Control of the cylinder wake in the laminar regime by Trust-Region Proper Orthogonal Decomposition.

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Two dimensional flow around a circular cylinder at $Re = 200$
Viscous, incompressible and Newtonian fluid
Cylinder oscillation with a tangential sinusoidal velocity $\gamma(t)$

$$\gamma(t) = \frac{V_T}{U_\infty} = A \sin(2\pi St_f t)$$

Find the control parameters $c = (A, St_f)^T$ such that the mean drag coefficient is minimized

$$\langle C_D \rangle_T = \frac{1}{T} \int_0^T \int_0^{2\pi} 2 p n_x R d\theta dt - \frac{1}{T} \int_0^T \int_0^{2\pi} \frac{2}{Re} \left( \frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y \right) R d\theta dt,$$
Variation of the mean drag coefficient with $A$ and $St_f$.
Numerical minimum $(A_{min}, St_{f_{min}}) = (4.3, 0.74)$. 
**Introduction**

Mean drag coefficient & steady unstable base flow

**Fig.** Variation with the Reynolds number of the mean drag coefficient. Contributions and corresponding flow patterns of the base flow and unsteady flow.

Reduced Order Model (ROM) and optimization problems

Initialization

High-fidelity model

Recourse to detailed model (TRPOD)

Approximation model

Optimization

Optimization on simplified model

\[ f(x), \text{grad } f(x) \]

\[ a(x), \text{grad } a(x) \]

\[ \Delta x \]
Proper Orthogonal Decomposition (POD)

Introduced in turbulence by Lumley (1967).
Method of information compression

Look for a realization $\Phi(X)$ which is closer, in an average sense, to realizations $u(X)$. ($X = (x, t) \in \mathcal{D} = \Omega \times \mathbb{R}^+$)

$\Phi(X)$ solution of the problem:

$$\max_{\Phi} \langle |(u, \Phi)|^2 \rangle \quad \text{s.t.} \quad \|\Phi\|^2 = 1.$$

Snapshots method, Sirovich (1987):

$$\int_{T} C(t, t') a^{(n)}(t') \, dt' = \lambda^{(n)} a^{(n)}(t).$$

Optimal convergence in $L^2$ norm (energy) of $\Phi(X)$

$\Rightarrow$ Dynamical order reduction is possible.
Discussion of parameter sampling in an optimization setting (from Gunzburger, 2004).

- path to optimizer using full system, □ initial values, ■ optimal values, and • parameter values used for snapshot generation.
Necessity for a given reference flow to introduce new modes: either new operating conditions or shift-modes.

**Fig.**: Schematic representation of a dynamical transition with a non-equilibrium mode.
A robust POD surrogate for the drag coefficient

POD approximations consistent with our approach:

\[
U(x, t) = (u, v, p)^T = \sum_{i=0}^{N} a_i(t) \phi_i(x) + \sum_{i=N+1}^{N+M} a_i(t) \phi_i(x) + \gamma(c, t) U_c(x)
\]

- **Galerkin modes**
  - Dynamics of the reference flow \( I \)
  - Inclusion of new operating conditions \( II, III, IV, \cdots \)

- **non-equilibrium modes**

**Physical aspects**

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**POD ROM**

Temporal dynamics of the modes (eventually, the mode \( i = 0 \) is solved then \( a_0 \equiv a_0(t) \))
Galerkin projection of NSE onto the POD basis:

\[
\left( \phi_i, \frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) \right) = \left( \phi_i, -\nabla p + \frac{1}{Re} \Delta u \right).
\]

Reduced order dynamical system where only \((N + M + 1)(\ll N_{POD})\) modes are retained (state equations):

\[
\frac{d a_i(t)}{dt} = \sum_{j=0}^{N+M} B_{ij} a_j(t) + \sum_{j=0}^{N+M} \sum_{k=0}^{N+M} C_{ijk} a_j(t) a_k(t)
+ \sum_{j=0}^{N+M} \mathcal{F}_{ij} a_j(t) \gamma(c, t) + \mathcal{G}_i \gamma^2(c, t),
\]

\[
a_i(0) = (U(x, 0), \phi_i(x)).
\]

\(B_{ij}, C_{ijk}, D_i, E_i, F_{ij} \) and \(G_i\) depend on \(\phi_i, U_c\) and \(Re\).
Surrogate drag function and model objective function

Drag operator:

\[ C_D : \mathbb{R}^3 \rightarrow \mathbb{R} \]

\[ u \mapsto 2 \int_0^{2\pi} \left( u_3 n_x - \frac{1}{Re} \frac{\partial u_1}{\partial x} n_x - \frac{1}{Re} \frac{\partial u_1}{\partial y} n_y \right) R \, d\theta, \]  

Surrogate drag function:

\[ \widetilde{C}_D(t) = a_0(t) N_0 + \sum_{i=N+1}^{N+M} a_i(t) N_i + \sum_{i=1}^{N} a_i(t) N_i \quad \text{with} \quad N_i = C_D(\phi_i). \]

