Optimal rotary control of the cylinder wake using POD reduced order model

Michel Bergmann, Laurent Cordier & Jean-Pierre Brancher

Michel.Bergmann@ensem.inpl-nancy.fr

Laboratoire d’Énergétique et de Mécanique Théorique et Appliquée
UMR 7563 (CNRS - INPL - UHP)
ENSEM - 2, avenue de la Forêt de Haye
BP 160 - 54504 Vandoeuvre Cedex, France
Outline

I - Flow configuration and numerical methods
II - Optimal control
III - Proper Orthogonal Decomposition (POD)
IV - Reduced Order Model of the cylinder wake (ROM)
V - Optimal control formulation applied to the ROM
VI - Results of POD ROM
VII - General observations
VIII - Nelder Mead Simplex method

Conclusions and perspectives
I - Configuration and numerical method

- Two dimensional flow around a circular cylinder at $R_e = 200$
- Viscous, incompressible and Newtonian fluid
- Cylinder oscillation with a tangential velocity $\gamma(t)$

- Fractional steps method in time
- Finite Elements Method (FEM) in space

Numerical code written by M.Braza (IMFT-EMT2) & D.Ruiz (ENSEEIHT)
I - Configuration and numerical method

Iso pressure at $t = 100$.

Iso vorticity at $t = 100$.

Aerodynamics coefficients.

<table>
<thead>
<tr>
<th>Authors</th>
<th>$S_t$</th>
<th>$C_D$</th>
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<tbody>
<tr>
<td>Braza et al. (1986)</td>
<td>0.2000</td>
<td>1.4000</td>
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<td>Henderson et al. (1997)</td>
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<td>He et al. (2000)</td>
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<td>this study</td>
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Strouhal number and drag coefficient.
II - Optimal control Definition

Mathematical method allowing to determine without empiricism a control law starting from the optimization of a cost functional.

- State equation $\mathcal{F}(\phi, c) = 0$ ;
  $(\text{Navier-Stokes} + \text{C.I.} + \text{C.L.})$

- Control variables $c$ ;
  $(\text{Blowing/suction, design parameters ...})$

- Cost functional $\mathcal{J}(\phi, c)$.
  $(\text{Drag, lift, ...})$

Find a control law $c$ and state variable $\phi$ such that the cost functional $\mathcal{J}(\phi, c)$ reach an extremum under the constrain $\mathcal{F}(\phi, c) = 0$. 
Constrained optimization $\Rightarrow$ unconstrained optimization

- Introduction of Lagrange multipliers $\xi$.

- Lagrange functional:
  \[
  \mathcal{L}(\phi, c, \xi) = \mathcal{J}(\phi, c) - \langle \mathcal{F}(\phi, c), \xi \rangle
  \]

- Force $\mathcal{L}$ to be stationary $\Rightarrow$ look for $\delta \mathcal{L} = 0$:
  \[
  \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial c} \delta c + \frac{\partial \mathcal{L}}{\partial \xi} \delta \xi = 0
  \]

- Suppose $\phi$, $c$ and $\xi$ independant each other:
  \[
  \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi = \frac{\partial \mathcal{L}}{\partial c} \delta c = \frac{\partial \mathcal{L}}{\partial \xi} \delta \xi = 0
  \]
II - Optimal control *optimal system*

- **State equation** \( \frac{\partial L}{\partial \xi} \delta \xi = 0 \):
  \[ F(\phi, c) = 0 \]

- **Co-state equation** \( \frac{\partial L}{\partial \phi} \delta \phi = 0 \):
  \[ \left( \frac{\partial F}{\partial \phi} \right)^* \xi = \left( \frac{\partial J}{\partial \phi} \right)^* \]

- **Optimality condition** \( \frac{\partial L}{\partial c} \delta c = 0 \):
  \[ \left( \frac{\partial J}{\partial c} \right)^* = \left( \frac{\partial F}{\partial c} \right)^* \xi \]

⇒ Expensive method in CPU time and storage memory for large system!

