Super-convergence of the gradient for the Shortley-Weller method

Lisl Weynans

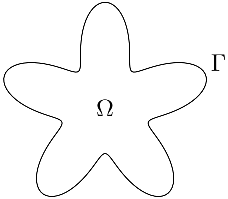
Bordeaux University and INRIA

Sanya, China, 4-8 June 2018

Shortley-Weller scheme (SW)

- SW scheme designed to solve on a cartesian grid

$$\begin{cases} -\triangle u = f \text{ in } \Omega, \\ u = g \text{ on } \Gamma. \end{cases}$$



- Applications to fluid dynamics, free surface problems...

Disclaimer : in the following we assume that a unique solution u exists and is smooth enough for our consistency error analyses to be valid.

Shortley-Weller scheme (SW)

The Shortley-Weller scheme :

- u_h numerical solution, for brevity $u_h(M_{i,j}) = u_{i,j}$
- Based on a dimension by dimension approach.

$$
\begin{aligned}
(A_h u_h)_{i,j} = & \frac{2}{(h_{i+1/2,j} + h_{i-1/2,j})} \left[\frac{(u_{i,j} - u_{i+1,j})}{h_{i+1/2,j}} + \frac{(u_{i,j} - u_{i-1,j})}{h_{i-1/2,j}} \right] \\
& + \frac{2}{(h_{i,j+1/2} + h_{i,j-1/2})} \left[\frac{(u_{i,j} - u_{i,j+1})}{h_{i,j+1/2}} + \frac{(u_{i,j} - u_{i,j-1})}{h_{i,j-1/2}} \right]
\end{aligned}
$$

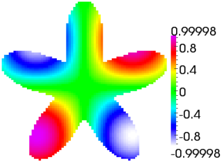
- Associated matrix : all diagonal entries strictly positive, all off-diagonal entries nonpositive, irreducibly diagonally dominant.

  $\Rightarrow$ Monotone matrix ($\equiv$ all coefficients of the inverse are positive).

- Approximation error : first-order near boundary, second-order elsewhere

- Second-order convergence for the solution u_h
- Second-order convergence too for the discrete gradient

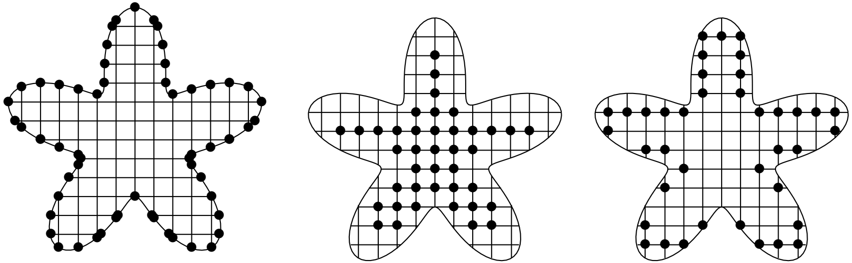
Example : exact solution $u(x, y) = sin(3x) * sin(3y)$



N	L^∞ error u_h	L^∞ error grad.
50	3.00E-004	1.01E-002
100	8.29E-005	2.59E-003
200	2.16E-005	6.58E-004
400	5.53E-006	1.68E-004

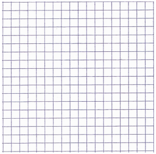
- It is commonly admitted that a first-order truncation error on a small subset does not destroy the 2nd-order convergence, but what about the gradient ?

Some notations



$$\Gamma_h = \text{ boundary nodes, } \underbrace{\Omega_h^{**} = \text{ regular grid nodes, } \Omega_h^{*} = \text{ irregular grid nodes}}_{= \Omega_h}$$

Strategies : FEM vs FV vs FD



On a cartesian grid the same discretization matrix can be obtained with the 3 classes of methods, but different strategies for convergence :

- FEM : coercivity + Cea lemma + interpolation thm $\Rightarrow H^1$-norm cv
- FV : coercivity + consistency + Poincaré inequality $\Rightarrow H^1$-norm cv
- FD : monotonicity + discrete maximum principle $\Rightarrow L^\infty$-norm cv

