A Sharp Cartesian Method For The Simulation Of Flows With High Density Ratios

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Incompressible flows with high density ratios?

Air-water interfaces
• NaSCar : 3D parallel incompressible code with fluid-structure interaction
  (Michel Bergmann, INRIA Bordeaux)
• Discretization on cartesian grids, level-set method
• Second-order for velocity near solid boundary : use of ghost cells
  (Mittal et al 2008, Bergmann et al 2014)
Goal: Fluid-structure interaction with waves
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Regularized method for interface treatment: CSF

- Loss of accuracy + stability issues
- How to improve the accuracy near the interface?

⇒ Use a sharp cartesian method to solve the pressure at the interface
Methodology

We work with:

- a discretization on cartesian grids,
- finite differences,
- a level-set function to represent the interface.

We want

- a second-order accuracy for the pressure
- a scheme easy to implement (and to parallelize)
Interface description

• The level-set function $\phi$ is advected with fluid velocity,
• Straightforward treatment of complex geometries and topological changes (fragmentation, coalescence)
• Convenient for discretization on cartesian grids
• Formulas for geometric quantities:

\[ n = \nabla \phi, \quad \kappa = \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right), \]

• In practice, $\phi$ is the signed distance to the interface
  \( \Rightarrow |\nabla \phi| = 1, \quad \kappa = \Delta \phi \).
Outline

1. Second-order cartesian method for elliptic problems with immersed interfaces
2. Application to incompressible bifluid flows
3. How to preserve high-order level-set along time?
Elliptic problem with immersed interface

\[ \nabla.(k \nabla u) = f \text{ on } \Omega = \Omega_1 \cup \Omega_2 \]

\[ [u] = \alpha \text{ on } \Sigma \]

\[ [k \frac{\partial u}{\partial n}] = \beta \text{ on } \Sigma \]

\[ u = g \text{ on } \delta \Omega \]
Discretization strategy

- Creation of additional unknowns on the interface
  - used to discretize the elliptic operator on each side of the interface
  - obtained by a discretization of jump conditions across the interface

* A method related to the large family of methods inspired by IIM *

- Cons: additional unknowns...
- Pros: additional unknowns!
Which accuracy near the interface?

To obtain second-order convergence (\( L^\infty \) norm), it is enough to have:

- a first-order truncation error for the elliptic operator near the interface  
  \( \Rightarrow \) avoid linear extrapolations

- a second-order truncation error for the flux discretization  
  \( \Rightarrow \) use of a larger stencil

\[
\begin{array}{cccccc}
  & j & j+1 & & & \\
  j & & & & & \\
  j-1 & & & & & \\
  i-1 & i & i+1 & i+2 & \\
\end{array}
\]
Theoretical convergence

- $A_h$ matrix of linear system, $U_h$ solution, $f_h$ source term

$$A_h U_h = f_h$$

- Local error $e_h$ and truncation error $\tau_h$ linked by

$$A_h e_h = \tau_h$$

- Naive estimate:

$$||e_h||_\infty \leq ||A_h^{-1}||_\infty ||\tau_h||_\infty$$

- Not accurate enough here because $||\tau_h||_\infty = O(h)$

$\Rightarrow$ we need bounds on $A_h^{-1}$ coefficients, summed by blocks
Theoretical convergence

• For each discretization point $Q$, define the discrete Green function $G_h(P, Q)$ as:

$$A_h G_h(P, Q) = \begin{cases} 
0, & P \neq Q \\
1, & P = Q \\
G_h(P, Q) = 0, & P \text{ on the boundary}.
\end{cases}$$

• Each array $G_h(\cdot, Q)$ is a column of $A_h^{-1}$

$$u_h(P) = \sum_Q G_h(P, Q) (A_h U_h)(Q) \quad \forall P$$

**Figure**: Examples of discrete Green functions
Theoretical convergence

Theorem (Ciarlet, 71):

$S$ is a subset of $\Omega_h$ and $W$ an array such that:

\[
\begin{align*}
W(P) &\geq 0 \quad \forall P \in \Omega_h, \\
(A_h W)(P) &\geq 0 \quad \forall P \in \Omega_h, \\
(A_h W)(P) &\geq h^{-i} \text{ for each } P \in S.
\end{align*}
\]

