Control of the cylinder wake in the laminar regime by Trust-Region methods and POD Reduced Order Models

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***Abstract*— In this paper we investigate the optimal control approach for the active control of the circular cylinder wake flow considered in the laminar regime ($Re = 200$). The objective is the mean drag minimization of the wake where the control function is the time harmonic angular velocity of the rotating cylinder. When the Navier-Stokes equations are used as state equation, the discretization of the optimality system leads to large scale discretized optimization problems that represent a tremendous computational task. In order to reduce the number of state variables during the optimization process, a *Proper Orthogonal Decomposition* (POD) Reduced Order Model (ROM) is then derived to be used as state equation. Since the range of validity of the POD ROM is generally limited to the vicinity of the design parameters in the control parameter space, we propose to use the Trust-Region Proper Orthogonal Decomposition (TRPOD) approach, originally introduced by Fahl (2000), to update the reduced order models during the optimization process. Benefiting from the trust-region philosophy, rigorous convergence results guarantee that the iterates produced by the TRPOD algorithm will converge to the solution of the original optimization problem defined with a high fidelity model. A lot of computational work is indeed saved because the optimization process is now based only on low-fidelity models. When the TRPOD is applied to the wake flow configuration, this approach leads to a relative mean drag reduction of 30% for reduced numerical costs.**

I. Introduction

A. *Interest of model reductions in flow control*

These last years, the aeronautics and automobile industries brought a renewed interest to the active control and aerodynamic shape design in both inviscid and viscous compressible flows. Formally, these problems reduce [1] to minimize or maximize an objective functional $\mathcal{J}$ (drag or lift coefficients, concentration of pollutant, emitted noise, mixing...) according to n parameters $\boldsymbol{c} = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$ (unsteady blowing/suction velocities, heat flows, design parameters...) under certain constraints (Navier-Stokes equations, geometric constraints ...). Coarsely, the various existing methods of resolution can be classified in two categories, on the one hand, the *methods of descent type* which at least require an approximation of the gradient of the objective functional, and, on the other hand, the *stochastic methods* whose principle consists in studying the evolution of a population of potential solutions during successive generations (genetic algorithms, simplex method...). Whatever the specific method considered, the numerical costs (CPU and memory) related to these methods of resolution are so important that they become unsuited to the applications of active flow control in closed loop, for which the controller needs to determine his action in real time. Consequently, an alternative approach is necessary.

In this communication, we propose to solve the aforementioned problem of optimization by an optimal control approach in which the state equations, usually the Navier-Stokes equations, are replaced by a *Proper Orthogonal Decomposition* (POD) Reduced Order Model (ROM) of the dynamics for the controlled flow. The POD was originally introduced in Turbulence [2] as an unbiased method of extraction of the Coherent Structures widely known to exist in a turbulent flow. Essentially, this technique leads to the evaluation of POD functions that define a flow basis, optimal in an energetic sense. Thereafter, these POD modes can be used through a Galerkin projection on the Navier-Stokes equations to derive a POD ROM of the controlled flow [3]. The POD basis is determined *a posteriori* using experimental or numerical data previously obtained for the configuration under study. In first approximation, the POD can be viewed as a method of information compression. As a consequence, the ability of POD modes to approximate any state of a complex system is totally dependent of the information originally contained in the snapshot set used to generate the POD functions. Then, despite the energetic optimality of the POD modes, it seems difficult to build once for all, at the beginning of the optimization process, a POD ROM able to approximate correctly the different controlled states encountered by the flow along the optimal path (see the discussion in [4]). Some kind of reactualization of the POD basis during the optimization process seems essential, the main difficulty consisting in determining the moment when a new resolution of the Navier-Stokes equations is necessary to evaluate a new POD basis. Thereafter, we will use an algorithm called TRPOD for *Trust-Region POD* [5] which couples a trust-region method of optimization and reduced order models based on POD (see section II). The principal advantages of this approach are, on the one hand, that the radius of the trust-region corresponding to the POD ROM does not have to be fixed by the user, but is evaluated automatically during the optimization process, and that on the other hand, there are results of convergence proving that the solution obtained for the problem of optimization, formulated with the POD ROM, converges towards the solution of the problem defined by the Navier-Stokes equations.

