Margaret Bilu
Office 604

bilu@cims.nyu.edu

Algebra homework 11
Normal subgroups, quotients, isomorphism theorems, classification of finite abelian groups

Exercise 1. Let G be a group and H a normal subgroup of G. Show that

1. If G is abelian, then G/H is abelian.

   Solution. Let aH and bH be two elements of G/H, for $a, b \in G$. Then by commutativity of G,

$$(aH)(bH) = abH = baH = (bH)(aH).$$

2. If G is cyclic then G/H is cyclic.

   Solution. Let g be a generator of G. We claim that the coset gH is a generator of G/H. Indeed, let aH be some coset, for $a \in G$. Since g is a generator, there exists $n \in \mathbf{Z}$ such that $a = g^n$. Then

$$aH = g^n H = (gH)^n.$$

Exercise 2. Let G be a group and H a normal subgroup of G such that H and G/H are abelian. Is it true that G is abelian?

Solution. No, this is not true. Take e.g. $G = \mathfrak{S}_3$, and H generated by a cycle of length 3. Then H is cyclic of order 3 so abelian and G/H is of order $6/3 = 2$, and therefore abelian. However, G is not abelian.

Exercise 3. A group G is said to be *simple* if it has no proper normal subgroups.

1. Give two examples of simple groups.

   Solution. The groups $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Z}/3\mathbf{Z}$ are simple.

2. What can you say about abelian simple groups?

   Solution. Since every subgroup of an abelian group is normal, a simple abelian group is an abelian group which has no proper subgroups. From the lectures, we know that such groups are of the form $\mathbf{Z}/p\mathbf{Z}$ for p a prime number.

3. For what values of n is $\mathfrak{S}_n$ simple?

   Solution. We know that $\mathfrak{S}_n$ always has a normal subgroup, namely $\mathfrak{A}_n$. Moreover, for $n \geq 3$, $\mathfrak{A}_n$ is non-trivial since it contains at least the cycles of length 3. Thus, $\mathfrak{S}_n$ is not simple for $n \geq 3$. On the other hand, $\mathfrak{S}_1 = \{\mathrm{id}\}$ and $\mathfrak{S}_2 = \{\mathrm{id}, (12)\} \simeq \mathbf{Z}/2\mathbf{Z}$ are simple.

4. Show that the set

$$H = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$$

   is a subgroup of $\mathfrak{A}_4$. Conclude that $\mathfrak{A}_4$ is not simple.

   Solution. All of the elements of H are even permutations, so H is a subset of $\mathfrak{A}_n$. It contains the identity, and we can check that all of the elements of H are their own inverses (all of the elements other than the identity are of order 2). Finally, we may check closure by computing the products of the elements other than the identity: you find that the product of any to elements other than the identity is equal to the third one:

$$(12)(34) \cdot (13)(24) = (14)(23)$$

   etc.

Thus, H is a subgroup of $\mathfrak{A}_4$. To disprove that $\mathfrak{A}_4$ is simple, we will show that H is in fact a normal subgroup of $\mathfrak{A}_4$. Let $\sigma \in \mathfrak{A}_4$ be a permutation. Then we must check that for all $h \in H$, $\sigma h \sigma^{-1} \in H$.

Note that

$$\sigma(12)(34)\sigma^{-1} = \sigma(12)\sigma^{-1}\sigma(34)\sigma^{-1} = (\sigma(1), \sigma(2))(\sigma(3), \sigma(4))$$

is also a product of two transpositions, and therefore is an element of H. Thus, H is a normal subgroup of $\mathfrak{A}_4$ and $\mathfrak{A}_4$ is not simple.

Remark: In fact, the smallest non-abelian simple group is $\mathfrak{A}_5$. which is of order 60. You can find a proof of the fact that $\mathfrak{A}_5$ is simple in Judson, section 10.2.

Exercise 4. Let G be a group and let $K \subset H$ be normal subgroups of G.

1. Show that K is a normal subgroup of H.

   Solution. Let $h \in H$ and $k \in K$. Then $hkh^{-1} \in K$ because K is a normal subgroup of G.

2. Show that H/K is a subgroup of G/K.

   Solution. The elements of H/K are cosets of the form aK where $a \in H$. Thus, H/K is a subset of G/K. Moreover, we check:

   Closure: For all $a, b \in H$, $(aK)(bK) = abK$ is an element of H/K by closure of H in G.

   Identity: We have $e \in H$ so the coset K is an element of H/K.

   Inverses: Let $a \in H$. Then the inverse of aK is $a^{-1}K$, which is in H/K because H is stable under taking inverses.

3. Let $a, b \in G$ be such that $aK = bK$. Show that then $aH = bH$.

   Solution. We have $b^{-1}a \in K$, and since $K \subset H$, this implies $b^{-1}a \in H$, so $aH = bH$.

4. The previous question shows that the map $\phi : G/K \to G/H$ sending a coset aK to the coset aH is well-defined. Show that it is a group homomorphism and determine its kernel and image.

   Solution. For all $a, b \in G$, we have

   $$\phi(aK)\phi(bK) = (aH)(bH) = abH = \phi(abK),$$

   so ϕ is a group homomorphism. We have $\phi(aK) = H$ if and only if $aH = H$, that is, if and only if $a \in H$. Thus, the kernel of ϕ is given by $\{aK, a \in H\} = H/K$.

   On the other hand, ϕ is surjective, since a coset aH is the image of the coset aK. Thus, $\mathrm{Im}\phi = G/H$.

5. Conclude that

   $$(G/K)/(H/K) \simeq G/H.$$

   (The group on the left-hand side is the quotient of G/K by H/K). This is the *Third Isomorphism Theorem.*

   Solution. We apply the First Isomorphism Theorem to the homomorphism ϕ.

Exercise 5. 1. How many abelian groups of order 36 are there up to isomorphism?

Solution. The only ways of writing 36 as a product of prime powers are

$$36 = 2^2 \times 3^2 = 2 \times 2 \times 3^2 = 2^2 \times 3 \times 3 = 2 \times 2 \times 3 \times 3.$$

Thus, up to isomorphism there are 4 abelian groups of order 36, given by

$$\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/9\mathbf{Z}, \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/9\mathbf{Z}, \quad \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z},$$

$$\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}.$$

2. Let p be a prime. How many abelian groups of order p^5 are there up to isomorphism?

Solution. We may write

$$p^5 = p \times p^4 = p \times p \times p^3 = p \times p^2 \times p^2 = p \times p \times p \times p^2 = p \times p \times p \times p \times p = p^2 \times p^3$$

in seven ways as a product of prime powers. Thus, there are 7 abelian groups of order p^5 up to isomorphism.