A Stochastic Look at Geodesics on the Sphere

Marc Arnaudon[1,2](✉) and Jean-Claude Zambrini[1,2]

[1] Inst. de Mathématiques de Bordeaux, Univ. de Bordeaux, Bordeaux, France
marc.arnaudon@math.u-bordeaux.fr
[2] Grupo de Física-Matemática Univ. Lisbon, Lisbon, Portugal
jczambrini@fc.ul.pt

Abstract. We describe a method allowing to deform stochastically the completely integrable (deterministic) system of geodesics on the sphere S^2 in a way preserving all its symmetries.

Keywords: Geodesic flow · Stochastic deformation · Integrable systems

1 Introduction

Free diffusions on a sphere S^2 are important case studies in applications, for instance in Biology, Physics, Chemistry, Image processing etc., where they are frequently analysed with computer simulations. However, as for most diffusions on curved spaces, no closed form analytical expressions for their probability densities are available for such simulations. Another way to express the kind of difficulties one faces is to observe that one cannot define Gaussian functions on S^2.

If, instead of free diffusions on S^2 we consider their deterministic counterpart, the classical geodesic flow, a famous integrable system whose complete solution dates back to the 19th century, the situation is much simpler. Indeed, one can use the conservation of angular momentum and energy to foliate the phase space (the cotangent bundle of its configuration space).

We describe here a method allowing to construct free diffusions on S^2 as stochastic deformations of the classical geodesic flow, including a probabilistic counterpart of its conservation laws.

2 Classical Geodesics

The problem of geodesic on the sphere S^2 is a classical example of completely integrable elementary dynamical system [1].

For a unit radius sphere and using spherical coordinates $(q^i) = (\theta, \phi) \in]0, \pi[\times [0, 2\pi]$ where ϕ is the longitude, the Lagrangian L of the system is the scalar defined on the tangent bundle TS^2 of the system by

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = (\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2) \tag{1}$$

F. Nielsen and F. Barbaresco (Eds.): GSI 2017, LNCS 10589, pp. 470–476, 2017.
https://doi.org/10.1007/978-3-319-68445-1_55

(where $\dot{\theta} = \frac{d\theta}{dt}$ etc. ...), since it coincides with $ds^2 = g_{ij}dq^i dq^j$, here $g = (g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$. The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \qquad i = 1, 2 \tag{2}$$

in these coordinates are easily solved. They describe the dynamics of the extremals (here minimal) curves of the action functional

$$S_L[q(\cdot)] = \int_{Q_1}^{Q_2} L(q, \dot{q}) dt \tag{3}$$

computed, for instance, between two fixed configurations $Q_1 = (\theta_1, \phi_1)$ and $Q_2 = (\theta_2, \phi_2)$ in the configuration space. Those equations are

$$\ddot{\theta} = \dot{\phi}^2 \sin\theta \cos\theta, \qquad \ddot{\phi} = -2\dot{\theta}\dot{\phi} \text{ cotg } \theta \tag{4}$$

Defining the Hamiltonian $H : T^*S^2 \to \mathbb{R}$ as the Legendre transform of L, we have $H = \frac{1}{2} g^{ij} p_i p_j$, where $p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij}\dot{q}^j$ denote the momenta, here

$$H(\theta, \phi, p_\theta, p_\phi) = \frac{1}{2}(p_\theta^2 + \frac{1}{\sin^2\theta} p_\phi^2), \tag{5}$$

with $p_\theta = \dot{\theta}, p_\phi = \sin^2\theta\dot{\phi}$.

It is clear that the energy H is conserved during the evolution. There are three other first integrals for this system, corresponding to the three components of the angular momentum $\mathcal{L}$. They can be expressed as differential operators of the form $X_j^\theta \frac{\partial}{\partial\theta} + X_j^\phi \frac{\partial}{\partial\phi}, j = 1, 2, 3$, namely

$$\mathcal{L}_1 = \sin\phi \frac{\partial}{\partial\theta} + \frac{\cos\phi}{\tan\theta} \frac{\partial}{\partial\phi}$$

$$\mathcal{L}_2 = -\cos\phi \frac{\partial}{\partial\theta} + \frac{\sin\phi}{\tan\theta} \frac{\partial}{\partial\phi}, \qquad \mathcal{L}_3 = -\frac{\partial}{\partial\phi} \tag{6}$$

In geometrical terms, written as $\mathcal{L}_j = (X_j^\theta, X_j^\phi), j = 1, 2, 3$, they are the three Killing vectors for S^2, forming a basis for the Lie algebra of the group of isometries $SO(3)$ of S^2. $\mathcal{L}_3$ corresponds to the conservation of the momentum p_ϕ.

