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# Stochastic Tools on Hilbert Manifolds: Interplay with Geometry and Physics

**M. Arnaudon**[1], **S. Paycha**[2]

[1] Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7, rue René Descartes 67084 Strasbourg Cedex, France
[2] Laboratoire de mathématiques appliquées, Complexe Universitaire des Cézeaux, Université Blaise Pascal, 63188 Aubière Cedex, France. E-mail: paycha@ucfma.univ-bpclermont.fr

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**Abstract:** Projections via the action of a Hilbert Lie group of a class of semi-martingales (given by Itô fields) defined on Hilbert manifolds are investigated. Using Itô calculus, we show that the drift term arising in the projected process can be interpreted in terms of a regularised trace of the second fundamental form of the orbits. For group actions with finite dimensional orbit space, we introduce a notion of strongly harmonic functions resp. regularised Brownian motion, which project onto harmonic functions resp. onto Brownian motion, whenever the orbits are minimal (in a regularised sense). We relate this projection procedure of semi-martingales to the Faddeev-Popov procedure in gauge field theory.

## Introduction

We investigate here the interplay between the geometry of orbit spaces for a class of infinite dimensional group actions, the projections of a class of semi-martingales from the total space to the orbit space for such group actions and the Faddeev-Popov procedure in gauge field theory. A cornerstone for building up links between these topics in the fields of geometry, stochastic analysis and physics are heat-kernel operators which arise in the regularisation procedure to define geometric notions such as minimality of orbits, to describe projected processes and the so called “Faddeev-Popov" determinant in gauge field theory [1, 2].

The heat-kernel regularisation methods involved have been studied and compared with other regularisation procedures in [2]. The aim of the present work is to shed light on the relationships between the three topics mentioned above, using apropriate stochastic and geometric tools, which we define as we go along. We shall in particular define the Stratonovich differential for a class of semi-martingales on Hilbert manifolds and discuss how they project on a class of principal bundles defined by the action of a Hilbert Lie group on a Hilbert manifold.

Let us briefly describe the contents of this paper. We first generalise the notion of Stratonovich differential to a class of semi-martingales defined by an Itô field on a Hilbert manifold in the sense of [3], using a notion of weighted divergence for 1-forms. We apply this framework to project such semi-martingales onto the space of orbits for the action of a Hilbert Lie group on the Hilbert manifold.

We also write up an Itô formula for this class of martingales which involves a weighted trace of the Hessian. We show that smooth functions with vanishing weighted trace of the Hessian transform martingales defined by an Itô field into martingales.

We then interpret the drift term arising in the projection of such semi-martingales in terms of a regularised trace of the second fundamental form, already discussed in [2]. We prove that for a group action with strongly minimal (in the sense of [2]) orbits, semi-martingales defined by a class of Itô fields $(a, \mathcal{A})$ (the choice of which depends on the action of the group) project onto semi-martingales defined by the corresponding projected Itô field $(\overline{a}, \overline{\mathcal{A}})$.

When the space of orbits is a finite dimensional Riemannian manifold, we can choose a family of Itô fields entirely determined by the group action. They give rise to a class of martingales which we call regularised Brownian motions associated to the group action. We call smooth functions which transform these regularised Brownian motions into martingales strongly harmonic functions. We prove that when the orbits are strongly minimal, these regularised Brownian motions project onto Brownian motions and invariant strongly harmonic functions onto harmonic functions on the manifold of orbits.

We finally investigate a class of group actions arising in gauge field theories to which we apply the above framework. As we go along, we shall illustrate the abstract geometric framework with a toy model, given by the coadjoint action of a loop group on the corresponding infinite dimensional Lie algebra. We shall also make comments on the way as to the technical difficulties one comes across when investigating other models such as Yang-Mills actions and actions of diffeomorphism groups on metric spaces arising in string theory.

Similar techniques have already been applied before to give a stochastic interpretation of ill defined integrals on path space in the context of Yang-Mills theory [4]. The corresponding processes were built directly on the orbit space there, whereas here we are interested in the behavior of processes defined on the whole path space when projecting them onto the orbit space. This projection procedure for a class of processes (which we call renormalised Brownian motions) is relevent from the physical point of view since it mimics the projection and renormalisation procedure for formal volume measure, an essential ingredient of the Faddeev-Popov procedure in gauge field theory. Using results of [2] that relate the regularised trace of the second fundamental form of the orbits to the horizontal variation of their (heat-kernel) regularised volume, we interpret the projections of renormalised Brownian motions as a paradigma for the Faddeev-Popov procedure. We discuss the choice of the underlying Riemannian structure on the manifold $\mathcal{P}$.

In what follows, $\mathcal{P}$ is an infinite dimensional Hilbert manifold modelled on a Hilbert space $(H, < \cdot, \cdot >)$. We equip $\mathcal{P}$ with a smooth strong Riemannian structure given by a positive symmetric two tensor $g : T\mathcal{P} \times T\mathcal{P} \to T\mathcal{P}$ which induces a scalar product $< \cdot, \cdot >_p$ on the tangent space $T_p\mathcal{P}$. We shall denote by $\nabla$ the corresponding Riemannian connection.

## 1. Some Tools for Stochastic Calculus on Hilbert Manifolds

*1.1. Weighted divergence and weighted traces of Hessians.* Let $K_p$, $p \in \mathcal{P}$ be a family of Hilbert spaces and

$$A(p) : K_p \to T_p\mathcal{P}, \quad p \in \mathcal{P}$$

be a family of self-adjoint Hilbert Schmidt operators.

For a field of bounded $k+2$-multilinear maps $\mathcal{B}$ on $T\mathcal{P}$, we can define the weighted trace of $\mathcal{B}$ as the $k$-multilinear map $\mathrm{tr}_A\mathcal{B}$ defined by:

$$(\mathrm{tr}_A\mathcal{B})(X_1, \cdots, X_k) = \sum_{n \in \mathbb{N}} \mathcal{B}(Ae_n, Ae_n, X_1, \cdots, X_k)$$

where $X_i$ are vector fields on $\mathcal{P}$, where for $p \in \mathcal{P}$, $(e_n(p))_{n\in\mathbb{N}}$ is any complete orthonormal basis of $K_p$ and $(Ae_n)(p) \equiv A(p)e_n(p)$. This infinite sum is well defined since $A(p)$ is a Hilbert-Schmidt operator on $T_p\mathcal{P}$. For fixed $p \in \mathcal{P}$, $\sum_{n\in\mathbb{N}} \mathcal{B}(Ae_n, Ae_n, X_1, \cdots, X_k)$ is independent of the choice of the orthonormal basis $(e_n(p))$ of $K_p$.

The covariant differential $\nabla$ on vector fields on $\mathcal{P}$ extends in the usual way (see e.g [5]) to an endomorphism of the space of tensors on $\mathcal{P}$, setting $DT(X; X_1, \cdots, X_k) = \nabla_X T(X_1, \cdots, X_k)$. If $T$ is a smooth covariant $k$-tensor, since the Riemannian structure on $\mathcal{P}$ is smooth, the map $DT$ is a smooth covariant $k+1$-tensor which induces a field of bounded multilinear maps on $T\mathcal{P}$. In particular, for a smooth $k$-form on $\mathcal{P}$, $D\alpha(\cdot, \cdot, X_1, \cdots, X_{k-1})$ is a field of bilinear bounded maps on $T\mathcal{P}$ and we can define its weighted trace $\mathrm{tr}_A(D\alpha(\cdot, \cdot, X_1, \cdots, X_{k-1}))$ and a notion of weighted divergence which generalises the finite dimensional notion (obtained by setting $A(p) = I$, see e.g [6], par. 3.135).

