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Super-Convergence in Maximum Norm of the Gradient for the Shortley–Weller Method

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Abstract We prove in this paper the second-order super-convergence in L^∞-norm of the gradient for the Shortley–Weller method. Indeed, this method is known to be second-order accurate for the solution itself and for the discrete gradient, although its consistency error near the boundary is only first-order. We present a proof in the finite-difference spirit, using a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix. The proof is based on a discrete Poisson equation for the discrete gradient, with second-order accurate Dirichlet boundary conditions. The advantage of this finite-difference approach is that it can provide pointwise convergence results depending on the local consistency error and the location on the computational domain.

Keywords Finite-difference · Poisson equation · Super-convergence · Discrete Green's function · Shortley–Weller method

1 Introduction

The Shortley–Weller method is a classical finite-difference method to solve the Poisson equation with Dirichlet boundary conditions in irregular domains. It is known to converge with second-order accuracy, although the consistency error of the numerical scheme is only first-order near the boundary. Furthermore, it has been numerically observed that the gradient of the numerical solution also converges with second-order accuracy. Recently, Yoon and Min raised in [7] the issue that mathematical justifications of this super-convergence phenomenon were lacking. Then they provided in [8] a proof of this super-convergence in a discrete L^2-norm.

Here we present a proof of the super-convergence of the gradient in a discrete L^∞-norm, with a finite-difference technique. To our knowledge, all proofs of the super-convergence of

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the gradient for the Shortley–Weller method, either using finite-differences as in [8], or a finite-element formalism as in [3] and [4], have been established for discrete L^2-norms as we will discuss in Sect. 6.

Our proof is based on the use of the discrete maximum principle, for monotone matrices, following the method presented by Ciarlet [1]. This discrete maximum principle, applied to ad-hoc functions, leads us to obtain bounds on the coefficients of the inverse matrix. We first provide some notations, recall the Shortley–Weller method and present our results in Sect. 2. Then we present the technique of Ciarlet [1] adapted to our case in Sect. 3. In Sect. 4 we recall the second-order convergence of the numerical solution in the whole domain, and the third-order convergence near the boundary. We use this property in Sect. 5 to formulate a discrete Poisson equation for the discrete gradient, with Dirichlet boundary conditions that are second-order accurate, and we finally prove the second-order convergence of the gradient. We compare our approach to the literature in Sect. 6.

2 Notations and Statement of Results

In the following, we consider a domain Ω belonging to $\mathbb{R}^2$ or $\mathbb{R}^3$, with a boundary Γ. The Shortley–Weller method is aimed to solve the Poisson equation in the domain Ω with Dirichlet conditions on Γ:

$$\begin{cases} -\triangle u = f \text{ in } \Omega, \\ u = g \text{ on } \Gamma. \end{cases} \tag{1}$$

For our analysis, which is based on a finite-differences formulation, we need:

a) that a unique solution u of (1) exists and is smooth enough for our consistency error analyses to be valid.
b) that the solution of problem (1) with $f = 1$ and $g = 0$ is C^1 near the boundary, because this property provides us estimates of the discrete Green functions in Sect. 4.

Consequently, for the sake of simplicity, we assume in the whole paper that the source term f, the boundary Γ, and the boundary condition g are such that these two properties are satisfied. In this context, the boundary Γ may not necessarily be smooth. For instance, it may be only piecewise smooth and have corners, as soon as conditions a) and b) are satisfied. However it is known that if the boundary has corners with angles greater than a limit value, then the solution may lose its regularity near these corners. In this case, our analysis is not valid anymore. This behavior is illustrated in the appendix. Let us notice that the case of convergence when singularities occur near the interface has been handled in [3] with a finite-element approach, obtaining a $O(h^{1.5})$ convergence in a discrete H^1-norm.

The problem (1) is discretized on a uniform cartesian grid, see Fig. 1. For the sake of clarity, the figures will represent the discretization points in two dimensions only, but the formulation of the problem and the proofs of convergence will be presented in three dimensions. The grid spacing is denoted by h, and the coordinates of the points on the grid are defined by $(x_i, y_j, z_k) = (i\,h, j\,h, k\,h)$. The points on the cartesian grid are named either with letters such as P or Q, or with letters and indices such as $M_{i,j,k} = (x_i, y_j, z_k)$ if we need to have informations about the location of the point.

The set of grid points located inside the domain Ω is denoted by Ω_h. These points are called interior nodes. The set of points located at the intersection of the axes of the grid and the boundary Γ is denoted by Γ_h. These points are called boundary nodes and are used for imposing the boundary conditions in the numerical scheme, see Fig. 1 for an illustration.