Most convergence studies for FD methods with complex boundaries obtained with FEM techniques (exception : Li and Ito 2001)

- But loss of the "explicit character" of FD methods !
- L^∞-norm provides informations about accuracy near boundaries

Strategy

- Approximation error :

$$\tau(P) = \begin{cases} O(h^2) \text{ for regular nodes,} \\ O(h) \text{ for irregular nodes.} \end{cases}$$

- The local error satisfies the linear system :

$$A_h e_h(P) = \tau(P) \quad \forall P \in \Omega_h$$

- Naive estimate : $||e_h||_\infty \leq ||A_h^{-1}||_\infty \underbrace{||\tau_h||_\infty}_{=O(h)}$ only

- Look for estimates by blocks of A_h^{-1}

$$\begin{pmatrix} \overbrace{\cdots\cdots\cdots\cdots}^{\sum a^{-1}_{i,j}=O(1)} & \overbrace{\cdots}^{\sum a^{-1}_{i,j}=O(h^2)} \\ \cdots\cdots\cdots\cdots & \cdots \\ \cdots\cdots\cdots\cdots & \cdots \\ \cdots\cdots\cdots\cdots & \cdots \\ \cdots\cdots\cdots\cdots & \cdots \\ \cdots\cdots\cdots\cdots & \cdots \end{pmatrix} \cdot \overbrace{\begin{pmatrix} O(h^2) \\ \vdots \\ O(h^2) \\ \hline O(h) \\ \vdots \\ O(h) \end{pmatrix}}^{\tau} = O(h^2)$$

What we can prove

Result

For the Shortley-Weller method, the local error $e_h(P)$ at node P satisfies

$$|e_h(P)| \leq O(h^2) \quad \forall P \in \Omega_h,$$
$$|e_h(P)| \leq O(h^3), \quad \textit{for nodes at a distance } O(h) \textit{ from the boundary .}$$

.

Result

The local error on the discrete x-derivative is second-order accurate :

$$|\partial_x u(P) - D_x^h u_h(P)| \leq O(h^2)$$

Outline

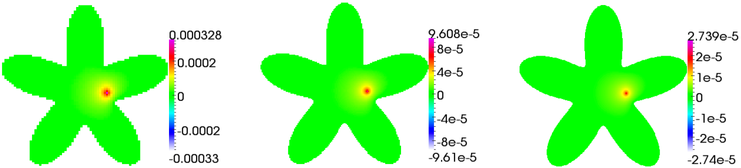
1. Discrete Green functions
2. Convergence of u_h
3. Convergence of the gradient

Discrete Green functions

For each $Q \in \Omega_h$, define the discrete Green's function $G_h(:,Q) = \Big(G_h(P,Q)\Big)_{P\in\Omega_h\cup\Gamma_h}$ as the solution of :

$$\begin{cases} A_h G_h(:,Q)(P) = \begin{cases} 0, & P \neq Q \\ 1, & P = Q \end{cases} & P \in \Omega_h, \\ G_h(P,Q) = 0, \quad P \in \Gamma_h. \end{cases}$$

- Each $G_h(:,Q)$ represents a column of the inverse matrix A_h^{-1}.
- A_h being monotone, all values of $G_h(:,Q)$ are positive.



Discrete Green functions, 100^2, 200^2 and 400^2 grid points

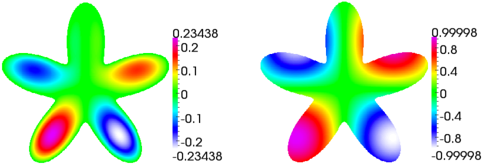
- If $u \equiv 0$ on Γ, the numerical solution can be expressed as

$$u_h(P) = \sum_{Q \in \Omega_h} G_h(P,Q)\, f(Q)$$

$\Rightarrow$ Discrete equivalent of the convolution

- If non-homogeneous Dirichlet BC, add them in the source terms for irregular nodes

$$u_h(P) = \sum_{Q \in \Omega_h} G_h(P,Q) \left[f(Q) + BC \text{ term} \right], \quad \forall P \in \Omega_h.$$