**Definition .** *For a smooth $k$- form on $\mathcal{P}$, the weighted divergence $div_A\alpha$ is the $k-1$-form:*

$$(div_A\alpha)(p)(X_1, \cdots, X_{k-1}) \equiv tr_A(D\alpha(\cdot; \cdot, X_1, \cdots, X_{k-1})).$$

*If $X$ is a smooth vector field on $\mathcal{P}$, let $X^\flat$ be the smooth one form on $\mathcal{P}$ defined by $X^\flat(u)(p) =< X, u >_p$ for any vector $u$ at point $p$. We define its weighted divergence as the real valued function on $\mathcal{P}$:*

$$div_A(X) = tr_A(DX^b).$$

*We also define a* **weighted trace of the Hessian** *for smooth vector-valued functions which coincides in the finite dimensional case with the ordinary trace of the Hessian when taking $A(p) = I$ by:*

$$tr_A Hess(f) = div_A \nabla f$$

*1.2. Semi-martingales defined by Itô fields .* We shall consider semi-martingales $\xi_t$ on $\mathcal{P}$ defined by a locally Lipschitz Itô field $(a, \mathcal{A})$ in the sense of [3] chap.4., i.e $\xi_t$ is locally described as a solution of a stochastic equation

$$d^\nabla \xi = \mathcal{A}(\xi)dB + a(\xi)dt, \tag{1.1}$$

where $B$ is a $H$-valued Wiener process (in fact $B$ takes its values in the completion of $H$ with respect to a norm $\|\cdot\|_- = \|S(\cdot)\|$ defined by an injective self-adjoint Hilbert-Schmidt operator $S : H \to H$ with densely defined inverse, see [3] p.6 and p.91),

$$p \mapsto a(p) \in T_p\mathcal{P} \qquad \text{and} \qquad p \mapsto \mathcal{A}(p) \in HS(H, T_p\mathcal{P})$$

are locally Lipschitz maps ($HS(H,K)$ denotes the space of Hilbert-Schmidt operators from the Hilbert space $H$ to the Hilbert space $K$). For the second map, this means that for any local chart $(U,\phi)$, there is an open subset $V \subset U$ such that for $p, p' \in V$,

$$\|\phi_* \circ \mathcal{A}(p) - \phi_* \circ \mathcal{A}(p')\|_{H.S} \leq c\|\phi(p) - \phi(p')\|_H$$

for some strictly positive constant $c$ and where $\|\cdot\|_{H.S}$ is the Hilbert-Schmidt norm on operators defined on $H$.

Equation (1.1) is to be understood locally as follows. Let $\phi : U \subset \mathcal{P} \to H$, be a smooth local chart. Let $S, T$ be two stopping times such that $S \leq T$, and on $\{S < \infty\}$, $\xi_t(\omega)$ belongs to $U$ if $t \in [S(\omega), T(\omega)[$. Setting $\xi_\phi \equiv \phi(\xi)$, on the random interval $[S, T[$, Eq. (1.1) reads:

$$\begin{aligned} d\xi_\phi &= (\phi_* \circ \mathcal{A})(\phi^{-1}(\xi_\phi))dB + \phi_* a(\phi^{-1}(\xi_\phi))dt + \frac{1}{2}(\mathrm{tr}_{\mathcal{A}}\mathrm{Hess}\phi)\phi^{-1}(\xi_\phi)dt \\ &= \phi_*(d^\nabla \xi) + \frac{1}{2}(\mathrm{tr}_{\mathcal{A}}\mathrm{Hess}\phi)\phi^{-1}(\xi_\phi)dt, \end{aligned} \tag{1.2}$$

where $\phi_*(d^\nabla\xi)$ stands for $(\phi_* \circ \mathcal{A})(\phi^{-1}(\xi_\phi))dB + \phi_* a(\phi^{-1}(\xi_\phi))dt$.

The last term in the drift is well defined as the weighted trace of the Hessian of a smooth $H$ valued function on $\mathcal{P}$. Equation (1.2) is a stochastic equation on the Hilbert space $H$ as described in [3] Eq. (2.29), Chapter 3.

The formal differential $d^\nabla\xi$ written in (1.1) will be called the Itô differential of $\xi$, the terms $\mathcal{A}(\xi)dB$ and $a(\xi)dt$ will be called respectively its martingale part and its drift.

We shall say that the semimartingale $\xi$ defined by (1.1) is a martingale if $a \equiv 0$. Note that manifold-valued martingales correspond to local martingales if the manifold is $\mathbb{R}$.

*1.3. Stratonovich differentials and the Itô formula.* Let $\alpha$ be any smooth one form on $\mathcal{P}$. The Itô integral on the Hilbert space $H$ ( see e.g [7] for Itô integrals on Hilbert spaces) defined via charts $\phi$ on $\mathcal{P}$ and via time localizations with stopping times by $\int(\phi^{-1*})\alpha(\phi_* d^\nabla\xi)$ is independent of the choice of the charts and the localizations and yields an Itô integral on $\mathcal{P}$:

$$\int \alpha(d^\nabla\xi) = \int(\phi^{-1*}\alpha)(\phi_* d^\nabla\xi), \tag{1.3}$$

where $\phi_*(d^\nabla\xi)$ is given by (1.2).

Generalising the definition of the Stratonovich differential $\delta\xi$ for a diffusion $\xi$ on a finite dimensional Riemannian manifold, solution of a stochastic differential equation of the above type to the infinite dimensional case, we set:

**Definition .** *The Stratonovich differential $\delta\xi$ of $\xi$ is caracterised by the following relation:*

$$\int \alpha(\delta\xi) = \int \alpha(d^\nabla\xi) + \frac{1}{2}\int div_{\mathcal{A}(\xi)}(\alpha)dt$$

*where $\alpha$ is a smooth 1-form on $\mathcal{P}$.*

*Remark .* Notice that this is the usual relation between Stratonovich and Itô differentials for semi-martingales solution of $d^\nabla\xi = \mathcal{A}(\xi)dB + a(\xi)dt$ in the finite dimensional case.

The Itô formula on Hilbert spaces [7] yields the Itô formula on $\mathcal{P}$ via (1.2). For a smooth function $f$ on $\mathcal{P}$ and a semi-martingale $\xi$ chosen as above, we have:

$$\begin{aligned} df(\xi) &= df(d^{\nabla}\xi) + \frac{1}{2}\mathrm{tr}_{\mathcal{A}}\mathrm{Hess}(f)(\xi)dt \\ &= df(d^{\nabla}\xi) + \frac{1}{2}\mathrm{div}_{\mathcal{A}}\nabla f(\xi)dt \end{aligned} \tag{1.4}$$

*Remark* . 1) This type of Itô formula was used by Asorey and Mitter [4] in the context of Yang-Mills theory.

2) One clearly recovers the usual Itô formula in the finite dimensional case.

As a consequence, we have that for a smooth function $f$ and an Itô field $(0, \mathcal{A})$, the following statements are equivalent:

i) for all $p \in \mathcal{P}$, a martingale locally defined by $d^{\nabla}\xi^p = \mathcal{A}(\xi^p)dB, \xi^p(0) = p$ transforms into a local martingale $f(\xi^p)$

ii) for all $p \in \mathcal{P}$, $\mathrm{tr}_{\mathcal{A}}\mathrm{Hess} f(p) = 0$.

The proof goes as in the finite dimensional case [8, 5.28].

## 2. Projecting Semi-Martingales

Let now $\mathbf{G}$ be a (right-) semi-Hilbert Lie group (in the sense of [9], i.e. the usual definition of a Hilbert Lie group holds up to the fact that only the multiplication on the right is required to be smooth, see also [10] for a detailed discussion concerning the differential structure on the group) acting smoothly on $\mathcal{P}$ on the right:

$$\begin{aligned} \mathcal{P} \times \mathbf{G} &\rightarrow \mathcal{P} \\ (p, g) &\mapsto p \cdot g \end{aligned}$$

in such a way that $X = \mathcal{P}/\mathbf{G}$ is a smooth manifold and the canonical projection $\pi : \mathcal{P} \rightarrow \mathcal{P}/\mathbf{G}$ yields a smooth principal fibre bundle. We shall assume that the Riemannian structure on $\mathcal{P}$ is $\mathbf{G}$-invariant so that it induces a Riemannian connection on the fibre bundle:

$$T_p\mathcal{P} = H_p\mathcal{P} + V_p\mathcal{P},$$

where $V_p\mathcal{P}$ is the vertical tangent space at point $p$ and $H_p\mathcal{P}$ (the horizontal space) is the orthogonal of $V_p\mathcal{P}$. This induces also a Riemmannian metric on $X$, such that the restriction of $\pi_*$ to each horizontal space $H_p\mathcal{P}$ is an isometry. Its Levi-Civita connection will be denoted by $\overline{\nabla}$.