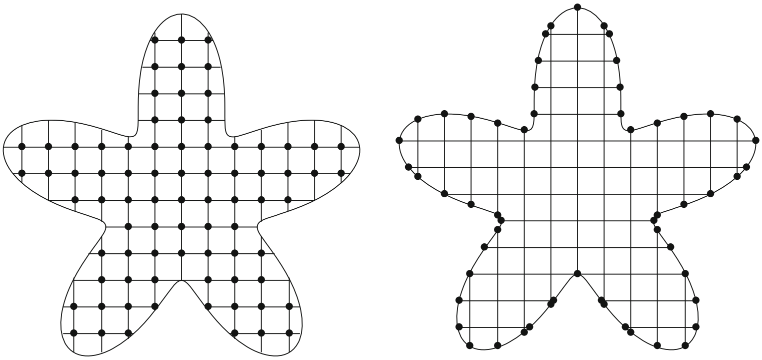

Fig. 1 *Left*: interior nodes, belonging to Ω_h, *right*: boundary nodes, belonging to Γ_h

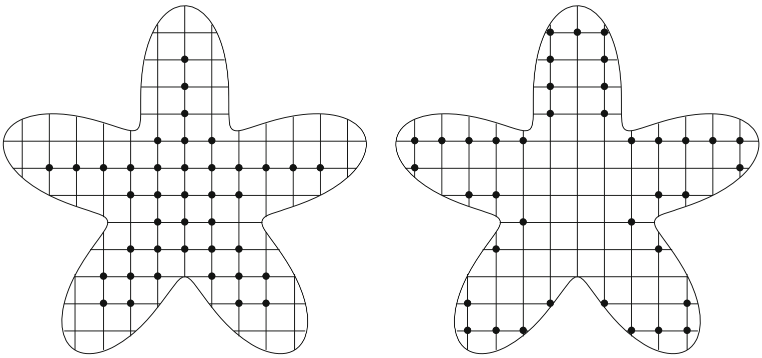

Fig. 2 *Left*: regular nodes, i.e. belonging to Ω_h^{**}, *right*: irregular nodes, i.e. belonging to Ω_h^{*}.

We say that a grid node is regular if none of its direct neighbors is on the boundary Γ_h, and that it is irregular if at least one of its neighbors belongs to Γ_h. The set of regular grid nodes is denoted by Ω_h^{**}, and the set of irregular grid nodes is denoted by Ω_h^{*}. See Fig. 2 for an illustration. The Shortley–Weller scheme for solving the Poisson equation with Dirichlet boundary conditions is based on a dimension by dimension approach. In the following, for the sake of clarity we use the same notations as in the paper of Yoon and Min [7].

Let the six neighboring nodes of a grid node P inside the domain be named as $P_i, \ 1 \leq i \leq 6$ and the distances between P and these nodes as $h_i, \ 1 \leq i \leq 6$. If P is a regular node then $h_i = h$ for all $1 \leq i \leq 6$. If P is an irregular node then at least one of the h_i is different from h.

The discretization of the Laplace operator with the Shortley–Weller method reads:

$$
\begin{aligned}
-\triangle_h u_h(P) = &\left(\frac{2}{h_1 h_2} + \frac{2}{h_3 h_4} + \frac{2}{h_5 h_6}\right) u_h(P) - \frac{2}{h_1(h_1+h_2)} u_h(P_1) \\
&- \frac{2}{h_2(h_1+h_2)} u_h(P_2) - \frac{2}{h_3(h_3+h_4)} u_h(P_3) - \frac{2}{h_4(h_3+h_4)} u_h(P_4) \\
&- \frac{2}{h_5(h_5+h_6)} u_h(P_5) - \frac{2}{h_6(h_5+h_6)} u_h(P_6).
\end{aligned}
$$

The matrix associated with this linear system has all its diagonal entries strictly positive, all off-diagonal entries nonpositive (or negative or zero) and is irreducibly diagonally dominant.

Consequently it is a monotone matrix. Therefore, all coefficients of the inverse matrix are positive. This property will allow us to apply a discrete maximum principle useful to bound the coefficients of the inverse matrix.

We denote by u_h the numerical solution of problem (1) with the Shortley–Weller method. The local error on a node P is defined by $e_h(P) = u(P) - u_h(P)$. We denote by $\phi(P)$ the distance between a node P and the boundary Γ. The following result, presented in [5] and in [8], will be useful for our purpose, because it provides second-order boundary conditions for a discrete Laplace operator applied to the components of the gradient.

