A proof in the finite-difference spirit of the superconvergence of the gradient for the Shortley-Weller method.

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Abstract: We prove in this paper the second-order super-convergence of the gradient for the Shortley-Weller method. Indeed, with this method the discrete gradient is known to converge with second-order accuracy even if the solution itself only converges with second-order. We present a proof in the finite-difference spirit, inspired by the paper of Ciarlet [2] and taking advantage of a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix. This reasoning leads us to prove third-order convergence for the numerical solution near the boundary of the domain, and then second-order convergence for the discrete gradient in the whole domain. The advantage of this finite-difference approach is that it can provide locally pointwise convergence results depending on the local truncation error and the location on the computational domain, as well as convergence results in maximum norm.

Key-words: Finite-difference, Poisson equation, super-convergence, discrete Green's function, Shortley-Weller method

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Une preuve par différences finies de la superconvergence du gradient avec la méthode de Shortley-Weller.

Résumé : Nous présentons dans ce rapport une preuve de la super-convergence à l'ordre deux du gradient pour la méthode de Shortley-Weller. En effet, avec cette méthode le gradient discret converge à l'ordre deux même si la solution elle-même converge aussi à l'ordre deux seulement. La preuve est réalisée avec des techniques de différences finies, inspirées par l'article de Ciarlet [2], et utilisant un principe du maximum discret pour obtenir des estimations des coefficients de la matrice inverse. Ce raisonnement nous permet de prouver que la solution numérique converge à l'ordre trois près du bord du domaine, puis que le gradient discret converge à l'ordre deux dans tout le domaine. Cette approche par différences finies permet d'obtenir des résultats de convergence locaux, en fonction des différentes valeurs de l'erreur de troncature et de la position du point considéré sur le domaine de calcul. Elle permet aussi d'obtenir des résultats en norme du maximum.

Mots-clés : Différences finies, super-convergence, fonction de Green discrète, méthode de Shortley-Weller

Contents

1 Introduction

The Shortley-Weller method is a classical finite-difference method to solve the Poisson equation with Dirichlet boundary conditions in irregular domains. It is known to converge with second-order accuracy, although the truncation error of the numerical scheme is only first-order near the boundary. Furthermore, it has been numerically observed that the gradient of the numerical solution also converges with second-order accuracy. Recently, Yoon and Min raised in [8] the issue that mathematical justifications of this super-convergence phenomenon were lacking. Indeed, to our knowledge, only finite-element analyses of this phenomenon are available in the literature, as we will discuss in §6.

Here we prove the super-convergence of the gradient with a finite-difference technique. We propose a variant of the method introduced by Ciarlet in [2].This method is based on the use of the discrete maximum principle, for monotone matrices, leading us to obtain bounds on the coefficients of the inverse matrix. We first provide some notations, recall the Shortley-Weller method and present our results in §2. Then we present the technique of Ciarlet [2] adapted to our case in §3. We prove with this technique that the numerical solution converges with second-order accuracy in the whole domain, and with third-order accuracy near the boundary in §4. This intermediate result leads us to formulate in §5 a discrete Poisson equation for the discrete gradient, with Dirichlet boundary conditions that are second-order accurate, and finally to prove the second-order convergence of the gradient. We compare our approach to the literature in §6.

2 Notations and statement of results

The Shortley-Weller method is aimed to solve the Poisson equation in a domain Ω with an arbitrary shape, satisfying Dirichlet conditions on the boundary Γ:

$$\begin{cases} -\triangle u = f \text{ on } \Omega, \\ u = g \text{ on } \Gamma. \end{cases} \tag{1}$$

In the whole paper, we assume that the source term f is such that the solution u exists and is smooth enough so that our truncation error analyses are valid. For instance, f belongs to $C^2(\bar{\Omega})$ and u belongs to $C^3(\bar{\Omega})$.