Model objective function:

\[ m = \langle \widetilde{C}_D(t) \rangle_T = \frac{1}{T} \int_0^T \left( a_0(t) N_0 + \sum_{i=N+1}^{N+M} a_i(t) N_i \right) \, dt. \]
Surrogate drag function

Test case $A = 2$ and $St = 0.5$

- Comparison of real drag coefficient $C_D$ (symbols) and model function $\tilde{C}_D$ (lines) at the design parameters.
Robustness of the model objective function

Test case $A = 2$ and $St = 0.5$

Fig. : Comparison of the real and the model objective functions associated to the mean drag coefficient.
Range of validity of the POD ROM restricted to a vicinity of the design parameters

**Objective**

1. Automatic restriction of the range of validity
2. Global convergence

**Solution**

- Embed the POD technique into the concept of trust-region methods:
  Trust-Region Proper Orthogonal Decomposition (Fahl, 2000)

Trust-Region Proper Orthogonal Decomposition (TRPOD) Algorithm

Initialization: $c_0$, Navier-Stokes resolution, $J_0$. $k = 0$.

Construction of the POD ROM and evaluation of the model objective function $m_k$

Solve the optimality system based on the POD ROM under the constraints $\Delta_k$

$c_{k+1}$ and $m_{k+1}$

Solve the Navier-Stokes equations and estimate a new POD basis

Evaluation of the performance $(J_{k+1} - J_k)/(m_{k+1} - m_k)$

$\Delta_{k+1} \lesssim \Delta_k$

$\Delta_{k+1} < \Delta_k$

$\Delta_{k+1} > \Delta_k$

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Numerical results

Initial control parameters: $A = 1.0$ et $St = 0.2$

Optimal control parameters: $A = 4.25$ et $St = 0.74$

Mean drag coefficient: $\mathcal{J} = 0.993$

8 resolutions of the Navier-Stokes equations
Initial control parameters: $A = 6.0$ et $St = 0.2$

Optimal control parameters: $A = 4.25$ and $St = 0.74$

Mean drag coefficient: $\mathcal{J} = 0.993$

6 resolutions of the Navier-Stokes equations
Initial control parameters: $A = 6.0$ et $St = 1.0$

Optimal control parameters: $A = 4.25$ and $St = 0.74$

Mean drag coefficient: $\mathcal{J} = 0.993$

4 resolutions of the Navier-Stokes equations
Initial control parameters: \( A = 1.0 \) et \( St = 1.0 \)

Optimal control parameters: \( A = 4.25 \) and \( St = 0.74 \)

Mean drag coefficient: \( J = 0.993 \)

5 resolutions of the Navier-Stokes equations
Optimal control law: $\gamma_{opt}(t) = A \sin(2\pi St t)$ avec $A = 4.25$ et $St = 0.74$

Relative drag reduction of 30% ($J_0 = 1.4 \Rightarrow J_{opt} = 0.99$)
Uncontrolled flow, $\gamma = 0$.

Controlled flow, $\gamma = \gamma_{\text{opt}}$.

Fig.: Iso-values of vorticity $\omega_z$.

Controlled flow: near wake strongly unsteady, far wake (after 5 diameters) steady and symmetric $\rightarrow$ steady unstable base flow.
Optimal control of NSE by He et al. (2000):
⇒ 30% drag reduction for $A = 3$ and $S_t = 0.75$.

Optimal control POD ROM by Bergmann et al. (2005) with no reactualization of the POD ROM:
⇒ 25% drag reduction for $A = 2.2$ and $S_t = 0.53$.

- Reduction costs compared to NSE:
  - CPU time: 100
  - Memory storage: 600
  but no mathematical proof concerning the Navier-Stokes optimality.

TRPOD (this study):
⇒ More than 30% of drag reduction for $A = 4.25$ and $S_t = 0.738$.

- Reduction costs compared to NSE:
  - CPU time: 4
  - Memory storage: 400
  but global convergence.

⇒ "Optimal" control of 3D flows becomes possible!
Conclusions and perspectives

Conclusions on TRPOD
- Important relative drag reduction: more than 30% of relative drag reduction
- Global convergence: mathematical assurance that the solution is identical to the one of the high-fidelity model
- TRPOD compared to NSE ⇒ important reduction of numerical costs:
  - Reduction factor of the CPU time: 4
  - Reduction factor of the memory storage: 400

"OPTIMAL" CONTROL OF 3D FLOWS POSSIBLE BY POD ROM

Perspectives
- Test mode interpolations for controlled flows (Morzynski and Tadmor’s talks, GAMM 2006)
- Test other reduced basis method than classical POD
  - Centroidal Voronoi Tessellations (Gunzburguer, 2004): "intelligent" sampling in the control parameter space
  - Model-based POD (Willcox, 2005): modify the definitions of the POD modes
- Optimal control of the channel flow at $Re_\tau = 180$