Margaret Bilu
Office 604
bilu@cims.nyu.edu

Algebra homework 11
Normal subgroups, quotients, isomorphism theorems, classification of finite abelian groups

Exercise 1. Let G be a group and H a normal subgroup of G. Show that

1. If G is abelian, then G/H is abelian.
2. If G is cyclic then G/H is cyclic.

Exercise 2. Let G be a group and H a normal subgroup of G such that H and G/H are abelian. Is it true that G is abelian?

Exercise 3. A group G is said to be *simple* if it has no proper normal subgroups.

1. Give two examples of simple groups.
2. What can you say about abelian simple groups?
3. For what values of n is $\mathfrak{S}_n$ simple?
4. Show that the set
$$H = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$
is a subgroup of $\mathfrak{A}_4$. Conclude that $\mathfrak{A}_4$ is not simple.

Exercise 4. Let G be a group and let $K \subset H$ be normal subgroups of G.

1. Show that K is a normal subgroup of H.
2. Show that H/K is a subgroup of G/K.
3. Let $a, b \in G$ be such that $aK = bK$. Show that then $aH = bH$.
4. The previous question shows that the map $\phi : G/K \to G/H$ sending a coset aK to the coset aH is well-defined. Show that it is a group homomorphism and determine its kernel and image.
5. Conclude that
$$(G/K)/(H/K) \simeq G/H.$$
(The group on the left-hand side is the quotient of G/K by H/K). This is the *Third Isomorphism Theorem.*

Exercise 5.

1. How many abelian groups of order 36 are there up to isomorphism?
2. Let p be a prime. How many abelian groups of order p^5 are there up to isomorphism?