We shall also assume there is an orthogonal splitting of $H = H' \oplus H''$ and that there are fields of operators $\mathcal{A}^h(p) : H' \rightarrow H_p\mathcal{P}$, $\mathcal{A}^v(p) : H'' \rightarrow V_p\mathcal{P}$, such that

$$\mathcal{A}(p) = \mathcal{A}^h(p) \oplus \mathcal{A}^v(p) \tag{2.1}$$

$$\mathcal{A}(p \cdot g) = R_{g*}\mathcal{A}(p) \quad g \in \mathbf{G}, p \in \mathcal{P}. \tag{2.2}$$

Of course, since $\mathcal{A}(p)$ is Hilbert-Schmidt, so are $\mathcal{A}^v(p)$ and $\mathcal{A}^h(p)$, where the upper indexes "$h$" and "$v$" stand for the horizontal part and the vertical part.

*2.1. The weighted trace of the second fundamental form.* Let us define the second fundamental form (see e.g [11, 12]):

$$\begin{aligned} S^p : V_p\mathcal{P} \times V_p\mathcal{P} &\to H_p\mathcal{P} \\ (Y, Y') &\mapsto (\nabla_Z Z')^h(p), \end{aligned}$$

where $Z$, $Z'$ are vertical vector fields such that $Z(p) = Y$, $Z'(p) = Y'$. This definition is independent of the choice of the extensions of $Y$ and $Y'$.

Since the Riemannian structure on $\mathcal{P}$ is smooth, $S^p$ defines a bounded bilinear form on $V_p\mathcal{P}$ and we can define a weighted trace of the second fundamental form by $\mathrm{tr}_{\mathcal{A}^v} S$. Since $\mathcal{A}^v$ is right invariant, we have $R_{g*}S^p(\mathcal{A}^v(p)\cdot, \mathcal{A}^v(p)\cdot) = S^{p\cdot g}(\mathcal{A}^v(p\cdot g)\cdot, \mathcal{A}^v(p\cdot g)\cdot)$, and $\pi_*(\mathrm{tr}_{\mathcal{A}^v} S(p))$ is independent on the point $p$ chosen in the orbit of $\pi(p)$ so that we shall denote it by $\pi_*(\mathrm{tr}_{\mathcal{A}^v} S)(\pi(p))$.

**Lemma 2.1.** *For $\mathcal{A}$ satisfying conditions (2.1) and (2.2) and a smooth $1$-form $\bar\alpha$ on the quotient space $X$, we have:*

$$\bar\alpha(\pi_*(tr_{\mathcal{A}^v} S)) = div_{\bar{\mathcal{A}}}\bar\alpha - div_{\mathcal{A}}(\pi^*(\bar\alpha)). \tag{2.3}$$

*Proof.* Since $\bar\alpha$ is a 1-form on $X$, $\alpha \equiv \pi^*(\bar\alpha)$ defines a smooth 1-form on $\mathcal{P}$. Let us first notice that for right invariant horizontal vector fields $U$ and $V$, we have: $\pi_*(\nabla_U V) = \bar\nabla_{\bar U}\bar V$ (see e.g [11]). Hence, if $u$ and $v$ are horizontal vectors at point $p \in \mathcal{P}$, letting $U$, resp. $V$ be right invariant horizontal vector fields extending $u$, resp. $v$, we can write, setting $x = \pi(p) = \bar p$:

$$\begin{aligned} \nabla(\pi^*(\bar\alpha))(p)(u, v) &= u(\pi^*(\bar\alpha)(V)) - \pi^*(\bar\alpha)(p)(\nabla_U V) \\ &= \bar u(\bar\alpha(\bar V)) - \bar\alpha(\pi_*(p)(\nabla_U V)) \\ &= \bar u(\bar\alpha(x)(\bar V)) - \bar\alpha(x)(\bar\nabla_{\bar U} \bar V) \\ &= \bar\nabla\bar\alpha(x)(\bar u, \bar v). \end{aligned}$$

Let now $u, v$ be vertical vectors at point $p$ and $U$, resp. $V$ right invariant vertical fields extending $u$, resp. $v$. We have:

$$\begin{aligned} \nabla(\pi^*(\bar\alpha))(p)(u, v) &= -(\bar\alpha(\pi_*\nabla_U V))(x) \quad \text{since} \quad \bar V = 0 \\ &= -\bar\alpha(x)(\pi_* S^p(u, v)). \end{aligned}$$

Combining these two equalities, and using the fact that $\mathcal{A} = \mathcal{A}^h \oplus \mathcal{A}^v$, we find that:

$$\mathrm{tr}_{\mathcal{A}}(\nabla(\pi^*(\bar\alpha)))(p) = (\mathrm{tr}_{\bar{\mathcal{A}}}\bar\nabla\bar\alpha)(x) - \bar\alpha(x)(\pi_*(\mathrm{tr}_{\mathcal{A}^v} S)),$$

so that

$$\begin{aligned} \bar\alpha\pi_*((\mathrm{tr}_{\mathcal{A}^v} S)) &= \mathrm{tr}_{\bar{\mathcal{A}}}\bar\nabla\bar\alpha - \mathrm{tr}_{\mathcal{A}}(\nabla(\pi^*(\bar\alpha))) \\ &= \mathrm{div}_{\bar{\mathcal{A}}}\bar\alpha - \mathrm{div}_{\mathcal{A}}\pi^*(\bar\alpha). \qquad \square \end{aligned}$$

*2.2. The projected semi-martingale.* We investigate here how a semi-martingale defined by an Itô field $(a, \mathcal{A})$, where $\mathcal{A}$ satisfies conditions (2.1) and (2.2) projects onto the space of orbits.

**Proposition 2.2.** *Under assumptions (2.1) and (2.2), the process $\xi_t$ defined by the locally Lipschitz Itô field $(a, \mathcal{A})$ projects onto a process $x_t$ on $X$ defined by the locally Lipschitz Itô field $(\overline{a} - \frac{1}{2}\pi_*(tr_{\mathcal{A}^v} S), \overline{\mathcal{A}})$. In other words, the projected process $x$ is described as a solution of the stochastic differential equation:*

$$d^{\bar{\nabla}} x = \overline{\mathcal{A}}(x) d\overline{B} + \overline{a}(x) dt - \frac{1}{2}\pi_*(tr_{\mathcal{A}^v(x)} S)(x) dt,$$

*where $\overline{B}$ is the orthogonal projection of $B$ on $H'$, $\overline{\mathcal{A}}$ is the canonical projection of $\mathcal{A}$, $\overline{\mathcal{A}}(x) = \pi_* \mathcal{A}(p) = \pi_* \mathcal{A}^h(p)$ with $\pi(p) = x$.*

*Proof.* Let $f$ be a smooth function on $X$. Applying the Itô formula to $\xi$ for $f \circ \pi$, we have:

$$\begin{aligned} d(f(\overline{\xi})) &= d(f \circ \pi)(\xi) = d(f \circ \pi)(d^{\nabla}\xi) + \frac{1}{2}\mathrm{div}_{\mathcal{A}} d(f \circ \pi) dt \\ &= d(f \circ \pi)(\mathcal{A}(\xi) dB + a(\xi) dt) + \frac{1}{2}\mathrm{div}_{\mathcal{A}} d(f \circ \pi) dt. \end{aligned}$$

Using Lemma 2.1 applied to $\overline{\alpha} = df$, this yields :

$$df(d^{\bar{\nabla}}\overline{\xi}) = df(\overline{\mathcal{A}}(\overline{\xi}) d\overline{B} + \overline{a}(\overline{\xi}) dt)) + \frac{1}{2}\left(\mathrm{div}_{\overline{\mathcal{A}}} d(f) dt - df(\pi_* \mathrm{tr}_{\mathcal{A}^v} S) dt\right) dt.$$

Choosing $f$ to be the $k^{th}$ coordinate $\phi_k$ in a local chart $(U, \phi)$, and letting $k$ run in $I\!N$, we find that the Itô differential of the projected process is of the type $d^{\bar{\nabla}}\overline{\xi} = \overline{\mathcal{A}} d\overline{B} + \overline{a} dt - \frac{1}{2}\pi_* \mathrm{tr}_{\mathcal{A}^v} S dt$. □

**Corollary 2.3.** *For $p \in \mathcal{P}$, let $\xi^p$ be defined by a stochastic differential equation of the type:*

$$d^{\nabla}\xi^p = \mathcal{A}(\xi^p) dB, \qquad \xi^p(0) = p,$$

*with the same assumptions as above. Then*

1) *for a smooth* **G***-invariant function $f$ on $\mathcal{P}$, there is an equivalence between i) and ii):*
   *i) for any $p \in \mathcal{P}$ $f(\xi^p)$ is a local martingale,*
   *ii)$tr_{\overline{\mathcal{A}}} Hess \overline{f}(x) - df(\pi_*(tr_{\mathcal{A}^v} S)(x)) = 0, \quad \forall x \in X$;*
2) *The Stratonovich differential commutes with the canonical projection, i.e. $\delta\overline{\xi} = \overline{\delta\xi}$, where $\int \overline{\alpha}(\overline{\delta\xi}) = \int \pi^* \overline{\alpha}(\delta\xi)$.*

*Proof.* 1) This follows from the shape of the projected process $\overline{\xi_t^p} = x_t^{\bar{p}}$ from which we see that $f(\xi^p) = \overline{f}(x^{\bar{p}})$ is a local martingale for all $p \in \mathcal{P}$ if and only if $\mathrm{tr}_{\overline{\mathcal{A}}} \mathrm{Hess} \overline{f}(x) - \pi_*(\mathrm{tr}_{\mathcal{A}^v} S)(x) = 0, \quad \forall x \in X$ (see [8] (5.28)).