⇒ Ensure only a local (*generally not global*) minimum
"without an inexpensive method for reducing the cost of flow computation, it is unlikely that the solution of optimization problems involving the three dimensional unsteady Navier-Stokes system will become routine"

M. Gunzburger, 2000
II - Proper Orthogonal Decomposition (POD)

- Introduced in fluid mechanics (turbulence context) by Lumley (1967).
- Look for a realization $\phi(X)$ which is closer, in an average sense, to the realizations $u(X)$. ($X = (x, t) \in \mathcal{D} = \Omega \times \mathbb{R}^+$)
- $\phi(X)$ solution of the problem: $\max_{\phi} \langle |u, \phi|^2 \rangle \quad \text{s.t.} \quad \|\phi\|^2 = 1$.
- Snapshots method, Sirovich (1987):

$$\int_T C(t, t') a^{(n)}(t') \, dt' = \lambda^{(n)} a^{(n)}(t).$$
- Optimal convergence $L^2$ norm (energy) of $\phi(X)$
  \Rightarrow Dynamical order reduction is possible.
- Decomposition of the velocity field:

$$u(x, t) = \sum_{i=1}^{N_{POD}} a^{(i)}(t) \phi^{(i)}(x).$$
Galerkin’s projection of $NSE$ on the POD basis:

\[
\left( \phi^{(i)}, \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = \left( \phi^{(i)}, -\nabla p + \frac{1}{Re} \Delta u \right).
\]

Integration by parts (Green’s formula) leads:

\[
\left( \phi^{(i)}, \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = (p, \nabla \cdot \phi^{(i)}) - \frac{1}{Re} \left( (\nabla \otimes \phi^{(i)})^T, \nabla \otimes u \right) \\
- [p \phi^{(i)}] + \frac{1}{Re} [(\nabla \otimes u) \phi^{(i)}].
\]

with $[a] = \int_{\Gamma} a \cdot n \, d\Gamma$ and $(A, B) = \int_{\Omega} A : B \, d\Omega = \sum_{i,j} \int_{\Omega} A_{ij} B_{ji} \, d\Omega.$
III - Reduced Order Model of the cylinder wake (ROM)

- Velocity decomposition with $N_{POD}$ modes:

$$
u(x, t) = u_m(x) + \gamma(t) u_c(x) + \sum_{k=1}^{N_{POD}} a^{(k)}(t) \phi^{(k)}(x).$$

- Reduced order dynamical system where only $N_{gal} (\ll N_{POD})$ modes are retained (state’s equation):

$$\begin{cases}
\frac{d a^{(i)}(t)}{dt} = A_i + \sum_{j=1}^{N_{gal}} B_{ij} a^{(j)}(t) + \sum_{j=1}^{N_{gal}} \sum_{k=1}^{N_{gal}} C_{ijk} a^{(j)}(t) a^{(k)}(t) \\
+ D_i \frac{d}{dt} \gamma + \sum_{j=1}^{N_{gal}} F_{ij} a^{(j)}(t) \gamma + G_i \gamma^2 \\
a^{(i)}(0) = (u(x, 0), \phi^{(i)}(x)).
\end{cases}$$

$A_i, B_{ij}, C_{ijk}, D_i, E_i, F_{ij}$ and $G_i$ depend on $\phi, u_m, u_c$ and $Re$. 
Integration and (optimal) stabilization of the POD ROM for
\[ \gamma = A \sin(2\pi S_t t), \ A = 2 \text{ et } S_t = 0.5. \]

POD reconstruction errors ⇒ temporal modes amplification

- **Causes:**
  - Extraction of large energetic eddies
  - Dissipation takes place in small eddies

- **Solution:**
  - Optimal addition of artificial viscosity on each POD mode

Temporal evolution of the first 6 POD temporal modes.

projection (Navier Stokes)
prediction before stabilisation (POD ROM)
prediction after stabilisation (POD ROM).
IV - Reduced Order Model of the cylinder wake Stabilization

Comparison of energetic spectrum.

- Good agreements between POD and DNS spectrum
- Reduced reconstruction error between predicted (POD) and projected (DNS) modes

⇒ Validation of the POD ROM
Objective functional:

\[ J(a, \gamma(t)) = \int_0^T J(a, \gamma(t)) \, dt = \int_0^T \left( \sum_{i=1}^{N_{gal}} a^{(i)}^2 + \frac{\alpha}{2} \gamma(t)^2 \right) \, dt. \]

\( \alpha \) : regularization parameter (penalization).

Co-state’s equations:

\[
\begin{align*}
\frac{d \xi^{(i)}(t)}{dt} &= -\sum_{j=1}^{N_{gal}} \left( B_{ji} + \gamma F_{ji} + \sum_{k=1}^{N_{gal}} (C_{jik} + C_{jki}) a^{(k)} \right) \xi^{(j)}(t) - 2a^{(i)} \\
\xi^{(i)}(T) &= 0.
\end{align*}
\]

Optimality condition (gradient):

\[
\delta \gamma(t) = -\sum_{i=1}^{N_{gal}} D_i \frac{d \xi^{(i)}}{dt} + \sum_{i=1}^{N_{gal}} \left( E_i + \sum_{j=1}^{N_{gal}} F_{ij} a^{(j)} + 2G_i \gamma(t) \right) \xi^{(i)} + \alpha \gamma.
\]
IV - Optimal control formulation based on reduced order model

