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Algebra homework 1
Set theory, equivalence relations

Due September 18th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Let $f : \mathbf{R} \to \mathbf{R}$ be the map $f : x \mapsto (x+1)^2$. Compute the inverse image sets $f^{-1}(A)$ of the following sets A:

(a) $\{-9\}$,

Solution. This is empty, as -9 is never a square.

(b) $\{-1, 0, 4\}$,

Solution. We have

$$\begin{aligned} f^{-1}(\{-1,0,4\}) &= \{x \in \mathbf{R} : \ (x+1)^2 = -1, 0 \text{ or } 4\} \\ &= \{x \in \mathbf{R} : \ x+1 = 0, 2 \text{ or } -2\} \\ &= \{-1, 1, -3\}. \end{aligned}$$

(c) $[0, +\infty) = \{x \in \mathbf{R} : \ x \geq 0\}$.

Solution. $f^{-1}([0,+\infty)) = \{x \in \mathbf{R}, \ (x+1)^2 \geq 0\} = \mathbf{R}$.

Exercise 2. Let $f : X \to Y$ be a map between sets.

1. For any two subsets A, B of Y, show that

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \quad \text{and} \quad f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B).$$

Solution.

$$\begin{aligned} f^{-1}(A) \cup f^{-1}(B) &= \{x \in X \text{ such that } x \in f^{-1}(A) \text{ or } x \in f^{-1}(B)\} \\ &= \{x \in X \text{ such that } f(x) \in A \text{ or } f(x) \in B\} \\ &= \{x \in X \text{ such that } f(x) \in A \cup B\} \\ &= f^{-1}(A \cup B). \end{aligned}$$

In exactly the same way,

$$\begin{aligned} f^{-1}(A) \cap f^{-1}(B) &= \{x \in X \text{ such that } x \in f^{-1}(A) \text{ and } x \in f^{-1}(B)\} \\ &= \{x \in X \text{ such that } f(x) \in A \text{ and } f(x) \in B\} \\ &= \{x \in X \text{ such that } f(x) \in A \cap B\} \\ &= f^{-1}(A \cap B). \end{aligned}$$

2. For any two subsets A, B of X, show that

$$f(A) \cup f(B) = f(A \cup B).$$

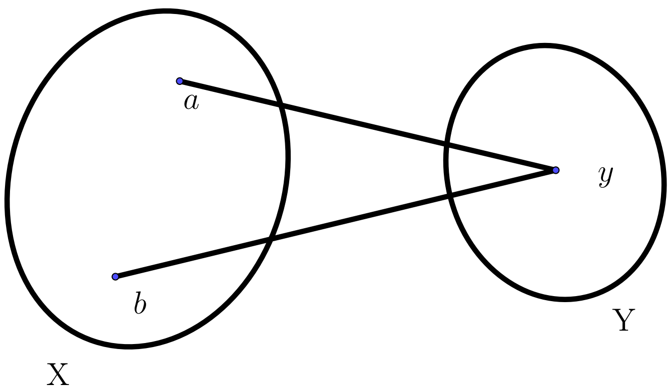
Solution. Let $y \in Y$. We have $y \in f(A) \cup f(B)$ if and only if y is of the form $f(x)$ where $x \in A$ or $x \in B$. This is the case if and only if $y = f(x)$ with $x \in A \cup B$, that is, if and only if $y \in f(A \cup B)$, whence the result.

3. (a) Show that in general

$$f(A) \cap f(B) \neq f(A \cap B) \quad (1)$$

by giving a counterexample. (Hint: draw a picture)

Solution. The following picture



shows that if $X = \{a, b\}$ is a set with two elements, $Y = \{y\}$ is a singleton (that is, a set with one element), and f is defined to be the constant map, we have, putting $A = \{a\}$ and $B = \{b\}$, that

$$f(A) \cap f(B) = \{y\}$$

whereas

$$f(A \cap B) = f(\varnothing) = \varnothing.$$

Note that the inclusion

$$f(A) \cap f(B) \supseteq f(A \cap B)$$

is nevertheless always true. Indeed, if $y \in f(A \cap B)$, then we can write $y = f(x)$ with $x \in A \cap B$ (that is, $x \in A$ and $x \in B$), which means in particular that $y \in f(A)$ and $y \in f(B)$, that is, $y \in f(A) \cap f(B)$.