B. A model problem: the laminar wake flow

In this study, we are interested to control the laminar regime of the unsteady wake flow downstream from a circular cylinder (Fig. 1). The objective is the mean drag minimization of the wake flow by rotary oscillation of the cylinder. The flow is considered as incompressible and the fluid is supposed to be viscous and Newtonian. Wake flows dynamics are characterized [6] by the Reynolds number Re and by the natural Strouhal number St_n at which vortices are shed in the wake of the cylinder (Fig. 2). Traditionally, the Reynolds number is defined as $Re = U_\infty D/\nu$ where D is the cylinder diameter, U_∞ the uniform velocity of the incoming flow and ν the kinematic viscosity of the fluid. The natural Srouhal number is defined as $St_n = fD/U_\infty$ where f is the frequency characteristic of the periodic behavior of the flow. The rotary control is characterized by the non dimensional velocity $\gamma(t)$ defined as the ratio of the tangential velocity V_T to the upstream velocity U_∞ *i.e.* $\gamma(t) = V_T(t)/U_\infty$. For $\gamma = 0$, the flow is naturally said uncontrolled. Hereafter, the control function $\gamma(t)$ is sought for $Re = 200$ as an harmonic function of the form:

$$\gamma(t) = A \sin(2\pi St_f t)$$

where the amplitude A and the *forcing* Strouhal number St_f correspond to two degrees of freedom for the control.

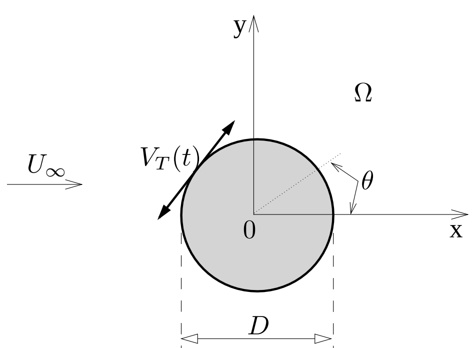


Fig. 1. Configuration of controlled flow.

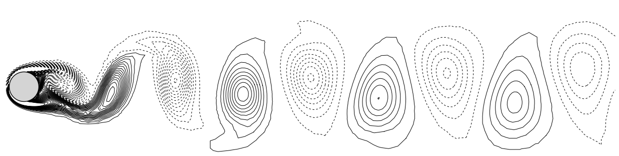


Fig. 2. Vorticity contour plot of the wake for the uncontrolled flow ($\gamma = 0$) at $t = 100$. Dashed lines correspond to negative values.

II. Optimization by Trust-Region methods and POD Reduced Order Models

In this section, only the principle of the Trust-Region Proper Orthogonal Decomposition approach for flow control is exposed. For all the details of the algorithms and in particular the proofs of convergence, the reader is referred to [5], [4].

We consider that the flow control problem discussed in section I-A can be formulated as an unconstrained optimization problem

$$\min_{\boldsymbol{c} \in \mathbb{R}^n} \mathcal{J}(\boldsymbol{\phi}_{NS}(\boldsymbol{c}), \boldsymbol{c}) \tag{1}$$

where $\mathcal{J} : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R}$ represents the objective functional and where $\boldsymbol{\phi}_{NS}$ and $\boldsymbol{c}$ respectively represent the state variables obtained by numerical resolution of the state equations and the control variables. The subscript NS means that the state equations which connect the control variables $\boldsymbol{c}$ to the state variables are the Navier-Stokes equations. Since an accurate computation of the state variables $\boldsymbol{\phi}$ for given $\boldsymbol{c}$ is computationally expensive when the Navier-Stokes equations are used as the state equations, the evaluation of $\mathcal{J}$ during the solution of the optimization process (1) is computationally expensive. A reduction of numerical cost can be achieved by employing a POD ROM as the state equation. In such a way an approximate solution $\boldsymbol{\phi}_{POD}$ of the state variables $\boldsymbol{\phi}$ is obtained and the optimization problem (1) is then replaced by a succession of subproblems of the form

$$\min_{\boldsymbol{c} \in \mathbb{R}^n} \mathcal{J}(\boldsymbol{\phi}_{POD}(\boldsymbol{c}), \boldsymbol{c}). \tag{2}$$