Left : Source term, without BC
Right : Source term + BC imposed on irregular nodes

Application of a discrete maximum principle

Theorem

(Ciarlet, 70) Let S be a subset of grid nodes, $\alpha > 0$ and W a discrete function s.t. :

$$\left\{ \begin{array}{c} W(P) \equiv 0 \quad \forall P \in \Gamma_h, \\ (A_h W)(P) \geq 0 \quad \forall P \in \Omega_h, \\ (A_h W)(P) \geq \alpha^{-i} \textit{ for all } P \in S. \end{array} \right.$$

Then

$$\sum_{Q \in S} G_h(P, Q) \leq \alpha^i W(P).$$

Application of a discrete maximum principle

Proof :

$$A_h\Big(\sum_{Q\in S} G_h(:,Q)\Big)(P) = \begin{cases} 1 & \text{if } P\in S, \\ 0 & \text{if } P\notin S. \end{cases}$$

Therefore,

$$A_h\Big(W-\alpha^{-i}\sum_{Q\in S} G_h(:,Q)\Big)(P)\geq 0 \quad \forall P\in\Omega_h.$$

Because A_h is monotone

$$W(P)-\alpha^{-i}\sum_{Q\in S} G_h(P,Q)\geq 0 \quad \forall P\in\Omega_h,$$

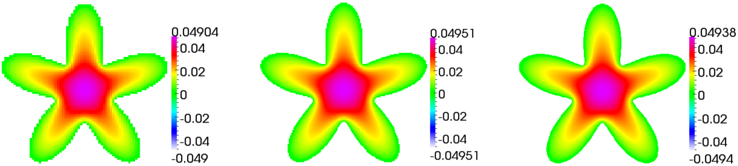
Therefore

$$\sum_{Q\in S} G_h(P,Q)\leq \alpha^i W(P).$$

Outline

1. Discrete Green functions
2. Convergence of u_h
3. Convergence of the gradient

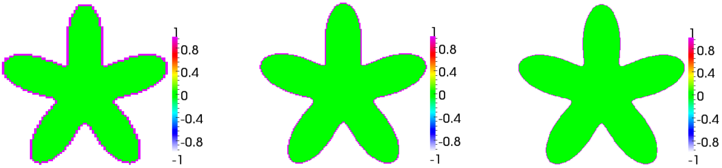
Green functions on regular nodes



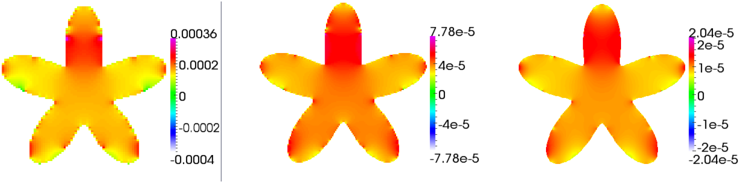
Sum of discrete Green functions for regular nodes, $N = 100, 200, 400$

$\Rightarrow$ amplitude scales like $\mathrm{O}(1)$

Green functions on irregular nodes



Source terms for discrete Green functions for irregular nodes, $N = 100, 200, 400$



Sum of discrete Green functions for irregular nodes, $N = 100, 200, 400$

$\Rightarrow$ amplitude scales like $\mathrm{O}(\mathrm{h}^2)$

- Estimate for discrete Green functions for regular nodes :
  Let $M = (x_M, y_M, z_M) \in \Omega$.

$$W(Q) = \frac{2diam(\Omega)^2 - (x_Q - x_M)^2 - (y_Q - y_M)^2 - (z_Q - z_M)^2}{6},$$

$$\begin{cases} W(P) \geq 0, & \\ A_h W(P) = 1, & \forall P \in \Omega_h. \end{cases}$$

$$\sum_{Q \in \Omega_h} G_h(P,Q) \quad \leq \quad W(P), \quad \forall P \in \Omega_h.$$

$$\Rightarrow \sum_{Q \in \Omega_h^{**}} G_h(P,Q) \leq \frac{(diam(\Omega))^2}{6}, \quad \forall P \in \Omega_h$$

