Imagery for 3D geometry design: application to fluid flows.

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An imagery soft

C++ with wxWidgets, OpenGL (Mesa) VTK.
Tools:

- Contrast.
- **Ginzburg-Landau**.
- Connected component (Scanning, Front propagation).
- **Skeleton**.

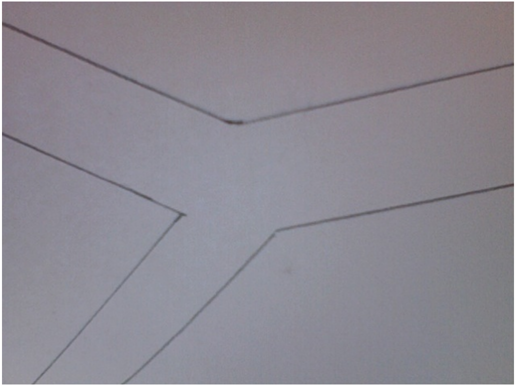
Ginzburg-Landau.

$$\partial_t u - L^2 \Delta u = -u(u-1)(u-\theta).$$

with Neuman boundary conditions.

- forcing black ($u = 0$) and white ($u = 1$).
- thickening $\{u = 0\}$: choose $\theta > \frac{1}{2}$.
- until $\{u = 0\}$ is connected
- sliming $\{u = 0\}$:(choose $\theta < \frac{1}{2}$).

Example



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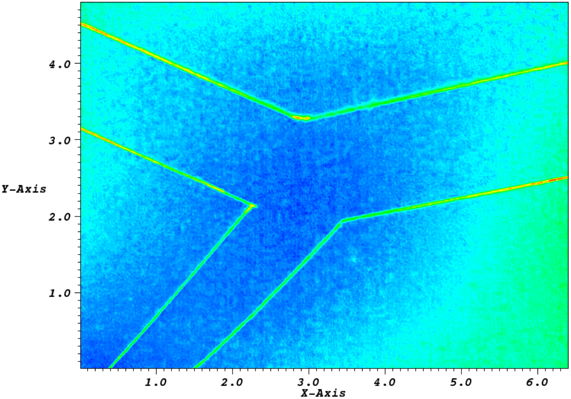
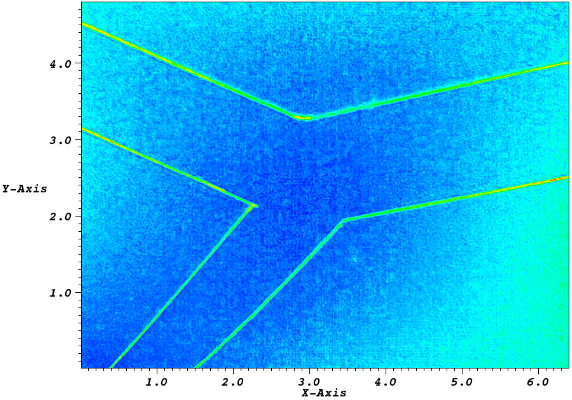
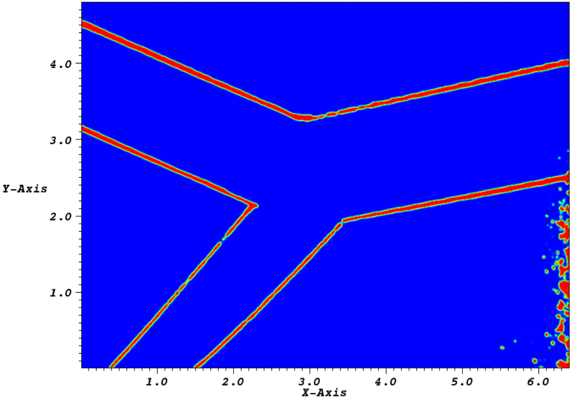


