Optimal rotary control of the cylinder wake using POD Reduced Order Model

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Outline

I - Flow configuration and numerical methods
II - Optimal control approach
III - Proper Orthogonal Decomposition (POD)
IV - Reduced Order Model of the cylinder wake (ROM)
V - Optimal control formulation applied to the ROM
VI - Results of POD ROM
VII - Nelder-Mead Simplex method
Conclusions and perspectives
Motivations Cylinder wake flow?

- Prototype configuration of separated flow
- Experimental study of Tokumaru and Dimotakis (JFM 1991) $Re = 15000$
  - Unforced flow

- Forced flow $\rightarrow$ 80% of drag reduction
I - Configuration and numerical method

- Two dimensional flow around a circular cylinder at $Re = 200$
- Viscous, incompressible and Newtonian fluid

\[ \nabla \cdot u = 0 ; \quad \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla P + \frac{1}{Re} \Delta u \]

- Cylinder oscillation with a tangential velocity $\gamma(t)$

- Control parameter:

\[ \alpha(t) = \frac{\gamma(t)}{U_\infty} = \frac{R \dot{\theta}(t)}{U_\infty} = \frac{\text{Tangential velocity}}{\text{Upstream velocity}} \]
I - Configuration and numerical method

- Fractional step method in time (pressure correction)
- Finite Element Method (FEM) in space \((P_1, P_1)\)
  - Numerical domain \(\Omega = \{-10 \leq x \leq 20; -10 \leq y \leq 10\}; D = 1\)
  - Mesh: 25042 triangles, 12686 vertices

Numerical code written by M.Braza (IMFT-EMT2) & D.Ruiz (ENSEEIHT)
I - Configuration and numerical method

Iso pressure at $t = 100$.

Iso vorticity at $t = 100$.

Aerodynamic coefficients.

<table>
<thead>
<tr>
<th>Authors</th>
<th>$S_t$</th>
<th>$C_D$</th>
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<tr>
<td>Braza et al. (1986)</td>
<td>0.2000</td>
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<td>this study</td>
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Strouhal number and drag coefficient.
II - Optimal control Definition

Mathematical method allowing to determine without a priori knowledge a control law based on the optimization of a cost functional.

- State equations $F(\phi, c) = 0$;
  
  (Navier-Stokes + I.C. + B.C.)

- Control variables $c$;
  
  (Blowing/suction, design parameters ...)

- Cost functional $J(\phi, c)$.
  
  (Drag, lift, target function, ...)

Find a control law $c$ and state variables $\phi$ such that the cost functional $J(\phi, c)$ reach an extremum under the constraint $F(\phi, c) = 0$. 
II - Optimal control Lagrange multipliers

Constrained optimization ⇒ unconstrained optimization

- Introduction of Lagrange multipliers \( \xi \) (adjoint variables).
- Lagrange functional:

\[
\mathcal{L}(\phi, c, \xi) = \mathcal{J}(\phi, c) - \langle \mathcal{F}(\phi, c), \xi \rangle
\]

- Force \( \mathcal{L} \) to be stationary ⇒ look for \( \delta \mathcal{L} = 0 \):

\[
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial c} \delta c + \frac{\partial \mathcal{L}}{\partial \xi} \delta \xi = 0
\]

- Hypothesis: \( \phi, c \) and \( \xi \) assumed to be independent of each other:

\[
\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi = \frac{\partial \mathcal{L}}{\partial c} \delta c = \frac{\partial \mathcal{L}}{\partial \xi} \delta \xi = 0
\]

where

\[
\frac{\partial \mathcal{L}}{\partial x} = \lim_{\epsilon \to 0} \frac{\mathcal{L}(x + \epsilon \delta x) - \mathcal{L}(x)}{\epsilon} = 0 \quad \forall \delta x \quad \text{(Fréchet derivative)}
\]
II - Optimal control *Optimality system*

- **State equations** \( \frac{\partial L}{\partial \xi} \delta \xi = 0 \):
  \[ F(\phi, c) = 0 \]

- **Co-state (adjoint) equations** \( \frac{\partial L}{\partial \phi} \delta \phi = 0 \):
  \[
  \left( \frac{\partial F}{\partial \phi} \right)^* \xi = \left( \frac{\partial J}{\partial \phi} \right)^*
  \]

- **Optimality condition** \( \frac{\partial L}{\partial c} \delta c = 0 \):
  \[
  \left( \frac{\partial J}{\partial c} \right)^* = \left( \frac{\partial F}{\partial c} \right)^* \xi
  \]

⇒ Expensive method in CPU time and storage memory for large system!

