume now that $d \geq 2$ is such that the result holds for $d-1$. Let $\lambda_1, \cdots, \lambda_d$ in $\mathrm{Frac}(\mathbf{B}_K^I\{\{u_K\}\}_n)$ be such that $\sum_{i=1}^d \lambda_i u_i = 0$. By induction, we may assume that $\lambda_1 \neq 0$. Dividing by λ_1, we obtain that $u_1 + \sum_{i=2}^d \lambda_i' u_i = 0$ where $\lambda_i' = \frac{\lambda_i}{\lambda_1}$. Applying ∇, we get that $\sum_{i=2}^d \nabla(\lambda_i')u_i = 0$ so that the λ_i' are all zero by induction.

This now implies that α is injective, so that $\mathbf{B}_K^I\{\{u_K\}\}_n \otimes_{(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}} \mathcal{D}_n^I(V)$ is finitely generated. By lemma 7.4 and descent for faitfully flat modules, we deduce that $\mathcal{D}_n^I(V)$ is finitely generated, and thus free since it is torsion free over $(\mathbf{B}_K^I\{\{u_K\}\}_n)^{\nabla=0}$ which is a PID. □

Corollary 7.7. — *If V is an F-analytic E-representation of $\mathcal{G}_K$, then $\mathcal{D}(V) = \bigcup_{n\geq 1}(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}}$ is a free $E\langle\langle x_K\rangle\rangle$-module of finite type, and the map*

$$\alpha_V : \widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger\{\{u_K\}\} \otimes_{E\langle\langle x_K\rangle\rangle} \mathcal{D}(V) \to \widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger\{\{u_K\}\} \otimes_E V$$

is injective.

Proof. — This follows from the fact that if we let $\mathcal{D}_n(V) = (\mathbf{D}_{\mathrm{rig},K}^\dagger\{\{u_K\}\}_n)^{\nabla=0}$ then $\mathcal{D}_n(V) = \varprojlim_I \mathcal{D}_n^I(V)$. The injectivity of α_V follows from the same arguments as in the proof of proposition 7.6 above. □

7.2. Frobenius regularity. — We now explain how to use the fact that our rings are embedded with a Frobenius in order to show some regularity property. Namely, we will show that F-analytic representation which are $\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger\{\{u_K\}\}_n$-admissible are actually $\mathbf{B}_{\mathrm{tri},K}^{\mathrm{an}}$-admissible. This section is in the same spirit as [**Ber02**, §3.1 and §3.2].

Lemma 7.8. — *Let h be a positive integer. Then*

$$\bigcap_{s=0}^{+\infty} \pi^{-hs}\widetilde{\mathbf{A}}^{\dagger,q^{-s}r} = \widetilde{\mathbf{A}}^+ \quad \textit{and} \quad \bigcap_{s=0}^{+\infty} \pi^{-hs}\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,q^{-s}r} \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^+.$$

Proof. — This is [**Ber02**, Lemm. 3.1] when $K = \mathbf{Q}_p$. The generalization when K is a finite extension of $\mathbf{Q}_p$ is straightforward. □

Lemma 7.9. — *Let h be a positive integer and let n be an integer ≥ 1. Then*

$$\bigcap_{s=0}^{+\infty} \pi^{-hs}\widetilde{\mathbf{A}}^{\dagger,q^{-s}r}\{\{u_K\}\}_n = \widetilde{\mathbf{A}}^+\{\{u_K\}\}_n \quad \textit{and} \quad \bigcap_{s=0}^{+\infty} \pi^{-hs}\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,q^{-s}r}\{\{u_K\}\}_n \subset \mathbf{B}_{\mathrm{tri},K}^{n,\mathrm{an}}.$$

Proof. — It just follows from the definitions of 3.6 for $\mathbf{Z}_p$-algebras, and from lemma 7.8. $\square$

Proposition 7.10. — *Let r, v be two positive integers, and let $A \in \mathrm{M}_{v\times r}(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n)$. Assume that there exists $P \in \mathrm{GL}_v(K\langle\langle x_K\rangle\rangle)$ such that $A = P\varphi_q^{-1}A$. Then $A \in \mathrm{M}_{v\times r}(\mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K})$.*

Proof. — Write A as (a_{ij}) and P as (p_{ij}). The assumption on the relation between P and A can be translated as:

$$p_{i1}\varphi^{-1}(a_{1j}) + \cdots + p_{iv}\varphi^{-1}(a_{vj}) = a_{ij} \quad \forall i \leq v, j \leq r.$$

Let us first show that, if $f \in K\langle\langle x_K\rangle\rangle$ and if $r_0 \geq 0$, then there exists $h \geq 0$ such that $\pi^h \cdot f \in \widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}\{\{u_K\}\}_n$ for all $0 \leq r \leq r_0$. Indeed, write f as $f = \sum_{k\geq 0} a_k x_K^k$. This can be also written as $f = \sum_{k\geq 0} b_k u_K^k$, with $b_k = \frac{(-1)^k}{k!}\sum_{j\geq 0} j^k a_j t_\pi^j$. We know that the a_j tend to 0 exponentially and we also know that $-\infty < V([r_0;r_0],t) \leq V([r;r],t_\pi)$ for $0 \leq r \leq r_0$, hence the result follows.

Now let c, r_0 be such that all the a_{ij} belong to $\pi^{-c}\widetilde{\mathbf{A}}^{\dagger,r_0}_{\mathrm{rig}}\{\{u_K\}\}_n$ and let h be as above, so that the $\pi^h p_{ij}$ belong to $\widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}\{\{u_K\}\}_n$ for all $r \leq r_0$. Since $\varphi_q^{-1}(a_{ij}) \in \pi^{-c}\widetilde{\mathbf{A}}^{\dagger,r_0/q}_{\mathrm{rig}}$, we get that the a_{ij} belong to $\pi^{-h-c}\widetilde{\mathbf{A}}^{\dagger,r_0/q}_{\mathrm{rig}}\{\{u_K\}\}_n$. By iterating, we see that the a_{ij} actually belong to $\cap_{s=0}^{+\infty}\pi^{-hs-c}\widetilde{\mathbf{A}}^{\dagger,r_0q^{-s}}_{\mathrm{rig}}\{\{u_K\}\}_n$ and thus, by lemma 7.9, this proves the proposition. $\square$

Proposition 7.11. — *Let V be an E-representation of $\mathcal{G}_K$. Then the morphism*

$$\mathbf{D}^{\mathrm{an}}_{\mathrm{tri},K}(V) \to (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}}$$

induced by the inclusion $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}\{\{u_K\}\}_n \subset \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n$, is an isomorphism of (φ_q,∇_u)-modules on $E\langle\langle x_K\rangle\rangle$.