2) Let $\overline{\alpha}$ be a smooth one form on $X$. We have, setting $\alpha = \pi^* \overline{\alpha}$:

$$\begin{aligned} \int \overline{\alpha}(\delta\overline{\xi}) &= \int \overline{\alpha}(d^{\bar{\nabla}}\overline{\xi}) + \frac{1}{2}\int div_{\overline{\mathcal{A}}}(\overline{\alpha}) dt \quad \text{(definition of the Stratonovich differential)} \\ &= \int \alpha(d^{\nabla}\xi) - \frac{1}{2}\int \overline{\alpha}(\pi_*(\mathrm{tr}_{\mathcal{A}^v} S)) dt + \frac{1}{2}\int div_{\overline{\mathcal{A}}}(\overline{\alpha}) dt \quad \text{by Prop. 2.2} \\ &= \int \alpha(\delta\xi) - \frac{1}{2}\int \mathrm{div}_{\mathcal{A}}\alpha dt + \frac{1}{2}\int div_{\overline{\mathcal{A}}}(\overline{\alpha}) dt - \frac{1}{2}\overline{\alpha}(\pi_*(\mathrm{tr}_{\mathcal{A}^v} S)) dt \\ &= \int \alpha(\delta\xi) \quad \text{by Lemma 2.1.} \end{aligned}$$

## 3. Interplay with Geometry

*3.1. A class of group actions .* Let **G**, $\mathcal{P}$ be as in Sect. 2. We equip the group **G** with a family of equivalent $\mathrm{Ad}_g$ invariant Riemannian metrics indexed by $p \in \mathcal{P}$. The scalar product induced on the Lie algebra $\mathcal{G}$ by the Riemannian metric on **G** indexed by $p$ will be denoted by $(\cdot, \cdot)_p$. The closure of $\mathcal{G}$ for these scalar products is independent of $p$ since they are equivalent and we shall denote it by $\overline{\mathcal{G}}$.

We make additional assumptions on the group action imposing conditions on the operator $\tau_p$ defined by

$$\begin{aligned} \tau_p : \mathcal{G} &\to T_p\mathcal{P} \\ u &\mapsto \frac{d}{dt}(p \cdot e^{tu})_{t=0}. \end{aligned}$$

Since $\mathcal{G}$ is dense in the Hilbert space $\overline{\mathcal{G}}$, the operator $\tau_p$ is a densely defined operator on $\mathcal{G}$ and we can therefore define its adjoint operator $\tau_p^*$ w.r. to the scalar products $(\cdot, \cdot)_p$ and $< \cdot, \cdot >_p$. We make the following assumptions:

**1)** for any $p \in \mathcal{P}$, $\tau_p \tau_p^*$ is a self adjoint operator on a dense subspace of $V_p\mathcal{P}$

**2)** and that for any $\varepsilon > 0$, for any $p \in \mathcal{P}$, the operator $e^{-\varepsilon \tau_p \tau_p^*}$ is a Hilbert-Schmidt operator on $V_p\mathcal{P}$,

**3)** the map $p \to e^{-\varepsilon \tau_p \tau_p^*}$ is $C^1$.

Let $\varepsilon > 0$ and let us define for $p \in \mathcal{P}$ the operator:

$$A_\varepsilon^v(p) = e^{-\frac{1}{2}\varepsilon \tau_p \tau_p^*}$$

acting on $V_p\mathcal{P}$. It is right invariant since $\tau_{p \cdot g} = R_{g*}\tau_p \mathrm{Ad}g$.

For $p \in \mathcal{P}$, let $A^h(p)$ be a right invariant Hilbert-Schmidt operator acting on $H_p\mathcal{P}$. The operator

$$A_\varepsilon(p) \equiv A^h(p) \oplus A_\varepsilon^v(p)$$

is also a right invariant Hilbert-Schmidt operator in $T_p\mathcal{P}$.

Since $\mathcal{P}$ is a smooth Hilbert manifold, it is parallelisable [13] (we in fact only need a local parallelisation). Let $\mathcal{I}(p) : H \to T_p\mathcal{P}$ be a smooth field of isometries induced by the parallelisation. We shall assume that the model space $H$ splits into two orthogonal spaces $H = H' \oplus H''$, that the family of isometries $\mathcal{I}(p)$ is right invariant and splits into a sum of isometries $\mathcal{I}(p) = \mathcal{I}'(p) \oplus \mathcal{I}''(p)$ with $\mathcal{I}'(p) : H' \to H_p\mathcal{P}, \mathcal{I}''(p) : H'' \to V_p\mathcal{P}$.

Let us set:

$$\mathcal{A}_\varepsilon \equiv \mathcal{A}^h \oplus \mathcal{A}_\varepsilon^v \tag{3.1}$$

with

$$\mathcal{A}^h \equiv A^h \circ \mathcal{I}', \quad \mathcal{A}_\varepsilon^v \equiv A_\varepsilon^v \circ \mathcal{I}''. \tag{3.2}$$

For any $\varepsilon > 0$, the operator $\mathcal{A}_\varepsilon$ satisfies conditions (2.1) and (2.2) of Sect. 2.

*3.2. Strong minimality of orbits.* We briefly recall here the notion of strong minimality we introduced in [2] for a group action with the above properties.

**Definition .** *The orbit $O_p$ is strongly minimal if and only if the family of pre-regularised traces indexed by $\varepsilon > 0$ of the second fundamental form of the orbit*

$$(tr_\varepsilon(S))(p) \equiv (tr_{\mathcal{A}_\varepsilon^v}(S))(p) \tag{3.3}$$

*vanishes.*

*Remark* . 1) Notice that the regularisation $\varepsilon$ introduced via the operator $A^v_\varepsilon$ is entirely determined by the group action.

2) The coadjoint action of a loop group on the corresponding loop algebra as described in [14] gives rise to strongly minimal orbits as was pointed out in [2] Appendix 1.

From Proposition 2.2 easily follows:

**Proposition 3.1.** *Whenever the orbits of the group action in the above class are strongly minimal, any semi-martingale defined by an Itô field of the type* $(a, \mathcal{A}_\varepsilon)$*, with* $\varepsilon > 0$ *and* $\mathcal{A}_\varepsilon$ *as in (3.1), projects onto a semi-martingale defined by the projected Itô field* $(\bar{a}, \bar{\mathcal{A}} = \bar{A}^h)$.

*3.3. The case when the orbit space is finite dimensional.* When the orbit space is a finite dimensional Riemannian manifold, we can set $A^h(p) = I$ for $p \in \mathcal{P}$, and we have only to assume that $\mathcal{I}'(p)\mathcal{I}'(p)^*$ is equal to the identity of $H_p\mathcal{P}$ for all $p \in \mathcal{P}$ and remove the assumption that $\mathcal{I}'(p)$ is injective. The family of Itô fields $(0, \mathcal{A}_\varepsilon)$ then defines a one parameter family of martingales entirely determined in law by the group action which we shall call a family of *regularised Brownian motions associated to the group action*.

We shall call a *strongly harmonic function* on $\mathcal{P}$, a smooth function that takes any regularised Brownian motion onto a martingale.

From the above results follows that:

**Proposition 3.2.** *Whenever the orbit space is finite dimensional and the orbits of the action are strongly minimal,*

*1) a regularised Brownian motion projects onto a Brownian motion on the orbit space,*

*2) a strongly harmonic* **G***-invariant function projects onto a harmonic function on the orbit space.*

*Proof.* 1) This follows from Proposition 3.1, since here $A^h(p) = I$.