Bewley et al. (2000) : \(10^8\) grid points
⇒ Ensure only a local *(generally not global)* minimum
II - Optimal control *Iterative method*

- $c^{(0)}$ given; for $n = 0, 1, 2, ...$ and while a convergence criterium is not satisfied, do:

1. From $t = 0$ to $t = T$ solve the state equations with $c^{(n)}$;
   $\leadsto$ *state variables* $\phi^{(n)}$

2. From $t = T$ to $t = 0$ solve the co-state equations with $\phi^{(n)}$;
   $\leadsto$ *co-state variables* $\xi^{(n)}$

3. Solve the optimality condition with $\phi^{(n)}$ and $\xi^{(n)}$;
   $\leadsto$ *objective gradient* $\delta c^{(n)}$

4. New control law $\leadsto c^{(n+1)} = c^{(n)} + \omega^{(n)} \delta c^{(n)}$

- End do.
"without an inexpensive method for reducing the cost of flow computation, it is unlikely that the solution of optimization problems involving the three dimensional unsteady Navier-Stokes system will become routine"

M. Gunzburger, 2000
III - Proper Orthogonal Decomposition (POD)

► Introduced in fluid mechanics (turbulence context) by Lumley (1967).

► Look for a realization \( \phi(X) \) which is closer, in an average sense, to the realizations \( u(X) \). \( X = (x, t) \) \( \in \mathcal{D} = \Omega \times \mathbb{R}^+ \)

► \( \phi(X) \) solution of the problem: \( \max_{\phi} \langle |(u, \phi)|^2 \rangle \) s.t. \( ||\phi||^2 = 1 \).

► Snapshots method, Sirovich (1987):

\[
\int_T C(t, t') a^{(n)}(t') dt' = \lambda^{(n)} a^{(n)}(t).
\]

► Optimal convergence \( L^2 \) norm (energy) of \( \phi(X) \)

\( \Rightarrow \) Dynamical order reduction is possible.

► Decomposition of the velocity field:

\[
u(x, t) = \sum_{i=1}^{N_{POD}} a^{(i)}(t) \phi^{(i)}(x).\]
III - POD POD modes: uncontrolled flow ($\gamma = 0$)

- First POD mode.
- Second POD mode.
- Third POD mode.
- Fourth POD mode.
Galerkin projection of NSE on the POD basis:

\[
\left( \phi^{(i)}, \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = \left( \phi^{(i)}, -\nabla p + \frac{1}{Re} \Delta u \right).
\]

Integration by parts (Green’s formula) leads:

\[
\left( \phi^{(i)}, \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = \left( p, \nabla \cdot \phi^{(i)} \right) - \frac{1}{Re} \left( (\nabla \otimes \phi^{(i)})^T, \nabla \otimes u \right)
\]

\[- [p \phi^{(i)}] + \frac{1}{Re} [(\nabla \otimes u) \phi^{(i)}].\]

with \([a] = \int_{\Gamma} a \cdot n \, d\Gamma\) and \((A, B) = \int_{\Omega} A : B \, d\Omega = \sum_{i,j} \int_{\Omega} A_{ij} B_{ji} \, d\Omega\).
III - Reduced Order Model of the cylinder wake (ROM)

- Velocity decomposition with $N_{POD}$ modes:

$$u(x, t) = u_m(x) + \gamma(t) u_c(x) + \sum_{k=1}^{N_{POD}} a^{(k)}(t) \phi^{(k)}(x).$$

- Reduced order dynamical system where only $N_{gal}$ ($\ll N_{POD}$) modes are retained (state equations):

$$\frac{d a^{(i)}(t)}{dt} = A_i + \sum_{j=1}^{N_{gal}} B_{ij} a^{(j)}(t) + \sum_{j=1}^{N_{gal}} \sum_{k=1}^{N_{gal}} C_{ijk} a^{(j)}(t)a^{(k)}(t)$$

$$+ D_i \frac{d \gamma}{dt} + \left( E_i + \sum_{j=1}^{N_{gal}} F_{ij} a^{(j)}(t) \right) \gamma + G_i \gamma^2$$

$$a^{(i)}(0) = (u(x, 0), \phi^{(i)}(x)).$$

$A_i, B_{ij}, C_{ijk}, D_i, E_i, F_{ij}$ and $G_i$ depend on $\phi, u_m, u_c$ and $Re$. 
Integration and "optimal" stabilization of the POD ROM for

\[ \gamma = A \sin(2\pi S_t t), \quad A = 2 \text{ and } S_t = 0.5. \]