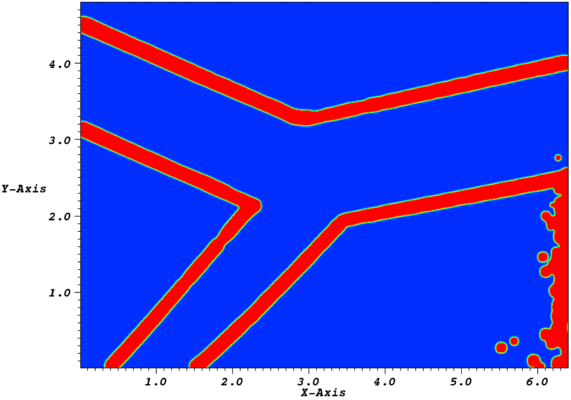
Contrast



Ginzburg-Landau



Ginzburg-Landau Thikening



Ginzburg-Landau Sliming

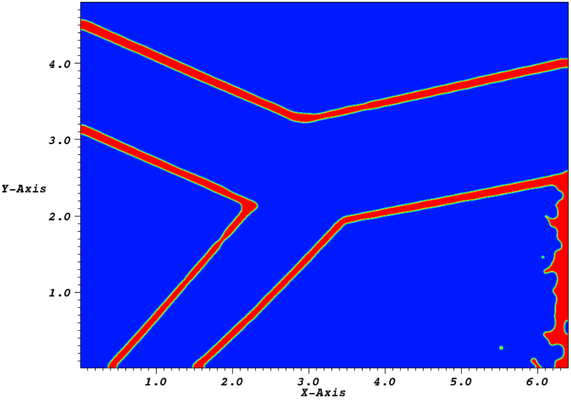
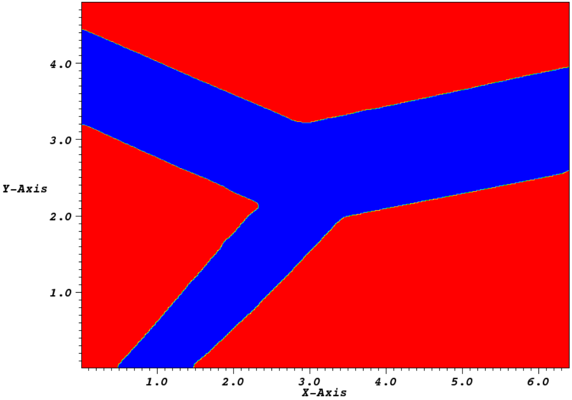


Image processing result



Defintion

Let $\Omega_d \cap \mathbb{R}^d$, the skelton of Ω_d is the smallest set S such that

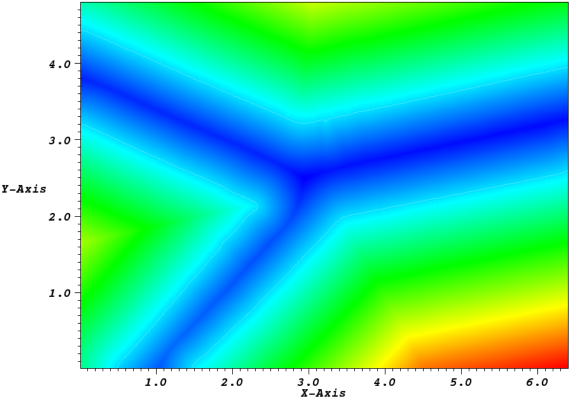
$$\Omega_d = \cup_{x \in S} B_d(x, r(x)),$$

with maximal radius $r(x)$.
$B_d(x, r(x))$ is the bigger ball centered in x included in Ω_d.

$$\Omega_d = \cup_{x \in \Omega_d} B_d(x, r(x)),$$

with $r(x) = dist(x, \partial\Omega_d)$.
If $x \in S$, then $B_d(x, r(x))$ is tangent to $\partial\Omega_d$ in two points at least

Example



2D-3D distance function

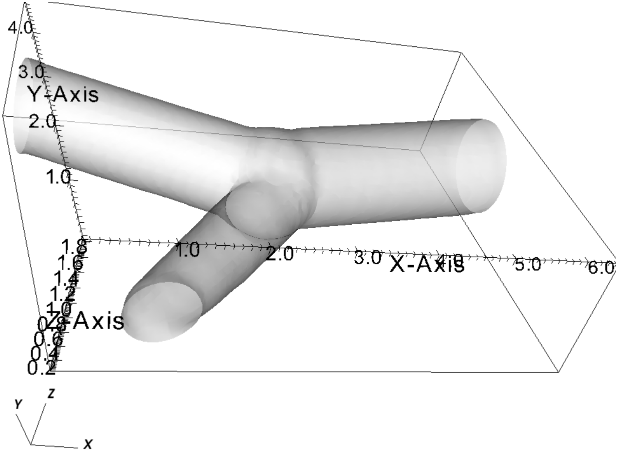
The following function ψ is associated to the skeleton:

$$\forall y \in \mathbb{R}^d, \ \psi(y) = \inf_{x \in S} \{\|x - y\| - r(x)\}, \quad (1)$$