Usually, a POD ROM is constructed for a specific flow configuration, e.g., for an uncontrolled flow or for a flow altered by a specified control. Therefore, the range of validity of a given POD ROM is generally restricted to a region located in the vicinity of the design parameters in the control parameter space, the so-called *trust-region*. Let $\Delta_k > 0$ be the trust-region radius and $\boldsymbol{c}_k$ be the control parameters obtained at an iterate k of the optimization process. To evaluate the function $\mathcal{J}(\boldsymbol{\phi}_{NS}(\boldsymbol{c}_k), \boldsymbol{c}_k)$, it is necessary to determine the variables $\boldsymbol{\phi}_{NS}(\boldsymbol{c}_k)$. These variables are obtained by resolution of the high-fidelity model, the Navier-Stokes equations. Then, we compute snapshots that correspond to the flow dynamics forced by $\boldsymbol{c}_k$. These snapshots form the input ensemble necessary [2] to generate a POD basis $\{\boldsymbol{\Phi}_i^k\}_{i=1,\ldots,N_{POD}}$ (here, N_{POD} corresponds to the number of POD modes). This POD basis can then be used via a Galerkin projection of the Navier-Stokes equation onto the POD eigenvectors to derive a POD ROM for $\boldsymbol{c}_k$ [3], [4]. After integration in time of this POD ROM, the state variables $\boldsymbol{\phi}_{POD}(\boldsymbol{c}_k)$ are estimated, and thus the function $\mathcal{J}(\boldsymbol{\phi}_{POD}(\boldsymbol{c}_k), \boldsymbol{c}_k)$ is evaluated. Since this POD ROM can be employed for an optimization cycle, we define

$$m_k(\boldsymbol{c}_k + \boldsymbol{s}_k) = \mathcal{J}(\boldsymbol{\phi}_{POD}(\boldsymbol{c}_k + \boldsymbol{s}_k), \boldsymbol{c}_k + \boldsymbol{s}_k), \tag{3}$$

as a model function for

$$f(\boldsymbol{c}_k + \boldsymbol{s}_k) = \mathcal{J}(\boldsymbol{\phi}_{NS}(\boldsymbol{c}_k + \boldsymbol{s}_k), \boldsymbol{c}_k + \boldsymbol{s}_k), \tag{4}$$

on the trust-region $\|\boldsymbol{s}_k\| \leq \Delta_k$ around $\boldsymbol{c}_k$.

One is then brought to solve approximately[1] the corresponding trust-region subproblem defined as

$$\min_{\boldsymbol{s}\in\mathbb{R}^n} m_k(\boldsymbol{c}_k+\boldsymbol{s}), \quad \text{s.t.} \quad \|\boldsymbol{s}\| \leq \Delta_k. \tag{5}$$

In order to estimate the quality of the presumed next control parameters $\boldsymbol{c}_{k+1} = \boldsymbol{c}_k + \boldsymbol{s}_k$ where $\boldsymbol{s}_k$ is an approximate solution of (5), we compare the actual reduction in the true objective, $ared_k = f(\boldsymbol{c}_k+\boldsymbol{s}_k) - f(\boldsymbol{c}_k)$, to the predicted reduction obtained with the model function $pred_k = m_k(\boldsymbol{c}_k+\boldsymbol{s}_k) - m_k(\boldsymbol{c}_k)$. Essentially, it is this comparison that gives a measure for the current models prediction capability. If the trial step $\boldsymbol{s}_k$ yields to a satisfactory decrease in the original objective functional, the iteration is called *successful*, in the opposite case we call the iteration *unsuccessful*. When the iteration is successful, the trial step $\boldsymbol{s}_k$ is accepted and the model m_k is updated *i.e.* a new POD ROM is derived that incorporates the flow dynamics as altered by the new control[2]. Furthermore, if the achieved decrease in f indicates a good behavior of the model m_k, the trust-region radius Δ_k can be increased. Now, if there is a limited predicted decrease compared to the actual decrease, we have the possibility to decrease slightly the value of the trust-region radius. For unsuccessful iterations, the trial step $\boldsymbol{s}_k$ is not accepted, the trust-region radius Δ_k is decreased and the trust-region subproblem (5) is solved again within a smaller trust-region. With the contraction of the trust-region it is more likely to have a good approximation to the true objective functional with the POD ROM. The corresponding TRPOD algorithm is schematically described in Fig. 3.