Theorem 1 *For the Shortley–Weller method, the local error $e_h(P)$ at node P satisfies*

$$|e_h(P)| \le O(h^2) \quad \forall P \in \Omega_h,$$
$$|e_h(P)| \le O(h^2)\Big(\phi(P) + \min(h_i)\Big), \quad \forall P \text{ such that } \phi(P) = O(h),$$

with $h_i,\ 1 \le i \le 6$ defined as above.We will briefly recall in Sect. 4 the proof of this theorem in the formalism of discrete Green functions, in spite of its redundancy with the references above, because it will help us to introduce our notations and to present the proof of convergence of the discrete gradient.

Concerning the convergence of the gradient, in practice, we will only study the convergence of the discrete version of $\partial_x u$, because the $x-$, $y-$ and $z-$directions have symmetric behaviors.

We define S_h as:

$$S_h = \{P,\ P \text{ middle of } [MN], M, N \in \Omega_h \cup \Gamma_h, M \text{ and } N \text{ adjacent in the x-direction.}\} \tag{2}$$

We define the discrete x-derivative $D_x u_h(P)$ at every point P in S_h as

$$D_x u_h(P) = \frac{u_h(M) - u_h(N)}{x_M - x_N}, \tag{3}$$

where M and N are the points belonging to $\Omega_h \cup \Gamma_h$ such that P is defined as the middle of $[MN]$.

We divide S_h into two new subsets of points (see Fig. 3):

$$\tilde{\Omega}_h = \{P \in S_h,\ \text{all direct neigbors of } P \text{ in } S_h \text{ are at distance } h \text{ from } P\},$$
$$\tilde{\Gamma}_h = S_h \setminus \tilde{\Omega}_h.$$

By construction, it is possible to apply the classical second-order seven points stencil for the Laplacian to all points belonging to $\tilde{\Omega}_h$. Remark also that the points in $\tilde{\Gamma}_h$ satisfy by construction the property

$$\phi(P) \le 3h \quad \forall P \in \tilde{\Gamma}_h. \tag{4}$$

Theorem 2 *For the Shortley–Weller method, the local error on the discrete x-derivative is second-order accurate:*

$$|\partial_x u(P) - D_x u_h(P)| \le O(h^2), \quad \forall P \in S_h.$$

3 Discrete Maximum Principle to Prove Convergence

Here we recall the principle of the method presented in [1] to prove high-order convergence for finite-differences operators with the help of the discrete maximum principle. As we do

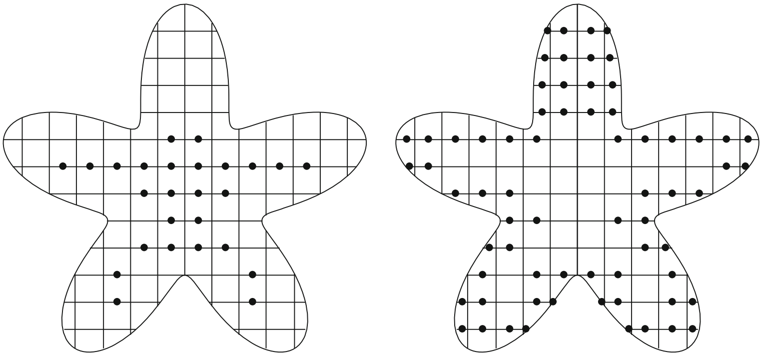

Fig. 3 *Left*: nodes belonging to $\tilde{\Omega}_h$, *right*: nodes belonging to $\tilde{\Gamma}_h$

not use exactly the same discretization matrix as in [1], due to the different way to account for boundary conditions, we present the reasoning in our case.

3.1 Discrete Green's Function

For each $Q \in \Omega_h$, define the discrete Green's function $G_h(:, Q) = \Big(G_h(P, Q)\Big)_{P\in\Omega_h\cup\Gamma_h}$ as the solution of the discrete problem:

$$\begin{cases} -\triangle_h G_h(:, Q)(P) = \begin{cases} 0, & P \neq Q \\ 1, & P = Q \end{cases} \quad P \in \Omega_h, \\ \qquad G_h(P, Q) = 0, \quad P \in \Gamma_h. \end{cases} \tag{5}$$

In fact, each discrete Green function $G_h(:, Q)$ represents a column of the inverse matrix of the discrete operator $(-\triangle_h)$. The matrix of $(-\triangle_h)$ being monotone, as we noticed in Sect. 2, it means that all values of $G_h(:, Q)$ are positive.