2) $\underline{f}$ is strongly harmonic whenever for any regularised Brownian motion $\xi_\varepsilon$, $f(\xi^\varepsilon_t) = \bar{f}(\xi^\varepsilon_t)$ is a martingale, which holds whenever $\bar{f}$ is harmonic, by Corollary 2.3, 1.

This proposition generalises similar well known results in the finite dimensional case, see e.g [15, 16].

Before we give an example to illustrate this proposition, let us first describe the general geometric framework in which examples will fit in naturally.

## 4. Projections of Martingales as a Paradigma for the Faddeev–Popov Procedure in Gauge Field Theory

The group actions arising from the action of the gauge group on the path space in gauge field theory fit in the class of actions described in Sect. 3.1 and give rise to what we called regularisable fibre bundles in [2]. The fibre bundles arising in gauge theories can be equipped with both weak and strong Riemannian structures which we now describe.

*4.1. The geometric setting for a class of gauge field theories.* We present a geometric framework which reflects the essential features of gauge field theory. This is to be seen as a simplified model of gauge field theory which we shall use below to describe a paradigma for the Faddeev-Popov procedure in gauge field theory. Some examples of actions which fit in the framework described in this section are:

- the action arising in string theory of the group of diffeomorphisms homotopic to identity of a compact Riemann surface of genus larger than 1 on the manifold of metrics of this surface,
- the action arising in Yang-Mills theory of automorphisms of a principal bundle built on a compact Riemannian manifold on the manifold of irreducible connections on this bundle,
- the (coadjoint) action arising in the representation of loop groups of pinned Lie group valued loops on the space of loops with values in the corresponding Lie algebra.

We shall illustrate the abstract setting with this last example since it offers a good toy model.

Let $M$ be a smooth compact boundaryless Riemannian manifold. For a smooth vector bundle $\mathcal{V}$ on $M$ with finite dimensional fibres, we can define the Sobolev spaces $H^k(\mathcal{V})$, $k \in I\!N$ using a partition of unity (see [17]). Let $\mathcal{E}$ and $\mathcal{F}$ be two smooth vector bundles on $M$ with finite dimensional fibres. In gauge field theory, the path space $\mathcal{P}$ is a smooth Hilbert manifold modelled on $H^k(\mathcal{E})$, for some $k > 1 + \frac{1}{2}\dim M, k \in I\!N$. The gauge group **G** is a right semi-Hilbert Lie group (i.e. it has the properties of a Hilbert group up to the fact that only multiplication on the right is smooth, see [9]) modelled on $H^{k+1}(\mathcal{F})$. We shall assume that both $\mathcal{E}$ and $\mathcal{F}$ are equipped with a fibre metric in such a way that $\mathcal{P}$ is equipped with a smooth weak Riemannian structure $g_0$ which induces an $L^2$ scalar product $< \cdot, \cdot >_{p,0}$ on $T_p\mathcal{P}$ and that $\mathcal{G}$ is equipped with a right invariant family of equivalent $L^2$ scalar products $(\cdot, \cdot)_{p,0}$ indexed by $p$. The closures of $\mathcal{G}$ w.r.t these scalar products coincide and will be denoted by $\bar{\mathcal{G}}$.

Let the operator $\tau_p : \mathcal{G} \to T_p\mathcal{P}$ be a differential operator. We shall assume it is injective.

*A toy model: the coadjoint action of loop groups*. We shall give only the general features of the model here and refer the reader to [14] and [2] for a detailed description. Let $G$ be a connected compact Lie group, **g** its Lie algebra. We set $\mathcal{P} = L^2([0,1], \mathbf{g})$ and $\mathbf{G} = \{g \in H^1([0,1], G), g(0) = g(1) = e\}$. **G** acts on $\mathcal{P}$ via a smooth free action (also called coadjoint action):

$$\begin{aligned} \mathbf{G} \times \mathcal{P} &\to \mathcal{P} \\ (g, \gamma) &\to g\gamma g^{-1} - g' g^{-1}. \end{aligned}$$

The orbit space is the Lie group $G$ and the map

$$\begin{aligned} \pi : L^2([0,1], \mathbf{g}) &\to G \\ \gamma &\to g(1) \text{ with } g^{-1}g = \gamma, \quad g(0) = e \end{aligned}$$

yields a fibre bundle structure on $L^2([0,1], \mathbf{g})$ with structure group **G**. The action is isometric for the natural Riemanian metric on $\mathcal{P}$ induced by a fixed Ad invariant inner product on **g**.

For a loop $p \in \mathcal{P}$, the operator $\tau_p$ is given by

$$\begin{aligned} \{u \in H^1([0,1], \mathbf{g}), u(0) = u(1) = 0\} &\to L^2([0,1], \mathbf{g}) \\ u &\to [u, p] - u'. \end{aligned}$$

This operator is clearly a first order differential operator which is injective, the action being free.

One can show [2] (Appendix A) that the orbits for this coadjoint action are strongly minimal in $L^2([0,1], \mathbf{g})$.

*Remark* . Let us at this point make a few comments about the other two examples we have in mind.

One can see the toy model described above as a one dimensional Yang-Mills (dual) action. It is therefore natural to enquire about Yang-Mills actions for manifolds of dimension 2,3 and 4. These were investigated in [18]. Strong minimality of orbits holds for smooth irreducible connections when the Lie group but otherwise, one can only hope for minimality (and not strong minimality) of orbits of a certain class of smooth irreducible connections. Going from strongly minimal to minimal requires a limit procedure which motivates paragraph 3 of this section.

Another example of group action arising in string theory is the action of a group of diffeomorphisms on a manifold of Riemannian metrics, for which the problem (yet unsolved) of deciding which of the orbits are minimal is made difficult by the fact that, unlike the above two models, the Riemannian structure on the group **G** (here a group of diffeomorphisms) depends on the parameter $p \in \mathcal{P}$, namely on a metric chosen on the manifold $M$.

*A comment on the choice of Riemannian structure $P$ and $G$*. Let us denote for the moment by $\tau_p^{*^0}$ the adjoint of $\tau_p$ w.r. to the $L^2$ scalar products $(\cdot,\cdot)_{p,0}$ and $< \cdot,\cdot >_{p,0}$.

In the context of gauge field theory, the operators $\tau_p^{*^0}\tau_p$ arise as positive self adjoint elliptic operators on $M$ of order 2. Their coefficients are not smooth in general, but they are regular enough to recover the properties of elliptic operators with smooth coefficients we shall use below (for details concerning this point, see [19, 20, 9]).

In particular, we shall asume that the scalar products on $H^k(\mathcal{F})$ defined by

$$(\cdot,\cdot)_{p,k} \equiv (\cdot,\cdot)_{p,0} + ((\tau_p^{*^0}\tau_p)^{\frac{k}{2}}\cdot, (\tau_p^{*^0}\tau_p)^{\frac{k}{2}}\cdot)_{p,0}$$

induce a strong Riemannian structure on **G** which we shall denote by $h_k$, as is the case when the coefficients of the elliptic operator are smooth [17].

Since $\tau_p^{*^0}\tau_p$ is a self adjoint elliptic operator on $C^\infty(\mathcal{F})$, the range R($\tau_p$), resp. the kernel Ker$\tau_p^{*0}$ are closed w.r.to the $L^2$ scalar products $< \cdot,\cdot >_{p,0}$, resp. $(\cdot,\cdot)_{p,0}$ as well as in the $H^k$ topology (see e.g [21], Sect. 6 and [20] (3.1.5) for a discussion in the Yang-Mills case ) and we have the $L^2$ orthogonal splitting

$$T_p\mathcal{P} = \mathrm{R}(\tau_p) \oplus \mathrm{Ker}(\tau_p^{*^0}), \tag{4.1}$$

where the orthogonal sum is taken w.r.to the scalar products $< \cdot,\cdot >_{p,0}$.

We now specialize to a class of gauge theories with finite dimensional orbit space. In particular Ker$\tau_p^*$ is finite dimensional. This assumption is in particular satisfied for the toy model described above.

We make the assumption that the scalar products on $T_p\mathcal{P}$ defined by

$$< \cdot,\cdot >_{p,k} \equiv < \cdot,\cdot >_{p,0} + < (\tau_p\tau_p^{*^0})^{\frac{k}{2}}\cdot, (\tau_p\tau_p^{*^0})^{\frac{k}{2}}\cdot >_{p,0}$$

induce a strong Riemannian structure on $\mathcal{P}$ which we shall denote by $g_k$.