The problem (1) is discretized on a uniform cartesian grid, see Fig. 1. The grid spacing is denoted h, and the coordinates of the points on the grid are defined by $(x_i, y_j) = (i\,h, j\,h)$. The points on the cartesian grid are named either with letters such as P or Q, or with letters and indices such as $M_{i,j} = (x_i, y_j)$ if one wants to have informations about the location of the point.

The set of grid points located inside the domain Ω is denoted Ω_h. The set of points located at the intersection of the axes of the grid and the boundary Γ is denoted Γ_h. These points are used for imposing the boundary conditions in the numerical scheme. See Fig. 1 for an illustration. We say that a grid node is regular if none of its direct neighbors is on the boundary Γ_h, and that it is near the boundary if at least one of its neighbors belongs to Γ_h. The set of regular grid nodes is denoted Ω_h^{**}, and the set of grid nodes near the boundary is denoted Ω_h^{*}. See Fig. 2 for an illustration.

The Shortley-Weller scheme for solving the Poisson equation with Dirichlet boundary conditions is based on a dimension by dimension approach. In the following, for the sake of clarity we use the same notations as in the paper of Min [8].

Let the four neighboring nodes of a grid node P inside the domain be named as P_1,P_2,P_3,P_4, and the distances to the neighbors as h_1,h_2, h_3 and h_4. Some of the neighboring nodes may be the points on the boundary, thus the distances may be different to each other near the interface

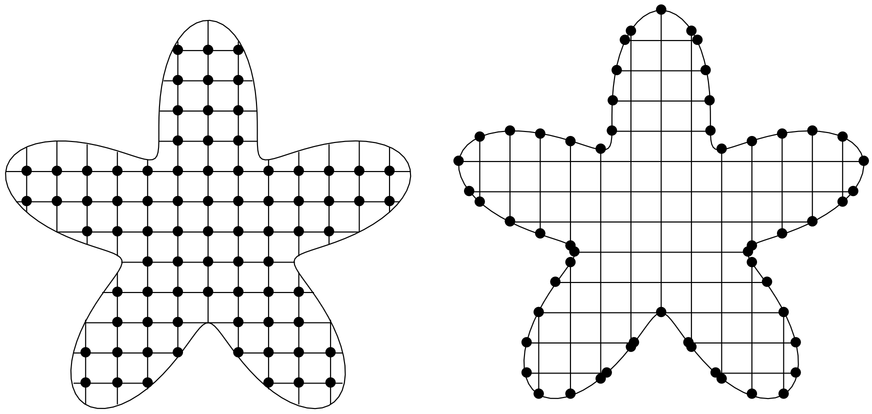

Figure 1: Left: nodes belonging to Ω_h, right: nodes belonging to Γ_h.

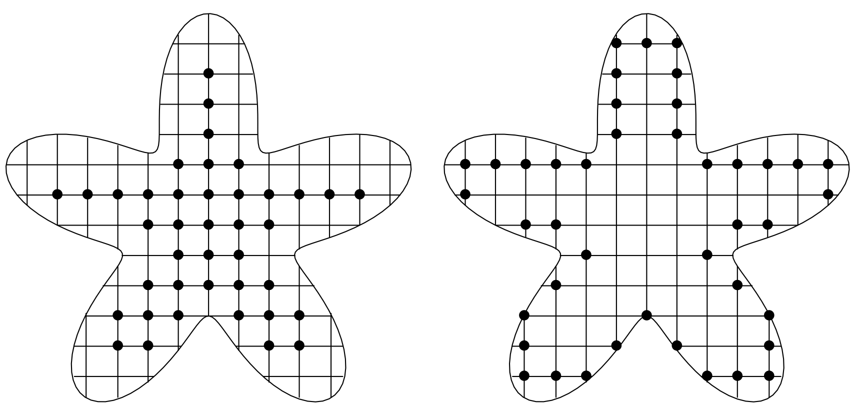

Figure 2: Left: nodes belonging to Ω_h^{**}, right: nodes belonging to Ω_h^{*}.

