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Algebra homework 1
Set theory, equivalence relations

Due September 18th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Let $f : \mathbf{R} \to \mathbf{R}$ be the map $f : x \mapsto (x+1)^2$. Compute the inverse image sets $f^{-1}(A)$ of the following sets A:

(a) $\{-9\}$,

(b) $\{-1, 0, 4\}$,

(c) $[0, +\infty) = \{x \in \mathbf{R} : \ x \geq 0\}$.

Exercise 2. Let $f : X \to Y$ be a map between sets.

1. For any two subsets A, B of Y, show that
$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \quad \text{and} \quad f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B).$$
2. For any two subsets A, B of X, show that
$$f(A) \cup f(B) = f(A \cup B).$$
3. (a) Show that in general
$$f(A) \cap f(B) \neq f(A \cap B) \tag{1}$$
by giving a counterexample. (Hint: draw a picture)

   (b) Show that we do get equality in (1) if we furthermore assume that f is injective.

Exercise 3. Let $f : X \to Y$ and $g : Y \to Z$ be maps between sets.

1. Show that if $g \circ f$ is injective, then f is injective.
2. Show that if $g \circ f$ is surjective, then g is surjective.

Exercise 4. For an element $x = (x_1, x_2)$ of the plane $\mathbf{R}^2$, we denote by $||x|| = \sqrt{x_1^2 + x_2^2}$ its Euclidean norm. Let $\sim$ be the relation on the plane $\mathbf{R}^2$ given by
$$x \sim y \quad \text{if} \quad ||x|| = ||y||.$$
Show that $\sim$ is an equivalence relation and describe its equivalence classes.

Exercise 5. We define a relation R on $\mathbf{Z}$ by $a\,R\,b$ if a divides $2b$.

1. Is R reflexive?
2. Is it symmetric?
3. Is it transitive?