The adjoint of the operator $\tau_p$ w.r. to the induced scalar products $< \cdot,\cdot >_{p,k}$ and $(\cdot,\cdot)_{p,k}$ coincide with $\tau_p^{*^0}$. Indeed, we have for $u \in \mathcal{G}$ and $h \in T_p\mathcal{P}$:

$$\begin{aligned}< \tau_p u, h >_{p,k} &= < (\tau_p \tau_p^{*^0})^{\frac{k}{2}} \tau_p u, (\tau_p \tau_p^{*^0})^{\frac{k}{2}} h >_{p,0} + < \tau_p u, h >_{p,0} \\ &= < \tau_p (\tau_p^{*^0} \tau_p)^{\frac{k}{2}} u, (\tau_p \tau_p^{*^0})^{\frac{k}{2}} h >_{p,0} + < \tau_p u, h >_{p,0} \\ &= ((\tau_p^{*^0} \tau_p)^{\frac{k}{2}} u, \tau_p^{*^0} (\tau_p \tau_p^{*^0})^{\frac{k}{2}} h)_{p,0} + (u, \tau_p^* h)_{p,0} \\ &= (u, \tau_p^{*^0} h)_{p,k}\end{aligned}$$

We shall henceforth uniformise the notation denoting by $\tau_p^*$ the adjoints w.r. to the scalar products induced by the scalar products $< \cdot, \cdot >_{p,0}$ and $< \cdot, \cdot >_{p,k}$.

The orthogonal splitting (4.1) on $\mathcal{P}$ also holds w.r.to the scalar products $< \cdot, \cdot >_{p,k}$. Since the spaces Im$\tau_p$ and Ker$\tau_p^*$ are closed in the $H^k$ topology, (4.1) yields a connection associated to the metric $g_k$. We shall henceforth not specify which of the metrics we choose to define horizontality.

As a heat-operator built from a self adjoint elliptic operator on a compact boundaryless manifold, the operator $e^{-\varepsilon \tau_p^* \tau_p}$ (resp. $e^{-\varepsilon \tau_p \tau_p^*}$) is trace-class (see [17]) and hence Hilbert-Schmidt. An easy computation shows that its trace taken w.r.to $(\cdot, \cdot)_{p,0}$ (resp. $< \cdot, \cdot >_{p,0}$) coincides with its trace taken w.r.to $(\cdot, \cdot)_{p,k}$ (resp. $< \cdot, \cdot >_{p,k}$ ). We will therefore not specify which of the two scalar products we choose to define these traces.

*Back to the toy model*. Combining this with Proposition 3.2 yields that regularised Brownian motion on the loop algebra $L^2([0,1], \mathbf{g})$ projects onto Brownian motion on the Lie group $G$ via the coadjoint action since the orbits are strongly minimal.

*Remark* . A similar statement holds for Yang-Mills action when the Lie group $G$ is abelian since the orbits are also strongly minimal in that case.

*4.2. Minimal orbits as orbits with extremal volume.* In Sect. 3.2, we defined a notion of strong minimality without specifying the underlying Riemannian structure on $\mathcal{P}$. The aim of this section is to show that one can choose either the metric $g_k$ or the metric $g_0$.

Let us first briefly recall the generalisation of Hsiang's theorem relating minimality of orbits with the extremality of their volume, which we wrote down in [2] using heat-kernel regularisation methods. In [2], the underlying Riemannian structure was chosen strong or weak, as long as it induced a connection on $\mathcal{P}$ with the usual properties. We choose here the connection described by (4.1) and any of the two metrics $g_0$ or $g_k$.

We introduced in [2] a notion of heat-kernel pre-regularised volume of an orbit $O_p$, $p \in \mathcal{P}$, setting for $\varepsilon > 0$:

$$\text{vol}_\varepsilon(O_p) = \exp[-\frac{1}{2} \int_\varepsilon^{+\infty} t^{-1} \, \text{tr} e^{-t(\tau_p^* \tau_p)} dt] \tag{4.2}$$

which, by the above discussion, is independent of whether one chooses the scalar products $< \cdot, \cdot >_0$ or $< \cdot, \cdot >_{p,k}$ to define the trace.

In the context of gauge field theories, we can apply the results of [2] (i.e. assumptions (2.1)–(2.5 bis) of [2] are fulfilled in that context, see the discussion that follows (2.5 bis)) choosing any of the two metrics $g_k$ or $g_0$, and we see that whenever the Riemannian structure on $\mathbf{G}$ is independent of $p$, the following relations between a horizontal directional derivative of the pre-regularised volumes and the regularised second fundamental form hold:

$$< \text{tr}_\varepsilon^k S^k, X >_{p,k} = -\delta_X \log \text{vol}_\varepsilon(O_p), \tag{4.3}$$

$$< \mathrm{tr}_\varepsilon^0 S^0, X >_{p,0} = -\delta_X \log \mathrm{vol}_\varepsilon(O_p) \qquad (4.3\text{bis})$$

for any horizontal vector $X$ at any point $p \in \mathcal{P}$ and where $S^k$ (resp. $S^0$) is the second fundamental form defined using the Levi-Civita connection associated to the Riemannian metric $g_k$ (resp. $g_0$).

From this follows that for any horizontal vector $X$, we have:

$$< \mathrm{tr}_\varepsilon^k S^k, X >_{p,k} = < \mathrm{tr}_\varepsilon^0 S^0, X >_{p,0}$$

Hence, since $< \cdot, \cdot >_{p,k}$ and $< \cdot, \cdot >_{p,0}$ coincide horizontally, we have:

$$\mathrm{tr}_\varepsilon^k S^k = \mathrm{tr}_\varepsilon^0 S^0.$$

*Remark* . 1) From this follows that the notion of strong minimality does not depend on whether one chooses the $g_k$ or the $g_0$ structure. Moreover relations (4.3) and (4.3 bis) tell us that an orbit $O_p$ is strongly minimal whenever its pre-regularised volume $\mathrm{vol}_\varepsilon(O_p)$ is extremal among other orbits.

2) When applied to the coadjoint loop group action on the corresponding Lie algebra, relation (4.3) tells us that all orbits in the “toy model” have extremal pre-regularised volume.

3) As was pointed out above, the assumption on the independence w.r.to the parameter $p$ of the Riemannian structure on **G** excludes the case of string theory where the parameter $p$ is a metric on a finite dimensional manifold $M$ which arises in the definition of the scalar products on the Lie algebra $\mathcal{G}$ given by the space of smooth vector fields on $M$.

As a consequence, when the Riemannian structure on **G** is independent of $p$, we have the following:

**Proposition 4.1.** *Whenever the orbits have constant pre-regularised volume, the process defined by a locally Lipschitz Itô field $(\underline{a}, \mathcal{A}_\varepsilon)$, with $\mathcal{A}_\varepsilon$ as in (3.1), projects onto a process defined by the projected Itô field $(\overline{a}, \mathcal{A}^h)$ which is independent of $\varepsilon$.*

*Proof.* This follows directly from Proposition 3.1 and (4.3). Let us stress here that the regularised trace of the second fundamental form that appears in the drift is the one $(\mathrm{tr}_\varepsilon^k S^k)$ taken w.r.to the metric $g_k$. □

*4.3. Projected renormalised Brownian motions.* The pre-regularised volume $\mathrm{vol}_\varepsilon(O_p)$ diverges when $\varepsilon$ goes to zero. Using the asymptotic expansion for heat-kernels of elliptic operators on compact manifolds (see [17] and [2] for a more detailed description of these asymptotic expansions), we have:

$$\mathrm{tr}(e^{-\varepsilon \tau_p^* \tau_p}) \simeq_0 \sum_{j=-J}^{\infty} b_j(p) \varepsilon^{\frac{j}{2}}$$

(recall that the operator is of order 2) for some real valued coefficients $b_j(p)$, $J = \dim M$. We can then define a notion of heat-kernel regularised volume $\mathrm{vol}_{reg}(O_p)$ and a notion of regularised trace of the second fundamental form which boils down to taking the $\varepsilon \to 0$ limit after having got rid of the divergences of the corresponding pre-regularised volumes and traces. This regularisation was compared in [2] with the zeta-regularisation method. A minimal orbit is an orbit the second fundamental form of which has vanishing regularised trace. A strongly minimal orbit is minimal [2].