while they are all equal to h away from the boundary. The discretization of the Laplace operator with the Shortley-Weller method reads:

$$-\triangle_h u_h(P) =(\frac{2}{h_1 h_3} + \frac{2}{h_2 h_4})u_h(P) - \frac{2}{h_1(h_1+h_3)}u_h(P_1) - \frac{2}{h_3(h_1+h_3)}u_h(P_3)$$
$$- \frac{2}{h_2(h_2+h_4)}u_h(P_2) - \frac{2}{h_4(h_2+h_4)}u_h(P_4).$$

The matrix associated with this linear system has all diagonal terms strictly positive, all extra-diagonal terms negative and is irreductibly diagonally dominant. Consequently it is a monotone matrix. Therefore, all coefficients of the inverse matrix are positive. This property will allow us to apply a discrete maximum principle useful to bound the coefficients of the inverse matrix.

In the paper of Min, the second-order convergence of the solution itself was adressed. Here we propose another version of the convergence proof, leading notably to third order estimates for the convergence on points belonging to Ω_h^*. This third-order convergence will be useful in the convergence proof of the discrete gradient of the solution, because it will provide second-order boundary conditions for an Laplace operator applied to the components of the gradient.

We note u_h the numerical solution of problem (1) with the Shortley-Weller method. The local error on a node P is defined by $e_h(P) = u(P) - u_h(P)$.

Theorem 1. *For the Shortley-Weller method, the local error $e_h(P)$ satisfies*

$$|e_h(P)| \leq O(h^2) \quad \forall P \in \Omega_h.$$

Theorem 2. *For the Shortley-Weller method, the local error $e_h(P)$ satisfies*

$$|e_h(P)| \leq O(h^3) \quad \forall P \in \Omega_h^*.$$

This result is still valid for points P such that $\phi(P)$ the distance to the interface is lower than $2h$.

Note in passing that this result had already been presented, with a different proof, in [6] and in [1].

Concerning the convergence of the gradient, in practise, we will only study the convergence of the discrete version of $\partial_x u$, because the $x-$ and $y-$ directions have symmetric behaviours. Let us define where and how the discrete version of $\partial_x u$ is defined. We consider two adjacent points belonging to Ω_h: $M_{i,j} = (x_i, y_j)$ and $M_{i+1,j} = (x_{i+1}, y_j)$. We note $M_{i+1/2,j}$ the middle of the segment $[M_{i,j}, M_{i+1,j}]$:

$$M_{i+1/2,j} = \frac{M_{i,j} + M_{i+1,j}}{2}.$$

For the sake of clarity we also denote $M_{i+1/2,j}$ the middle of the segment $[M_{i,j}, Q]$, if Q, the nearest point on the right side of $M_{i,j}$ belongs to Γ_h. Similarly, we also denote $M_{i+1/2,j}$ the middle of the segment $[Q, M_{i+1,j}]$, if Q the nearest point on the left side of $M_{i+1,j}$ belongs to Γ_h, see Figure 3 for an illustration.

We define the discrete x-derivative $D_x u(M_{i+1/2,j})$ on the point $M_{i+1/2,j}$ as

$$D_x u(M_{i+1/2,j}) = \frac{u_h(M) - u_h(N)}{x_M - x_N},$$

where M and N are the points belonging to $\Omega_h \cup \Gamma_h$ such that $M_{i+1/2,j}$ is defined as the middle of $[MN]$.

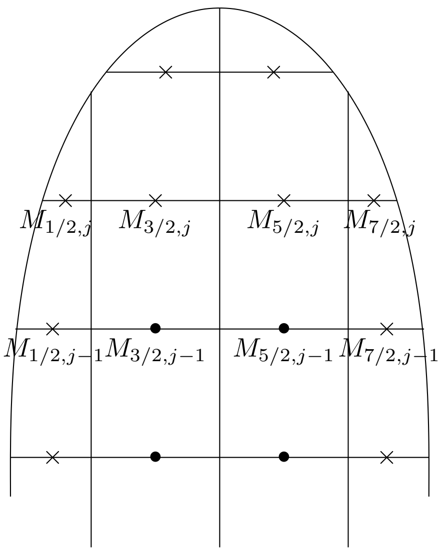

Figure 3: Example of geometrical configuration and numerotation. Nodes belonging to $\tilde{\Omega}_h$ are plotted with $\bullet$, nodes belonging to $\tilde{\Gamma}_h$ with $\times$.