► $\gamma^{(0)}(t)$ done; for $n = 0, 1, 2, \ldots$ and while a convergence criterium is not satisfied, do:

1. From $t = 0$ to $t = T$ solve the state’s equations with $\gamma^{(n)}(t)$;
   $\leftarrow$ state’s variables $a^{(n)}(t)$

2. From $t = T$ to $t = 0$ solve the co-state’s equations with $a^{(n)}(t)$;
   $\leftarrow$ co-state’s variables $\xi^{(n)}(t)$

3. Solve the optimality condition with $a^{(n)}(t)$ and $\xi^{(n)}(t)$;
   $\leftarrow$ objective gradient $\delta \gamma^{(n)}(t)$

4. New control law $\leftarrow \gamma^{(n+1)}(t) = \gamma^{(n)}(t) + \omega^{(n)} \delta \gamma^{(n)}(t)$

► End do.
V - Closed loop results  *Generalities*

- No reactualization of the POD basis.

- The energetic representativity is *a priori* different to the dynamical one:

  - possible inconvenient for control,

  - a POD dynamical system represents *a priori* only the dynamics (and its vicinity) used to build the low dynamical model.

- Construction of a POD basis representative of a large range of dynamics:

  - excitation of a great number of degrees of freedom scanning $\gamma(t)$ in amplitudes and frequencies.
\[ \gamma_e(t) = A_1 \sin(2\pi S_{t1} t) \times \sin(2\pi S_{t2} t - A_2 \sin(2\pi S_{t3} t)) \]

with \( A_1 = 4, \ A_2 = 18, \ S_{t1} = 1/120, \ S_{t2} = 1/3 \) and \( S_{t3} = 1/60. \)

- 0 \leq \text{amplitudes} \leq 4 \text{ and Fourier analysis } \Rightarrow 0 \leq \text{frequencies} \leq 0.65
- \( \gamma_e \) initial control law in the iterative process.
Stationary cylinder $\gamma = 0 : \leftrightarrow \text{2 modes out of 100 are sufficient to restitute 97\% of the kinetic energy.}$

Controlled cylinder $\gamma = \gamma_e : \leftrightarrow \text{40 modes out of 100 are then necessary to restitute 97\% of the kinetic energy}$

$\implies$ Robustness evolution with dynamical evolutions.
Reduction of the wake instationarity. $\gamma_{opt} \simeq A \sin(2\pi S_t t)$ with $A = 2.2$ and $S_t = 0.53$

$$\mathcal{J}(\gamma_e) = 9.81 \implies \mathcal{J}(\gamma_{opt}) = 5.63.$$
V - Closed loop results *Comparison of wakes’ organization*

- No mathematical proof concerning the Navier Stokes optimality.

- No control: $\gamma = 0$ \Rightarrow Asymmetrical flow.
  \leftarrow Large and energetic eddies.

- Optimal control: $\gamma = \gamma_{opt}$ \Rightarrow Symmetrization of the (near) wake.
  \leftarrow Smaller and lower energetic eddies.
V - Closed loop results Aerodynamics coefficients

- Very consequent drag reduction:
  \[ C_{D0} = 1.40 \text{ for } \gamma = 0 \quad \text{et} \quad C_D = 1.04 \text{ for } \gamma = \gamma_{opt} \]
  \[ C_D/C_{D0} = 0.74 \Rightarrow \text{more than 25\%}. \]

- Decrease of the lift amplitude:
  \[ C_L = 0.68 \text{ for } \gamma = 0 \quad \text{et} \quad C_L = 0.13 \text{ for } \gamma = \gamma_{opt} \]
V - Closed loop results Numerical costs

- Optimal control of NSE by He et al. (2000):
  - harmonic control law with $A = 3$ and $S_t = 0.75$.
  - $30\%$ drag reduction.

- Optimal control POD ROM (this study):
  - harmonic control law with $A = 2.2$ and $S_t = 0.53$.
  - $25\%$ drag reduction.

- Less energetic costs (greater energetic gain?)

- Calculus time costs: 100 less using POD ROM than NSE! (for co-states equations and optimality conditions too)

- Memory storage: 600 less variables using POD ROM than NSE!