Proof. — Let $(v_1,\ldots,v_r)$ and $(d_1,\ldots,d_v)$ be respectively an E-basis of V and an $E\langle\langle x_K\rangle\rangle$-basis of $(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}}$. There exists $A \in \mathrm{M}_{r\times v}$ such that $(d_i) = A(v_i)$. Let $P \in \mathrm{GL}_v(E\langle\langle x_K\rangle\rangle)$ be the matrix of φ_q in the basis (d_i). We then have $\varphi_q(A) = PA$ and thus $A = \varphi_q^{-1}(P)\varphi_q^{-1}(A)$. By proposition 7.10, we have $A \in M_{r\times v}(\mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K})$ and hence

$$(\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}} \subset (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}\{\{u_K\}\}_n \otimes_E V)^{\mathcal{G}_{K_n}} = \mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V).$$

$\square$

7.3. φ-modules on $L\langle\langle t_\pi\rangle\rangle$. — By proposition 7.2, to any E-representation V of $\mathcal{G}_K$ we can attach a module on $E\langle\langle t_\pi\rangle\rangle$, which is endowed with a Frobenius φ_q and an operator ∇_u. Note that φ_q and Γ_K act on $E\langle\langle t_\pi\rangle\rangle$ by

$$\varphi_q(t_\pi) = \pi t_\pi, \quad g(t_\pi) = \chi_\pi(g)t_\pi.$$

We can also define an operator ∇ on $E\langle\langle t_\pi\rangle\rangle$ by $\nabla_u = t_\pi\frac{d}{dt_\pi}$.

As a matter of fact, φ-modules on $E\langle\langle t_\pi\rangle\rangle$ were already studied by Colmez in [**Col14**, 3.1] and the results proved by Colmez show that φ-modules on $E\langle\langle t_\pi\rangle\rangle$ are not as bad as one may think. Be careful that what we call $E\langle\langle t_\pi\rangle\rangle$ corresponds in the notations of

Colmez to $E\{\{t_\pi\}\}$ (which to us means something different!). In this section, we recall Colmez's results on φ-modules on $E\langle\langle t_\pi\rangle\rangle$.

Definition 7.12. — A (φ_q, Γ_K)-module on $E\langle\langle t_\pi\rangle\rangle$ is a finite free $E\langle\langle t_\pi\rangle\rangle$-module, endowed with semilinear actions of φ_q and Γ_K which commute one to another and such that φ_q is an isomorphism.

A (φ_q, ∇)-module on $E\langle\langle t_\pi\rangle\rangle$ is a finite free $E\langle\langle t_\pi\rangle\rangle$-module, endowed with semilinear actions of φ_q and ∇ which commute one to another and such that φ_q is an isomorphism.

A (φ_q, Γ_K)-module on $E\langle\langle t_\pi\rangle\rangle$ gives rise to a (φ_q, ∇)-module on $E\langle\langle t_\pi\rangle\rangle$ by taking the same φ-structure and taking ∇ to be the operator $\frac{\log(g)}{\log \chi_\pi(g)}$ for g close enough to 1.

The ring $E\langle\langle t_\pi\rangle\rangle$ can be interpreted *via* analytic functions, as it is the projective limit of the rings of analytic functions on the disks $v_p(x) \geq -ne$ for $n \in \mathbf{N}$. Those rings are principal Banach rings and therefore $E\langle\langle t_\pi\rangle\rangle$ is a Fréchet-Stein ring, which in particular implies that any closed submodule of a free module of rank d if free of rank $\leq d$ and that a submodule of finite type of a free finite type module is closed and thus free. Moreover, Newton polygons theory show that an element $f \in E\langle\langle t_\pi\rangle\rangle$ does not vanish if and only if $f \in E^\times$, so that $(E\langle\langle t_\pi\rangle\rangle)^\times = E^\times$.

Lemma 7.13. — *A finite type ideal of $E\langle\langle t_\pi\rangle\rangle$ which is stable by either φ_q or a finite index subgroup of Γ_K is of the form (t_π^k) with $k \in \mathbf{N}$.*

Proof. — The proof in the cyclotomic case is done in [**Col14**, Lemm. 3.1] and the extension of the proof to K is straightforward. □

Lemma 7.14. — *If $\alpha \in E$, then $\varphi_q - \alpha$ induces an isomorphism on $E\langle\langle t_\pi\rangle\rangle$, unless $\alpha = \pi^i$ with $i \in \mathbf{N}$, in which case the kernel of $\varphi_q - \alpha$ is Et_π^i and $\varphi_q - \alpha$ induces an isomorphism on $\{\sum_{k\geq 0} a_k t_\pi^k, a_i = 0\}$.*

Proof. — See [**Col14**, Lemm. 3.2]. □

Lemma 7.15. — *Let M be a rank d φ_q-module on $E\langle\langle t_\pi\rangle\rangle$ and let $v \in M$ be such that there exists $\alpha \in E^\times$ such that $\varphi_q(v) = \alpha v$. Then there exists $k \in \mathbf{N}$ such that $t_\pi^{-k} v \in M$ and $M/E\langle\langle t_\pi\rangle\rangle t_\pi^{-k} v$ is free of rank $d-1$ on $E\langle\langle t_\pi\rangle\rangle$.*

Proof. — See [**Col14**, Lemm. 3.4]. □

Let M be a φ_q-module on $E\langle\langle t_\pi\rangle\rangle$ and let $\overline{M} = M/t_\pi M$. If $P \in E[X]$ is unitary of degree d and irreducible, then we let M_P (resp. $\overline{M}_P$) denote the set of elements $v \in M$ (resp. in $\overline{M}$) such that $P(\varphi_q)^n \cdot v = 0$ for $n \gg 0$ and if $k \in \mathbf{N}$, we let $P[k]$ be the polynomial $\pi^{kd} P(X/\pi^k)$.

Lemma 7.16. — *The natural map $M_P \to \overline{M}_P$ is surjective.*

Proof. — See [**Col14**, Lemm. 3.5]. □

Theorem 7.17. — *If M is a φ_q-module of rank d on $E\langle\langle t_\pi\rangle\rangle$, there exists a basis $e_1, \ldots, e_d$ of M in which the matrix of φ is $A + N$, where $A \in \mathrm{GL}_d(L)$ is semisimple and invertible, and N is nilpotent and commutes with A. Moreover, N splits into $N = N_0 + t_\pi N_1 + \ldots$, where $N_i \in M_d(L)$ sends the kernel M_P of $P(A)$ into the one $M_{P[-i]}$ of $P(\pi^i A)$ for all P (and thus in particular the sum is finite).*

Proof. — See [**Col14**, Thm. 3.6]. □

Given a (φ_q, Γ_K)-module D on $E\langle\langle t_\pi \rangle\rangle$, we say that an element v of D is proper for the action of φ_q and Γ_K if there exists $\delta \in \widehat{\mathfrak{T}}(E)$ such that $\varphi(v) = \delta(\pi)v$ and $g(v) = \delta(\chi_\pi(g))v$ for all $g \in \Gamma_K$.

Lemma 7.18. — *Given a (φ_q, Γ_K)-module D of rank 1 on $E\langle\langle t_\pi \rangle\rangle$, F-analytic, with basis e, then there exists $\delta \in \widehat{\mathfrak{J}}(E)$ such that e is proper for δ.*

Proof. — This just follows from the fact that a rank 1 (φ_q, Γ_n)-module on $E\langle\langle t_\pi \rangle\rangle$ has a unique basis e, up to multiplication by an element of $(E\langle\langle t_\pi \rangle\rangle)^\times = E^\times$. □

7.4. The modules $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}}$. — We now explain what is $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}}$ and prove that its rank as an $E\langle\langle x_K \rangle\rangle$-module is too small in general.

Given a (φ, Γ)-module $\mathbf{D}$ over E (in the cyclotomic setting), Colmez has defined [**Col14**, §3.3] a module denoted by $\mathbf{D} \boxtimes \{0\}$ by $\cap_{n \geq 0} \varphi^n(\mathbf{D})$, which is a (φ, Γ)-module over $E\langle\langle t \rangle\rangle$.