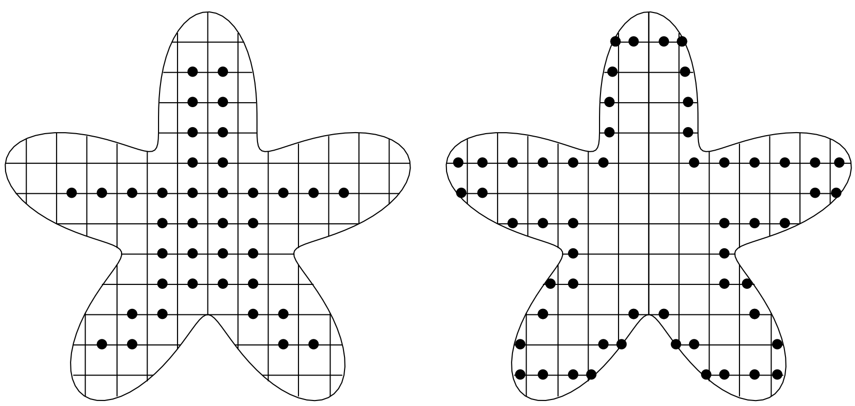

Figure 4: Left: nodes belonging to $\tilde{\Omega}_h$, right: nodes belonging to $\tilde{\Gamma}_h$

We consider the set S of all points $M_{i+1/2,j}$ located inside the domain Ω. We define $\tilde{\Omega}_h$ the subset containing all points of S where we can define the discrete Laplace operator $(-\triangle_h)$ with all points of the five-point stencil belonging to S. The subset $\tilde{\Gamma}_h$ contains all other points of S and satisfies by construction the property (see an illustration on Fig. 3)

$$\phi(P) \leq 2\,h \quad \forall P \in \tilde{\Gamma}_h. \tag{2}$$

The subset $\tilde{\Omega}_h$ is divided into two subsets of points: we say that a grid node in $\tilde{\Omega}_h$ is regular if none of its direct neighbors is on the boundary $\tilde{\Gamma}_h$, and that it is near the boundary if at least one if its neighbors belongs to $\tilde{\Gamma}_h$. The set of regular grid nodes is denoted $\tilde{\Omega}_h^{**}$, and the set of grid nodes near the boundary is denoted $\tilde{\Omega}_h^{*}$.

Theorem 3. *For the Shortley-Weller method, the local error on the discrete x-derivative is second-order accurate*

$$|\partial_x u(M_{i+1/2,j}) - D_x u(M_{i+1/2,j})| \leq O(h^2) \quad \forall M_{i+1/2,j} \in S.$$

3 Estimating convergence with Ciarlet's technique

Here we review the principle of the method presented in [2] to prove high-order convergence for finite-differences operators with the help of the discrete maximum principle. We do not use exactly the same type of discretization matrix as in [2], due to the different way to account for boundary conditions, so here we present the reasoning in the case of our discretization matrix.

3.1 Discrete Green's function

For each $Q \in \Omega_h$, define the discrete Green's function $G_h(P,Q), P \in \Omega_h$ as the solution of the discrete problem:

$$\left\{ \begin{array}{c} -\triangle_h u_h(P) = \left\{ \begin{array}{ll} 0, & P \neq Q \\ 1, & P = Q \end{array} \right. \qquad P \in \Omega_h, \\ u_h(P) = 0, \quad P \in \Gamma_h. \end{array} \right. \tag{3}$$

In fact, each array $(G_h(P,Q)_{P\in\Omega_h}$ represents a column of the inverse matrix of the discrete operator $(-\triangle_h)$. For the sake of brevity we note $G_h(:,Q)$ the column corresponding to the grid node Q. The matrix of $(-\triangle_h)$ being monotone, as we noticed in §2, it means that all values of $G_h(P,Q)$ are positive.