The function ψ: the signed distance Level Set function to $\partial\Omega_d$.

- $d = 2$: ψ is known
- the Skeleton S is computed (set points where $\nabla\psi$ is singular and $\psi < 0$)
- $d = 3$: ψ is evaluated.

Example continued



Computation of skeleton

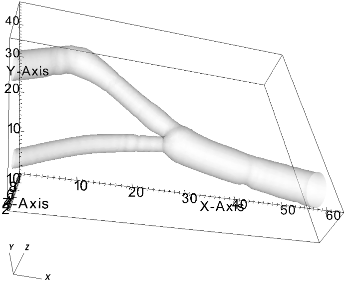
- Imagery processing $\Rightarrow$ black (0) and white (1) image
- $u_0 = 0$ in black region, $u_0 = 1$ in white region
- Solve Eikonal equation with Fast Marching Method:

$$|\nabla \psi| = 1 \text{ where } u_0 = 1$$
$$\psi = 0 \text{ where } u_0 = 0.$$

- Compute discrete gradients: $\nabla_{++}\psi$, $\nabla_{+-}\psi$, $\nabla_{-+}\psi$, $\nabla_{--}\psi$
- Normalize discrete gradients and compute minimal scalar product PS.
- if $PS < 0.8$ the point belongs to Skeketon!

Example


File Edit Improvement Processing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Flow rate or pressure

Dirichlet boundary conditions:

- the user precises the flow rate near inlet or outlet
- the flow rate is associated to points on skeleton
- the velocity field is given with parabolic profile depending on ψ ("Poiseuille flow"):

The direction of the skeketon and $\nabla\psi$ give the velocity field direction

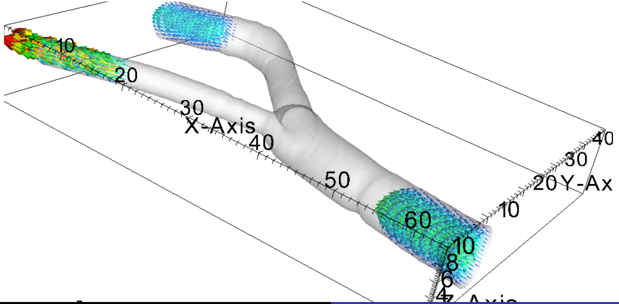
Pressure can be imposed instead of velocities.

Skeleton direction

The direction of the skeleton, for $x \in S$:

$$ds(x) = \frac{\nabla_1 \psi}{|\nabla_1 \psi|} + \frac{\nabla_2 \psi}{|\nabla_2 \psi|} \text{ or } ds(x) = (\nabla_1 \psi)^{\perp} + (\nabla_2 \psi)^{\perp}$$

$\nabla \psi$ is singular, ∇_1, ∇_2 are such that $\frac{|\nabla_1 \psi \cdot \nabla_2 \psi|}{|\nabla_1 \psi||\nabla_2 \psi|}$ is minimal.



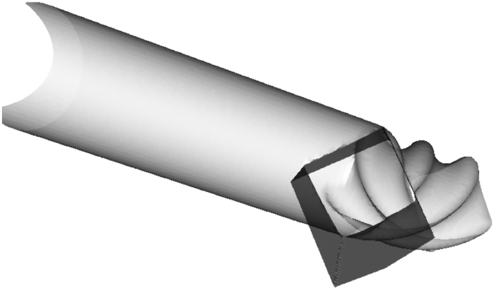
Super-Skeleton

position: $x \in \mathbb{R}^3$
radius: $r(x) \in \mathbb{R}^+$
flow rate: $d(x) \in \mathbb{R}$
direction: $ds(x) \in \mathbb{R}^3$
transverse direction: $dst(x) \in \mathbb{R}^3$
choice of L^p norm: $p(x) \geq 1$

The L^p norm is the chosen distance in the transverse direction:
$p = 2$ for circular section
$p = 1$ or $p = \infty$ for square section
... (design of the unit ball in $\mathbb{R}^2$).