The integrability of this dynamical system relies on the existence of the two first integrals H and p_ϕ. They allow to foliate the phase space by a two-parameter family of two-dimensional tori. Let us recall that the list of first integrals of the system is the statement of Noether's Theorem, according to which the invariance of the Lagrangian L under the local flow of vector field

$$v^{(1)} = X^i(q, t)\frac{\partial}{\partial q^i} + \frac{dX^i}{dt}\frac{\partial}{\partial \dot{q}^i} \tag{7}$$

associated with the group of transformations

$$(q^i, t) \to (Q_\alpha^i = q^i + \alpha X^i(q, t), \tau_\alpha = t + \alpha T(t))$$

for α a real parameter, provides a first integral along extremals of S_L of the form

$$\frac{d}{dt}(X^i p_i - TH) = 0. \tag{8}$$

The coefficients X^i, T must, of course, satisfy some relations between them called "determining equations" of the symmetry group of the system [2]. For instance, for our geodesics on S^2,

$T = 1, X = (X^\theta, X^\phi) = (0, 0)$ corresponds to the conservation of the energy H, and $T = 0, X = (0, -1)$ to the conservation of p_ϕ In fact, the three vectors X_j must satisfy the Killing equations in the (θ, ϕ) coordinates,

$$\nabla^\theta X_j^\phi + \nabla^\phi X_j^\theta = 0, j = 1, 2, 3 \tag{9}$$

where $\nabla^\cdot$ denotes the covariant derivatives.

3 Stochastic Deformation of the Geodesics on the Sphere

Many ways to construct diffusions on S^2 are known. In the spirit of K. Itô [3], we want to deform the above classical dynamical system in a way preserving the essential of its qualitative properties.

Let us start from the backward heat equation for the Laplace-Beltrami "Hamiltonian" operator H (without potentials). in local coordinates (q^i) it can be written $\frac{\partial \eta}{\partial t} = H\eta$, where $g = \det(g_{ij})$ and

$$H = -\frac{1}{2}\Delta_{LB} = -\frac{1}{2\sqrt{\det g}}\frac{\partial}{\partial q^i}\Big(\sqrt{\det g}\; g^{ij}\frac{\partial}{\partial q^j}\Big). \tag{10}$$

A more revealing form in terms of the Christoffel symbols of the Riemannian connection is

$$-\frac{1}{2}\Delta_{LB} = -\frac{g^{ij}}{2}\frac{\partial^2}{\partial q^i \partial q^j} + \frac{1}{2}\Gamma^i_{jk}(q) g^{jk}(q)\frac{\partial}{\partial q^i}. \tag{11}$$

Indeed, the extra first order term, of purely geometric origin, will coincide with the drift of the simplest diffusion on our manifold, the Brownian motion; this was observed by K. Itô, as early as 1962 [3]. In our spherical case, one finds

$$\Gamma^\theta_{jk} g^{jk} = -\text{cotg}\,\theta, \Gamma^\phi_{jk} g^{jk} = 0. \tag{12}$$

Now we shall consider general diffusions z^i on S^2 solving SDEs of the form

$$dz^i(\tau) = (B^i - \frac{1}{2}\Gamma^i_{jk} g^{jk})d\tau + dW^i(\tau), \quad \tau > t \tag{13}$$

for B^i an unspecified vector field, where $dW^i(\tau) = \sigma^i_k d\beta^k(\tau)$ with σ^i_k the square root of g^{ij}, i.e. $g^{ij} = \sigma^i_k \sigma^j_k$, in our case $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin\theta} \end{bmatrix}$ and β is a two dimensional Wiener process.

Here is the stochastic deformation of the extremality condition for dynamical trajectories in terms of the classical action S_L. It will be convenient to consider S_L as a function of starting configurations q at a time t. For convenience, we shall add a final boundary condition to S_u to S_L.

Let $S_L(q,t)$ be defined now by $S_L(q,t) = -\ln \eta(q,t)$, where $\eta(q,t)$ is a positive solution of the backward heat equation for a (smooth) final boundary condition $S_u(q), u > t$. Let B^i in (13) be adapted to the increasing filtration $\mathcal{P}_\tau$, bounded but otherwise arbitrary. Then

$$S_L(q,t) \leq E_{qt}\{\int_t^u \frac{1}{2} B^i B_i(z(\tau),\tau) d\tau + S_u(z(u))\} \tag{14}$$

where E_{qt} denotes the conditional expectation given $z(t) = q$. The equality holds on the extremal diffusion on S^2, of drift

$$B^i(q,t) = \frac{\partial_i \eta}{\eta}(q,t) = -\nabla^i S_L. \tag{15}$$

This means, on S^2, that S_L minimizes the r.h.s. functional of (14) for the Lagrangian

$$L = \frac{1}{2}[(\frac{\partial_\theta \eta}{\eta})^2 + \sin^2\theta (\frac{\partial_\phi \eta}{\eta})^2] \tag{16}$$

where, manifestly, $(\frac{\partial_\theta \eta}{\eta})$ and $(\frac{\partial_\phi \eta}{\eta})$ plays the roles of $\dot{\theta}$ and $\dot{\phi}$ in the deterministic definition (1).