With this definition of $G_h(P,Q)$ we can write the solution of the numerical problem as a sum of the source terms multiplied by the local values of the discrete Green function:

$$u_h(P) = \sum_{Q\in\Omega_h} G_h(P,Q)\,(-\triangle_h u_h)(Q), \quad \forall P \in \Omega_h.$$

3.2 Estimating the coefficients of the discrete Green's function

The following is an adaptation of the proof presented in [2].

Theorem 4. *Let S be a subset of grid nodes (thus corresponding also to a subset of the indices of the matrix), and W an array such that:*

$$\left\{ \begin{array}{c} W(P) \geq 0 \quad \forall P \in \Omega_h, \\ (-\triangle_h W)(P) \geq 0 \quad \forall P \in \Omega_h, \\ (-\triangle_h W)(P) \geq h^{-i} \textit{ for all } P \in S. \end{array} \right.$$

Then

$$\sum_{Q\in S} G_h(P,Q) \leq h^i W(P).$$

Proof. Using the definition of the discrete Green function, we can write

$$(-\triangle_h \sum_{Q\in S} G_h(:,Q))(P) = \begin{cases} 1 & \text{if } P \notin S, \\ 0 & \text{if } P \in S. \end{cases}$$

Therefore,

$$-\triangle_h (W - h^{-i} \sum_{Q\in S} G_h(:,Q))(P) \geq 0 \quad \forall P \in \Omega_h.$$

As all coefficients of the inverse of $-\triangle_h$ are positive, it leads to

$$W(P) - h^{-i} \sum_{Q\in S} G_h(P,Q) \geq 0 \quad \forall P \in \Omega_h,$$

and finally we obtain an estimate of the coefficients of $\sum_{Q\in S} G_h(:,Q)$ in terms of the coefficients of W:

$$\sum_{Q\in S} G_h(P,Q) \leq h^i W(P).$$

4 Convergence study of the solution

In this section we look for adequate subsets S and functions W in order to prove convergence. We first prove that the numerical solution converges with second-order accuracy in §4.1, then that the convergence is in fact third-order for the grid nodes near the boundary in §4.2. This third-order convergence will be used to obtain second-order boundary conditions for a similar problem involving the discrete gradient of the numerical solution in §5.

4.1 Second-order convergence of the solution in the whole domain

Proof. We denote by $\tau(P)$ the truncation error of the Shortley-Weller method on a point P belonging to Ω_h. With a classical Taylor expansion one can show that $\tau(P) = O(h^2)$ if P belongs to Ω_h^{**}, and $\tau(P) = O(h)$ only a priori if P belongs to Ω_h^*. The local error satisfies the same linear system as the numerical solution $u_h(P)$, but with the truncation error as a source term:

$$-\triangle_h e_h(P) = \tau(P) \quad \forall P \in \Omega_h.$$

We want to obtain some bounds on $\sum_{Q\in\Omega_h^{**}} G(P,Q)$ and $\sum_{Q\in\Omega_h^{*}} G(P,Q)$ with the method described in §3. We handle these two subsets of Ω_h separately because they do not have the same truncation error.

We consider a point $M = (x_M, y_M)$ inside Ω. We define the discrete function:

$$W(Q) = \frac{C - (x_Q - x_M)^2 + (y_Q - y_M)^2}{4},$$

with (x_Q, y_Q) the coordinates of the point Q, and C such that $W(Q) \geq 0$ for all $Q \in \Omega_h$. For instance we take $C = 2\,(diam(\Omega))^2$. On every grid node P belonging to Ω_h,

$$(-\triangle_h W)(P) = 1.$$

Thus

$$\sum_{Q \in \Omega_h} (-\triangle_h G(:,Q))(P) - (-\triangle_h W(P)) \leq 0 \quad \forall P \in \Omega_h.$$