- Estimate for Green functions for irregular nodes

$$\tilde{W}(Q) = \begin{cases} 0 & \text{if } Q \in \Gamma_h, \\ 1 & \text{otherwise.} \end{cases}$$

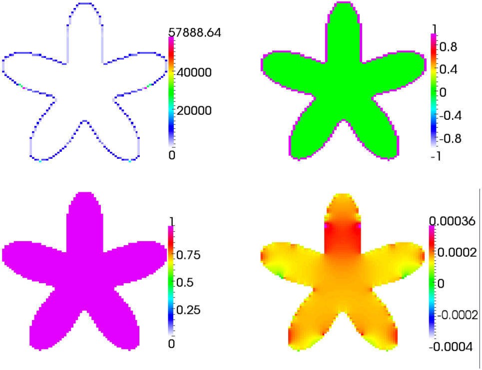
$\tilde{W}$ satisfies

$$\begin{cases} A_h \tilde{W}(Q) \geq \frac{1}{h^2} & \text{if } Q \in \Omega_h^* \text{ (irregular grid points)}, \\ A_h \tilde{W}(Q) = 0 & \text{otherwise.} \end{cases}$$

Therefore, using the lemma

$$\sum_{Q \in \Omega_h^*} G_h(P, Q) \leq h^2 \tilde{W}(P) \leq h^2, \quad \forall P \in \Omega_h.$$

Second-order convergence of u_h



$\tilde{W}$ and Green functions for irregular nodes

Second-order convergence of u_h

- Obtained estimates :

$$\sum_{Q\in\Omega_h^{**}} G_h(P,Q) \leq O(1) \text{ and } \sum_{Q\in\Omega_h^{*}} G_h(P,Q) \quad \leq h^2 \quad \forall P \in \Omega_h.$$

- Second-order estimate of the local error :

$$e(P) \quad = \quad (A_h^{-1}\tau)(P)$$

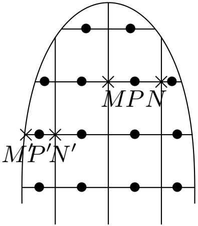
$$\begin{aligned} |e(P)| &= |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)| \\ &\leq \sum_{Q\in\Omega_h^{**}} G_h(P,Q)\underbrace{O(h^2)}_{=\tau(Q)} + \sum_{Q\in\Omega_h^{*}} G_h(P,Q)\underbrace{O(h)}_{=\tau(Q)} \\ &\leq O(h^2) \end{aligned}$$

Outline

1. Discrete Green functions
2. Convergence of u_h
3. Convergence of the gradient

Discrete gradient : notations

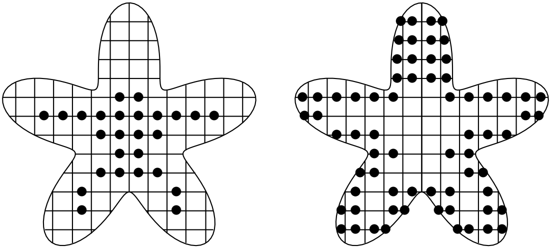
$$S_h = \{P,\ P \text{ middle of } [MN], M \text{ and } N \in \Omega_h \cup \Gamma_h, \\ M \text{ and } N \text{ adjacent in the x-direction.}\}$$



<u>Discrete x-derivative</u>

$$D_x^h u_h(P) = \frac{u_h(M) - u_h(N)}{x_M - x_N}, \quad \forall P \in S_h$$

- S_h splitted into two subsets of points



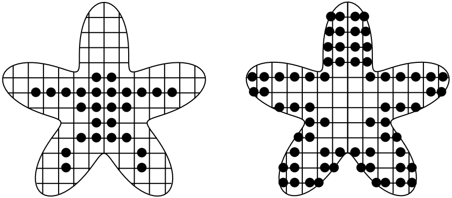
$\tilde{\Omega}_h$ $\tilde{\Gamma}_h$

- $\forall P \in \tilde{\Omega}_h$ the classical discrete Laplacian operator can be applied
- $\forall P \in \tilde{\Gamma}_h, \phi(P) \leq 3\,h$

