Contrôle optimal par réduction de modèle POD et méthode à région de confiance du sillage laminaire d’un cylindre circulaire.

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Two dimensional flow around a circular cylinder at $Re = 200$

Viscous, incompressible and Newtonian fluid

Cylinder oscillation with a tangential sinusoidal velocity $\gamma(t)$

$$\gamma(t) = \frac{V_T}{U_\infty} = A \sin(2\pi St_f t)$$

Find the control parameters $c = (A, St_f)^T$ such that the mean drag coefficient is minimized

$$\langle C_D \rangle_T = \frac{1}{T} \int_0^T \int_0^{2\pi} 2p n_x R d\theta dt - \frac{1}{T} \int_0^T \int_0^{2\pi} \frac{2}{Re} \left( \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) R d\theta dt,$$
Variation of the mean drag coefficient with $A$ and $St_f$.

Numerical minimum $(A_{min}, St_{fmin}) = (4.3, 0.74)$. 

Contrôle optimal par réduction de modèle PODet méthode à région de confiance du sillage laminaire d’un cylindre circulaire. – p.3/22
**Introduction** Mean drag coefficient & steady unstable base flow

![Variation with the Reynolds number of the mean drag coefficient. Contributions and corresponding flow patterns of the base flow and unsteady flow.](image)

Reduced Order Model (ROM) and optimization problems

- Initialization
- High-fidelity model
  - $f(x)$, $\nabla f(x)$
- Approximation model
  - $a(x)$, $\nabla a(x)$
- Optimization on simplified model
  - $\Delta x$
- Optimization
  - Recourse to detailed model (TRPOD)
Reduced Order Model (ROM) *Proper Orthogonal Decomposition (POD)*

- Introduced in turbulence by Lumley (1967).
- Method of information compression
- Look for a realization $\Phi(X)$ which is closer, in an average sense, to realizations $u(X)$. ($X = (x, t) \in D = \Omega \times \mathbb{R}^+$)
- $\Phi(X)$ solution of the problem:
  \[
  \max_{\Phi} \langle |(u, \Phi)|^2 \rangle \quad \text{s.t.} \quad \|\Phi\|^2 = 1.
  \]
- Snapshots method, Sirovich (1987):
  \[
  \int_{T} C(t, t') a^{(n)}(t') \, dt' = \lambda^{(n)} a^{(n)}(t).
  \]
- Optimal convergence *in* $L^2$ *norm* (energy) of $\Phi(X)$
  $\Rightarrow$ Dynamical order reduction is possible.
Discussion of parameter sampling in an optimization setting (from Gunzburger, 2004).

--- path to optimizer using full system, □ initial values, ■ optimal values, and • parameter values used for snapshot generation.
A simple configuration, a rich dynamical behavior

\[ St_f = 0,1 \quad C_D = 4,25. \]

\[ St_f = 0,2 \quad C_D = 2,24. \]

\[ St_f = 0,3 \quad C_D = 1,57. \]

\[ St_f = 0,4 \quad C_D = 1,25. \]

\[ St_f = 0,5 \quad C_D = 1,09. \]

\[ St_f = 0,6 \quad C_D = 1,02. \]

\[ St_f = 0,7 \quad C_D = 1,03. \]

\[ St_f = 0,8 \quad C_D = 1,07. \]

\[ St_f = 0,9 \quad C_D = 1,13. \]

\[ St_f = 1,0 \quad C_D = 1,18. \]

**Fig.**: Iso-values of the vorticity fields $\omega_z$ for $A = 3$
Non-equilibrium modes (Noack et al. 2004)

- Necessity for a given reference flow to introduce new modes: either new operating conditions or shift-modes

Fig.: Schematic representation of a dynamical transition with a non-equilibrium mode
A robust POD surrogate for the drag coefficient

POD approximations consistent with our approach:

\[
U(x, t) = (u, v, p)^T = \sum_{i=0}^{N} a_i(t) \phi_i(x) + \sum_{i=N+1}^{N+M} a_i(t) \phi_i(x) + \gamma(c, t) U_c(x)
\]

- **Galerkin modes**
  - Dynamics of the reference flow \( I \)
  - \( i = 1 \)
  - \( i = 2 \)
  - \( \cdots \)
  - \( i = N \)

- **POD ROM**
  - Temporal dynamics of the modes (eventually, the mode \( i = 0 \) is solved then \( a_0 \equiv a_0(t) \))

- **non-equilibrium modes**
  - Inclusion of new operating conditions \( II, III, IV, \cdots \)
  - \( i = N + 1 \)
  - \( \cdots \)
  - \( i = N + M \)

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Galerkin projection of NSE onto the POD basis:

\[
\begin{align*}
(\phi_i, \frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u)) &= (\phi_i, -\nabla p + \frac{1}{Re} \Delta u).
\end{align*}
\]

Reduced order dynamical system where only \((N + M + 1) \ll N_{POD}\) modes are retained (state equations):

\[
\begin{align*}
\frac{d a_i(t)}{dt} &= \sum_{j=0}^{N+M} B_{ij} a_j(t) + \sum_{j=0}^{N+M} \sum_{k=0}^{N+M} C_{ijk} a_j(t) a_k(t) \\
&+ D_i \frac{d \gamma}{dt} + \left( E_i + \sum_{j=0}^{N+M} F_{ij} a_j(t) \right) \gamma(c, t) + G_i \gamma^2(c, t),
\end{align*}
\]

\(a_i(0) = (U(x, 0), \phi_i(x)).\)

\(B_{ij}, C_{ijk}, D_i, E_i, F_{ij}\) and \(G_i\) depend on \(\phi_i, U_c\) and \(Re\).
Surrogate drag function and model objective function