Therefore, we can directly write, using the fact that the matrix is monotone:

$$\sum_{Q \in \Omega_h} G(P,Q) \quad \leq \quad W(P) \leq \frac{(diam(\Omega))^2}{2}, \quad \forall P \in \Omega_h. \tag{4}$$

To prove the second-order convergence of the solution, it remains to prove an appropriate estimate for $Q \in \Omega_h^*$. We define the discrete function

$$\tilde{W}(Q) = \begin{cases} 0 & \text{if } Q \in \Omega_h^*, \\ 1 & \text{otherwise.} \end{cases}$$

This function satisfies

$$\begin{cases} -\triangle_h(\tilde{W})(Q) \geq \frac{1}{h^2} & \text{if } Q \in \Omega_h^*, \\ -\triangle_h(\tilde{W})(Q) = 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{Q \in \Omega_h^*} (-\triangle_h G(:,Q))(P) - h^2(-\triangle_h \tilde{W})(P) \leq 0,$$

and we can directly write, using the fact that the matrix is monotone

$$\sum_{Q \in \Omega_h^*} G(P,Q) \quad \leq \quad h^2 \tilde{W}(P) \leq h^2. \tag{5}$$

Finally, combining (4) and (5), we obtain an estimate of the local error on every point $P \in \Omega_h$

$$\begin{aligned} |u(P) - u_h(P)| &= |\sum_{Q \in \Omega_h} G_h(P,Q)\tau(Q)|, \\ &\leq |\sum_{Q \in \Omega_h^{**}} G_h(P,Q)\tau(Q)| + |\sum_{Q \in \Omega_h^*} G_h(P,Q)\tau(Q)|, \\ &\leq \sum_{Q \in \Omega_h^{**}} G_h(P,Q) O(h^2) + \sum_{Q \in \Omega_h^*} G_h(P,Q) O(h), \\ &\leq \frac{diam(\Omega)}{4} O(h^2) + h^2 O(h) = O(h^2). \end{aligned}$$

which proves that the numerical solution converges with second-order accuracy to the exact solution.

4.2 Third-order convergence near the boundary

Proof. In order to prove that the numerical solution converges with third order accuracy on nodes in Ω_h^*, we aim to obtain a bound within $O(h)$ for the sum of the coefficients on the rows of the inverse matrix used for the solution on nodes belonging to Ω_h^*. That is, we want to prove that:

$$\sum_{Q\in\Omega_h} G(P,Q) \quad\leq\quad O(h), \forall P\in\Omega_h^*.$$

We use the function

$$\tilde{W}(x,y) = 1 - e^{-A\phi(x,y)},$$

where $\phi(x,y)$ is the (positive) distance of the point (x,y) to the boundary. This function satisfies

$$\begin{aligned}
-\triangle\tilde{W}(x,y) &= e^{-A\phi(x,y)}\Big(A^2((\partial_x\phi(x,y))^2 + (\partial_y\phi(x,y))^2) - A\triangle\phi(x,y)\Big),\\
&= e^{-A\phi(x,y)}\Big(A^2 - A\triangle\phi(x,y)\Big),
\end{aligned}$$

because the function distance ϕ satisfies $(\partial_x\phi(x,y))^2 + (\partial_y\phi(x,y))^2 = 1$. We choose A large enough so that $\Big(A^2 - A\triangle\phi(x,y)\Big) \geq 1$ for all $(x,y)\in\Omega$. Such a value of A depends only of $\triangle\phi(x,y)$ and thus, of the geometry of the domain Ω. As the discretization of the Laplace operator is consistent on every grid node, for h small enough we can write

$$-\triangle_h\tilde{W}(P) \geq \frac{e^{-A\phi(P)}}{2}, \quad \forall P\in\Omega_h.$$

Therefore,

$$\sum_{Q\in\Omega_h}(-\triangle_h G(:,Q))(P) \leq \frac{2}{e^{-A||\phi||_\infty}}(-\triangle_h\tilde{W})(P), \quad \forall P\in\Omega_h,$$

where $||\phi||_\infty = \sup_{x,y\in\Omega}\phi(x,y)$. Then, because the matrix is monotone,

$$\sum_{Q\in\Omega_h} G(P,Q) \leq \frac{2}{e^{-A||\phi||_\infty}}\tilde{W}(P) \leq \frac{2}{e^{-A||\phi||_\infty}} \quad A\,\phi(P), \quad \forall P\in\Omega_h.$$