Proposition 7.19. — *Let V be an F-analytic representation of $\mathcal{G}_K$. Then*

$$(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{H_K, \Gamma-\mathrm{la}} = \mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}} = \cap_{n \geq 0} \varphi_q^n(\mathbf{D}^\dagger_{\mathrm{rig},K}(V)).$$

Proof. — The first equality follows from applying proposition 3.8 to proposition 7.5. We now prove the second equality.

If $x \in \mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}}$, then x belongs to a free $K\langle\langle t_\pi \rangle\rangle$-module which is φ-stable, so that $\varphi^{-1}(x) = \mathrm{Mat}(\varphi)^{-1} \cdot x$ and therefore x belongs to $\varphi^{-1}(\mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}})$. By induction, this shows that x belongs to $\cap_{n \geq 0} \varphi_q^n(\mathbf{D}^\dagger_{\mathrm{rig},K}(V))$.

If $x \in \cap_{n \geq 0} \varphi_q^n(\mathbf{D}^\dagger_{\mathrm{rig},K}(V))$, then x belongs to a free $K\langle\langle t_\pi \rangle\rangle$-module which is Γ_K-stable so that for $g \in \mathrm{Gal}(K_\infty/K)$, $g(x) = \mathrm{Mat}(g) \cdot x$ where $\mathrm{Mat}(g) \in \mathrm{GL}_d(K\langle\langle t_\pi \rangle\rangle)$, so that the Galois action on x is locally analytic. □

In particular, the following result of Colmez shows that in general the module $\mathbf{D}^\dagger_{\mathrm{rig},K}(V)^{\mathrm{la}}$ is too small:

Proposition 7.20. — *Let V be a two dimensional irreducible representation of $\mathcal{G}_{\mathbf{Q}_p}$. If V is not trianguline then $\mathbf{D}^\dagger_{\mathrm{rig},K}(V) \boxtimes \{0\} = 0$.*

Proof. — This is item (i) of theorem 3.23 of [**Col14**]. □

Remark 7.21. — Theorem 3.23 of [**Col14**] also says that if V is a semistable, noncrystalline 2-dimensional representation, then $\mathbf{D}^\dagger_{\mathrm{rig},K}(V) \boxtimes \{0\}$ is a (φ, Γ)-module of rank 1 over $E\langle\langle t \rangle\rangle$.

8. Applications to trianguline representations

We now explain how some of the rings previously introduced provide some results towards the question of the existence of a ring of periods for trianguline representations. We will start by recalling the notions of trianguline representations and refinements.

In a previous version of this paper, we claimed that trianguline representations were admissible for the ring $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\{\{u\}\}$ but there was a gap in the proof and the claim is actually not true. We do expect though that if such a ring exists then it has to be some intermediate ring B between $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\{\{u\}\}$ and the rings $\widetilde{\mathbf{B}}^I\{\{u\}\}$, but it is not clear at all "how much periods we have to add" to $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}\{\{u\}\}$. More generally, we extend these constructions to the F-analytic Lubin-Tate case.

8.1. Trianguline representations and refinements. — We start by recalling the definitions of trianguline representations and some associated properties. The notion of trianguline representations was introduced by Colmez in [**Col08b**]. Here we choose to follow Berger's and Chenevier's definitions [**BC10**] instead of Colmez's.

Definition 8.1. — We say that an E-representation V of $\mathcal{G}_K$ is split trianguline if $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is a successive extension of (φ,Γ_K)-modules of rank 1 over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^\dagger_{\mathrm{rig},K}$.

We say that an L-representation V of $\mathcal{G}_K$ is trianguline if there exists a finite extension E of L such that the E-representation $E\otimes_L V$ is split trianguline.

We say that an E-representation V of $\mathcal{G}_K$ is potentially split trianguline (resp. potentially trianguline) if there exists a finite extension K' of K such that $V_{|\mathcal{G}_{K'}}$ is split trianguline (resp. trianguline).

Remark 8.2. — Definition 8.1 can be equivalently stated in terms of B-pairs: an E-representation V of $\mathcal{G}_K$ is split trianguline if the attached $\mathbf{B}^{\otimes E}_{|K}$-pair is a successive extension of rank 1 $\mathbf{B}^{\otimes E}_{|K}$-pairs.

Lemma 8.3. — *Let V be an F-analytic representation. Then the following are equivalent:*

1. *V is split trianguline.*
2. *The Lubin-Tate (φ_q,Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is a successive extension of F-analytic Lubin-Tate (φ_q,Γ_K)-modules of rank 1.*

Proof. — See [**Poy20**, Thm. 4.11]. □

For a $\mathbf{B}_{e,E}$-representation, we say that it is split triangulable if it is a successive extension of rank 1 $\mathbf{B}_{e,E}$-representations.

Lemma 8.4. — *An E-representation V of $\mathcal{G}_K$ is split trianguline if and only if the corresponding $\mathbf{B}_{e,E}$-representation is split triangulable as a $\mathbf{B}_{e,E}$-representation of $\mathcal{G}_K$.*

Proof. — See [**BDM19**, Coro. 3.2]. □

Proposition 8.5. — *The categories of split trianguline representations and of trianguline representations are stable by subobjects, quotients, direct sums and tensor products.*

Proof. — The fact that it is stable by quotients and subobjects follows from [**BDM19**, Prop. 3.3]. For direct sums and tensors products it is a straightforward consequence of definition 8.1. □

Let D be a (φ,Γ_K)-module of rank d over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and equipped with a strictly increasing filtration $(\mathrm{Fil}_i(D))_{i=0\dots d}$:

$$\mathrm{Fil}_0(D):=\{0\}\subsetneq \mathrm{Fil}_1(D)\subsetneq\cdots\subsetneq \mathrm{Fil}_i(D)\subsetneq\cdots\subsetneq \mathrm{Fil}_{d-1}(D)\subsetneq \mathrm{Fil}_d(D):=D,$$

of (φ,Γ_K)-submodules which are direct summand as $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-modules. We call such a D a triangular (φ,Γ_K)-module over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, and the filtration $\mathcal{T}:=(\mathrm{Fil}_i(D))$ a triangulation of D over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

Let D be a triangular (φ,Γ_K)-module. By proposition 3.1 of [**Col08b**], each

$$\mathrm{gr}_i(D):=\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D),\ \ 1\leq i\leq d,$$

is isomorphic to the (φ,Γ_K)-module on $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ attached to a character δ_i for some unique $\delta_i:K^\times\to E^\times$. Following [**BC09a**, 2.3.2], we define the parameter of the triangulation to be the continuous homomorphism

$$\delta:=(\delta_i)_{i=1,\cdots,d}:K^\times\to(E^\times)^d.$$

When $K=\mathbf{Q}_p$, the parameter of a triangular (φ,Γ_K)-module refines the data of its Sen polynomial:

Proposition 8.6. — *Let D be a triangular (φ,Γ)-module over $E\otimes_{\mathbf{Q}_p}\mathbf{B}^{\dagger}_{\mathrm{rig},\mathbf{Q}_p}$ and δ the parameter of a triangulation of D. Then the Sen polynomial of D is*

$$\prod_{i=1}^{d}(T-w(\delta_i)).$$

Proof. — See [**BC09a**, Prop. 2.3.3]. □

We now recall the notion of refinements for crystalline trianguline representations of $\mathcal{G}_{\mathbf{Q}_p}$ as in [**BC09a**, §2.4]. Let V be finite, d-dimensional, continuous, E-representation of $\mathcal{G}_{\mathbf{Q}_p}$. We will assume that V is crystalline and that the crystalline Frobenius φ acting on $\mathbf{D}_{\mathrm{crys}}(V)$ has all its eigenvalues in $E^\times$.