Convergence of discrete gradient

Steps of the proof :

1) Prove that $D_x^h u_h$ is second-order accurate for every point P in $\tilde{\Gamma}_h$
2) Use $D_x^h u_h$ on $\tilde{\Gamma}_h$ as BCs for a discrete Laplace equation on S_h
3) Obtain linear system similar to the one for u_h and use similar technique for convergence



$\tilde{\Omega}_h$ $\tilde{\Gamma}_h$

More accurate estimates for u_h

Going more into the details, we can obtain

$$\sum_{Q\in\Omega_h^*} G_h(P,Q) \leq C\,h\,\min(h_{i\pm 1/2,j}, h_{i,j\pm 1/2}) \quad \forall P \in \Omega_h^*$$

$$\sum_{Q\in\Omega_h^{**}} G_h(P,Q) \leq O(\phi(P)), \quad \forall P \text{ s.t. } \phi(P) = O(h).$$

Thus $\forall P$ such that $\phi(P) = O(h)$

$$|e_h(P)| = |\sum_{Q\in\Omega_h} G_h(P,Q)\tau(Q)|$$

$$|e_h(P)| \leq O(h^2)(\phi(P) + \min(h_{i\pm 1/2,j}, h_{i,j\pm 1/2})).$$

$\Rightarrow$ Estimate depending on local geometry

1) Consider a point P in $\tilde{\Gamma}_h$

$$|\partial_x u(P) - D_x^h u_h(P)| \leq \left| \frac{e_h(M) - e_h(N)}{x_M - x_N} \right| + O(x_M - x_N)^2$$

For instance, $M \in \Gamma_h$ and $N \in \Omega_h$:

$$\begin{cases} e_h(M) = 0, \\ |e_h(N)| \leq O(h^2)\Big(\phi(N) + \min(h_i)\Big) \end{cases}$$

$$|\partial_x u(P) - D_x^h u_h(P)| \leq \left| \frac{O(h^2)\Big(\phi(N) + \min(h_i)\Big)}{x_M - x_N} \right| + O(x_M - x_N)^2 \leq O(h^2)$$

$D_x^h u_h$ is second-order accurate for every point P in $\tilde{\Gamma}_h$

2) Now consider $P \in \tilde{\Omega}_h$

$$D_x^h(A_h u_h)(P) = \frac{(A_h u_h)(M) - (A_h u_h)(N)}{x_M - x_N} = D_x^h f(P), \quad \forall P \text{ in } \tilde{\Omega}_h.$$

The operators A_h and D_x^h commute

$$D_x(A_h u_h)(P) = A_h(D_x^h u_h)(P), \quad \forall P \in \tilde{\Omega}_h.$$

Consequently, the array $v_h = (D_x^h u_h(P))_{P \in S_h}$ satisfies a discrete version of the Laplace operator :

$$\begin{cases} A_h v_h(P) = D_x^h f(P) & \forall P \in \tilde{\Omega}_h, \\ v_h(P) = D_x^h u_h(P) & \forall P \in \tilde{\Gamma}_h, \end{cases}$$

Convergence of discrete gradient

$$\begin{cases} A_h v_h(P) = D_x^h f(P) & \forall P \in \tilde{\Omega}_h \\ v_h(P) = D_x^h u_h(P) & \forall P \in \tilde{\Gamma}_h \end{cases}$$

3) The consistency errors are :

- $\tau(P) = O(h^2)$ for nodes in $\tilde{\Omega}_h$
- $\tau(P) = O(h^2)$ for nodes in $\tilde{\Gamma}_h$

$\Rightarrow$ With similar reasoning as for u_h we obtain second-order convergence

Take home message

- Discrete Green functions offer the possibility to study convergence of FD methods with boundaries/interfaces
- Various approximation errors at various locations can be taken into account accurately
- Use of discrete maximum principle : easier if the discretization matrix is monotone...
- Applicable to other convergence studies : interface problems, elliptic schemes on octrees...