III. Application to the drag reduction of the cylinder wake flow

The objective of this section is to implement the TRPOD approach presented at the section II for minimizing the mean drag coefficient of the cylinder wake flow.

A. Objective functional and model function

In order to simplify the future notations, one introduces the *drag operator* $\mathcal{C}_\mathcal{D}$ defined as:

$$\begin{aligned}\mathcal{C}_\mathcal{D} &: \mathbb{R}^3 \mapsto \mathbb{R}\\ &\boldsymbol{U} \mapsto 2\int_{\Gamma_c}\left(pn_x - \frac{1}{Re}\frac{\partial u}{\partial x}n_x - \frac{1}{Re}\frac{\partial u}{\partial y}n_y\right)\,d\Gamma,\end{aligned} \tag{6}$$

where $\boldsymbol{U} = (u,\, v,\, p)^T$ denotes the vector corresponding to the velocity and pressure fields. By definition, $\mathcal{C}_\mathcal{D}(\boldsymbol{U}) = C_D(t)$ where C_D represents the instantaneous drag coefficient. The velocity component u and pressure p present in the relation (6) can be obtained either by resolution of the Navier-Stokes equations, or by estimation using a POD ROM. In this study, a special care is taken to the development of the POD ROM. First, a POD basis $\boldsymbol{\Phi}_i$ representative of the velocity fields u and v, as of the pressure field p was determined [4]. In addition, in order to improve the robustness of the POD ROM, the POD basis functions determined for a given control parameter $\boldsymbol{c}$, was increased by adding N_{neq} *non-equilibrium modes*, following the procedure described in [8]. Finally, the *control function method* introduced in [9] is used to determine POD basis functions with homogeneous boundary conditions. The velocity and pressure fields can then be expanded into the POD basis functions $\boldsymbol{\Phi}_i$ as:

$$\begin{aligned}\boldsymbol{U}(\boldsymbol{x},\,t) = &\underbrace{\sum_{i=0}^{N_{gal}} a_i(t)\,\boldsymbol{\Phi}_i(\boldsymbol{x})}_{\text{POD Galerkin modes}} + \underbrace{\sum_{i=N_{gal}+1}^{N_{gal}+N_{neq}} a_i(t)\,\boldsymbol{\Phi}_i(\boldsymbol{x})}_{\text{non-equilibrium modes}}\\ &+ \gamma(\boldsymbol{c},\,t)\,\boldsymbol{U}_c(\boldsymbol{x}),\end{aligned} \tag{7}$$

where N_{gal} is the number of Galerkin modes and where $\boldsymbol{U}_c$ is called the control function. Mathematically, $\boldsymbol{U}_c$ is determined as a solution of the Navier-Stokes equations satisfying specific boundary conditions such that the POD eigenfunctions $\boldsymbol{\Phi}_i$ satisfy homogeneous boundary conditions [3], [4].