By a refinement of V, using the definition of [**Maz00**, §3], we mean the data of a full φ-stable E-filtration $\mathcal{F}=(\mathcal{F}_i)_{i=0,\dots,d}$ of $\mathbf{D}_{\mathrm{crys}}(V)$:

$$\mathcal{F}_0=0\subsetneq\mathcal{F}_1\subsetneq\cdots\subsetneq\mathcal{F}_d=\mathbf{D}_{\mathrm{crys}}(V).$$

As in [**BC09a**, 2.4.1], we remark that any refinement $\mathcal{F}$ determines two orderings:

1. It determines an ordering $(\varphi_1,\cdots,\varphi_d)$ of the eigenvalues of φ, defined by the formula

   $$\det(T-\varphi_{|\mathcal{F}_i})=\prod_{j=1}^{i}(T-\varphi_j).$$

   If all these eigenvalues are distinct then such an ordering conversely determines $\mathcal{F}$.

2. It determines also an ordering $(s_1,\cdots,s_d)$ on the set of Hodge-Tate weights of V, defined by the property that the jumps of the weight filtration of $\mathbf{D}_{\mathrm{crys}}(V)$ induced on $\mathcal{F}_i$ are $(s_1,\cdots,s_i)$

The theory of refinements has a simple interpretation in terms of (φ,Γ)-modules: let D be a crystalline (φ,Γ)-module as above and let $\mathcal{F}$ be a refinement of D. We can construct from $\mathcal{F}$ a filtration $(\mathrm{Fil}_i(D))_{i=0,\cdots,d}$ of D by setting

$$\mathrm{Fil}_i(D) := (E \otimes_{\mathbf{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbf{Q}_p}[1/t]\mathcal{F}_i) \cap D,$$

which is a finite type saturated $E \otimes_{\mathbf{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbf{Q}_p}$-submodule of D.

Proposition 8.7. — *The map defined above $(\mathcal{F}_i) \mapsto (\mathrm{Fil}_i(D))$ induces a bijection between the set of refinements of D and the set of triangulations of D, whose inverse is $\mathcal{F}_i := \mathrm{Fil}_i(D)[1/t]^{\Gamma}$. In the bijection above, for $i = 1,\dots,d$, the graded piece $\mathrm{Fil}_i(D)/\mathrm{Fil}_{i-1}(D)$ is isomorphic to the (φ,Γ)-module on $E \otimes_{\mathbf{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},\mathbf{Q}_p}$ attached to δ_i where $\delta_i(p) = \varphi_i p^{-s_i}$ and $\delta_{i|\Gamma} = \chi^{-s_i}$, where the φ_i and s_i are defined by items 1 and 2 above.*

Proof. — See [**BC09a**, Prop. 2.4.1]. □

Remark 8.8. — In particular, Proposition 8.7 shows that crystalline representations are trianguline, and that the set of their triangulations is in natural bijection with the set of their refinements.

We now finish this section with a result regarding trianguline representations that we were not able to find in the litterature.

Proposition 8.9. — *Let V be an L-representation of $\mathcal{G}_K$. Then V is trianguline if and only if the underlying $\mathbf{Q}_p$-representation of V is trianguline.*

Proof. — Let V be an L-representation of $\mathcal{G}_K$ and let E be a finite extension of L, containing all the images of the embeddings $\tau : L \to \overline{K}$ and such that $E \otimes_L V$ is split trianguline. Then $E \otimes_{\mathbf{Q}_p} V = (E \otimes_{\mathbf{Q}_p} L) \otimes_L V = \oplus_{\tau\in\Sigma}(E \otimes_L V)_\tau$ where $\Sigma = \mathrm{Emb}(L,\overline{K})$.

In particular, $E \otimes_L V$ is a subrepresentation of $E \otimes_{\mathbf{Q}_p} V$ and this concludes the first half of the proof by proposition 8.5. For the other direction, let $W = W_e(E \otimes_L V)$ the corresponding $\mathbf{B}_{e,E}$-representation and let $W_0 = 0 \subset W_1 \subset \dots W_d = W$ a triangulation of W. For $\tau \in \Sigma$, let $\mathbf{B}_{e,E,\tau} = E \otimes_{L,\tau} \mathbf{B}_{e,E}$. For $\tau \in \Sigma$ and $1 \leq i \leq d$, let $W_{i,\tau} = \mathbf{B}_{e,E,\tau} \otimes_{\mathbf{B}_{e,E}} W_i$. By construction

$$0 \subset W_{1,\tau} \subset \dots \subset W_{d,\tau}$$

is a triangulation of $W((E \otimes_L V)_\tau)$ and thus $E \otimes_{\mathbf{Q}_p} V$ is trianguline. □

8.2. Discussion on a ring of periods for trianguline representations. — By proposition 8.5, we know that the category of (split) trianguline representations of $\mathcal{G}_K$ is a Tannakian category. Because of this and because of proposition 8.9, it appears reasonable to look for a ring B such that trianguline representations are exactly the representations which are B-admissible in the sense of Fontaine.

Recall that the notion of admissibility in the sense of Fontaine is defined for what he called regular rings and is as follows (we only recall the definitions of [**Fon94b**] in the particular case of $\mathbf{Q}_p$-representations because that's all we need here).

Let B be a topological $\mathbf{Q}_p$-algebra endowed with an action of a group G. For any $\mathbf{Q}_p$-representation of G, we let $D_B(V) := (B \otimes_{\mathbf{Q}_p} V)^G$. We let $\alpha_B(V)$ denote the B-linear map $B \otimes_{B^G} D_B(V) \longrightarrow B \otimes_{\mathbf{Q}_p} V$ deduced from the inclusion $D_B(V) \subset B \otimes_{\mathbf{Q}_p} V$ by extending the scalars to B. The ring B is said to be G-regular if the following hold:

1. B is reduced;
2. for any p-adic representation V of G, the map α_V is injective;
3. any element b of B which is nonzero and is such that the $\mathbf{Q}_p$-line generated by B is G-stable is invertible.

The last condition implies in particular that B^G is a field. If B is G-regular, a representation V of G is said to be B-admissible if $\alpha_B(V)$ is an isomorphism, which is equivalent as saying that $\dim_{B^G} D_B(V) = \dim_{\mathbf{Q}_p} V$.

Unfortunately, it seems to us that in the case we consider, the last condition is too strong and thus we extend the notion of G-regularity as follows: we say that B is G-regular if the following conditions are met:

1. B is reduced;
2. for any p-adic representation V of G, $D_B(V)$ is a free B^G-module;
3. the map α_V is injective.

It is clear that G-regular rings in the sense of Fontaine are G-regular for us, but that the converse does not hold.

In the rest of the paper, G-regularity and admissibility are to be understood in our sense.

We now explain exactly what we mean by a ring of trianguline periods.

Definition 8.10. — A $\mathcal{G}_K$-regular ring B is said to be a trianguline periods ring for $\mathcal{G}_K$ if trianguline representations of $\mathcal{G}_K$ are B-admissible, and if B-admissible representations of $\mathcal{G}_K$ are trianguline.

