Margaret Bilu
Office 604

Fall 2019

bilu@cims.nyu.edu

Algebra homework 4
Laws of composition, groups, subgroups

Due October 9th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. We define a law of composition $*$ on $\mathbf{R}$ by $x * y = x^2 + y^2$.

(a) Is it associative?

(b) Is it commutative?

(c) Does it have an identity?

Exercise 2. We consider the set $\mathcal{F}(\mathbf{R}, \mathbf{R})$ of functions from $\mathbf{R}$ to $\mathbf{R}$. We saw in lectures that composition of functions $\circ$ is an associative law of composition on this set, with identity the function $\mathrm{id} : \mathbf{R} \to \mathbf{R}$ defined by $\mathrm{id}(x) = x$. For the following elements of $\mathcal{F}(\mathbf{R}, \mathbf{R})$, determine if they have an inverse for $\circ$, and if yes, give it.

(a) The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 5x + 2$.

(b) The function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^2 - 3$.

(c) The function $h : \mathbf{R} \to \mathbf{R}$ given by $h(x) = 2e^x$.

Exercise 3. Let G be a group. Show that

$$Z(G) = \{x \in G,\ xg = gx \text{ for all } g \in G\}$$

is a subgroup of G. It is called the *center* of G.

Exercise 4. Let G be a group such that for every $x \in G$ we have $x^2 = e$, where e is the identity element of g. Show that G must be abelian.

Exercise 5. Prove that

$$G = \{a + b\sqrt{3},\quad a, b \in \mathbf{Q},\ a, b \text{ not both zero}\}$$

is a subgroup of the group $(\mathbf{R}^\times, \cdot)$.

Exercise 6.

1. Let $O_n(\mathbf{R})$ be the set of matrices $A \in M_n(\mathbf{R})$ satisfying ${}^tAA = I_n$, where ${}^tA$ denotes the transpose of A and I_n denotes the identity matrix. Show that any $A \in O_n(\mathbf{R})$ is an invertible matrix.

2. Show that $O_n(\mathbf{R})$ is a subgroup of $(GL_n(\mathbf{R}), \cdot)$.