Let us take $\mathcal{P}/\mathbf{G}$ finite dimensional and compact and let us investigate the projections of families indexed by $\varepsilon$ of renormalised Brownian motions and their limit when $\varepsilon$ goes to zero. By *family of renormalised Brownian motions* associated to the group action, we mean a family of processes $(\xi^\varepsilon)$ such that for all $\varepsilon$, $\xi^\varepsilon$ is defined by a locally Lipschitz Itô field $(-\frac{1}{2}\sum_{j=-J}^{-1} \text{grad}\frac{b_j}{j}\varepsilon^{\frac{j}{2}} - \frac{1}{4}\text{grad}b_0\log\varepsilon, \mathcal{A}_\varepsilon)$ with $\mathcal{A}_\varepsilon = \mathcal{A}^h + \mathcal{A}^v_\varepsilon$, $\mathcal{A}^v_\varepsilon$ as described in (3.2), and for all $p \in \mathcal{P}$, $\mathcal{A}^h(p) = \mathcal{I}'(p) : H' \to H_p\mathcal{P}$ defined on an Euclidian space $H'$, not necessarily injective and such that $\mathcal{I}'(p)\mathcal{I}'(p)^*$ is the identity of $H_p\mathcal{P}$. We assume furthermore that there exists a $H'$-valued Brownian motion $B'$ such that for all $\varepsilon$, the horizontal and martingale part of $d^\nabla \xi^\varepsilon$ is $\mathcal{I}'(\xi^\varepsilon)dB'$.

**Proposition 4.2.** *Let us assume that* $\mathbf{G}$ *is equipped with a fixed Riemannian metric* $(\cdot,\cdot)_k$ *independent of* $p \in \mathcal{P}$. *Let* $(\xi^\varepsilon_t)$ *be a family of renormalised Brownian motions associated to the group action, such that for all* $\varepsilon$, $\xi^\varepsilon_0 = p_0 \in \mathcal{P}$. *If the following assumptions are fulfilled:*

1) *The gradients of the coefficients* $b_j(p)$ *in the heat-kernel expansion of* $e^{-t\tau_p^*\tau_p}$ *are Lipschitz and invariant under the action of the group.*
2) *There is a constant* $C > 0$ *such that:*

$$sup_{p\in\mathcal{P}} \|grad\ trF_p(t)\| \le Ct,$$

*where* $F_p(t) = tre^{-t\tau_p^*\tau_p} - \sum_{j=-J}^{1} b_j(p)t^{\frac{j}{2}}$.

3) *The maps* $p \mapsto$ *grad log vol*$_\varepsilon(O_p) - \sum_{j=-J}^{-1} grad\frac{b_j(p)}{j}\varepsilon^{\frac{j}{2}} - \frac{1}{2}gradb_0(p)log\varepsilon$ *and* $p \mapsto$ *grad log vol*$_{reg}(O_p)$ *are locally Lipschitz.*

*Then the projected process* $x^\varepsilon_t \equiv \overline{\xi^\varepsilon_t}$ *satisfies* $x^\varepsilon_0 = \overline{p}_0$ *and*

$$\begin{aligned} d^{\bar\nabla} x^\varepsilon_t &= \mathcal{I}'(x_t)dB'_t + \frac{1}{2}grad\ log\ vol_\varepsilon O_{x_t}dt \\ &\quad -\frac{1}{2}\sum_{j=-J}^{-1} grad\frac{b_j(x_t)}{j}\varepsilon^{\frac{j}{2}}dt - \frac{1}{4}gradb_0(x_t)log\varepsilon dt \\ &= \mathcal{I}'(x_t)dB'_t - \frac{1}{2}tr_\varepsilon S^{x_t}dt - \frac{1}{2}\sum_{j=-J}^{-1} grad\frac{b_j(x_t)}{j}\varepsilon^{\frac{j}{2}}dt - \frac{1}{4}gradb_0(x_t)log\varepsilon dt \end{aligned}$$

*and converges in* $L^2$ *uniformly on compact sets (i.e for any* $T > 0$, $E[sup_{t\le T}d^2(x^\varepsilon_t, x_t)]$ *converges to zero) to the solution starting at* $\overline{p}_0$ *of the stochastic equation:*

$$d^{\bar\nabla} x_t = \mathcal{I}'(x_t)dB'_t + \frac{1}{2}grad\ log\ vol_{reg}O_{x_t}dt.$$

*If the fibres are minimal, the limit process is a Brownian motion.*

*Remark* . The uniform upper bound on $\mathcal{P}$ in assumption 2) is in fact a uniform upper bound on the quotient since everything is $\mathbf{G}$ invariant. This uniform upper bound requirement is in particular fulfilled when the quotient manifold is compact and the map $p \mapsto \text{tr}e^{-t\tau_p^*\tau_p}$ is $C^1$.

*Proof.* We shall set $B_p = \tau_p^* \tau_p$ and refer the reader to [2] for notations.

(i) Let us first prove that the drift term of the process $x_t^\varepsilon$ converges uniformly to that of $x_t$, namely $\frac{1}{2}$grad log vol$_{reg}(O_p)$.
Setting $A_\varepsilon = -\int_\varepsilon^{+\infty} \frac{e^{-tB_p}}{t} dt$:

$$\begin{aligned}
&\text{grad log vol}_\varepsilon(O_p) - \sum_{j=-J}^{-1} \frac{\text{grad}\, b_j}{j} \varepsilon^{\frac{j}{2}} - \frac{1}{2}\text{grad}\, b_0 \log\varepsilon - \text{grad}\, b_1 \varepsilon^{\frac{1}{2}} \\
&= \frac{1}{2}\text{grad}\,(\text{tr} A_\varepsilon - \sum_{j=-J, j\neq 0}^{1} \frac{2b_j}{j}\varepsilon^{\frac{j}{2}} - b_0 \log\varepsilon) \\
&= -\sum_{j=-J, j\neq 0}^{1} \frac{\text{grad} b_j}{j} - \frac{1}{2}\text{grad} \int_\varepsilon^1 \frac{F_p(t)}{t} dt - \frac{1}{2}\text{grad} \int_1^\infty \text{tr}\frac{e^{-tB_p}}{t} dt
\end{aligned}$$

by (1.9) in [2]

with $F_p(t) = \text{tr} e^{-tB_p} - \sum_{j=-J}^{1} b_j t^{\frac{j}{2}}$.
Furthermore:

$$\|\frac{\text{grad} F_p(t)}{t}\| \leq C$$

by assumption 2) of the proposition. Hence grad $\int_\varepsilon^1 \frac{F_p(t)}{t} dt = \int_\varepsilon^1 \frac{\text{grad} F_p(t)}{t} dt$ and converges uniformly to $\int_0^1 \frac{\text{grad} F_p(t)}{t} dt$ in $p$. Thus grad log vol$_\varepsilon(O_p) - \sum_{j=-J}^{-1}$ grad$\frac{b_j}{j}\varepsilon^{\frac{j}{2}} - \frac{1}{2}$ grad$b_0$ log$\varepsilon$ converges uniformly in $p$ to grad log vol$_{reg}(O_p)$ (by (1.6) of [2]).

(ii) Let us now show the uniform $L^2$ convergence on compact sets of the processes $x_t^\varepsilon$ to the process $x_t$. We first show that proving this convergence property boils down to proving the $L^2$ convergence on compact sets of a diffusion on $\mathbb{R}^n$ defined by $dy_\varepsilon = A(y_\varepsilon)dB' + a_\varepsilon(y_\varepsilon)dt$ and $y_\varepsilon(0) = y_0 \in \mathbb{R}^n$ to a diffusion on $\mathbb{R}^n$ locally defined by $dy = A(y)dB' + a(y)dt$ and $y_\varepsilon(0) = y_0$, where $A$, $a_\varepsilon$ and $a$ are Lipschitz and $a_\varepsilon$ converges uniformly to $a$. Indeed, we can choose $n$ large enough for $X$ to be embedded in $\mathbb{R}^n$ via an embedding $u : X \to \mathbb{R}^n$. Using the compactness of $X$, we extend $u^{-1}$ to a map $\psi$ defined on a neighborhood of $u(X)$ so that it makes sense to look at the process $y_\varepsilon = u(x_\varepsilon)$ as defined by the stochastic differential equation:

$$dy_\varepsilon = \mathcal{I}'_u(y_\varepsilon)dB' + \alpha_\varepsilon^u(y_\varepsilon)dt, \qquad y_\varepsilon(0) = y_0 = u(\bar{p}_0)$$

