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Algebra homework 10
Index, Lagrange's theorem, normal subgroups

Due December 4th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Compute the indexes of the following subgroups H_i of the following groups G_i. Which of these subgroups H_i are normal subgroups of G_i?

1. $H_1 = \langle 3 \rangle$ (subgroup generated by 3) in $G_1 = \mathbf{Z}/81\mathbf{Z}$.
2. $H_2 = 23\mathbf{Z}$ in $G_2 = \mathbf{Z}$.
3. $H_3 = \{\mathrm{id}, (1,2,3), (1,3,2)\}$ in $G_3 = \mathfrak{S}_3$.
4. $H_4 = \{\mathrm{id}, (1,3)\}$ in $G_4 = \mathfrak{S}_3$.

Exercise 2. Let $f : \mathbf{Z}/9\mathbf{Z} \to \mathbf{Z}/9\mathbf{Z}$ given by $f(x) = 3x$.

1. Prove that f is a group homomorphism.
2. Compute $\operatorname{Ker} f$ and $\operatorname{Im} f$.
3. Check that $[\mathbf{Z}/9\mathbf{Z} : \operatorname{Ker} f] = |\operatorname{Im} f|$.

Exercise 3.

1. Give a list of all the subgroups of $\mathbf{Z}/14\mathbf{Z}$ together with their orders.
2. Check that
$$14 = \sum_{d|14} \phi(d)$$
where ϕ is Euler's function.

Exercise 4. Let $\phi : G \to G'$ be a group homomorphism. Assume that G is of order 18, G' is of order 15 and that ϕ is not the trivial homomorphism. What is the order of $\operatorname{Ker} \phi$?

Exercise 5.

1. Find an integer x such that $x^2 \equiv -1 \pmod 5$.
2. Find an integer x such that $x^2 \equiv -1 \pmod{13}$.
3. Let p be a prime congruent to 3 modulo 4. Show that there is no solution to the equation $x^2 \equiv -1 \pmod p$.

Exercise 6. For every integer $n \geq 0$, show that 13 divides $11^{12n+6} + 1$.

Exercise 7. Find the remainder of 11^{1213} in the Euclidean division by 26.

Exercise 8. Let G be a group, and H, K normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G.