If $A_h$ is monotonic then

\[
\sum_{Q \in S} G_h(P, Q) \leq h^i W(P).
\]
Theoretical convergence

• Prove that the matrix is monotonic, that is \((A_h U_h \geq 0 \Rightarrow U_h \geq 0)\):

requires to prove that if the minimum of \(U_h\) is located on the interface, then the discrete flux on this point is negative

• Use discrete maximum principle and ad hoc test functions to obtain bound on the coefficients of \(A_h^{-1} = G_h\):

\[
\sum_{Q \in \Omega_h^* \cup \Sigma_h} G_h(P, Q) \leq O(1),
\]

\[
\sum_{Q \in \Omega_h^{**}} G_h(P, Q) \leq O(h^2).
\]

Figure 3: Left: regular nodes (belonging to \(\Omega_h^{**}\)) described by bullets \(\bullet\), irregular nodes (belonging to \(\Omega_h^{**}\)) described by circles \(\circ\), right: nodes belonging to \(\Sigma_h\).
Theoretical convergence

- Multiply the truncation error array by $A_h^{-1}$, block by block:

$$|e_h(P)| \leq \sum_{Q \in \Omega^{**}_h} |G_h(P, Q)\tau_h(Q)| + \sum_{Q \in \Omega^*_h \cup \Sigma_h} |G_h(P, Q)\tau_h(Q)|,$$

$$\leq O(h^2)O(1) + O(1)O(h^2) = O(h^2)$$

- In our case:

  - Proof ok in 1D, 2D order 1
  
  - 2D order 2: the monotonicity of the matrix depends on the direction of the normal to the interface compared to the direction of the normal to the cartesian cell

  - But monotonicity ensured if normal aligned with the axis of the grid

  $\Rightarrow$ useful in the bifluid case!
2D convergence test

Interface $\Sigma$:

$$
\left(\frac{x}{18/27}\right)^2 + \left(\frac{y}{10/27}\right)^2 = 1.
$$

Exact solution:

$$
u(x, y) = \begin{cases} 
  e^x \cos(y), & \text{inside } \Sigma \\
  5e^{-x^2-y^2}/2, & \text{outside.}
\end{cases}
$$

$k = 1$ outside $\Sigma$ and 10 or 1000 inside.

**Figure**: Left : $k = 10$, right : $k = 1000$, convergence in $L^\infty$ norm
Parallel 2D convergence test

\[ k = \begin{cases} k^- \text{ inside } \Sigma \\ 1 \text{ outside} \end{cases} \]

\[ u = \begin{cases} e^x \cos(y), \text{ inside } \Sigma \\ 5e^{-x^2 - \frac{y^2}{2}}, \text{ outside.} \end{cases} \]

**Figure:** Convergence tests with \( \omega = 5, r_0 = 0.5, k^- = 1000 \) (left), and \( \omega = 12, r_0 = 0.4, k^- = 100 \) (right).
Outline

1. Second-order cartesian method for elliptic problems with immersed interfaces
2. Application to incompressible bifluid flows
3. How to preserve high-order level-set along time?
Notations

\[ \Omega^- \ (\phi < 0) \]

\[ \Omega^+ \ (\phi > 0) \]

\[ \Gamma \ (\phi = 0) \]

\[ \rho^-, \mu^- \]

\[ \rho^+, \mu^+ \]
Fluid model

- Incompressible Navier-Stokes equations in each fluid:
  \[
  \rho (u_t + (u \cdot \nabla) u) = -\nabla p + (\nabla \cdot \tau)^T + \rho g, \\
  \nabla \cdot u = 0
  \]

- Jump conditions on \( \Gamma \):
  \(\star\) Continuity of velocity and divergence of velocity
  \[
  [u] = [v] = 0, \\
  [(u_n, v_n) \cdot n] = 0.
  \]
  \(\star\) Balance between normal stresses and surface tension
  \[
  [\mu(u_n, v_n) \cdot \eta + \mu(u_\eta, v_\eta) \cdot n] = 0, \\
  [p] = \sigma \kappa + 2[\mu](u_n, v_n) \cdot n.
  \]
  \(\star\) Material derivative of velocity continuity
  \[
  \left[ \frac{\nabla p}{\rho} \right] = \left[ \frac{\nabla \cdot \tau}{\rho} \right].
  \]

* How to use them? *
Numerical scheme in the fluid

Predictor-corrector scheme (Chorin-Temam):