With this definition we can write the solution of the numerical problem as a sum of the source terms multiplied by the local values of the discrete Green function:

$$u_h(P) = \sum_{Q\in\Omega_h} G_h(P, Q)\,(-\triangle_h u_h)(Q), \quad \forall P \in \Omega_h.$$

In this formula we assume that $u_h \equiv 0$ on Γ_h. However, if one wants to impose non-homogeneous Dirichlet boundary conditions, it is possible to take them into account by modifying the source terms for the nodes belonging to Ω_h^*.

3.2 Estimating the Coefficients of the Discrete Green's Function

Lemma 1 *Let S be a subset of grid nodes (thus corresponding also to a subset of the indices of the matrix), W a discrete function with $W \geq 0$ on Γ_h, and $\alpha > 0$ such that:*

$$\begin{cases} (-\triangle_h W)(P) \geq 0 \quad \forall P \in \Omega_h, \\ (-\triangle_h W)(P) \geq \alpha^{-i} \text{ for all } P \in S. \end{cases}$$

Then

$$\sum_{Q\in S} G_h(P, Q) \leq \alpha^i W(P).$$

Proof of Lemma 1: Using the definition of the discrete Green function, we can write

$$-\triangle_h\Big(\sum_{Q\in S} G_h(:,Q)\Big)(P)=\begin{cases}1 & \text{if } P\in S,\\ 0 & \text{if } P\notin S.\end{cases}$$

Therefore,

$$-\triangle_h\Big(W-\alpha^{-i}\sum_{Q\in S} G_h(:,Q)\Big)(P)\geq 0 \quad \forall P\in \Omega_h.$$

As all coefficients of the inverse of $-\triangle_h$ are positive, and $W-\alpha^{-i}\sum_{Q\in S} G_h(P,Q)$ is positive on Γ_h, it leads to

$$W(P)-\alpha^{-i}\sum_{Q\in S} G_h(P,Q)\geq 0 \quad \forall P\in \Omega_h,$$

and finally we obtain an estimate of the coefficients of $\sum_{Q\in S} G_h(:,Q)$ in terms of the coefficients of W:

$$\sum_{Q\in S} G_h(P,Q)\leq \alpha^i W(P).$$

□

4 Reminder of the Proof of High-Order Convergence of the Solution

This section is devoted to a short reminder of the proof of Theorem 1.Adequate subsets S and functions W are used to prove second-order convergence in L^∞-norm in the whole numerical domain, and third-order convergence for the grid nodes whose distance to the boundary is $O(h)$.

Proof of Theorem 1: We denote by $\tau(P)$ the consistency error of the Shortley–Weller method on a point P belonging to Ω_h. With a classical Taylor expansion one can prove that

$$\tau(P)=\begin{cases} O(h^2) \text{ if } P\in\Omega_h^{**},\\ O(h) \text{ if } P\in\Omega_h^{*}.\end{cases}$$

The local error satisfies the same linear system as the numerical solution $u_h(P)$, but with the consistency error as a source term:

$$-\triangle_h e_h(P)=\tau(P) \quad \forall P\in\Omega_h.$$

We consider a point $M=(x_M,y_M,z_M)$ inside Ω.

$$W(Q)=\frac{C-(x_Q-x_M)^2-(y_Q-y_M)^2-(z_Q-z_M)^2}{6},$$

with (x_Q,y_Q,z_Q) the coordinates of the point Q, and C such that $W(Q)\geq 0$ for all $Q\in\Omega_h$. For instance we take $C=2\,(diam(\Omega))^2$. We can write

$$-\triangle_h W(P)=1, \quad \forall P\in\Omega_h.$$

□

Therefore, using Lemma 1:

$$\sum_{Q\in\Omega_h^{**}} G_h(P,Q) \le \sum_{Q\in\Omega_h} G_h(P,Q) \le W(P) \le \frac{(diam(\Omega))^2}{6}, \quad \forall P \in \Omega_h. \tag{6}$$

Now we define the discrete function

$$\tilde{W}(Q) = \begin{cases} 0 & \text{if } Q \in \Gamma_h, \\ 1 & \text{otherwise.} \end{cases}$$