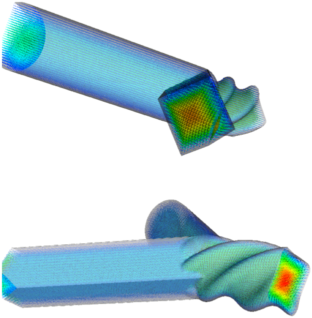
Geometry from Skeleton

Rotation on transverse direction of Super-Skeleton with $p = 1$:



Geometry from Skeleton

Rotation on transverse direction of Super-Skeleton with $p = 1$:



Model

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}.\nabla\vec{u}\right)-\nabla.(2\eta D\vec{u})+\nabla p=\vec{F}\quad\forall(t,\vec{x})\in\mathbb{R}^{+}\times\Omega, \tag{2}$$

With the incompressibility condition :

$$\nabla.\vec{u}=0\quad\forall(t,\vec{x})\in\mathbb{R}^{+}\times\Omega, \tag{3}$$

where the field $\vec{u}=(u,v,w)$ is the velocity, p the pressure, ρ the density, η the viscosity, $\vec{F}$ any body force detailed hereafter and $D\vec{u}=(\nabla\vec{u}+\nabla^{T}\vec{u})/2$.

Bifluid Model

Incompressible two-phase flows (Sussman, Smereka and Osher (94)) for two-phase flows:
Phases are located by the sign of a Level Set function ϕ.

$$\frac{\partial \phi}{\partial t} + \vec{u}.\nabla\phi = 0 \quad \forall(t,\vec{x}) \in \mathbb{R}^{+} \times \Omega. \tag{4}$$

Forces are gravity and surface tension:

$$\vec{F}_{\sigma} = \rho\vec{g} + \sigma\kappa\delta(\phi)\vec{n} \tag{5}$$

Complete model

$$\rho(\phi)\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}.\nabla\vec{u}\right) - \nabla.(2\eta D\vec{u}) + \frac{1}{\varepsilon}H(-\psi)\vec{u} + \nabla p$$
$$= \rho\vec{g} + \sigma\kappa\nabla H(\phi), \quad \forall(t,\vec{x}) \in \mathbb{R}^+ \times B$$
$$\nabla.\vec{u} = 0 \quad \forall(t,\vec{x}) \in \mathbb{R}^+ \times B$$
$$\vec{u} = 0 \text{ if } \psi \geq 0, \quad \forall(t,\vec{x}) \in \mathbb{R}^+ \times \partial B$$
$$\vec{u} = \vec{u_b} \text{ if } \psi < 0, \quad \forall(t,\vec{x}) \in \mathbb{R}^+ \times \partial B, \tag{6}$$

where $\vec{u_b}$ is defined thanks to direction of Skeleton.

Discretization

Time Discretization

$$\rho(\varphi^n)\frac{\vec{u}^{n+1}-\vec{u}^n}{\Delta t} - div(\nu(\varphi^n)(\nabla\vec{u}^{n+1}+(\nabla\vec{u}^{n+1})^t) + \rho(\varphi^n)\vec{u}^n\cdot\nabla\vec{u}^n$$
$$+\nabla p^{n+1} = -\sigma\kappa^n\nabla(H(\varphi^n)),$$

$$div\ \vec{u}^{n+1} = 0,$$

$$\kappa^n = div\ \frac{\nabla\varphi^n}{|\nabla\varphi^n|},$$

$$\frac{\varphi^{n+1}-\varphi^n}{\Delta t} + \vec{u}^{n+1}\cdot\nabla\varphi^n = 0.$$

Augmented Lagrangian for incompressibility.

Space Discretization

- Cartesian uniform grid on a box containing the domain.
- Discretization on uniform (MAC) staggered grid for fluid solver.
- WENO5 (G-S. JIANG, D. PENG) scheme for transport of smooth function (signed distance Level Set function) on grids 3 times thiner.