  $\Rightarrow$ "Optimal" control of 3D flows becomes possible!
VI - General observations Numerical experimentation

Observations

- Minimum is located in a smooth valley
  - Global minimum: around $A = 4.4$ and $St = 0.76$

- Maximum is located in a sharp peak
  - Global maximum: near $St = 0.2$, the natural frequency: lock-on flow

Finding the global minimum with an optimization algorithm may lead to difficulties because of the smooth valley.
VI - General observations  

Maximal rotation angle

- Maximum rotation angle:  
  \[ \theta = \frac{A}{\pi St} \]

**Observations**

- No drag reduction possible near natural frequency
- Maximum drag reduction around \( \theta = \theta_{opt} = 95^\circ \)

\[ \leftrightarrow \text{For all frequencies } g.t. \text{ natural frequency} \]

\[ \leftrightarrow \text{Minimum drag:} \]

\[ CD_{min} = 0.71CD_0 = 0.98 \]

Existence of an "optimal" maximal rotation angle.
VI - General observations  

**Maximal rotation angle**

![Graph showing comparison between $C_{D_{\text{min}}}$ and $C_{D_{\text{opt}}}$ vs $St$]

**Notations**

\[ C_{D_{\text{min}}}(St) = \min_{A \in \mathbb{R}} C_D(\theta, St) \]

\[ C_{D_{\text{opt}}}(St) = C_D(95^\circ, St) \]

**Observations**

- Good agreements between $C_{D_{\text{min}}}$ and $C_{D_{\text{opt}}}$ for $St > St_{\text{Nat}}$
- $\theta_{\text{opt}}$ is not optimal for $St < St_{\text{Nat}}$

In order to minimise the drag one couldn’t choose $A$ and $St$ independently. It seems that $A/St = 5.2$ ($\theta_{\text{opt}} = 95^\circ$).
VI - General observations  

Discussion

- Control law obtained by POD ROM is not optimal for drag minimisation

  - Parameters obtained: $A = 2.2$ and $St = 0.53 \ (\theta_{max} = 76^\circ)$
    
    \[ CD = 1.04 \]

  - Optimal parameters: $A = 4.4$ and $St = 0.76 \ (\theta_{max} = 105^\circ \neq \theta_{opt})$
    
    \[ CD = 0.98 \]

- Results in $(A, \ St)$ quite different but not so far in $CD$ terms

  - The smooth valley is reached

- Couple an efficient new optimisation algorithm for smooth functions

  - Take results obtained by POD ROM as initial conditions
VII - Nelder Mead Simplex method *Generalities*

**Avantages**

- Numerical simplicities
- Adaptive topology
- Gradients calculations not necessary
- Good results with smooth functions

**Disadvantages**

- No proof of optimality for simplex dimensions greater than two
- Need to fix free parameters
- Maybe more iterations than gradient based optimisation algorithms...
VII - Nelder Mead Simplex method \textit{Results}

\begin{itemize}
  \item Topology adaptation function of the curve of the valley
  \item Minimum found by Nelder-Mead simplex method:
    \[ A = 4.5 \text{ and } St = 0.76 \Rightarrow \theta = 107^\circ \]
    \( \Leftarrow \) Seems to be the global minimum
  \item 30 NSE resolutions \( \Rightarrow \) 5\% additive drag reduction compared to POD ROM
\end{itemize}

Relative drag reduction by POD ROM: 25\% (1 NSE resolution)
Utility of coupling a new algorithm?
Conclusions

- Significative drag reduction minimizing the wake instationnarity of the reduced order model: more than 25% of the relative drag
- But this is not a minimum (global or local) of the drag function
- Low calculus cost ⇒ numerical cost negligible.
  - Calculus time costs: 100 less using POD ROM than NSE! (for co-states equations and optimality conditions too)
  - Memory storage: 600 less variables using POD ROM than NSE!

"OPTIMAL" CONTROL OF 3D FLOWS POSSIBLE BY POD ROM

- Existence of an optimal maximal rotation angle for effective drag reduction, $\theta = 105^\circ$
- Couple POP ROM to Nelder Mead simplex method leads a priori the global minimum of the drag functional
- But the gain on the drag function is quite small compared to the result obtained by POD ROM
Perspectives

- Improve the representativity of the low order model
  - "Optimize" the temporal excitation $\gamma_e$
  - Mix snapshots corresponding to several different dynamics (temporal excitations)

- Look for harmonic control $\gamma(t) = A \sin(2\pi S t \, t)$ with POD basis reactualization (close loop on NSE and not only on POD ROM)

- Couple this optimal system with trust region methods (TRPOD) → proof of convergence under weak conditions

- Couple pressure with the POD dynamical system
  - pressure contribution to drag coefficient : 80%

- Optimal control of the Navier Stokes equations