This function satisfies

$$\begin{cases} -\triangle_h \tilde{W}(Q) \ge \frac{1}{h^2} & \text{if } Q \in \Omega_h^*, \\ -\triangle_h \tilde{W}(Q) = 0 & \text{otherwise.} \end{cases}$$

and we can directly write, using Lemma 1,

$$\sum_{Q\in\Omega_h^{*}} G_h(P,Q) \le h^2 \tilde{W}(P) \le h^2, \quad \forall P \in \Omega_h. \tag{7}$$

Combining (6) and (7), we obtain a second-order estimate of the local error on every point $P \in \Omega_h$:

$$\begin{aligned} |e(P)| = |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)| &\le \sum_{Q\in\Omega_h^{**}} G_h(P,Q)O(h^2) \\ &+ \sum_{Q\in\Omega_h^{*}} G_h(P,Q)O(h) \le O(h^2). \end{aligned}$$

We define $V_h = \sum_{Q\in\Omega_h^*} G_h(:,Q)$. Let us consider a point P in Ω_h^*. The discretization of the Laplace operator with the Shortley–Weller method reads:

$$\begin{aligned} -\triangle_h V_h(P) = &\left(\frac{2}{h_1h_2} + \frac{2}{h_3h_4} + \frac{2}{h_5h_6}\right) V_h(P) - \frac{2}{h_1(h_1+h_2)} V_h(P_1) \\ &- \frac{2}{h_2(h_1+h_2)} V_h(P_2) \\ &- \frac{2}{h_3(h_3+h_4)} V_h(P_3) - \frac{2}{h_4(h_3+h_4)} V_h(P_4) - \frac{2}{h_5(h_5+h_6)} V_h(P_5) \\ &- \frac{2}{h_6(h_5+h_6)} V_h(P_6). \end{aligned}$$

We assume for instance that only P_1 belongs to Γ_h. Consequently, $h_1 < h$, $h_i = h$ for $2 \le i \le 6$ and $V_h(P_1) = 0$. We can write

$$\left(\frac{2}{h_1h} + \frac{4}{h^2}\right) V_h(P) = \frac{2V_h(P_2)}{(h_1+h)h} + \frac{V_h(P_3)}{h^2} + \frac{V_h(P_4)}{h^2} + \frac{V_h(P_5)}{h^2} + \frac{V_h(P_6)}{h^2} + 1.$$

Using (7) we obtain

$$\frac{2}{h_1h} V_h(P) \le \left(\frac{2}{h_1h} + \frac{4}{h^2}\right) V_h(P) \le 7.$$

We can rewrite this formula as

$$\sum_{Q\in\Omega_h^*} G_h(P,Q) \le \frac{7}{2}\min(h_i)h. \tag{8}$$

Similar inequalities can be obtained if more than one point of the stencil for P belongs to Γ_h.

Now we consider the elliptic problem

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases} \tag{9}$$

We denote v the solution of this problem. We have assumed in the introduction that v was C^1 near the boundary. Therefore, it satisfies $v(P) = O(\phi(P))$ for points P such that $\phi(P) = O(h)$.

The operator $-\Delta_h$ is consistent on every grid node, consequently for h small enough

$$-\Delta_h v(P) \ge \frac{1}{2}, \quad \forall P \in \Omega_h.$$

Therefore, using Lemma 1 and restricting ourselves to Ω_h^{**},

$$\begin{aligned} \sum_{Q\in\Omega_h^{**}} G_h(P,Q) &\le \sum_{Q\in\Omega_h} G_h(P,Q) \le 2\,v(P) \\ &\le O(\phi(P)), \quad \forall P \text{ such that } \phi(P) = O(h). \end{aligned} \tag{10}$$

Finally, combining (10) and (8) we can write for all points P such that $\phi(P) = O(h)$

$$\begin{aligned} |e_h(P)| = |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)| &\le |\sum_{Q\in\Omega_h^{**}} G_h(P,Q)\,O(h^2)| + |\sum_{Q\in\Omega_h^{*}} G_h(P,Q)\,O(h)|, \\ &\le O(h^2)(\phi(P) + \min(h_i)). \end{aligned} \tag{11}$$

5 Second-Order Convergence for the Discrete Gradient

This section is devoted to the proof of Theorem 2. We aim to prove that the numerical gradient defined by (3) converges with second-order accuracy in L^∞-norm.

Proof of Theorem 2: The proof is divided into two parts:

- First we prove that the discrete gradient is second-order accurate for every point P in $\tilde{\Gamma}_h$,
- then we prove that the discrete gradient is second-order accurate for all points P in S_h by means of a discrete Laplace equation applied to it.

