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Algebra homework 5
Product groups, cyclic groups

Due October 16th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Describe the following groups:

(a) The subgroup of $(\mathbf{Z}, +)$ generated by 6.

(b) The subgroup of $(M_n(\mathbf{R}), +)$ generated by $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(c) The subgroup of $\mathbf{Z}/12\mathbf{Z}$ generated by 3.

(d) The subgroup of $\mathbf{Z}/14\mathbf{Z}$ generated by 5.

(e) The subgroup of $(\mathbf{C}^*, \cdot)$ generated by i.

(f) The subgroup of $(GL_2(\mathbf{R}), \cdot)$ generated by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Exercise 2.

1. Compute the orders of the elements of the group $((\mathbf{Z}/8\mathbf{Z})^\times, \cdot)$. Is it cyclic?
2. Show that the group $((\mathbf{Z}/7\mathbf{Z})^\times, \cdot)$ is cyclic by finding a generator.

Exercise 3. Show that the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is an element of infinite order of the group $(GL_2(\mathbf{R}), \cdot)$ of invertible 2×2 matrices with real coefficients. Give a formula for A^n in terms of $n \in \mathbf{Z}$.

Exercise 4.

1. Let r, s be positive integers. Let x be an element of order r in a group G and y an element of order s in a group H. Show that the order of the element (x, y) in the group $G \times H$ is the least common multiple $\operatorname{lcm}(r, s)$ of r and s, that is, the smallest positive integer divisible both by r and by s.
2. Show that $\mathbf{Z}/5\mathbf{Z} \times \mathbf{Z}/7\mathbf{Z}$ is cyclic. Give an explicit generator.
3. More generally, show that if $m, n \geq 2$ are relatively prime integers, then $\mathbf{Z}/m\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$ is cyclic.

Exercise 5. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

be elements of $GL_2(\mathbf{R})$. Show that A and B have finite orders but that their product AB has infinite order.