Let us observe that after the above logarithmic change of variable, it follows from the backward heat equation that the scalar field S_L solves

$$-\frac{\partial S_L}{\partial t} + \frac{1}{2}\|\nabla S_L\|^2 - \frac{1}{2}\nabla^i \nabla_i S_L = 0, \tag{17}$$

with $t < u$ and $S_L(q,u) = S_u(q)$.

This is an Hamilton-Jacobi-Bellman equation, whose relation with heat equations is well known and used in stochastic control [4]. The Laplacian term represents the collective effects of the irregular trajectories $\tau \to z^i(\tau)$ solving (13).

A second order in time dynamical law like (4) requires the definition of the parallel transport of our velocity vector field B^i.

In [3] Itô had already mentioned that there is some freedom of choice in this, involving the Ricci tensor R^i_k on the manifold. One definition is known today in Stochastic Analysis as "Damped parallel transport" [5]. Then the generator of the diffusion z^i acting on a vector field V on S^2 is given by

$$D_t V^i = \frac{\partial V^i}{\partial t} + B^k \nabla_k V^i + \frac{1}{2}(\varDelta V)^i \tag{18}$$

where, instead of the Laplace-Beltrami operator, one has now

$$\varDelta V^i = \nabla^k \nabla_k V^i + R^i_k V^k \tag{19}$$

When acting on scalar fields φ, D_t reduces to the familiar form

$$D_t\varphi = \frac{\partial \varphi}{\partial t} + B^k \nabla_k \varphi + \frac{1}{2}\nabla^k \nabla_k \varphi \tag{20}$$

When $\varphi = q^k$, $D_t\varphi^k = B^k(z(t),t) = -\nabla^k S_L$, so the r.h.s. Lagrangian of (14) is really $\frac{1}{2}\|D_t z\|^2$, for $\|\cdot\|$ the norm induced by the metric, as it should. For the vector field B^i, we use (17) and the integrability condition $\frac{\partial}{\partial t}\nabla^i S_L = \nabla^i \frac{\partial S_L}{\partial t}$, following from the definition of S_L, to obtain

$$D_t D_t z^i = 0 \tag{21}$$

i.e., the stochastic deformation of both O.D.E.s (4) when $z(t) = (\theta(t), \phi(t))$ solve Eq. (13) namely, in our case,

$$d\theta(t) = (\frac{\partial_\theta \eta}{\eta} + \frac{\text{cotg}\theta}{2})dt + dW^\theta(t), \quad d\phi(t) = \frac{1}{\sin^2\theta}(\frac{\partial_\phi \eta}{\eta})dt + dW^\phi(t) \tag{22}$$

The bonus of our approach lies in the study of the symmetries of our stochastic system. The symmetry group of the heat equation, in our simple case with constant positive curvature, is generated by differential operators of the form [6]

$$\hat{N} = X^i(q)\nabla_i + T\frac{\partial}{\partial t} + \alpha \tag{23}$$

where T and α are constants, and the X^i are three Killing vectors on (S^2, g). Besides a one dimensional Lie algebra generated by the identity, another one corresponds to $T = 1$ and $X = (X^\theta, X^\phi) = (0,0)$. This provides the conservation of energy defined here, since $S_L = -\ln\eta$, by $h(\theta(t),\phi(t)) = -\frac{1}{\eta}\frac{\partial\eta}{\partial t}$ or, more explicitly,

$$h = \frac{1}{2}g^{ij}B_iB_j + \frac{1}{2}g^{ij}\frac{\partial}{\partial q^i}B_j - \frac{1}{2}\Gamma^i_{jk}g^{jk}B_i \tag{24}$$

Using (20), one verifies that

$$D_t h(z(t),t) = 0 \tag{25}$$

in other words, h is a martingale of the diffusion $z(t)$ extremal of the Action functional in (14). This is the stochastic deformation of the corresponding classical statement (8) when $X = (0,0), T = 1$. Analogously, our (deformed) momentum p_ϕ is a martingale. In these conditions, one can define a notion of integrability for stochastic systems (not along Liouville's way, but inspired instead by Jacobi's classical approach) and show that, in this sense, our stochastic problem of geodesics on the sphere is as integrable as its deterministic counterpart. This will be done in [7].