- **Prediction** (we take $p = 0$)

  \[
  \frac{u^* - u^n}{\Delta t} = -[(u \cdot \nabla)u]^n + \frac{(\nabla \cdot \tau^n)^T}{\rho} - g
  \]

  centered second-order

  WENO 5

- **Correction**

  \[
  u^{n+1} = u^* - \frac{\Delta t \nabla p^{n+1}}{\rho}
  \]

  centered second-order

- **Resolution of an elliptic equation** :

  \[
  \nabla \cdot \left( \frac{1}{\rho} \nabla p^{n+1} \right) = \frac{\nabla \cdot u^*}{\Delta t}
  \]

  centered second-order
Numerical scheme in the fluid

- Prediction

\[
\frac{u^* - u^n}{\Delta t} = -[(u \cdot \nabla)u]^n + \frac{(\nabla \cdot \tau^n)^T}{\rho} - g
\]

- Elliptic equation:

\[
\nabla \cdot \left( \frac{1}{\rho} \nabla p^{n+1} \right) = \frac{\nabla \cdot u^*}{\Delta t}
\]

- Correction:

\[
u^{n+1} = u^* - \frac{\Delta t}{\rho} \nabla p^{n+1}
\]
Discretization near the interface

Prediction:

\[
\frac{u^* - u^n}{\Delta t} = -[(u \cdot \nabla)u]^n + \frac{(\nabla \cdot \tau^n)^T}{\rho} - g
\]

- **Diffusion**: discontinuous velocity derivatives
  \[\Rightarrow\text{ lack of consistency}\]

- **Convection**: WENO 5 naturally adaptative continuous velocity
  \[\Rightarrow\text{ less worrying, to a certain extent}\]
Discretization near the interface

- **Elliptic equation:**
  \[
  \nabla \cdot \left( \frac{1}{\rho} \nabla p^{n+1} \right) = \frac{\nabla \cdot \mathbf{u}^*}{\Delta t}
  \]
  Discontinuity for \( \rho \), jump conditions
  \[\Rightarrow \text{lack of consistency} \]

- **Correction:**
  \[
  \mathbf{u}^{n+1} = \mathbf{u}^* - \frac{\Delta t}{\rho} \nabla p^{n+1}
  \]
  Discontinuity of \( \rho \) and \( \phi \)
  \[\Rightarrow \text{lack of consistency} \]
State of the art for methods on cartesian grids

**CSF method** (Brackbill et al. 91):
the classical one
regularization of values near the interface, surface tension effect
re-formulated as the limit of a volumic force

**Methods without regularization:**
- VOF methods: Sussman et al, Luo et al., Le Chenadec and Pitsch ...
- Kang, Fedkiw and Liu 2000: application of Ghost Fluid method
- Raessi and Pitsch 2012: cut-cell type method
Ghost fluid method

Turbulent atomization of a liquid Diesel jet (Desjardins et al.)

⇒ Easy to implement, nice results, but stability issues due to erroneous momentum transfers between fluids

Dam break test case: propagation of interface
Raessi and Pitsch method

Use of conservative equations for mass and momentum near the interface, solved consistently with the same flux density

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0
\]

\[
\frac{\partial (\rho U)}{\partial t} + \nabla \cdot (\rho U U) = -\nabla p + \nabla \cdot \tau + F_B,
\]

**Figure**: Left: geometrical reconstruction near the interface, right: dam break, comparison between Ghost-Fluid method and conservative method of Raessi and Pitsch
New method: discretization near the interface

To solve accurately the pressure:

- Creation of interface unknowns for \( u^* \) and \( p \) on the interface

- Regularization of \( \mu \) and \( \rho \) only to take into account viscous effects
  \( \Rightarrow \) no more discontinuity for viscous terms
Elliptic problem

- In the fluid:
  \[ \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) = \frac{\nabla \cdot \mathbf{u}^*}{\Delta t} \]
  \[(u^* \text{ extrapolated on interface)} \]

- On interface:
  \[ [p] = \sigma \kappa, \]
  \[ \left[ \frac{\nabla p}{\rho} \right] = 0. \]
Discretization near the interface

Elliptic problem

- In the fluid:
  \[ \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) = \frac{\nabla \cdot \mathbf{u}^*}{\Delta t}. \]

- On interface:
  \[ [p] = \sigma \kappa, \]
  either \[ \left[ \frac{p_x}{\rho} \right] = 0, \]
  or \[ \left[ \frac{p_y}{\rho} \right] = 0. \]

* Elimination of interface variables
* Convergence analysis valid since the derivatives across the interface are aligned with the axis of the grid
Bubble at rest: parasitic oscillations