The Galerkin projection of the Navier-Stokes equations on the space spanned by the first $N_{gal} + N_{neq} + 1$ POD eigenfunctions yields [4] to

$$\begin{aligned}\frac{d\,a_i(t)}{d\,t} = &\sum_{j=0}^{N_{gal}+N_{neq}} \mathcal{B}_{ij}\,a_j(t) + \sum_{j,k=0}^{N_{gal}+N_{neq}} \mathcal{C}_{ijk}\,a_j(t)a_k(t)\\ &+ \mathcal{D}_i\,\frac{d\,\gamma}{d\,t} + \left(\mathcal{E}_i + \sum_{j=0}^{N_{gal}+N_{neq}} \mathcal{F}_{ij}\,a_j(t)\right)\gamma(\boldsymbol{c},\,t)\\ &+ \mathcal{G}_i\gamma^2(\boldsymbol{c},\,t),\end{aligned} \tag{8a}$$

with the following initial conditions:

$$a_i(0) = (\boldsymbol{u}(\boldsymbol{x},\,0),\,\boldsymbol{\Phi}_i(\boldsymbol{x})). \tag{8b}$$

The coefficients $\mathcal{B}_{ij}$, $\mathcal{C}_{ijk}$, $\mathcal{D}_i$, $\mathcal{E}_i$, $\mathcal{F}_{ij}$ and $\mathcal{G}_i$ depend explicitly on $\boldsymbol{\Phi}_i$ and $\boldsymbol{U}_c$. Their expression are given in [4].

Let $\boldsymbol{\phi}_{NS}(\boldsymbol{c}) = (u_{NS},\, v_{NS},\, p_{NS})^T$ represent the state variables obtained by resolution of the Navier-Stokes equations and $\boldsymbol{\phi}_{POD}(\boldsymbol{c}_{POD}) = (u_{POD},\, v_{POD},\, p_{POD})^T$ be the corresponding values estimated with the POD ROM (8), the objective functional is

$$f(\boldsymbol{c}) = \mathcal{J}(\boldsymbol{\phi}_{NS}(\boldsymbol{c})) = \frac{1}{T}\int_0^T \mathcal{C}_D(\boldsymbol{\phi}_{NS}(\boldsymbol{c}))\,dt,$$

and the model function, introduced and justified in [4], is

$$m_k(\boldsymbol{c}) = \widetilde{\mathcal{J}}(\boldsymbol{\phi}_{POD}(\boldsymbol{c})) = \frac{1}{T}\int_0^T \sum_{i=0}^{N_{gal}+N_{neq}} a_i(t)N_i\,dt,$$

[1] Following the trust-region philosophy [7], it is sufficient to compute a trial step $\boldsymbol{s}_k$ that achieves only a certain amount of decrease for the model function.

[2] Since a new snapshot ensemble is available, a new POD basis can then be determined, and finally, a new POD ROM can be derived.