**Generalities**

Drag operator:

\[
C_D : \mathbb{R}^3 \rightarrow \mathbb{R}
\]

\[
\mathbf{u} \mapsto 2 \int_0^{2\pi} \left( u_3 n_x - \frac{1}{Re} \frac{\partial u_1}{\partial x} n_x - \frac{1}{Re} \frac{\partial u_1}{\partial y} n_y \right) R \, d\theta,
\]

Surrogate drag function:

\[
\widetilde{C}_D(t) = a_0(t) N_0 + \sum_{i=N+1}^{N+M} a_i(t) N_i + \sum_{i=1}^{N} a_i(t) N_i \quad \text{with } N_i = C_D(\varphi_i).
\]

Model objective function:

\[
m = \langle \widetilde{C}_D(t) \rangle_T = \frac{1}{T} \int_0^{T} \left( a_0(t) N_0 + \sum_{i=N+1}^{N+M} a_i(t) N_i \right) dt.
\]
Surrogate drag function Test case $A = 2$ and $St = 0.5$

- Comparison of real drag coefficient $C_D$ (symbols) and model function $\widetilde{C}_D$ (lines) at the design parameters.
Robustness of the model objective function

Test case $A = 2$ and $St = 0.5$

Fig. : Comparison of the real and the model objective functions associated to the mean drag coefficient.
Range of validity of the POD ROM restricted to a vicinity of the design parameters

**Objective**: Use ROMs to solve large-scale optimization problems with assurance of:

1. Automatic restriction of the range of validity
2. Global convergence

**Solution**

- Embed the POD technique into the concept of trust-region methods: Trust-Region Proper Orthogonal Decomposition (Fahl, 2000)

Trust-Region Proper Orthogonal Decomposition (TRPOD) Algorithm

Initialization: $c_0$, Navier-Stokes resolution, $J_0$, $k = 0$.

- Construction of the POD ROM and evaluation of the model objective function $m_k$
- Solve the optimality system based on the POD ROM under the constraints $\Delta_k$
- Solve the Navier-Stokes equations and estimate a new POD basis $c_{k+1}$ and $m_{k+1}$
- Evaluation of the performance $(J_{k+1} - J_k)/(m_{k+1} - m_k)$

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Initial control parameters: $A = 1.0$ et $St = 0.2$

Optimal control parameters: $A = 4.25$ et $St = 0.74$

Mean drag coefficient: $\mathcal{J} = 0.993$

8 resolutions of the Navier-Stokes equations
Numerical results

Initial control parameters : $A = 6.0$ et $St = 0.2$

Optimal control parameters : $A = 4.25$ and $St = 0.74$

Mean drag coefficient : $\mathcal{J} = 0.993$

6 resolutions of the Navier-Stokes equations
Numerical results

Initial control parameters: \( A = 1.0 \) et \( St = 1.0 \)

Optimal control parameters: \( A = 4.25 \) and \( St = 0.74 \)

Mean drag coefficient: \( J = 0.993 \)

5 resolutions of the Navier-Stokes equations
Numerical results

Initial control parameters: $A = 6.0$ et $St = 1.0$

Optimal control parameters: $A = 4.25$ and $St = 0.74$

Mean drag coefficient: $\mathcal{J} = 0.993$

4 resolutions of the Navier-Stokes equations
Optimal control law: $\gamma_{opt}(t) = A \sin(2\pi St t)$ avec $A = 4.25$ et $St = 0.74$.

Relative drag reduction of 30% ($J_0 = 1.4 \Rightarrow J_{opt} = 0.99$).
Uncontrolled flow, $\gamma = 0$.

Controlled flow, $\gamma = \gamma_{opt}$.

Fig. : Iso-values of vorticity $\omega_z$.

Controlled flow : near wake strongly unsteady, far wake (after 5 diameters) steady and symmetric $\rightarrow$ steady unstable base flow.
Optimal control of NSE by He et al. (2000):
⇒ 30% drag reduction for $A = 3$ and $S_t = 0.75$.

Optimal control POD ROM by Bergmann et al. (2005) with no reactualization of the POD ROM:
⇒ 25% drag reduction for $A = 2.2$ and $S_t = 0.53$.

Reduction costs compared to NSE:
- CPU time: 100
- Memory storage: 600

but no mathematical proof concerning the Navier-Stokes optimality.

TRPOD (this study):
⇒ More than 30% of drag reduction for $A = 4.25$ and $S_t = 0.738$.

Reduction costs compared to NSE:
- CPU time: 4
- Memory storage: 400

but global convergence.

"Optimal" control of 3D flows becomes possible!
Conclusions and perspectives

Conclusions on TRPOD
- Important relative drag reduction: more than 30% of relative drag reduction
- Global convergence: mathematical assurance that the solution is identical to the one of the high-fidelity model
- TRPOD compared to NSE ⇒ important reduction of numerical costs:
  ‣ Reduction factor of the CPU time: 4
  ‣ Reduction factor of the memory storage: 400

"OPTIMAL" CONTROL OF 3D FLOWS POSSIBLE BY POD ROM

Perspectives
- Optimal control of the channel flow at $Re_\tau = 180$
- Test other reduced basis method than classical POD
  ‣ Centroidal Voronoi Tessellations (Gunzburguer, 2004): "intelligent" sampling in the control parameter space
  ‣ Balanced POD (Rowley, 2004)
  ‣ Model-based POD (Willcox, CDC-ECC 2005): modify the definitions of the POD modes
Contrôle partiel (3 paramètres de contrôle)

⇒ Effet propulsif, signe écoulement moyen inversé

Fig. : *Contrôle sur la partie amont : $-120^\circ \leq \theta \leq 120^\circ$*