- Parasitic oscillations caused by approximated values of the curvature
- More or less amplified by the numerical scheme for the pressure

![Diagram of a bubble at rest with dimensions labeled as 2L, ext, int, and R]

\[ L = 2 \text{ cm}, \]
\[ R = 1 \text{ cm}, \]
\[ \rho_{\text{int}} = 1000 \text{ kg.m}^{-3}, \]
\[ \mu_{\text{int}} = 10^{-3} \text{ Pa.s}, \]
\[ \rho_{\text{ext}} = 1 \text{ kg.m}^{-3}, \]
\[ \mu_{\text{ext}} = 10^{-5} \text{ Pa.s}, \]
\[ \sigma = 0.1 \text{ N.m}^{-1} \]

<table>
<thead>
<tr>
<th>N</th>
<th>Ghost Fluid method</th>
<th>CSF</th>
<th>new method</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$8.08 \times 10^{-3}$</td>
<td>$3.55 \times 10^{-2}$</td>
<td>$5.21 \times 10^{-3}$</td>
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<tr>
<td>32</td>
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<td>64</td>
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<td>$1.36 \times 10^{-5}$</td>
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<tr>
<td>128</td>
<td>$2.79 \times 10^{-5}$</td>
<td>$6.44 \times 10^{-3}$</td>
<td>$2.22 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

**Table:** Error in $L^\infty$ norm at time $t = 1$. 
Bubble at rest: parasitic oscillations

**Figure:** Left: 32*32 grid, horizontal velocity after 1 iteration, right: horizontal velocity after 1s.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>error $L^\infty$ for VOF (Sussman et al)</th>
<th>error $L^\infty$ for new method</th>
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</thead>
<tbody>
<tr>
<td>2.5/16</td>
<td>7.3 $\times 10^{-4}$</td>
<td>7.48 $\times 10^{-5}$</td>
</tr>
<tr>
<td>2.5/32</td>
<td>4.5 $\times 10^{-6}$</td>
<td>4.7 $\times 10^{-6}$</td>
</tr>
<tr>
<td>2.5/64</td>
<td>5.5 $\times 10^{-8}$</td>
<td>1.26 $\times 10^{-6}$</td>
</tr>
</tbody>
</table>

Error at non-dimensional time $t = 250$ for the VOF method of Sussman et al. and new method.
Small air bubble into water

Figure: Water: $\rho = 1000 \text{ kg/m}^3$, $\mu = 1,137 \cdot 10^{-3} \text{ kg/ms}$, air: $\rho = 1\text{ kg/m}^3$, $\mu = 1,78 \cdot 10^{-5} \text{ kg/ms}$, $\sigma = 0.0728 \text{ kg/s}^2$, bubble radius $1/300 \text{ m}$, $T_f = 0.05 \text{s}$. 
Small air bubble into water

**Figure:** Comparison between CSF method (left) and new method (right)
Larger air bubble into water

**Figure:** Water: \( \rho = 1000 \text{ kg/m}^3, \mu = 1,137 \times 10^{-3} \text{ kg/ms}, \) air: \( \rho = 1 \text{ kg/m}^3, \mu = 1,78 \times 10^{-5} \text{ kg/ms}, \sigma = 0.0728 \text{ kg/s}^2, \) bubble radius 0.025 m
Small water droplet in air

**Figure:** Water: $\rho = 1000 \text{ kg/m}^3$, $\mu = 1,137.10^{-3} \text{ kg/ms}$, air: $\rho = 1\text{ kg/m}^3$, $\mu = 1,78.10^{-5} \text{ kg/ms}$, $\sigma = 0.0728 \text{ kg/s}^2$, bubble radius $1/300 \text{ m}$, $Tf = 0.05\text{s}$.
Water: $\rho = 1000 \, \text{kg/m}^3$, $\mu = 1,137 \times 10^{-3} \, \text{kg/ms}$,
Air: $\rho = 1,226 \, \text{kg/m}^3$, $\mu = 1,78 \times 10^{-5} \, \text{kg/ms}$,
$\sigma = 0.0728 \, \text{kg/s}^2$, water column $h = 5.715 \, \text{cm}$, domain $40 \times 10 \, \text{cm}$
Propagation of front: comparison between the conservative method of Raessi and Pitsch, the Ghost-Fluid method and our new method
Outline

1 Second-order cartesian method for elliptic problems with immersed interfaces
2 Application to incompressible bifluid flows
3 How to preserve high-order level-set along time? (with F. Luddens and M. Bergmann)
Motivations for a high order level-set

