Introduction à l'analyse numérique
Année : 2017-2018
Formation : L2 Mathématiques

TP7 : Exercices sur les EDOs

1 Numerical integration : Euler method

We consider three differential equations :

$$\begin{aligned} x'(t) &= a \\ x'(t) &= a\,t \\ x'(t) &= a\,x(t) \end{aligned}$$

with a a real number, and with the initial condition $x(0) = 1$.

1. *What are the exact solutions of these equations ?*
The exact solutions are respectively :

$$\begin{aligned} x(t) &= a\,t + 1 \\ x(t) &= \frac{a\,t^2}{2} + 1 \\ x(t) &= e^{a\,t} \end{aligned}$$

2. *For each of these equations, compute the numerical solution at time $t = 0.3$ using the time step $\Delta t = 0.1$, with the Euler method.*
First equation :

$$\begin{aligned} x_0 &= 1 \\ x_1 &= x_0 + \Delta t a = 1 + \Delta t\ a = 1 + 0.1 \times a \\ x_2 &= x_1 + \Delta t a = 1 + 2\Delta t\ a = 1 + 0.2 \times a \\ x_3 &= x_2 + \Delta t a = 1 + 3\Delta t\ a = 1 + 0.3 \times a \end{aligned}$$

Second equation :

$$\begin{aligned} x_0 &= 1 \\ x_1 &= x_0 + \Delta t\ a \times 0 = 1 \\ x_2 &= x_1 + \Delta t\ a \times \Delta t = 1 + \Delta t^2\ a = 1 + 0.01 \times a \\ x_3 &= x_2 + \Delta t\ a \times (2\Delta t) = 1 + 3\Delta t^2\ a = 1 + 0.03 \times a \end{aligned}$$

Third equation :

$$\begin{aligned} x_0 &= 1 \\ x_1 &= x_0 + \Delta t\ a x_0 = x_0(1 + \Delta t a) = 1 + \Delta t\ a = 1 + 0.1 \times a \\ x_2 &= x_1 + \Delta t\ a x_1 = x_1(1 + \Delta t a) = (1 + \Delta t\ a)^2 = (1 + 0.1 \times a)^2 \\ x_3 &= x_2 + \Delta t\ a x_2 = x_2(1 + \Delta t a) = (1 + \Delta t\ a)^3 = (1 + 0.1 \times a)^3 \end{aligned}$$

2 Stability of Euler method

We consider the differential equation

$$y' = \lambda y$$

with the initial condition $y(0) = y_0$*. We denote* y_k *the sequence approximating with the explicit Euler method the values of the exact solution* $y(t_k)$ *with* $t_k = k\,dt$*, with* k *a positive integer and* dt *a (small) positive real.*

1. *Write the formula expressing* y_{k+1} *as a function of* y_k*.*

$$y_{k+1} = y_k + dt \times \lambda\, y_k = y_k(1 + dt\,\lambda)$$

2. *Using the latter formula, express* y_k *as a function of the initial condition* y_0*.*
   By a recurrence, one can prove that

$$y_k = y_0(1 + dt\,\lambda)^k$$

3. *Under which condition is the Euler method stable ? (That is, under which condition the numerical solution does not tend to infinity when* $k \to +\infty$ *?)*
   The Euler method is stable is stable if $(1 + dt\,\lambda)^k$ does not tend to infinity when k tends to infinity, that is, if $|1 + dt\,\lambda|$ is smaller than 1, which is equivalent to $-1 < 1 + dt\,\lambda < 1$ or to $\lambda < 0$ and $|dt\,\lambda| < 2$.

3 Non-explosion of a solution

We consider the differential equation

$$y' = y^2 - x$$

1. We denote $V_{x,y} = y^2 - x$.

$$\begin{aligned} V_{x,y} &= 0 \Leftrightarrow y^2 = x \Leftrightarrow x \geq 0 \text{ and } |y| = \sqrt{x}, \\ V_{x,y} &\leq 0 \Leftrightarrow y^2 \leq x \Leftrightarrow x \geq 0 \text{ and } |y| \leq \sqrt{x}, \\ V_{x,y} &\geq 0 \Leftrightarrow y^2 \geq x \Leftrightarrow (x \geq 0 \text{ and } |y| \geq \sqrt{x}) \text{ or } x \leq 0. \end{aligned}$$

2. Let $M_0 = (x_0, y_0)$ be a point where the slope of the tangent field is negative, and u the solution satisfying $u(x_0) = y_0$. Then following the tangent field, the graph of u will always remain in the part of the plane where the tangent slope is negative, which is bounded for bounded values of x. Therefore, u can not explode.

4 Barriers and limit of solutions when x tend to $+\infty$

We consider the differential equation

$$y' = -y - \frac{y}{x} \tag{1}$$

for $x \in]0, +\infty[$.

1. We denote $f(x, y) = -y - \dfrac{y}{x}$.

$$f(x, g(x)) = -g(x)(1 + \frac{1}{x}) < -g(x) = g'(x) \text{ if } g \text{ is positive.}$$

2. Similarly

$$f(x, g(x)) = -g(x)(1 + \frac{1}{x}) > -g(x) = g'(x) \text{ if } g \text{ is negative.}$$

3. The solutions of the differential equation $y' = -y$ can all be written under the form $y(x) = y_0 e^{-(x-x_0)}$, with $y_0 = y(x_0)$.

4. We define by f the maximal solution of the differential equation (1), for the initial condition $f(x_0) = y_0$. We consider $y_1(x) = -|y_0|e^{-x}$ and $y_2(x) = |y_0|e^{-x}$. The function y_1 are both barriers for the differential equation (1) : y_1 is a subsolution, and y_2 a supersolution. We deduce that for all x on which f is defined, $y_1(x) \leq f(x) \leq y_2(x)$. Therefore, f does not explode on a finite interval. Consequently, f is defined on $]x_0, +\infty[$. As y_1 and y_2 both tend to 0 when x tends to $+\infty$, we conclude that f also tends to 0 when x tends to $+\infty$.