To appreciate better in what sense our approach is a stochastic deformation of the classical problem of geodesics in S^2, replace our metric (g_{ij}) by $\hbar(\sigma_{ij})$ for

σ_{ij} the Riemannian metric, where $\hbar$ is a positive constant, and take into account that our underlying backward heat equation now becomes

$$\frac{\partial \eta}{\partial t} = -\frac{\hbar}{2} \Delta_{LB} \eta \tag{26}$$

then, one verifies easily that, when $\hbar \to 0$, $D_t \to \frac{d}{dt}$, the Lagrangian of (14) reduces to the classical one (1) and the conditional expectation of the action (1) disappears. The Hamilton-Jacobi-Bellman equation (17) reduces to the one of the classical dynamical system and our martingales to its first integrals. In this respect, observe that general (positive) final conditions for Eq. (26) may depend as well on $\hbar$. They provide analogues of Lagrangian submanifolds in the semiclassical limit of Schrödinger equation (Cf Appendix 11 of [13]).

We understand better, now, the role of the future boundary condition S_u in (1): when S_u is constant, the extremal process $z(\cdot)$ coincides with the Brownian motion on S^2 but, of course, in general this is not the case anymore. Stochastic deformation on a Riemannian manifold was treated in [8]. For another approach c.f. [9].

In spite of what was shown here, our approach can be made invariant under time reversal, in the same sense as our underlying classical dynamical system. The reason is that the very same stochastic system can be studied as well with respect to a decreasing filtration and an action functional on the time interval $[s, t]$, with an initial boundary condition $S^*_s(q)$. This relates to the fact that to any classical dynamical systems like ours are associated, in fact, two Hamilton-Jacobi equations adjoint with respect to the time parameter. The same is true after stochastic deformation. So, a time-adjoint heat equation, with initial positive boundary condition, is involved as well. The resulting ("Bernstein reciprocal") diffusions, built from these past and future boundary conditions, are invariant under time reversal on the time interval $[s, u]$. C.f. [10, 12].

In particular, Markovian Bernstein processes are uniquely determined from the data of two (stictly positive) probability densities at different times s and u, here on S^2. They solve a "Schrödinger's optimization problem", an aspect very reminiscent of foundational questions of Mass Transportation theory [14]. The close relations between this theory and our method of Stochastic Deformation have been carefully analysed in [11], where many additional references can be found as well.

References

1. Dubrovin, B.A., Krichever, J.M., Novikov, S.P.: Integrable systems I. In: Arnold, V.I., Novikov, S.P. (eds.) Dynamical Systems IV. Encyclopaedia of Mathematical Sciences, vol. 4, pp. 177–332. Springer, Heidelberg (1990)
2. Olver, P.: Applications of Lie Groups to Differential Equations, 2nd edn. Springer, New York (1993)
3. Itô, K.: The Brownian motion and tensor fields on a Riemannian manifold. In: Proceedings of the International Congress Mathematical (Stockholm), pp. 536–539 (1962)

4. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions, 2nd edn. Springer, New York (2005)
5. Malliavin, P.: Stochastic Analysis. Springer, Heidelberg (1997)
6. Kuwabara, R.: On the symmetry algebra of the Schrödinger wave equation. Math. Japonica **22**, 243 (1977)
7. Léonard, C., Zambrini, J.-C.: Stochastic deformation of Jacobi's integrability Theorem in Hamiltonian mechanics. Preparation
8. Zambrini, J.-C.: Probability and quantum symmetries in a riemannian manifold. Prog. Probab. **45** (1999). Birkhäuser
9. Kolsrud, T.: Quantum and classical conserved quantities: martingales, conservation law and constants of motion. In: Benth, F.E., Di Nunno, G., Lindstrøm, T., Øksendal, B., Zhang, T. (eds.) Stochastic Analysis and Applications: Abel Symposium, pp. 461–491. Springer, Heidelberg (2005)
10. Zambrini, J.-C.: The research program of Stochastic Deformation (with a view toward Geometric Mechanics). In: Dalang, R., Dozzi, M., Flandoli, F., Russo, F. (eds.) BirkhäuserStochastic Analysis, A Series of Lectures (2015)
11. Léonard, C.: A survey of the Schrödinger problem and some of its connections with optimal transport. Discrete Contin. Dyn. Syst. A **34**(4) (2014)
12. Léonard, C., Roelly, S., Zambrini, J.-C.: Reciprocal processes. A measure-theoretical point of view. Probab. Surv. **11**, 237–269 (2014)
13. Arnold, V.: Méthodes Mathématiques de la Mécanique Classique. Mir, Moscou (1976)
14. Villani, C.: Optimal Transport, Old and New. Grundlehren der mathematischen Wissenschaften. Springer, Heidelberg (2009)