- Better description of the interface
- Mass conservation
- Need of a consistent $\kappa$ to compute surface tension effects:

$$[p^{n+1}] = \sigma \kappa$$

**Third-order accuracy** needed to compute consistently $\kappa$ from derivatives of the level-set!
Standard approach

- Transport of $\phi$ with $u$ (or with an extension velocity):

$$\phi^* = \phi^n - \Delta t \, u^n \nabla \phi^n,$$

- Every few time steps, re-initialize $\phi^*$ with:
  - a Fast-Marching algorithm
  - a Fast-Sweeping algorithm
  - a relaxation method

$$\partial_\tau \phi + sign(\phi^*) (|\nabla \phi| - 1) = 0,$$
$$\phi|_{\tau=0} = \phi^*.$$

- Very often, RK3-TVD scheme for $\tau$, $t$, WENO-5 scheme for $\nabla \phi$. 

Standard approach

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Main problems of standard approach:

- WENO-5 schemes for reinitialization not enough accurate near interface $\Rightarrow$ the interface moves at each reinitialization step
- Cost: too many reinitialization steps?
- With extension velocities: more accurate but even more costly
To reduce the interface moving

Constatation:
The WENO scheme uses information from the wrong side of the interface

Subcell fix (Russo & Smereka, 2000):
Use information on interface to modify the scheme (decentering)

Higher order extension (Du Chéné et al. 2008):

- Far from interface, WENO scheme,
- near interface, decentered ENO scheme, taking into account the interface position
Example: interface with strong gradients

\[ d = \sqrt{x^2 + y^2} - r_0 \]
\[ \phi_0 = \frac{d}{r_0} \left( \epsilon + (x - x_0)^2 + (y - y_0)^2 \right) \]
\[ \Omega = (-1, 1)^2 \]
\[ r_0 = 0.5 \]
\[ \epsilon = 0.1, \ x_0 = -0.7, \ y_0 = -0.4 \]
Example: interface with strong gradients

<table>
<thead>
<tr>
<th>$\frac{1}{h}$</th>
<th>$|\phi_h - d|_{L^1(\Omega)}$ err</th>
<th>coc</th>
<th>$|\phi_h - d|_{L^\infty(B_n)}$ err</th>
<th>coc</th>
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<tr>
<td>1280</td>
<td>6.82E-07</td>
<td>1.91</td>
<td>6.98E-08</td>
<td>3.12</td>
</tr>
</tbody>
</table>
Example: interface with strong gradients

\[ \|\kappa_h - \kappa\|_{L^\infty(\Gamma)} \]

<table>
<thead>
<tr>
<th>( \frac{1}{h} )</th>
<th>err</th>
<th>coc</th>
<th>( N_{it} )</th>
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Coupling with transport

We introduce the quantity $r_g(\nabla \phi) := \| |\nabla \phi| - 1 \|_{L^1(\Omega)}$, and choose a threshold $\delta > 0$.

Algorithm:

- **Initialization**: with $\phi_0 = d_0$, the signed distance function at interface $\Gamma_0$,
- **Transport**: While $r_g(\nabla \phi) < \delta$, compute the evolution of $\phi$ with transport equation
- **Re-initialization**: When $r_g(\nabla \phi) \geq \delta$, re-compute $\phi$ as the signed distance function $d$.
  - redistanciation with relaxation in a band around interface
  - second-order fast-sweeping elsewhere
Vortex test case

\[ \Omega = (0, 1)^2 \]

\[ \phi|_{t=0} = \sqrt{((x - 0.5)^2 + (y - 0.75)^2)} - 0.15 \]

\[ \mathbf{u} = \cos \left( \frac{\pi t}{T} \right) \nabla \perp \omega \]

\[ \omega = \sin(\pi x)^2 \sin(\pi y)^2 \]

\[ T = 4, t_{fin} = 4 \]

\[ \Gamma \text{ deforms then goes back to } \Gamma_0 \text{ at } t = t_{fin}. \]
Comparison between 4 cases, at time $t = 2$:

1. 5 iterations of relaxation method, every 5 time steps
2. 3 iterations of relaxation method, every time steps
3. new method, with $\delta = 0.1$,
4. new method, with $\delta = 0.01$. 
Vortex test case