with $\mathcal{I}'_u = u_* \circ \mathcal{I}' \circ \psi$, and setting $p = \psi(y)$,

$$\begin{aligned}
\alpha_\varepsilon^u(y) = u_* &\left( -\frac{1}{2}\text{tr} S_\varepsilon^p - \frac{1}{2}\sum_{j=-J}^{-1} \text{grad}\, \frac{b_j}{j}(p)\varepsilon^{\frac{j}{2}} - \frac{1}{4}\text{grad}\, b_0(p)\log\varepsilon \right) \\
&+ \frac{1}{2}\text{tr}(\text{Hess}\, u)(p)
\end{aligned}$$

which shows that the embedded process $u(x_\varepsilon)$ is a diffusion in $\mathbb{R}^n$. Since $\mathcal{I}'$ is Lipschitz, so is $\mathcal{I}'_u$. The drift $\alpha_\varepsilon^u$ is Lipschitz by assumption 3) using the fact that grad $b_j$ and tr$S^p_\varepsilon = -$grad logVol$_\varepsilon(O_p)$ are Lipschitz. From the first part of the proof follows that $\alpha^u_\varepsilon$ converge uniformly to $\alpha_u$ defined by

$$\alpha^u = -\frac{1}{2}u_* \mathrm{tr}_{reg} S^{\psi(y)} + \frac{1}{2}\mathrm{tr}(\mathrm{Hess}\, u)(\psi(y)).$$

Note that the second trace is a finite dimensional one. Hence $y_\varepsilon$ is a diffusion process on $I\!R^n$ locally defined by a stochastic equation of the form

$$dy_\varepsilon = A(y_\varepsilon)dB' + a_\varepsilon(y_\varepsilon)dt, \qquad y_\varepsilon(0) = y_0, \qquad (*)$$

where $A$, $a_\varepsilon$ are Lipschitz and $a_\varepsilon$ converges uniformly to $a$ Lipschitz.

(iii) Let us now prove the required convergence property for a diffusion process $y_\varepsilon$ in $I\!R^n$ of the type $(*)$. We shall denote by $(\cdot,\cdot)_{I\!R^n}$ the scalar product in $I\!R^n$, by $\|\cdot\|_{I\!R^n}$ the corresponding norm. Since $A$ is Lipschitz, there is a constant $C > 0$, such that $\mathrm{tr}(A(x) - A(y))^*(A(x) - A(y)) \le C\|x - y\|^2_{I\!R^n}$ and $\|a(x) - a(y)\|^2_{I\!R^n} \le C\|x - y\|^2_{I\!R^n}$. We have

$$\|y_\varepsilon(t) - y(t)\|^2$$
$$\le 2\|\int_0^t (A(y_\varepsilon(s)) - A(y(s)))dB'_s\|^2 + 2\|\int_0^t (a_\varepsilon(y_\varepsilon(s)) - a(y(s)))ds\|^2$$

with $y$ defined by $dy = A(y)dB' + a(y)dt$ and $y(0) = y_0$.

Set $f_\varepsilon(T) = E(\sup_{t\le T}\|y_\varepsilon(t) - y(t)\|^2)$. We want to bound $f_\varepsilon(T)$ from above. Using Doob's inequality for martingales, we obtain the following estimate:

$$E(\sup_{v\le t}\|\int_0^v (A(y_\varepsilon(s)) - A(y(s)))dB'_s\|^2$$
$$\le 4E(\|\int_0^t (A(y_\varepsilon(s)) - A(y(s)))dB'_s\|^2)$$
$$\le 4\int_0^t E\left[\mathrm{tr}(A(y_\varepsilon(s)) - A(y(s)))^*(A(y_\varepsilon(s)) - A(y(s)))\right] ds$$
$$\le 4C\int_0^t E\left[\|y_\varepsilon(s) - y(s)\|^2_{I\!R^n}\right] ds.$$

On the other hand

$$\begin{aligned} E(\sup_{v\le t}\|\int_0^v (a_\varepsilon(y_\varepsilon(s)) - a(y(s)))ds\|^2) &\le E\left(\sup_{v\le t}\int_0^v \|a_\varepsilon(y_\varepsilon(s)) - a(y(s))\|^2 ds\right)\\ &\le \int_0^t E\left(\|a_\varepsilon(y_\varepsilon(s)) - a(y(s))\|^2\right) ds\\ &\le 2\int_0^t E(\|a_\varepsilon(y_\varepsilon(s)) - a(y_\varepsilon(s)))\|^2_{I\!R^n})ds\\ &\quad + 2\int_0^t E(\|a(y_\varepsilon(s)) - a(y(s))\|^2_{I\!R^n})ds\\ &\le 2th(\varepsilon) + 2C\int_0^t E(\|y_\varepsilon(s) - y(s)\|^2_{I\!R^n})ds, \end{aligned}$$

where $h(\varepsilon) = \sup_x \|a_\varepsilon(x) - a(x)\|^2_{I\!R^n}$ is a function tending to zero at zero arising from the uniform convergence of $a_\varepsilon$ to $a$.

Finally, we find that for $t \leq T$,

$$f_\varepsilon(t) \leq 4Th(\varepsilon) + 12C \int_0^t E(\|y_\varepsilon(s) - y(s)\|^2_{\mathbb{R}^n})ds$$
$$\leq 4Th(\varepsilon) + 12C \int_0^t f_\varepsilon(s)ds,$$

which, using Gromwald's lemma yields:

$$f_\varepsilon(T) \leq 4Th(\varepsilon)e^{12CT},$$

which shows that $f_\varepsilon(T)$ goes to zero when $\varepsilon \to 0$.

(iv) We have shown that $E(\mathrm{Sup}_{t\leq T}\|u(x_\varepsilon)(t) - u(x)(t)\|^2_{\mathbb{R}^n})$ tends to zero when $\varepsilon \to 0$ from which follows that $E(\mathrm{Sup}_{t\leq T} d(x_\varepsilon(t), x(t))^2)$ tends to zero, since $X$ being compact $d(\cdot,\cdot)$ and $\|u(\cdot) - u(\cdot)\|_{\mathbb{R}^n}$ are equivalent. □

*Remark* . 1) For group actions with strongly minimal orbits such as the toy model described above, the limit procedure ($\varepsilon \to 0$) which is carried out in the above proposition is unecessary and Proposition 4.2 boils down to Proposition 3.2. However, this limit procedure is necessary in the case of Yang-Mills actions for example for which minimal orbits are in general not strongly minimal.

2) The drift of the "renormalised" projected process $x_t$ is expressed here in terms of the logarithmic variation of the volume. However, it can just as well be written in terms of the trace of the second fundamental form using (4.3). The proof above uses regularisation methods for regularised determinants from which we deduce a regularised trace of the second fundamental form. Because relation (4.3) does not hold when the metric on the group varies (see [2]), the proof of this proposition does not apply to the case of diffeomorphisms acting on metrics. However, since the conclusion of the proposition only involves the trace of the second fundamental form and not the variation of volume of orbits, the proof should extend to the case of diffeomorphisms acting on metrics, a setting which one of the authors (S. Paycha) is investigating together with S. Rosenberg.

*4.4. Conclusions.* This last proposition gives, in some restrictive setting, a stochastic interpretation of the formal procedure (the "Faddeev-Popov procedure") used in gauge field theory, by which one projects a formal volume measure defined on the path space onto the orbit space. Via a regularisation procedure for the Jacobian operator that arises from this projection, in gauge field theory one interprets the projected volume measure as one with formal density given by a regularised "Faddeev-Popov" determinant. This subintends a limit procedure that brings the family of pre-regularised jacobian determinants $\det_\varepsilon, \varepsilon > 0$ to regularised jacobian determinants $\det_{reg}$, by adding divergent terms (this procedure is also referred to as a "renormalisation procedure") that compensate the divergences of the pre-regularised determinants. The above proposition clarifies this formal "renormalisation procedure" from a stochastic point of view, when the orbit space is finite dimensional and compact.

We hope to have convinced the reader that the heat-kernel regularisation approach in gauge field theory, although not so widely used as the zeta function regularisation approach, is natural both from a geometric and stochastic point of view. It helps clarify the formal reduction procedure used in the functional quantisation of gauge theory to "reduce" the measures defined on the path space to measures on the orbit space for the action of the gauge group. It also leads to natural geometric notions on infinite dimensional manifolds, such as minimality which are of interest for their own sake.

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