Numerical stability

Proposition (C.G., P. Vigneaux 08) For low Reynolds, the above numerical scheme is stable under the condition:

$$\Delta t \leq min\left(\Delta t_c, \Delta t_\sigma\right), \text{ avec } \Delta t_c = c_0 \|\vec{u}\|^{-1}_{L^\infty(\Omega)} \Delta x \text{ et}$$

$$\Delta t_\sigma = \frac{1}{2}\left(c_2 \frac{\eta}{\sigma}\Delta x + \sqrt{(c_2 \frac{\eta}{\sigma}\Delta x)^2 + 4 c_1 \frac{\rho}{\sigma}\Delta x^3}\right)$$

where Δt is the time step, Δx the space step and c_0 c_1, c_2 do not depend on physical and numerical parameter.

Numerical stability

Known time step:

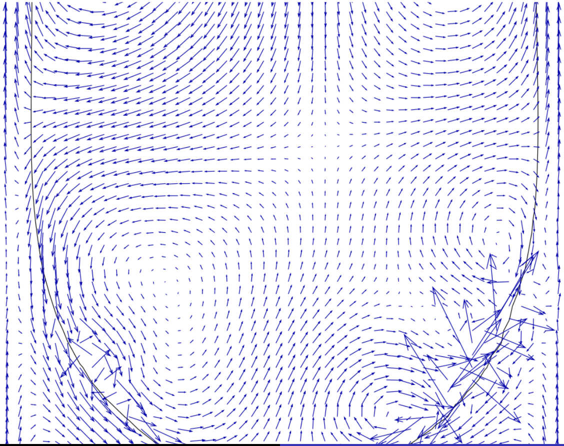
Brackbill (BKZ) capillary time step and Capillary time step for Stokes

$$\Delta t_{BKZ} = c_1 \sqrt{\frac{\rho}{\sigma} \Delta x^3} = \Delta t_\sigma(\rho, 0), \quad \Delta t_{STK} = c_2 \frac{\eta}{\sigma} \Delta x = \Delta t_\sigma(0, \eta).$$

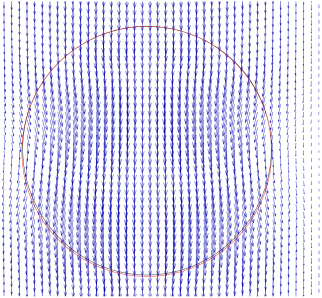
Remark:

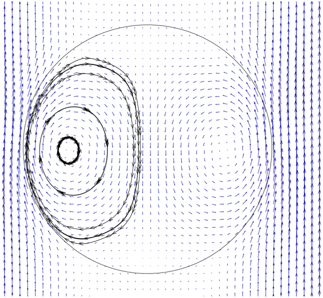
$$\Delta t_\sigma \leq \frac{1 + \sqrt{5}}{2} \max(\Delta t_{STK}, \Delta t_{BKZ}). \tag{7}$$