Moreover, if the point P belongs to Ω_h^*, then $\phi(P)\leq h$. As a consequence, for $P\in\Omega_h^*$

$$\sum_{Q\in\Omega_h} G(P,Q) \leq \frac{2}{e^{-A||\phi||_\infty}}Ah = O(h), \tag{6}$$

thus, restricting ourselves to the columns belonging to Ω_h^{**},

$$\sum_{Q\in\Omega_h^{**}} G(P,Q) \leq O(h), \quad \forall P\in\Omega_h^*. \tag{7}$$

Finally, combining (5) and (7) we can write for all $P\in\Omega_h^*$

$$\begin{aligned}
|u(P) - u_h(P)| &= |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)|,\\
&\leq |\sum_{Q\in\Omega_h^{**}} G_h(P,Q)\tau(Q)| + |\sum_{Q\in\Omega_h^*} G_h(P,Q)\tau(Q)|,\\
&\leq |\sum_{Q\in\Omega_h^{**}} G_h(P,Q)\,O(h^2)| + |\sum_{Q\in\Omega_h^*} G_h(P,Q)\,O(h)|,\\
&\leq O(h)O(h^2) + h^2O(h) = O(h^3),
\end{aligned}$$

which means that the numerical solution converges with third-order accuracy to the exact solution on grid nodes belonging to Ω_h^*.

Similarly, we could prove that for grid points P such that $\phi(P) \leq 2\,h$, the numerical solution is also third-order accurate: the only change in the proof would be to replace at (6) h by $2h$.

5 Second-order convergence for the discrete gradient

Proof. We assume that we know the values of u_h, as it is the numerical solution of the linear system (1). We have proven in §4.2 that u_h converges with third-order accuracy for points P such that $\phi(P) \leq 2\,h$. Therefore, if $M_{i+1/2,j}$ belongs to $\tilde{\Gamma}_h$, then the numerical solution on $M_{i+1/2,j}$ is a third-order approximation of the exact solution. As a consequence, if $M_{i+1/2,j}$ belongs to $\tilde{\Gamma}_h$, then $D_x u(M_{i+1/2,j})$ is an approximation with second-order accuracy of the x-derivative on point $M_{i+1/2,j}$.

Now we notice that we can build on points belonging to $\tilde{\Omega}_h$ a discretization $-\tilde{\triangle}_h$ of the Laplace operator with the Shortley-Weller method for the discrete x-derivative, with the discrete x-derivative of the function f as a source term, and the values of $D_x u$ on $\tilde{\Gamma}_h$ used as boundary conditions:

$$\begin{cases} (-\tilde{\triangle}_h v)(P) = D_x f(P) & \forall P \text{ on } \tilde{\Omega}_h, \\ v(P) = D_x u(P) & \forall P \text{ on } \tilde{\Gamma}_h. \end{cases}$$

This discretization has the truncation errors:

- $O(h^2)$ for the nodes belonging to $\tilde{\Omega}_h^{**}$,
- $O(1)$ for the nodes belonging to $\tilde{\Omega}_h^{*}$.

If the exact gradient was known on $\tilde{\Gamma}_h$, then the truncation error would be also $O(1)$ due to the fact that the Shortley-Weller operator does not commute with the discrete gradient D_x. The boundary conditions are defined with second-order accuracy, and thus lead to an additionnal $O(1)$ term in the truncation error. If the numerical solution only converged with second order accuracy near the boundary, then the discrete gradient would be only first-order near the boundary, and the truncation error would be $O(\frac{1}{h})$.