POD reconstruction errors \(\Rightarrow\) temporal modes amplification

**Reasons:**
- Extraction by POD only of the large energetic eddies
- Dissipation takes place in small eddies

**Solution:**
- Addition of an optimal artificial viscosity on each POD mode projection (Navier-Stokes) prediction before stabilization (POD ROM) prediction after stabilization (POD ROM).
IV - Reduced Order Model of the cylinder wake *Stabilization*

Comparison of energetic spectrum.

- Good agreements between POD ROM spectrum and DNS spectrum
- Reduction of the reconstruction error between predicted (POD ROM) and projected (DNS) modes

⇒ Validation of the POD ROM
V - Optimal control formulation based on ROM

► Objective functional:

\[
J(a, \gamma(t)) = \int_0^T J(a, \gamma(t)) \, dt = \int_0^T \left( \sum_{i=1}^{N_{gal}} a^{(i)}^2 + \frac{\alpha}{2} \gamma(t)^2 \right) \, dt.
\]

\(\alpha\) : regularization parameter (penalization).

► Co-state equations:

\[
\frac{d \xi^{(i)}(t)}{dt} = - \sum_{j=1}^{N_{gal}} \left( B_{ji} + \gamma F_{ji} + \sum_{k=1}^{N_{gal}} (C_{jik} + C_{jki}) a^{(k)} \right) \xi^{(j)}(t) - 2a^{(i)}
\]

\(\xi^{(i)}(T) = 0\).

► Optimality condition (gradient):

\[
\delta \gamma(t) = - \sum_{i=1}^{N_{gal}} D_i \frac{d \xi^{(i)}}{dt} + \sum_{i=1}^{N_{gal}} \left( E_i + \sum_{j=1}^{N_{gal}} F_{ij} a^{(j)} + 2G_i \gamma(t) \right) \xi^{(i)} + \alpha \gamma.
\]
VI - Results of POD ROM Generalities

- No reactualization of the POD basis.

- The energetic representativity is \textit{a priori} different to the dynamical one:
  - possible inconvenient for control,

- a POD dynamical system represents \textit{a priori} only the dynamics (and its vicinity) used to build the low dynamical model.

- Construction of a POD basis representative of a large range of dynamics:
  - \textit{excitation of a great number of degrees of freedom scanning $\gamma(t)$ in amplitudes and frequencies.}
$\gamma_e(t) = A_1 \sin(2\pi S_{t1} t) \times \sin(2\pi S_{t2} t - A_2 \sin(2\pi S_{t3} t))$

with $A_1 = 4$, $A_2 = 18$, $S_{t1} = 1/120$, $S_{t2} = 1/3$ and $S_{t3} = 1/60$.

- $0 \leq$ amplitudes $\leq 4$ and Fourier analysis $\Rightarrow 0 \leq$ frequencies $\leq 0.65$
- $\gamma_e$ initial control law in the iterative process.
VI - Results of POD ROM Energy

▶ Stationary cylinder $\gamma = 0$ : $\leftrightarrow$ 2 modes out of 100 are sufficient to restore 97% of the kinetic energy.

▶ Controlled cylinder $\gamma = \gamma_e$ : $\leftrightarrow$ 40 modes out of 100 are then necessary to restore 97% of the kinetic energy

⇒ Improvement of the POD ROM robustness to dynamical evolutions.
Reduction of the wake instationarity. $\gamma_{opt} \approx A \sin(2\pi S_t t)$ with $A = 2.2$ and $S_t = 0.53$

$$J(\gamma_e) = 9.81 \quad \implies \quad J(\gamma_{opt}) = 5.63.$$
VI - Results of POD ROM Comparison of wakes’ organization

- No mathematical proof concerning the Navier Stokes optimality.

- No control $\gamma = 0 \Rightarrow$ Asymmetric flow.
  $\leftarrow$ Large and energetic eddies.

