Margaret Bilu
Office 604

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bilu@cims.nyu.edu

Algebra homework 2
Arithmetic on the set of integers

Due September 25th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Prove the following properties:

(a) If a, b, c are integers such that $a|b$ and $b|c$ then $a|c$.

(b) If a, b are non-zero integers, then $a|b$ and $b|a$ implies $a = b$ or $a = -b$.

(c) If a, b, c are integers such that $a|b$ and $a|c$, then for all integers $u, v \in \mathbf{Z}$, a divides $ub + vc$.

Exercise 2. Let p be a prime number. Give the list of all the positive divisors of p^2 , then of p^3. More generally, describe, in terms of p and k, the list of positive divisors of p^k for any integer $k \geq 1$.

Exercise 3. For any integers a, b which are not both zero, prove the following properties of the greatest common divisor:

(a) For any non-zero integer k, $\gcd(ka, kb) = |k| \gcd(a, b)$.

(b) If $d = \gcd(a, b)$, then there exist relatively prime integers a', b' such that $a = da'$ and $b = db'$.

(c) $\gcd(a, b) = \gcd(a + b, b)$.

(d) $\gcd(a, a + 1) = 1$.

(e) For any integer $k \geq 1$, $\gcd(a, a + k)$ divides k.

Exercise 4.

1. Compute $\gcd(201, 694)$.
2. Find integers u and v such that $694u + 201v = \gcd(201, 694)$.

Exercise 5. Recall that for a set A, we denote by $|A|$ the number of its elements. The *Euler function* $\phi : \mathbf{N} \to \mathbf{N}$ is the function defined for every positive integer n by

$$\phi(n) = |\{k \in \{1, \ldots, n\},\ k \text{ relatively prime to } n\}|.$$

1. What is the value of $\phi(p)$ for a prime number p ?
2. Compute $\phi(n)$ for all integers n in the set $\{1, 2, \ldots, 12\}$.

Exercise 6.

1. Let n be an integer, and a, b non-zero relatively prime integers. Show that if both a and b divide n, then the product ab divides n. (Hint: Bézout's theorem)
2. Does this remain true if a and b are no longer assumed to be relatively prime?