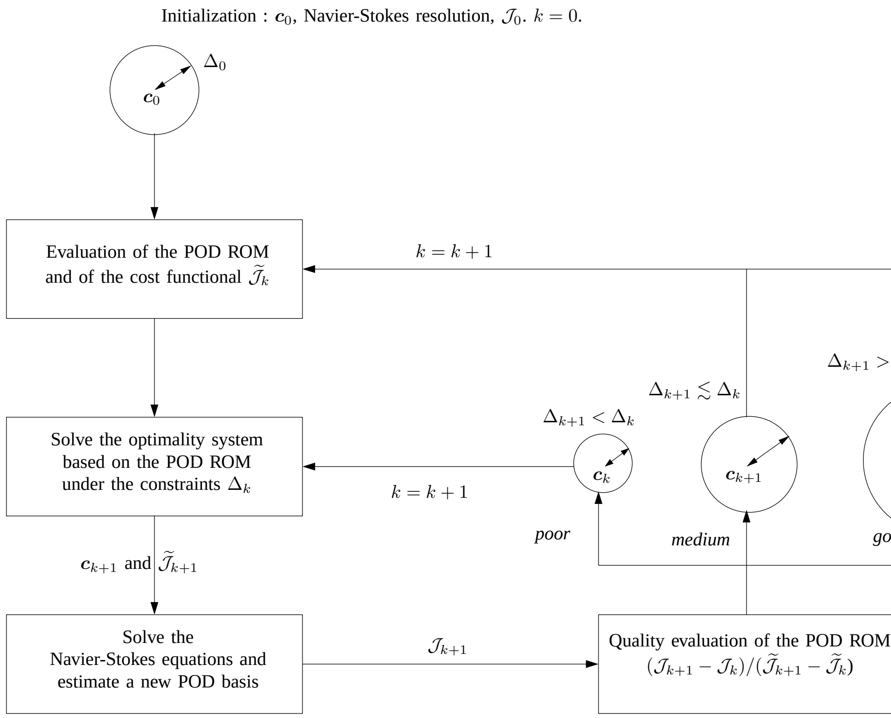


Fig. 3. TRPOD algorithm.

where $N_i = \mathcal{C}_D(\boldsymbol{\Phi}_i)$.

These two functions can then be used in a procedure of optimization coupling trust-region methods and POD reduced order models, following the method presented in section II.

To determine a solution of the subproblem of optimization (5), the simplest method consists in solving the optimality system based on the POD ROM. By definition [1], this optimality system is a system of three coupled partial differential equations [4] :

1) *the state equations* (8)
2) *the adjoint equations*

$$\frac{d\,\xi_i(t)}{dt} = -\sum_{j=0}^{N_{gal}+N_{neq}} \left(\mathcal{B}_{ji} + \gamma(\boldsymbol{c},\,t)\,\mathcal{F}_{ji}\right)\xi_j(t) - \sum_{j,k=0}^{N_{gal}+N_{neq}} \left(\mathcal{C}_{jik} + \mathcal{C}_{jki}\right)a_k(t)\xi_j(t) - \frac{1}{T}N_i, \tag{9a}$$

with terminal conditions :

$$\xi_i(T) = 0. \tag{9b}$$

3) *the optimality conditions*

$$\boldsymbol{\nabla}_{\boldsymbol{c}}\mathcal{L} = \int_0^T \left(\sum_{i=0}^{N_{gal}+N_{neq}} \mathcal{L}_i\right)\boldsymbol{\nabla}_{\boldsymbol{c}}\,\gamma\,dt, \tag{10}$$

with

$$\mathcal{L}_i = -\frac{d\xi_i}{dt}\mathcal{D}_i + \xi_i\left(\mathcal{E}_i + \sum_{j=0}^{N_{gal}+N_{neq}} \mathcal{F}_{ij}a_j + 2\gamma(\boldsymbol{c},\,t)\mathcal{G}_i\right),$$

where $\mathcal{L}$ is the Lagrangian functional introduced to enforce the constraints [4], [1] of the optimization problem.

This system can be solved using an iterative method described in [4]. In this study, the directions of descent are estimated using the Fletcher-Reeves version of the Conjugate Gradient Method [10]. The linear search parameter is computed at each iteration by the backtracking Armijo method [10], in which the length of the step, along each direction of descent, checks the constraint imposed by the trust-region approach.

B. Numerical results

The robustness of the TRPOD approach was evaluated in [4] by use of various initial control parameters $\boldsymbol{c}_0 = (A\,;\,St)$, namely $\boldsymbol{c}_0 = (1.0\,;\,0.2)$, $\boldsymbol{c}_0 = (1.0\,;\,1.0)$, $\boldsymbol{c}_0 = (6.0\,;\,0.2)$ and $\boldsymbol{c}_0 = (6.0\,;\,1.0)$. Figures 4 and 6 represent for two couples of initial control parameters taken at random[3], the evolutions of the values of the objective functional f during the optimization process. Finally, these results are synthesized in the control parameter space on Figs. 5 and 7.

[3]The results are identical for the two other couples of initial conditions.