<table>
<thead>
<tr>
<th>$\frac{1}{h}$</th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
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<td>err.</td>
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<td>-1.07</td>
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</tbody>
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**Table**: Error $L^\infty$ on curvature at $t = 2$
# Vortex test case

<table>
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<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
<th>case 4</th>
</tr>
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<tr>
<td>40</td>
<td>6.82E-03</td>
<td>1.06E-03</td>
<td>5.67E-03</td>
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<tr>
<td>80</td>
<td>9.39E-04</td>
<td>2.72E-04</td>
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<td>1.53E-04</td>
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<tr>
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<td>1.86E-04</td>
<td>2.19E-04</td>
<td>1.88E-05</td>
<td>1.30E-05</td>
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<tr>
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<td>7.38E-05</td>
<td>1.15E-04</td>
<td>9.66E-07</td>
<td>9.25E-07</td>
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<tr>
<td>640</td>
<td>3.66E-05</td>
<td>6.21E-05</td>
<td>4.30E-08</td>
<td>3.50E-08</td>
</tr>
</tbody>
</table>

**Table:** Volume loss at $t = 2$. 
Vortex test case

**Figure:** Left: $\delta = 0$ (i.e. redistanciation at time step), right: $\delta = 0.1$. Grid $80 \times 80$, $dt = dx/8$
Flow around a cylinder

\[ \Omega = [-3; 3]^2, \]
\[ \Gamma_0 = \{ x^2 + y^2 = 1 \}, \quad \phi_0 = d_0, \]
\[ U_r = \alpha c(r) \left( U_\infty - \frac{1}{r^2} \right) \cos(\theta), \]
\[ U_\theta = -\alpha c(r) \left( U_\infty + \frac{1}{r^2} \right) \sin(\theta), \]
\[ c(r) = \min \left( 1, \frac{r}{0.5} \right)^3, \]
\[ U_\infty = 1, \]
\[ \alpha \text{ such that } \| \mathbf{U} \|_{L^\infty(\Omega)} = 1, \]
\[ t_{fin} = 6. \]
Flow around a cylinder

Figure: Left : $\delta = +\infty$ (i.e. no redistanciation), right : $\delta = 0.1$, grid $80 \times 80$
Flow around a cylinder

| $\frac{1}{h}$ | $\delta = 0.01$ | | $\delta = 0.1$ | | $\delta = +\infty$. |  \\
|---|---|---|---|---|---|  \\
| | err. | coc | err. | coc | err. | coc |  \\
| 40 | 7.24E-02 | - | 8.35E-02 | - | 1.40E+00 | - |  \\
| 80 | 3.91E-02 | 0.87 | 1.68E-02 | 2.27 | 2.15E+00 | -0.61 |  \\
| 160 | 9.71E-03 | 1.99 | 5.09E-03 | 1.71 | 3.53E+00 | -0.71 |  \\
| 320 | 3.66E-03 | 1.40 | 2.51E-03 | 1.01 | 8.22E+01 | -4.52 |  \\
| 640 | 2.51E-03 | 0.54 | 1.88E-03 | 0.42 | 1.57E+02 | -0.93 |  \\

**Table:** $\|\kappa_h - \kappa\|_\infty(\Gamma)$ at $t = 6$ and convergence order.
Rising of a large air bubble into water: new redistanciation

**Figure:** Water: $\rho = 1000 \text{ kg/m}^3$, $\mu = 1,137.10^{-3} \text{ kg/ms}$, air: $\rho = 1 \text{ kg/m}^3$, $\mu = 1,78.10^{-5} \text{ kg/ms}$, $\sigma = 0.0728 \text{ kg/s}^2$, bubble radius 0.025 m
Rising of a large air bubble into water: new redistanciation

**Figure:** Water: $\rho = 1000 \text{ kg/m}^3$, $\mu = 1,137 \times 10^{-3} \text{ kg/ms}$, air: $\rho = 1\text{ kg/m}^3$, $\mu = 1,78 \times 10^{-5} \text{ kg/ms}$, $\sigma = 0.0728 \text{ kg/s}^2$, bubble radius 0.025 m
Conclusion

- New cartesian method for incompressible bifluid flows with high density ratios:
  - with second-order pressure resolution
  - compromise between accuracy and simplicity
- To obtain a third-order level-set method along time:
  DO NOT use redistanciation every few time steps!

In the future:

- Implementation in NasCar code
- Application to air-water interface + floating solid
- Development of an incremental form
- Comparisons with other families of methods: front-tracking, VOF, phase-field...