Proposition 8.11. — *Let B be a $\mathcal{G}_K$-regular ring and let V be an L-representation of $\mathcal{G}_K$. Then V is B-admissible if and only if there exists a finite extension E of L such that $V \otimes_L E$ is B-admissible.*

Proof. — It's clear that if V is B-admissible, then there exists a finite extension E of L such that $V \otimes_L E$ is B-admissible. To show the reverse, first note that the admissibility of an E representation V does not depend on wether one considers it as a $\mathbf{Q}_p$-representation or as an E-representation (the $B^{\mathcal{G}_K}$-module $D_B(V) = (V \otimes_{\mathbf{Q}_p} B)^{\mathcal{G}_K}$ is always the same). Now because the category of B-admissible representations is clearly stable by subobjects, it suffices to prove the proposition to note that V is a sub-$\mathbf{Q}_p$-representation of $V \otimes_L E$. $\square$

Unlike in the crystalline or semistable case, if such a ring exists, it has to depend on K:

Proposition 8.12. — *There is no ring B satisfying the properties above such that, for any finite extension K of $\mathbf{Q}_p$, B is a trianguline periods ring for $\mathcal{G}_K$.*

Proposition 8.12 is a consequence of the following result:

Proposition 8.13. — *Let L/K be any finite extension. Then there exists a representation V of $\mathcal{G}_K$ such that V is trianguline as a representation of $\mathcal{G}_L$ but is not trianguline as a representation of $\mathcal{G}_K$.*

Proof. — Let $\eta : \mathcal{G}_L \to L^\times$ be a character such that there exists $\tau_1 \neq \tau_2 \in \mathrm{Emb}(L, \overline{\mathbf{Q}}_p)$ with $(\tau_1)_{|K} = (\tau_2)_{|K}$, and such that η is τ_1-de Rham but not τ_2-de Rham in the sense of [**Din14**]. Our claim is that such a character can't possibly extend to $\mathcal{G}_K$ and neither can any of its conjugate, i.e. there is no character $\rho : \mathcal{G}_K \to L^\times$ such that $\rho_{|\mathcal{G}_L} = \sigma(\eta)$ for some $\sigma \in \mathrm{Emb}(L, \overline{\mathbf{Q}}_p)$. Indeed, if such a ρ existed, then the dimension of $\mathbf{D}_{\mathrm{dR},\sigma}(\sigma(\eta))$ would only depend of the dimension of $\mathbf{D}_{\mathrm{dR},\sigma_{|K}}(\rho)$, which is not the case because of the assumption on τ_1 and τ_2.

We now let $V = \mathrm{ind}_{\mathcal{G}_L}^{\mathcal{G}_K} \eta$. This is a p-adic representation of $\mathcal{G}_K$, whose restriction to $\mathcal{G}_L$ is the sum of the conjugates of η, so that it clearly is trianguline as a representation of $\mathcal{G}_L$. Let us assume that it also is trianguline as a representation of $\mathcal{G}_K$. Let W be the $\mathbf{B}_{|K}^{\otimes L}$-pair attached to V. As a $\mathbf{B}_{|L}^{\otimes L}$-pair, we can write

$$W = \oplus_\sigma W(\sigma(\eta)).$$

Since we assumed that W is trianguline as a $\mathbf{B}_{|K}^{\otimes L}$-pair, there exists $W_1 \subset W$ a direct summand of rank 1. For $\tau \in \mathrm{Emb}(L, \overline{\mathbf{Q}}_p)$, we have the following exact sequence

$$0 \to \oplus_{\tau \neq \sigma} W(\sigma(\eta)) \to W \to W(\tau(\eta)) \to 0$$

so that, since W_1 is a direct summand of rank 1 of W and by proposition 2.4 of [**BDM19**], we either have $W_1 = W(\tau(\eta))$ or $W_1 \subset \oplus_{\tau \neq \sigma} W(\sigma(\eta))$. By induction, W_1 has to be equal to one of the $W(\tau(\eta))$. Therefore, one of the conjugates of η has to extend to $\mathcal{G}_K$, which we have proven is not possible. $\square$

We can now give a proof of proposition 8.12:

Proof. — By the results of §5, the periods of any representation live in $\widetilde{\mathbf{B}}^I\{\{u\}\}$, for I any compact subinterval of $[r_0; +\infty[$, and in particular so do the periods of any trianguline representation V of $\mathcal{G}_K$, so that we can assume that $B \subset \widetilde{\mathbf{B}}^I\{\{u\}\}$. Since every unramified representation of $\mathcal{G}_K$ is trianguline, we can assume that $B \supset \widehat{\mathbf{Q}_p^{nr}}$. Moreover, if $L/K/\mathbf{Q}_p$ are unramified then it is easy to see that $(\widetilde{\mathbf{B}}^I\{\{u\}\})^{\mathcal{G}_L} = L \otimes_K (\widetilde{\mathbf{B}}^I\{\{u\}\})^{\mathcal{G}_K}$, and thus we can assume that $B^{\mathcal{G}_L} = L \otimes_K B^{\mathcal{G}_K}$.

Assume that B is a ring satisfying the properties, and such that for any finite extension $K/\mathbf{Q}_p$, B is a trianguline periods ring for $\mathcal{G}_K$. Let K be a finite unramified extension of $\mathbf{Q}_p$, let L be a finite unramified extension of K. Let V be a p-adic representation of $\mathcal{G}_K$ which is not trianguline as a representation of $\mathcal{G}_K$ but becomes trianguline over $\mathcal{G}_L$, which exists by the previous proposition. Let $D_L = (B \otimes V)^{\mathcal{G}_L}$. It is a $B^{\mathcal{G}_L}$-module, endowed with a semilinear action of $\mathrm{Gal}(L/K)$. Let $D = D_L^{\mathrm{Gal}(L/K)}$. By Speiser's lemma, $D_L \simeq L \otimes_K D$ and thus $D_L = B^{\mathcal{G}_L} \otimes_{B^{\mathcal{G}_K}} D$. Thus, V is B-admissible as a representation of $\mathcal{G}_K$. $\square$

8.3. F-analytic $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$-admissible representations. — We now explain why $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$ is a good starting candidate as a ring of trianguline periods. Note that by propositions 7.19 and 7.20 we already know that there are F-analytic trianguline representations of $\mathcal{G}_K$ which are not $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$-admissible.

Proposition 8.14. — *The ring $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$ is $\mathcal{G}_K$-regular for F-analytic representations.*

Proof. — This follows from proposition 7.11 and from corollary 7.7. □

We now define a notion of refinements for F-analytic representations of $\mathcal{G}_K$ which are $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$-admissible. We let V be a $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$-admissible L-representation of dimension d of $\mathcal{G}_K$. By a refinement of V, we mean the data of a full φ_q- and Γ_K-stable $L\langle\langle t_\pi\rangle\rangle$-filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0,\dots,d}$ of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \boxtimes \{0\}$:

$$\mathcal{F}_0 = 0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_d = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \boxtimes \{0\}.$$

Note that, as in the crystalline case studied in [**BC09a**], the theory of refinements has a simple interpretation in terms of (φ_q, Γ_K)-modules: let $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ be the triangulable (φ_q, Γ_K)-module over $L \otimes_K \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ attached to V and let $\mathcal{F}$ be a refinement of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) \boxtimes \{0\}$. We can construct from $\mathcal{F}$ a filtration $(\mathrm{Fil}_i(D))_{i=0,\cdots,d}$ of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ by setting

$$\mathrm{Fil}_i(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) := ((L \otimes_K \mathbf{B}^{\dagger}_{\mathrm{rig},K}) \otimes_{L\langle\langle t_\pi\rangle\rangle} \mathcal{F}_i)$$

which is a finite type saturated $E \otimes_K \mathbf{B}^{\dagger}_{\mathrm{rig},K}$-submodule of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$.