- Optimal control $\gamma = \gamma_{opt} \Rightarrow$ Symmetrization of the (near) wake.
  $\leftarrow$ Smaller and lower energetic eddies.
VI - Results of POD ROM Aerodynamic coefficients

Important drag reduction:

\[ C_{D0} = 1.40 \text{ for } \gamma = 0 \text{ and } C_D = 1.04 \text{ for } \gamma = \gamma_{opt} \]

\[ C_D/C_{D0} = 0.74 \Rightarrow \text{more than 25%}. \]

Decrease of the lift amplitude:

\[ C_L = 0.68 \text{ for } \gamma = 0 \text{ and } C'_L = 0.13 \text{ for } \gamma = \gamma_{opt}. \]
Optimal control of NSE by He et al. (2000) :

$\leftrightarrow$ harmonic control law with $A = 3$ and $S_t = 0.75$.

$\Rightarrow$ 30\% drag reduction.

Optimal control POD ROM (this study) :

$\leftrightarrow$ harmonic control law with $A = 2.2$ and $S_t = 0.53$.

$\Rightarrow$ 25\% drag reduction.

Reduction costs using POD ROM compared to NSE :

- CPU time : 100
- Memory storage : 600

$\leftrightarrow$ "Optimal" control of 3D flows becomes possible!

Does the POD ROM control law correspond to the global minimum?
VI - Results of POD ROM Numerical optimization

Observations

- Minimum is located in a smooth valley

  $\rightarrow$ Global minimum:
  around $A = 4.4$ and $St = 0.76$

- Maximum is located in a sharp peak

  $\rightarrow$ Global maximum:
  near $St = 0.2$, the natural frequency: lock-on flow

Finding the global minimum with an optimization algorithm may be difficult due to the smooth valley
VI - Results of POD ROM *Local versus global minimum*

- POD ROM control law does not correspond to the global minimum
  - POD ROM parameters: $A = 2.2$ and $St = 0.53 \Rightarrow C_D = 1.04$
  - Global minimum parameters: $A = 4.4$ and $St = 0.76 \Rightarrow C_D = 0.98$
- Results in $(A, St)$ quite different but not so far in terms of $C_D$
  - The smooth valley is reached
- Improvement: coupling to the POD ROM approach an efficient new optimization algorithm for smooth functions
  - Take results obtained by POD ROM as initial conditions
Advantages

- Numerical simplicities
- Adaptive topology
- Free gradient optimization method
- Good results with smooth functions

Drawbacks

- No proof of optimality for simplex dimensions greater than two
- Need to fix free parameters
- Maybe more iterations than gradient based optimisation algorithms.
VII - Nelder-Mead Simplex method Results

Observations

- Topology adaptation function of the curve of the valley
- Minimum found by the simplex method: $A = 4.5$ and $St = 0.76$
  $\Rightarrow$ Seems to be the global minimum
- 30 NSE resolutions $\Rightarrow$ 5% additive drag reduction compared to POD ROM

Relative drag reduction by POD ROM: 25% (1 NSE resolution)
Usefulness of coupling a new algorithm?
Conclusions

- Important drag reduction obtained by POD ROM: more than 25% of relative drag reduction
- This solution is not the global minimum of the drag function
- POD ROM compared to NSE ⇒ important reduction of numerical costs:
  - Reduction factor of the CPU time: 100
  - Reduction factor of the memory storage: 600

"OPTIMAL" CONTROL OF 3D FLOWS POSSIBLE BY POD ROM

- Coupling POD ROM with the Nelder-Mead simplex method leads a priori to the global minimum of the drag function
- But the gain on the drag function is quite small (5%) compared to results obtained by POD ROM
Perspectives

- Improve the representativity of the POD ROM
  "Optimize" the temporal excitation $\gamma_e$
  Mix snapshots corresponding to different dynamics (temporal excitations)
  Introduction of shift-mode?

- Look for harmonic control $\gamma(t) = A \sin(2\pi S_t t)$ with POD basis reactualization (closed loop on NSE and not only on POD ROM)

- Coupling the POD ROM approach with Trust Region POD method (TRPOD)
  proof of convergence under weak conditions

- Introducing the pressure into the POD dynamical system
  pressure contribution to drag coefficient: 80%

- Optimal control of the Navier-Stokes equations