We know from Sect. 4 that e_h, error for the numerical solution computed with the Shortley–Weller method, satisfies

$$|e_h(Q)| \le O(h^2)\Big(\phi(Q) + \min(h_i)\Big), \quad \forall Q \in \Omega_h \text{ such that } \phi(Q) \le 4\,h,$$

with h_i, $1 \le i \le 6$, the distances between the node Q and its direct neighbors in each direction. Because of (4), all points involved in the definition of the discrete gradient for points belonging to $\tilde{\Gamma}_h$ are at a distance to the boundary smaller than $4h$.

Let us consider a point P belonging to $\tilde{\Gamma}_h$. We denote by M and N the points such that P is their middle. To get estimates on the discrete x-derivative on P, one has to consider two possibilities: □

- One of them belongs to Γ_h:
Let us assume for instance that M belongs to Γ_h and N belongs to Ω_h. In this case, we can write that $\phi(N) \leq |x_M - x_N|$. It means that

$$e_h(M) = 0,$$
$$|e_h(N)| \leq O(h^2)\Big(\phi(N) + \min(h_i)\Big) \leq O(h^2)\Big(|x_M - x_N| + \min(h_i)\Big).$$

We use these estimates to bound the discrete x-derivative $D_x u_h(P)$:

$$\begin{aligned}|\partial_x u(P) - D_x u_h(P)| &= \left|\frac{u(M) - u(N)}{x_M - x_N} + O(x_M - x_N)^2 - \frac{u_h(M) - u_h(N)}{x_M - x_N}\right|,\\ &\leq \left|\frac{e_h(M) - e_h(N)}{x_M - x_N}\right| + O(x_M - x_N)^2,\\ &\leq \left|\frac{O(h^2)\Big(|x_M - x_N| + \min(h_i)\Big)}{x_M - x_N}\right| + O(x_M - x_N)^2.\end{aligned}$$

Moreover, because $h_i \leq |x_M - x_N| \leq h$ for all $i = 1, \ldots, 6$, for the h_i corresponding to the node N, we conclude that

$$|\partial_x u(P) - D_x u_h(P)| \leq O(h^2).$$

- Both of them belong to Ω_h:
In this case, $x_M - x_N = h$ and we have

$$|e_h(M)| \leq O(h^3),$$
$$|e_h(N)| \leq O(h^3).$$

We can again write

$$\begin{aligned}|\partial_x u(P) - D_x u_h(P)| &\leq \left|\frac{e_h(M) - e_h(N)}{x_M - x_N}\right| + O(x_M - x_N)^2,\\ &\leq \left|\frac{O(h^3)}{h}\right| + O(h^2),\\ &\leq O(h^2).\end{aligned}$$

Therefore, if P belongs to $\tilde{\Gamma}_h$, then the discrete x-derivative $D_x u_h(P)$ is a second-order accurate approximation of the x-derivative of u at point P:

$$|\partial_x u(P) - D_x u_h(P)| \leq O(h^2), \quad \forall P \in \tilde{\Gamma}_h. \tag{12}$$

Now we consider a point P belonging to $\tilde{\Omega}_h$. Let M and N be the points such that P is their middle. We apply the discrete x-derivative to the formula of the discrete elliptic operator on the points M and N:

$$D_x(-\triangle_h u_h)(P) = \frac{(-\triangle_h u_h)(M) - (-\triangle_h u_h)(N)}{x_M - x_N}.$$

Because u_h is the numerical solution of the linear system (1), we know that

$$D_x(-\triangle_h u_h)(P) = D_x f(P), \quad \forall P \text{ in } \tilde{\Omega}_h.$$

The node P belongs to $\tilde{\Omega}_h$, which means that in each direction the direct neighbors of P in S_h are at the same distance h from P. The discrete operator $-\triangle_h$ applied to points belonging to $\tilde{\Omega}_h$ thus reduces to the classical second-order seven-point stencil. Consequently, on this point, the discrete operators $-\Delta_h$ and D_x commute, and we can write

$$D_x(-\Delta_h u_h)(P) = -\Delta_h(D_x u_h)(P), \quad \forall P \in \tilde{\Omega}_h.$$

Consequently, the array $v_h = (D_x u_h(P))_{P\in S_h}$ satisfies the linear system

$$-\Delta_h v_h(P) = D_x f(P) \quad \forall P \in \tilde{\Omega}_h, \tag{13}$$

$$v_h(P) = D_x u_h(P) \quad \forall P \in \tilde{\Gamma}_h, \tag{14}$$

which is a discrete version of the Laplace operator applied to the $x-$ derivative of u solution of (1). The consistency errors for this linear system are the following:

- The discretization of the Laplace operator (13) has the consistency error $\tau(P) = O(h^2)$ for all nodes belonging to $\tilde{\Omega}_h$, because the Shortley–Weller scheme reduces for these nodes to the classical centered seven-points formula, and because $D_x f(P)$ is a second-order approximation of the $x-$derivative of f at point P.
- The formula (14) has the consistency error $\tau(P) = O(h^2)$ because we know from (12) that $D_x u_h(P)$ is a second-order accurate approximation of the x-derivative of u at a point P in $\tilde{\Gamma}_h$.