Proposition 8.15. — *The map defined above $(\mathcal{F}_i) \mapsto (\mathrm{Fil}_i(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)))$ induces a bijection between the set of refinements of V and the set of triangulations of $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, whose inverse is $\mathcal{F}_i := (\mathrm{Fil}_i(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V))) \boxtimes \{0\})$.*

Proof. — This is exactly as in the crystalline case. □

Proposition 8.16. — *Let M be a (φ_q, Γ_K)-module of rank d on $L\langle\langle t_\pi\rangle\rangle$. Then, up to extending the scalars to some finite extension E of L, there exists a filtration*

$$M_0 = 0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_d = M$$

of M by saturated sub-(φ_q, Γ_K)-modules.

Proof. — We prove the result by induction. If $d = 1$ there is nothing to prove. Assume now that $d \geq 2$ and that the result holds for $d - 1$.

By theorem 7.17 and lemma 7.15, upto replacing L by a finite extension E' of L, there exists e_1 proper for the action of φ and Γ_K such that $E'\langle\langle t_\pi\rangle\rangle \cdot e_1$ is saturated in M. By induction, $M/(E'\langle\langle t_\pi\rangle\rangle \cdot e_1)$ admits a full (φ_q, Γ_K)-stable filtration $(\mathcal{F}_i)_{i=1}^{d-1}$. We let M_{i+1} be a lift of $\mathcal{F}_i$ containing $E'\langle\langle t_\pi\rangle\rangle \cdot e_1$ and we put $M_1 = E'\langle\langle t_\pi\rangle\rangle \cdot e_1$. We then have that $(M_i)_{i=1}^{d}$ is a full (φ_q, Γ_K)-stable filtration of M. □

Theorem 8.17. — *Let V be a $\mathbf{B}^{\mathrm{an}}_{\mathrm{tri},K}$-admissible F-analytic p-adic representation V. Then V is trianguline.*

Proof. — By proposition 8.16 and proposition 8.15, there exists a finite extension L of K such that $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V \otimes_K L)$ is a triangulable (φ_q, Γ_K)-module over $L \otimes_K \mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

Moreover, we see from lemma 7.18 that the characters appearing in the triangulation are F-analytic. □

Lemma 8.18. — *Let V be an E-representation of $\mathcal{G}_K$ such that the attached (φ_q,Γ_K)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ is triangulable, and let $\delta:(K^\times)^d \to (E^\times)^d$ be the parameter of a triangulation of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$. Then in an adapted basis for the refinement of $\mathbf{D}^{\mathrm{an}}_{\mathrm{tri},K}(V)$ corresponding to δ by proposition 8.15 and via the isomorphism $\mathbf{D}^{\mathrm{an}}_{\mathrm{tri},K}(V) \simeq \mathbf{D}^\dagger_{\mathrm{rig}}(V)\boxtimes\{0\}$, the matrices of ∇_u and φ_q are respectively of the form:*

$$\begin{pmatrix} w(\delta_1) & * & \cdots & * \\ 0 & w(\delta_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\delta_d) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \delta_1(\pi) & * & \cdots & * \\ 0 & \delta_2(\pi) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_d(\pi) \end{pmatrix}.$$

Proof. — We prove it by induction on d. For $d=1$, by proposition 3.1 of [**Col08b**] there exists a basis e_δ of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ in which $g(e_\delta)=\delta(\chi_\pi(g))e_\delta$ and $\varphi_q(e_\delta)=\delta(\pi)e_\delta$, so that there exists $n\geq 0$ and $\alpha\in L^\times$ such that $e^{\alpha u_K}e_\delta \in \mathbf{D}^\dagger_{\mathrm{rig}}(V)\{\{u_K\}\}_n$ and is left invariant by Γ_{K_n}. Therefore $e^{\alpha u_K}e_\delta$ is a basis of $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)$ and this basis satisfies the result of the lemma. To see that it is unique note that since $(L\langle\langle x_K\rangle\rangle)^\times = L^\times$, the matrices of ∇_u and φ_q in an other basis of $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)$ would be the same.

Assume now that $d\geq 2$ is such that the result holds for $d-1$ and let $(\mathrm{Fil}_i(\mathbf{D}^\dagger_{\mathrm{rig}}(V)))_{i=0,\ldots,d}$ be the filtration of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ corresponding to the triangulation. Since our constructions are stable by saturated sub-objects, we get by induction that in an adapted basis for the refinement of $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(\mathrm{Fil}_{d-1}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)))$, the matrices of ∇_u and φ are respectively of the form:

$$\begin{pmatrix} w(\delta_1) & * & \cdots & * \\ 0 & w(\delta_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\delta_{d-1}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta_1(\pi) & * & \cdots & * \\ 0 & \delta_2(\pi) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_{d-1}(\pi) \end{pmatrix}.$$

Since our constructions are also stable by quotients by saturated sub-objects and using the proof in the rank 1 case, we know that the matrices of ∇_u and φ_q in a basis of $\mathbf{D}^\dagger_{\mathrm{rig}}(V)/\mathrm{Fil}_{d-1}(\mathbf{D}^\dagger_{\mathrm{rig}}(V)) \simeq E\otimes_K \mathbf{B}_{\mathrm{rig},K}(\delta_d)$ are respectively of the form $(w(\delta_d))$ and $(\delta_d(\pi))$. Therefore, in an adapted basis for the refinement of $\mathbf{D}^{\mathrm{an}}_{\mathrm{tri},K}(V)$ corresponding to δ, the matrices of φ_q and ∇_u are as we wanted. □

In particular, as in the crystalline case, a refinement defines an ordering on both the eigenvalues of φ_q and on the set of Hodge-Tate weights of V, and encodes the data of the Hodge-Tate weights of its parameter.

8.4. Crystabelian representations. — Recall that to any F-analytic representation V of $\mathcal{G}_K$, we can attach a filtered φ_q-module over K $\mathbf{D}_{\mathrm{crys}}(V)$, which is a finite dimensional K-vector space (whose dimension is $\leq \dim_K(V)$), endowed with a Frobenius map φ_q such that $\varphi_q^* D \simeq D$, and with a filtration by K subspaces, indexed by $\mathbf{Z}$, which is decreasing and separated.

Given the construction of our rings of periods, it is quite obvious that the modules $\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)$ attached to F-analytic p-adic representations of $\mathcal{G}_K$ contain its crystalline periods:

Proposition 8.19. — *Let V be an F-analytic p-adic representation. Then*

1. $\mathbf{D}_{\mathrm{crys}}^+(V) \subset \mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)^{\nabla_u=0}$;
2. $\mathbf{D}_{\mathrm{crys}}(V) \subset (\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)[1/x_K])^{\nabla_u=0}$.

Proof. — Item 1 just follows from the facts that ∇_u and the action of $\mathcal{G}_K$ commute, that $\mathbf{D}_{\mathrm{crys}}^+(V) = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ and that $(\mathbf{B}_{\mathrm{tri},K}^{n,\mathrm{an}})^{\nabla_u=0} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$.