(b) Show that we do get equality in (1) if we furthermore assume that f is injective.

Solution. The answer to the previous question illustrates the fact that non-injectivity makes the equality go wrong. Assume that f is injective. We already know that

$$f(A) \cap f(B) \supseteq f(A \cap B)$$

so it suffices to prove that

$$f(A) \cap f(B) \subseteq f(A \cap B).$$

If $y \in f(A) \cap f(B)$ then $y \in f(A)$ and $y \in f(B)$, that is, $y = f(a) = f(b)$ for some $a \in A$ and some $b \in B$. Since f is injective, we have $a = b$. Thus, $a \in B$, and so $a \in A \cap B$, which implies that $y \in f(A \cap B)$.

Exercise 3. Let $f : X \to Y$ and $g : Y \to Z$ be maps between sets.

1. Show that if $g \circ f$ is injective, then f is injective.

   Solution. Assume that $g \circ f$ is injective. Let $x, y \in X$ be such that $f(x) = f(y)$. Apply g to both sides of the equation, to get $g(f(x)) = g(f(y))$. By injectivity of $g \circ f$, we then get $x = y$. This proves f is injective.

2. Show that if $g \circ f$ is surjective, then g is surjective.

   Solution. Assume $g \circ f$ is surjective. Let $z \in Z$. By surjectivity of $g \circ f$, we have an element $x \in X$ such that $g(f(x)) = z$. Then $y = f(x)$ gives us an element of Y such that $g(y) = z$, so g is surjective.

Exercise 4. For an element $x = (x_1, x_2)$ of the plane $\mathbf{R}^2$, we denote by $||x|| = \sqrt{x_1^2 + x_2^2}$ its Euclidean norm. Let $\sim$ be the relation on the plane $\mathbf{R}^2$ given by

$$x \sim y \quad \text{if} \quad ||x|| = ||y||.$$

Show that $\sim$ is an equivalence relation and describe its equivalence classes.

Solution. For any $x \in \mathbf{R}^2$, we have $||x|| = ||x||$, so $x \sim x$, so $\sim$ is reflexive. For any $x, y \in \mathbf{R}^2$, if we have $||x|| = ||y||$ then we have $||y|| = ||x||$, and therefore $x \sim y$ implies $y \sim x$, which means that $\sim$ is symmetric. Finally, for any $x, y, z \in \mathbf{R}^2$, if $x \sim y$ and $y \sim z$, then we have $||x|| = ||y||$ and $||y|| = ||z||$, and therefore $||x|| = ||z||$, that is, $x \sim z$, so that $\sim$ is transitive.
Let $x \in \mathbf{R}^2$, and put $r = ||x||$. Then the equivalence class of x is the set

$$\{y \in \mathbf{R}^2;\ ||y|| = r\}$$

of all elements with norm r. If $r > 0$ this is the circle C_r of radius r centered in the origin. For $r = 0$, the only point of norm zero is the origin, so the corresponding equivalence class is just the singleton $\{(0, 0)\}$.

Exercise 5. We define a relation R on $\mathbf{Z}$ by $a\,R\,b$ if a divides $2b$.

1. Is R reflexive?

   Solution. Yes, since for every $a \in \mathbf{Z}$, we do have $a|2a$.

2. Is it symmetric?

   Solution. No: we have $2\,R\,8$, but we de not have $8\,R\,2$.

3. Is it transitive?

   Solution. If the relation were transitive, we would have that if a divides $2b$ and b divides $2c$, then a divides $2c$. Intuitively, this seems wrong as the assumption should a priori just imply that a divides $4c$, not $2c$. Let us build a counterexample based on this intuition, by trying c to be as small as possible while a is divisible by 4, so that we really need the factor 4. Thus, $a = 4$, $b = 2$, $c = 1$ gives us a counterexample.