The rows of the matrix associated with this linear system correspond either to the discretization of the Laplacian operator (13), or the identity (14). This matrix has therefore all its diagonal terms strictly positive, all off-diagonal entries nonpositive (or negative or zero) and is irreducibly diagonally dominant. Consequently it is a monotone matrix.

We apply the same reasoning as in Sect. 3 to obtain estimates on the coefficients of the inverse matrix. As in the previous subsection, we denote by $G_h(:, Q) = \Big(G_h(P, Q)\Big)_{P\in\tilde{\Omega}_h\cup\tilde{\Gamma}_h}$ the column of the inverse matrix of the linear system (13)–(14) corresponding to a point Q belonging to $\tilde{\Omega}_h \cup \tilde{\Gamma}_h$.

We consider a point $M = (x_M, y_M, z_M)$ inside Ω. We define the discrete function on all points P in S_h:

$$W(P) = \frac{C - (x_P - x_M)^2 - (y_P - y_M)^2 - (z_P - z_M)^2}{6},$$

with (x_P, y_P, z_P) the coordinates of the point P, and C such that $W(Q) \geq 1$ for all $Q \in \Omega_h$. For instance we take $C = 2\,(diam(\Omega))^2 + 6$. We can write

$$-\Delta_h W(P) = 1, \quad \forall P \in \tilde{\Omega}_h,$$

$$W(P) \geq 1, \quad \forall P \in \tilde{\Gamma}_h.$$

Therefore, using Lemma 1:

$$\sum_{Q\in\tilde{\Omega}_h\cup\tilde{\Gamma}_h} G_h(P, Q) \leq W(P) \leq \frac{(2diam(\Omega))^2 + 6}{6}, \quad \forall P \in S_h. \tag{15}$$

Therefore, the expression on the local error for the discrete x-derivative on a node P belonging to S_h reads

$$|\partial_x u(P) - D_x u_h(P)| = |\sum_{Q\in\tilde{\Omega}_h\cup\tilde{\Gamma}_h} G_h(P, Q)\tau(Q)|$$

$$\leq \frac{(2diam(\Omega))^2 + 6}{6} O(h^2), \quad \forall P \in S_h.$$

Consequently

$$|\partial_x u(P) - D_x u_h(P)| \leq O(h^2), \quad \forall P \in S_h, \tag{16}$$

which proves that the numerical gradient converges with second-order accuracy in L^∞-norm.

6 Discussion

This work was originally motivated by the remark in the paper of Yoon and Min [6] about the lack of mathematical analysis about the super-convergence of the Shortley–Weller method. This paper was followed by [8] where the autors provided a proof of this super-convergence in a discrete L^2-norm, using a discrete divergence theorem.

To our knowledge, few other works in the literature have studied the super-convergence of the gradient for elliptic finite-difference schemes, among them [2,4] and [3].

In [2] Ferreira and Grigorieff deal with more general elliptic operators, with variable coefficients and mixed derivatives, and prove second-order convergence in H^1 norm.The proof uses negative norms and is based on the fact that the finite difference scheme is a certain non-standard finite element scheme on triangular grids combined with a special form of quadrature.

In [4] Li et al. study the super-convergence of solution derivatives for the Shortley–Weller method for Poisson's equation, considering also this method as a special kind of finite element method. They obtained second-order convergence in H^1 norm for rectangular domains, and an order 1.5 for polygonal domains. The work in [3] adresses the case of unbounded derivatives near the boundary Γ, on polygonal domains.

Our approach differs from the latter because we do not use a finite-element approach. Instead we propose a proof based on a finite-difference analysis, which is a variant of the method presented in [1]: we use a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix, but in our case the bound on the coefficients can also vary with the rows of the inverse matrix. This variant is useful to obtain a specific bound for points located near the boundary and obtain the third-order convergence of the solution at these points, already presented in [5] and in [8]. This intermediate result leads us to formulate a discrete Poisson equation for the discrete gradient, with Dirichlet boundary conditions that are second-order accurate. Then the same maximum-principle methodology is applied to the discrete gradient, leading to second-order accuracy.