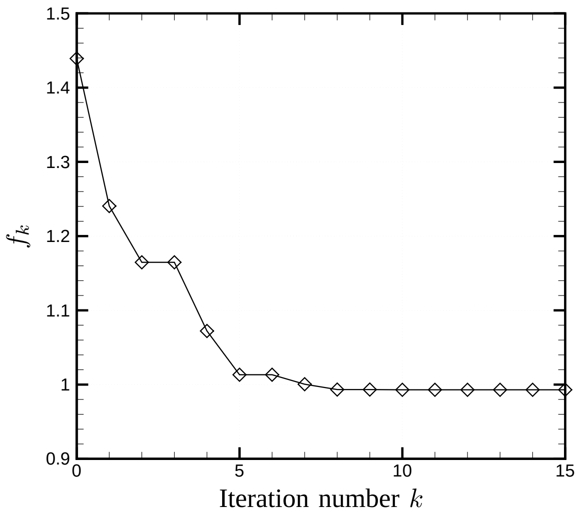


Fig. 4. Variation of the cost functional f with respect to the iteration number k. Initial conditions: $A = 1.0$ and $St = 0.2$.

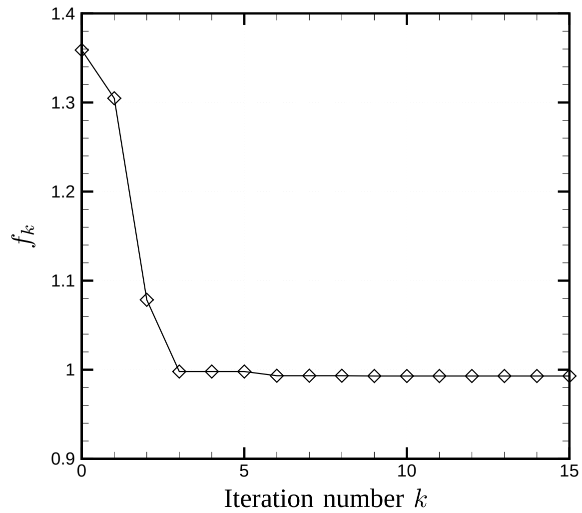


Fig. 6. Variation of the cost functional f with respect to the iteration number k. Initial conditions: $A = 1.0$ and $St = 1.0$.

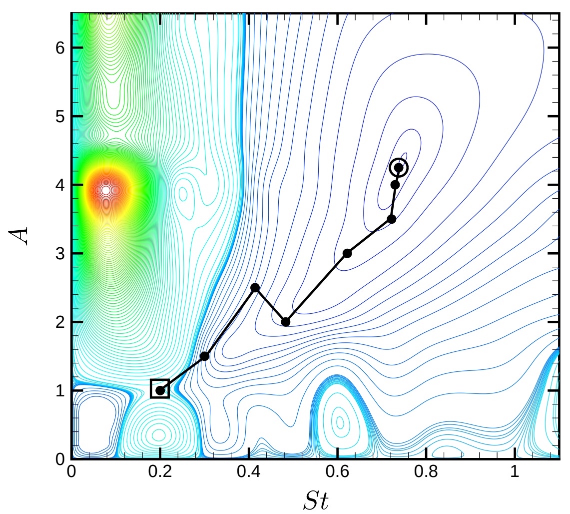


Fig. 5. Evolution of the control parameters during the optimization process. Initial conditions: $A = 1.0$ and $St = 0.2$.

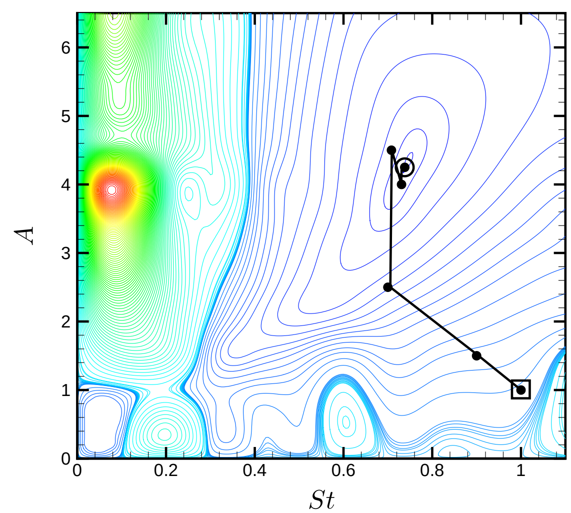


Fig. 7. Evolution of the control parameters during the optimization process. Initial conditions: $A = 1.0$ and $St = 1.0$.