Item 2 follows from the fact that $\mathbf{D}_{\mathrm{crys}}(V) = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t] \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ by lemma 3.8 of [**Por20**] and that $(\mathbf{B}_{\mathrm{tri},K}^{n,\mathrm{an}}[1/x_K])^{\nabla_u=0} = \widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t_\pi]$. □

A crystabelian representation of $\mathcal{G}_K$ is a representation of $\mathcal{G}_K$ which becomes crystalline over an abelian extension of K. In what follows, we will assume that V is a crystabelian F-analytic representation of $\mathcal{G}_K$. Since $K^{\mathrm{ab}} = K^{\mathrm{unr}} \cdot K_\infty$, and by the same argument as in the proof of proposition 8.12, it means that there exists $n \geq 0$ such that $V_{|G_{K_n}}$ is crystalline. We still let $\mathbf{D}_{\mathrm{crys}}(V)$ stand for the corresponding filtered φ_q-module, even though the filtration is defined over K_n.

Let $k \geq 0$ be such that $\mathbf{D}_{\mathrm{crys}}(V(\chi_\pi^{-k})) = \mathbf{D}_{\mathrm{crys}}^+(V(\chi_\pi^{-k}))$. A quick computation shows that $\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V(\chi_\pi^{-k})) = e^{ku_K}\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)$ and that $\mathbf{D}_{\mathrm{crys}}(V(\chi_\pi^{-k})) = t_\pi^k \mathbf{D}_{\mathrm{crys}}(V)$. In particular, this shows that

$$\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V) = e^{-ku_K}\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V) \supset \mathbf{D}_{\mathrm{crys}}^+(V(\chi_\pi^{-k})) = t_\pi^k \mathbf{D}_{\mathrm{crys}}(V)$$

so that $x_K^k \mathbf{D}_{\mathrm{crys}}(V) \subset \mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)$. Since we have $\varphi(x_K) = \pi \cdot x_K$, this shows that $\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)$ contains a copy of $\mathbf{D}_{\mathrm{crys}}(V)$, but with the filtration and Frobenius shifted by some power. Note that we can also recover the filtration on $K_\infty \otimes_K \mathbf{D}_{\mathrm{crys}}^+(V)$ as it was done for $\mathbf{D}_{\mathrm{dR}}^+(V)$ in §4.

Assume for now that V is a crystalline positive p-adic representation, which means that $\mathbf{D}_{\mathrm{crys}}(V) = \mathbf{D}_{\mathrm{crys}}^+(V)$. Let $\mathcal{F} = (\mathcal{F}_i)_{i=0,\dots,d}$ be a full φ- and ∇_u-stable $E\langle\langle x_K\rangle\rangle$-filtration of $\mathbf{D}_{\mathrm{tri},K}^{\mathrm{an}}(V)$. Since the filtration is stable by ∇_u, the filtration $\mathcal{F}^{\nabla_u=0} = (\mathcal{F}_i^{\nabla_u=0})_{i=0,\dots,d}$ is also a full φ-stable filtration of $\mathbf{D}_{\mathrm{tri},K}^{\mathrm{an}}(V)^{\nabla_u=0} = \mathbf{D}_{\mathrm{crys}}(V)$. In particular, refinements in our sense for trianguline representations give rise to refinements of positive crystalline representations. In the other direction, we can for example use proposition 8.7 in order to construct a filtration on $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ starting from a φ-stable filtration on $\mathbf{D}_{\mathrm{crys}}(V)$ and then use proposition 8.15 to recover the filtration on $\mathbf{D}_{\mathrm{tri},K}^{\mathrm{an}}(V)$. By compatibility of the constructions, we see that our definition of refinements is an extension of the one of [**BC09a**, §2.4].

We now explain how to recover our module $\mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)$ from $\mathbf{D}_{\mathrm{crys}}(V)$.

Lemma 8.20. — *Let V be an F-analytic crystabelian representation of $\mathcal{G}_K$. Then*

$$\mathbf{D}_{\mathrm{crys}}(V)\langle\langle x_K\rangle\rangle[1/x_K] = \mathbf{D}_{\mathrm{tri},K}^{n,\mathrm{an}}(V)[1/x_K].$$

Proof. — By [**KR09**, Prop. 2.2.6], we have $\mathbf{D}_{\mathrm{crys}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t_\pi])^{\mathcal{G}_{K_n}}$, and by proposition 7.5, we have $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$. In particular, if we let $D = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t_\pi]\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$, then D contains both $\mathbf{D}_{\mathrm{crys}}(V)$ and $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)$. Moreover, since $x_K = t_\pi e^{-u_K}$ and since e^{-u_K} is invertible in $\mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K}$, we have that $D = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/x_K]\{\{u_K\}\}_n)^{\mathcal{G}_{K_n}}$. Since $\frac{1}{x_K}$ is invariant under the action of $\mathcal{G}_{K_n}$, this implies that $D = \mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)[1/x_K]$. But since $\mathbf{D}_{\mathrm{crys}}(V)$ has the right dimension, we also know that $D = (\mathbf{D}_{\mathrm{crys}}(V)\otimes_K \mathbf{B}^{n,\mathrm{an}}_{\mathrm{tri},K}[1/x_K])^{\mathcal{G}_{K_n}} = \mathbf{D}_{\mathrm{crys}}(V)\otimes_K K\langle\langle x_K\rangle\rangle[1/x_K]$. □

It remains to see how one can recover $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)$ inside $\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V)[1/x_K]$. First note that we can define a filtration on $\mathbf{D}_{\mathrm{crys}}(V)\langle\langle x_K\rangle\rangle[1/x_K]$ by

$$\mathrm{Fil}^k(\mathbf{D}_{\mathrm{crys}}(V)\langle\langle x_K\rangle\rangle[1/x_K]) = \bigcup_{i+j=k} x^i\left(\mathrm{Fil}^j(\mathbf{D}_{\mathrm{crys}}(V)\right)\langle\langle x_K\rangle\rangle.$$

Proposition 8.21. — *We have*

$$\mathbf{D}^{n,\mathrm{an}}_{\mathrm{tri},K}(V) = \left\{\begin{matrix} z\in\mathbf{D}_{\mathrm{crys}}(V)\langle\langle x_K\rangle\rangle[\frac{1}{x_K}] \text{ such that } \varphi_q^{-n}(z)\in \mathrm{Fil}^0(\mathbf{D}_{\mathrm{crys}}(V)\langle\langle x_K\rangle\rangle[\frac{1}{x_K}]) \\ \hfill \text{for } n\gg 0\end{matrix}\right\}.$$

Proof. — This follows from the usual way one recovers $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ from $\mathbf{D}_{\mathrm{crys}}(V)$ (see for example [**Ber08**, §V.3]) and from the previous lemma. □

Because of proposition 8.13, one could be weary of the fact that our modules $\mathbf{D}^{\mathrm{an}}_{\mathrm{tri},K}(V)$ do not take into account what happens at a finite level inside the tower $K_\infty = \cup_{n\geq 0}K_n$. Regarding potential applications to trianguline representatons, this is already solved by theorem 8.17, but we also mention the following result, which adds some insight for why this is not a problem in the case we consider:

Proposition 8.22. — *Let V be a potentially crystalline representation of $\mathcal{G}_K$. If V is trianguline then V is crystabelian.*

Proof. — We prove the result by induction. First note that any p-adic representation of $\mathcal{G}_K$ of rank 1 which is potentially crystalline is crystabelian because such a representation factors through $\mathrm{Gal}(K^{\mathrm{ab}}/K)$.