Numerical instability example

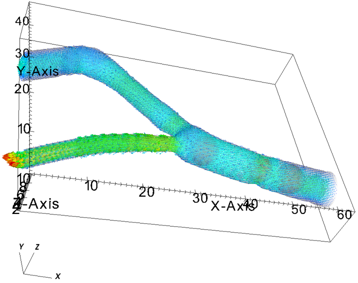


Stable flow





Examples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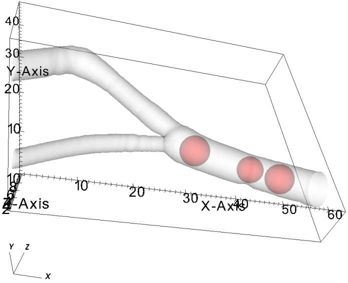
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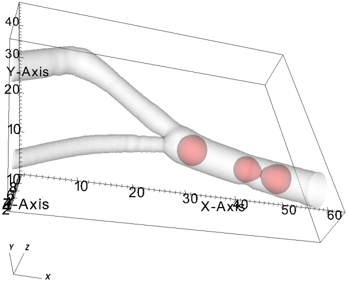
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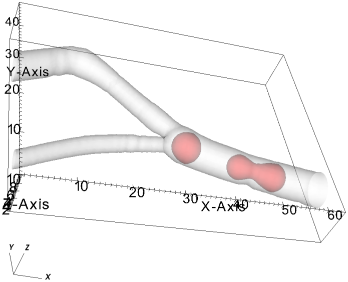
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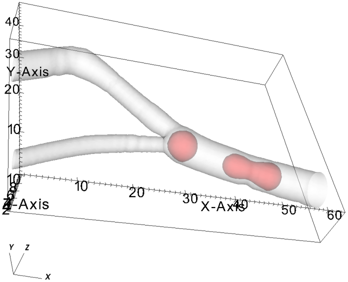
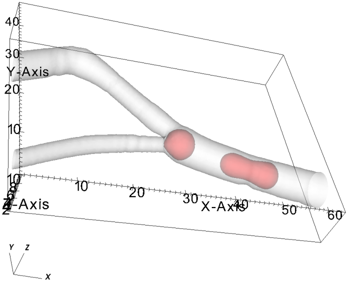
Examples





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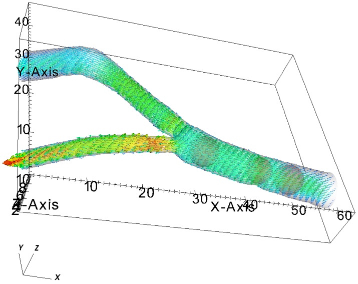
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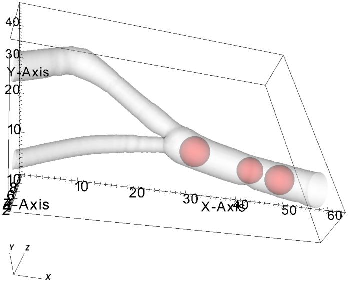
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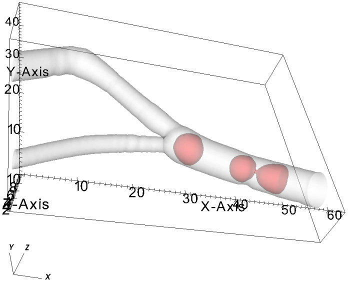
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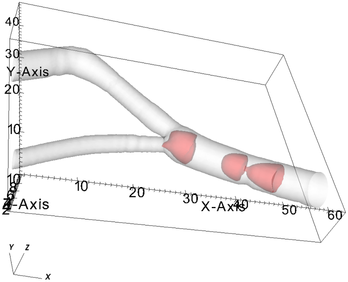
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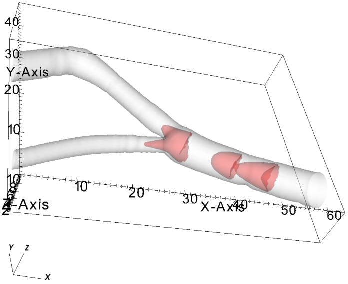
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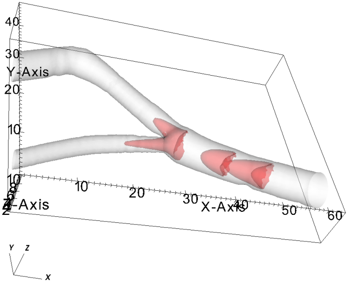
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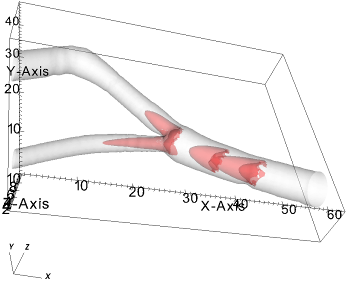
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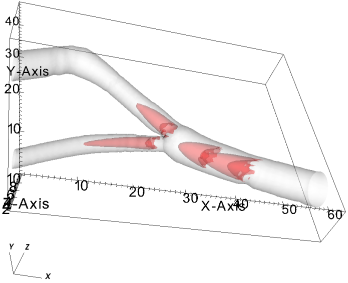
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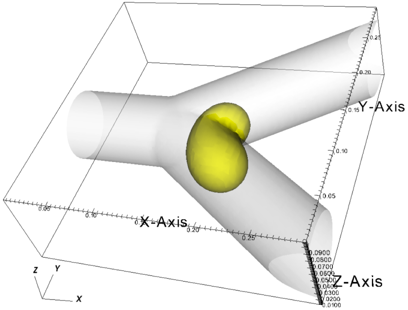
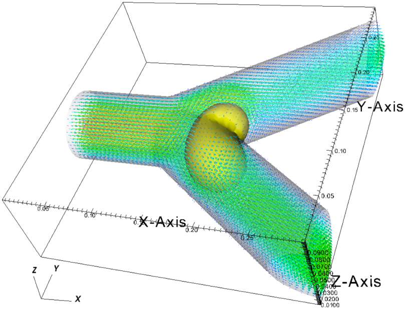
Examples