Now, similar estimates as in §4.1 can be obtained for the coefficients of the inverse matrix, because the matrix of the current linear system, being also a discretization with the Shortley-Weller method, has the same structure as the matrix of the linear system for u_h. If we denote $\tilde{G}_h(P,Q)$ the discrete Green's function corresponding to $-\tilde{\triangle}_h$, then it satisfies

$$\sum_{Q \in \tilde{\Omega}_h} \tilde{G}(P,Q) \leq \frac{(diam(\Omega))^2}{2}, \quad \forall P \in \tilde{\Omega}_h, \tag{8}$$

$$\sum_{Q \in \tilde{\Omega}_h^*} \tilde{G}(P,Q) \leq h^2, \quad \forall P \in \tilde{\Omega}_h. \tag{9}$$

Therefore, the expression on the local error on point P belonging to $\tilde{\Omega}_h$ reads

$$\begin{aligned}
|\partial_x u(M_{i+1/2,j}) - D_x u(M_{i+1/2,j})| &= |\sum_{Q\in\tilde{\Omega}_h} \tilde{G}_h(P,Q)\tilde{\tau}(Q)|, \\
&\leq |\sum_{Q\in\tilde{\Omega}_h^{**}} \tilde{G}_h(P,Q)\tilde{\tau}(Q)| + |\sum_{Q\in\tilde{\Omega}_h^{*}} \tilde{G}_h(P,Q)\tilde{\tau}(Q)|, \\
&\leq |\sum_{Q\in\tilde{\Omega}_h^{**}} \tilde{G}_h(P,Q)O(h^2)| + |\sum_{Q\in\tilde{\Omega}_h^{*}} \tilde{G}_h(P,Q)O(1)|, \\
&\leq O(1)O(h^2) + O(h^2)O(1) = O(h^2).
\end{aligned}$$

6 Discussion

This work was motivated by the remark in the recent paper of Yoon and Min [7] about the lack of mathematical analysis about the super-convergence of the Shortley-Weller method. To our knowledge, few works in the literature have studied the supraconvergence of the gradient for elliptic finite-difference schemes, among them [3], [5] and [4].

In [3] Ferreira and Grigorieff deal with more general elliptic operators, with variable coefficients and mixed derivatives, and prove second-order convergence in H^1 norm.The proof uses negative norms and is based on the fact that the finite difference scheme is a certain non-standart finite element scheme on triangular grids combined with a special form of quadrature.

In [5] Li et al. study the super-convergence of solution derivatives for the Shortley-Weller method for Poisson's equation, considering also this method as a special kind of finite element method. The work in [4] adresses the case of unbounded derivatives near the boundary Γ, on polygonal domains.

Our approach differs from the latter because we do not use a finite-element approach. Instead we propose a proof based on a finite-difference analysis, which is a variant of the method of Ciarlet [2]. As in this seminal paper, we use a discrete maximum principle to obtain estimates on the coefficients of the inverse matrix, but in our case the bound on the coefficients can also vary with the rows of the inverse matrix. This variant is useful to obtain a specific bound for points located near the interface and obtain the third-order convergence of the solution at these points.

The approach developed in this paper has the advantage to be simple to carry out, and to be able to provide locally pointwise estimates, instead of the usual convergence results in the L^2 or H^1 norms. This can especially be useful when one deals with methods where the amplitude of the truncation error varies with space, because these norms can not provide easily details about the effects of the variation of the truncation error on the local convergence.

7 Conclusion

We have proven that the discrete gradient obtained by the Shortley-Weller method for the Poisson equation converges with second-order accuracy. This is a super-convergence property because the numerical solution itself also converges with second order accuracy. This property is proven with a variant of Ciarlet's technique to obtain high-order convergence estimates for monotone finite-differences matrices. With carefuly chosen test functions we are able to bound the coefficients of the discrete Green function associated with the matrix of the Shortley-Weller method. One key ingredient is to prove that the solution converges with third-order accuracy near the boundary, so that the discrete gradient can be considered as the solution of another Poisson equation with

Dirichlet boundary conditions that have a second-order accuracy. A further development would be to extend this work to the case of more general elliptic operators.

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