The same arguments as in the proof of lemma 6.4 of [**Ber02**] shows that is W is a potentially crystalline B-pair which is an extension of E by E', where V and V' are two crystalline B-pairs, then W is crystalline.

Now if $W = W(V)$ is the B-pair attached to V, since V is trianguline there exists a finite extension L of $\mathbf{Q}_p$ such that $L\otimes_{\mathbf{Q}_p} W$ is an extension of E by E', where E is a rank 1 $\mathbf{B}^{\otimes L}_{|K}$-pair and E' is a rank $d-1$ $\mathbf{B}^{\otimes L}_{|K}$-pair. Since V is potentially crystalline, so are E and E', and by induction E and E' are both crystabelian, so that there exists $n\geq 0$ such that E and E' are crystalline $\mathbf{B}^{\otimes L}_{|K_n}$-pairs. Therefore $L\otimes_{\mathbf{Q}_p} W$ is a crystalline $\mathbf{B}^{\otimes L}_{|K_n}$-pair, so that V is crystabelian. □

References

[BC09a] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque **324** (2009), 1–314.

[BC09b] Olivier Brinon and Brian Conrad, *CMI summer school notes on p-adic hodge theory.*

[BC10] Laurent Berger and Gaëtan Chenevier, *Représentations potentiellement triangulines de dimension* 2, Journal de théorie des nombres de Bordeaux **22** (2010), no. 3, 557–574.

[BC16] Laurent Berger and Pierre Colmez, *Théorie de Sen et vecteurs localement analytiques*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 947–970.

[BDM19] Laurent Berger and Giovanni Di Matteo, *On triangulable tensor products of B-pairs and trianguline representations*, arXiv preprint arXiv:1906.04646 (2019).

[Ber02] Laurent Berger, *Représentations p-adiques et équations différentielles*, Inventiones mathematicae **148** (2002), no. 2, 219–284.

[Ber08] ______, *Équations differentielles p-adiques et* (φ, N)*-modules filtrés*, Astérisque **319** (2008), 13–38.

[Ber13] ______, *Multivariable Lubin-Tate* (φ, Γ)*-modules and filtered* φ*-modules*, Math. Res. Lett. **20** (2013), no. 3, 409–428.

[Ber16a] ______, *Iterated extensions and relative Lubin-Tate groups*, Ann. Math. Qué. **40** (2016), no. 1, 17–28.

[Ber16b] ______, *Multivariable* (φ, Γ)*-modules and locally analytic vectors*, Duke Math. J. **165** (2016), no. 18, 3567–3595.

[Bre98] Christophe Breuil, *Schémas en groupes et corps de normes*, 1998.

[CC98] Frédéric Cherbonnier and Pierre Colmez, *Représentations p-adiques surconvergentes*, Inventiones mathematicae **133** (1998), no. 3, 581–611.

[Col94] Pierre Colmez, *Sur un résultat de Shankar Sen*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique **318** (1994), no. 11, 983–985.

[Col02] ______, *Espaces de Banach de dimension finie*, Journal of the Institute of Mathematics of Jussieu **1** (2002), no. 3, 331–439.

[Col08a] ______, *Espaces Vectoriels de dimension finie et représentations de de Rham*, Astérisque **319** (2008), 117–186.

[Col08b] ______, *Représentations triangulines de dimension 2*, Astérisque **319** (2008), no. 213-258, 83.

[Col14] ______, *La série principale unitaire de* $GL_2(\mathbf{Q}_p)$ *: vecteurs localement analytiques*, Automorphic forms and Galois representations **1** (2014), 286–358.

[Col16] ______, *Représentations localement analytiques de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (φ, γ)*-modules*, Representation Theory of the American Mathematical Society **20** (2016), no. 9, 187–248.

[Din14] Yiwen Ding, *On some partially de rham galois representations*, arXiv preprint arXiv:1410.4597 (2014).

[Eme09] Matthew Emerton, *p-adic families of modular forms*, Séminaire Bourbaki (2009), 2009–2010.

[Eme17] ______, *Locally analytic vectors in representations of locally p-adic analytic groups*, Mem. Amer. Math. Soc. **248** (2017), no. 1175, iv+158.

[Fon90] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux (1ère partie)*, The Grothendieck Festschrift, Springer, 1990, pp. 249–309.

[Fon94a] ______, *Le corps des périodes p-adiques*, Astérisque (1994), no. 223, 59–102.

[Fon94b] ______, *Représentations p-adiques semi-stables*, Astérisque **223** (1994), 113–184.

[Fon04] ______, *Arithmétique des représentations galoisiennes p-adiques*, Astérisque **295** (2004), 1–115.

[Fou09] Lionel Fourquaux, *Applications* $\mathbf{Q}_p$*-linéaires, continues et Galois-équivariantes de* $\mathbf{C}_p$ *dans lui-même*, Journal of Number Theory **129** (2009), no. 6, 1246–1255.

[FX14] Lionel Fourquaux and Bingyong Xie, *Triangulable* $\mathcal{O}_F$*-analytic* (φ_q, Γ)*-modules of rank* 2, Algebra & Number Theory **7** (2014), no. 10, 2545–2592.

[Ked05] Kiran S Kedlaya, *Slope filtrations revisited*, Doc. Math **10** (2005), no. 447525.15.

[Kis03] Mark Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Inventiones mathematicae **153** (2003), no. 2, 373–454.

[Kis06] ______, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Springer, 2006, pp. 459–496.

[KR09] Mark Kisin and Wei Ren, *Galois representations and Lubin-Tate groups*, Doc. Math **14** (2009), 441–461.

[LT65] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Annals of Mathematics (1965), 380–387.

[M+95] Shigeki Matsuda et al., *Local indices of p-adic differential operators corresponding to Artin-Schreier-Witt coverings*, Duke Mathematical Journal **77** (1995), no. 3, 607–625.

[Maz00] Barry Mazur, *The theme of p-adic variation*, Mathematics: frontiers and perspectives (2000), 433–459.

[Nak09] Kentaro Nakamura, *Classification of two-dimensional split trianguline representations of p-adic fields*, Compositio Mathematica **145** (2009), no. 4, 865–914.

[Por20] Gal Porat, *Lubin-Tate theory and overconvergent Hilbert modular forms of low weight*, arXiv preprint arXiv:2010.14574 (2020).

[Por22] ______, *Locally analytic vector bundles on the fargues-fontaine curve*, arXiv preprint arXiv:2204.11354 (2022).

[Poy20] Léo Poyeton, *F-analytic B-pairs*, arXiv preprint arXiv:2011.04900 (2020).

[Sen80] Shankar Sen, *Continuous cohomology and p-adic Galois representations*, Inventiones mathematicae **62** (1980), no. 1, 89–116.

[ST03] Peter Schneider and Jeremy Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Inventiones mathematicae **153** (2003), no. 1, 145–196.

[Ven03] Otmar Venjakob, *On the Iwasawa theory of p-adic Lie extensions*, Compositio Mathematica **138** (2003), no. 1, 1–54.

[Win83] Jean-Pierre Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Annales scientifiques de l'Ecole Normale Supérieure, vol. 16, Société mathématique de France, 1983, pp. 59–89.

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