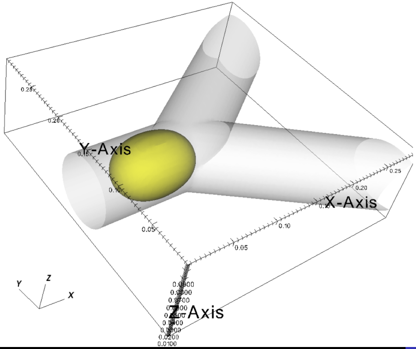
Asymetric flow in "Y"

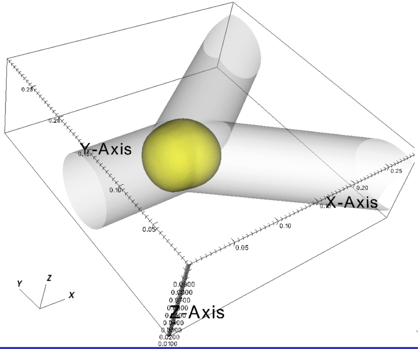




Examples

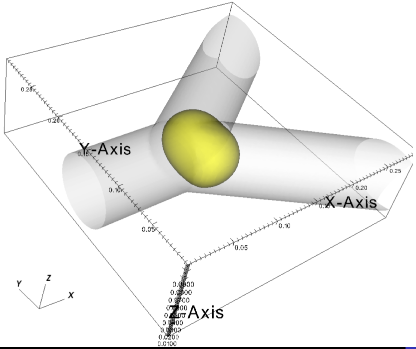
Symetric flow in "Y"

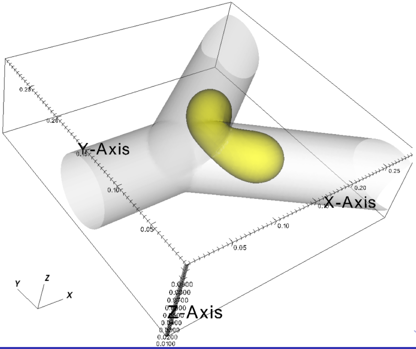




Examples

Symetric flow in "Y"





Conclusion

- Skeleton and Level Set for Geometries:
  A simple tool for complex geometries,
  no mesh to fit the geometries
- Level Set for fluid interfaces:
  simple and efficient
  thiner grid for Level Set than for flow (very low Reynolds)
- A "user friendly" interactive imagery software for geometry generation and flow informations

Improvement

AMR

- An adaptative mesh refinement to reduce computation cost
- A new fluid solver adapted to AMR
  (being developped with DDFV scheme)
- DDFV schemes (F. Hubert et al., F. Boyer...):
  - allow nonconforming meshes
  - verify a discrete variationnal formulation
  - verify exact discrete Green formula $(\nabla \cdot u, p)_{L^2} = -(u, \nabla p)_{L^2}$
  - MAC scheme generalization
  -verify flux continuity (good for viscosity discontinuity)

Modified boundary conditions...$-\psi \nabla u \cdot \nabla \psi + u = 0$ on walls...

Improvement

Vessel reconstruction

- Extend Vessel reconstruction to really 3D geometries
- use normal cuts of vessel on medical images
- connect the 3D Skeleton between normal cuts
- Skeleton defined by a Monge-Kantorovich problem
- Developpement of an imagery software (user friendly)

Non Newtonian flows...