C. Observations

Finally, the optimal control parameters, obtained when the numerical convergence of the iterative procedure is achieved, are $A = 4.25$ and $St_f = 0.738$ [4]. These values are obtained in at most ten iterations (10 resolutions of the Navier-Stokes equations). These results are similar to those predicted by numerical experimentation (open-loop control approach). The control parameters, obtained by the TRPOD approach, converge towards the optimal control parameters determined numerically, and this, whatever the initial values used for the control parameters. This proves the performance and the robustness of the TRPOD approach. Figure 8 represents the time evolution of the drag coefficient, for an uncontrolled flow and the flow forced by the optimal control parameters determined by TRPOD. These results are compared with the value obtained for the unstable stationary basic flow. Protas and Wesfreid argued in [11] that the basic flow generates the lowest coefficient of drag for the configuration under study. The mean drag coefficient varies from a value equal to 1.39 in the uncontrolled case to a value equal to 0.99 when the optimal control parameters are applied corresponding to a relative drag reduction of 30%. The value of the drag coefficient for the optimally controlled flow approaches that obtained for the unstable stationary basic flow ($C_D = 0.94$). In addition, the dynamics of the vorticity has also a similar behavior (see Fig. 9).

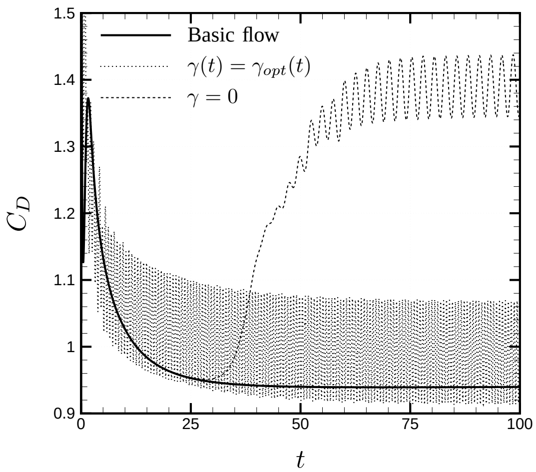
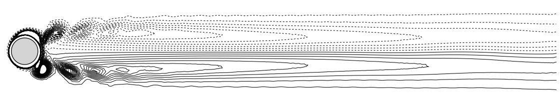
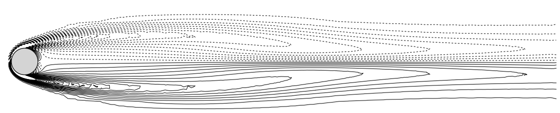


Fig. 8. Time evolution of drag for the uncontrolled ($\gamma = 0$, dashed line), optimally controlled ($\gamma(t) = \gamma_{opt}(t)$, dotted line) and basic ($\gamma = 0$, solid line) flows. Control was started at time $t = 0$.



(a) Optimally controlled flow ($\gamma(t) = A \sin(2\pi St_f\, t)$, $A = 4.25$ and $St_f = 0.738$).



(b) Basic flow ($\gamma = 0$).

Fig. 9. Vorticity contour plot of the wake for the optimally controlled and basic flows at $t = 100$. Dashed lines correspond to negative values.

IV. Conclusions

An optimization procedure coupling a trust-region method and POD Reduced Order Models was used in order to minimize the mean drag of the cylinder wake flow. The optimal control parameters obtained in this way are $A = 4.25$ and $St_f = 0.738$. The relative mean drag reduction is equal to 30%. In addition, the use of trust-region methods mathematically proves the convergence of the control parameters obtained with the reduced order models towards the optimal control parameters corresponding to the Navier-Stokes equations. Moreover, this approach leads to a significant reduction of the numerical costs. Indeed, when the optimality system is based on a POD ROM instead of the Navier-Stokes equations, a cost reduction factor of 600 is obtained for the memory and the optimization problem is solved approximately 4 times more quickly.

V. Acknowledgments

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