The approach developed in this paper has the advantage to be simple to carry out, and to be able to provide locally pointwise estimates, instead of the usual convergence results in the discrete L^2- or H^1-norms.

7 Conclusion

We have proven that the discrete gradient obtained by the Shortley–Weller method for the Poisson equation converges with second-order accuracy in L^∞-norm. This is a super-convergence property because the numerical solution itself converges only with second order accuracy in L^∞-norm. This property is proven with a variant of Ciarlet's technique to obtain high-order convergence estimates for monotone finite-differences matrices. With carefully chosen test functions we are able to bound the coefficients of the discrete Green functions associated with the matrix of the Shortley–Weller method. One key ingredient is the discrete gradient as the solution of another Poisson equation with Dirichlet boundary conditions that have a second-order accuracy. A further development would be to extend this work to the case of more general elliptic operators.

A Numerical Illustration: Corners and Regularity of the Solution

We illustrate here the possible loss of regularity of the solution in the case of corners, that was evocated in Sect. 2. Depending on the angles of this corners, the solution can indeed be less than C^1 near the boundary, even if the source term and the boundary conditions are very smooth.

On Fig. 4, we consider two domains that are only piecewise smooth: the diamond-shaped one, denoted by Ω_1, and its complementary, denoted by Ω_2. The angles of the first one do not exceed the value π, while some of the second one do actually.

We solve numerically the following problem

$$\begin{cases} -\triangle u = 1 \text{ on } \Omega, \\ u = 0 \text{ on } \Gamma. \end{cases} \tag{17}$$

in these two domains: $\Omega = \Omega_1$ and $\Omega = \Omega_2$, with the Shortley–Weller method. The numerical solution is in fact the sum of all the discrete Green functions associated to grid points in the numerical domain. In this paper, to obtain the estimate (10) on the discrete Green function, we make the assumption that the solution u of problem (17) is at least C^1 near the boundary, so that it satisfies $u(x) = O(h)$ for points located at a distance $O(h)$ of the domain boundary.

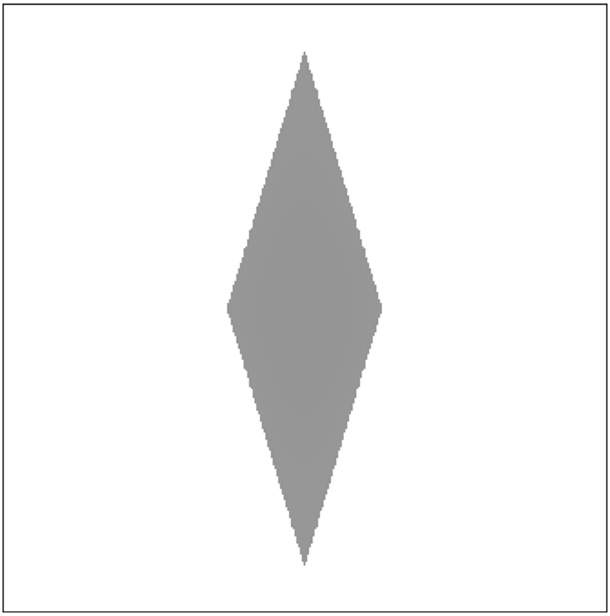

Fig. 4 Example of domains with corners

Table 1 Convergence to zero for irregular grid points for problem (17) in domain Ω_1

N	L^∞ norm	Convergence order
100	4.80E−003	–
200	2.45E−003	0.970
400	1.26E−003	0.965
600	8.43E−004	0.971
800	6.34E−004	0.97 3
1000	5.077E−004	0.976

Table 2 Convergence to zero for irregular grid points for problem (17) in domain Ω_2

N	L^∞norm	Convergence order
100	1.462E−002	–
200	8.104E−003	0.856
400	4.503E−003	0.853
600	3.195E−003	0.851
800	2.506E−003	0.850
1000	2.075E−003	0.850

We compute in both cases, the L^∞-norm of the numerical solution on grid points in Ω_h^* (that is, irregular grid points). The Tables 1 and 2 present these results. We observe that for domain Ω_1, the numerical solution on Ω_h^* converges to zero at order one. For domain Ω_2, the convergence order is strictly smaller than one, which means that the discrete Green functions do not satisfy the property that we need for our convergence estimates.

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