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Algebra homework 6
Homomorphisms, isomorphisms

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Please hand in your homework stapled, with your name written on it. All answers have to be justified.

Exercise 1. Show that the following maps are group homomorphisms and compute their kernels.

(a) $f : (\mathbf{R}^\times, \cdot) \to (GL_2(\mathbf{R}), \cdot)$ given by

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}.$$

Solution. For $x, y \in \mathbf{R},\ x, y \neq 0$, we have:

$$f(x) \cdot f(y) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & xy \end{pmatrix} = f(xy)$$

Therefore, f is a homomorphism. Moreover,

$$\text{Ker}(f) = \left\{ x \in \mathbf{R}^\times, f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{1\}.$$

(b) $g : (\mathbf{R}, +) \to (GL_2(\mathbf{R}), \cdot)$ given by

$$g(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Solution.

For $x, y \in \mathbf{R}$, we have:

$$g(x) \cdot g(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + y \\ 0 & 1 \end{pmatrix} = g(x + y)$$

Therefore, f is a homomorphism. Moreover,

$$\text{Ker}(g) = \left\{ x \in \mathbf{R}^\times, g(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{0\}.$$

(c) $h : (\mathbf{R}^2, +) \to (\mathbf{R}, +)$ given by $h(x, y) = y$.

Solution.

For $x, y, x', y' \in \mathbf{R}$, we have:

$$h(x, y) + h(x', y') = y + y' = h(x + x', y + y') = h((x + x') + (y, y'))$$

Therefore, h is a homomorphism. Its kernel is

$$\text{Ker}(h) = \{(x, y), h(x, y) = 0\} = \{(x, 0), x \in \mathbf{R}\} = \mathbf{R} \times \{0\}.$$

(d) The complex conjugation map $j : (\mathbf{C}, +) \to (\mathbf{C}, +)$, given by $j(x + iy) = x - iy$. *Solution.* For $x, y, x', y' \in \mathbf{R}$, we have:

$$j(x + iy) + j(x' + iy') = x - iy + x' - iy' = x + x' - i(y + y') = j((x + iy) + (x' + iy'))$$

Therefore, j is a homomorphism. Moreover,

$$\text{Ker}(j) = \{x + iy; x - iy = 0\} = \{0\}$$

Exercise 2. Let G and H be two groups. Show that $G \times H$ is isomorphic to $H \times G$.

Solution. You can check that the map $G \times H \to H \times G$ given by $(g, h) \mapsto (h, g)$ is an isomorphism. Indeed, it is easy to check that it is a homomorphism, and it has an inverse given by $H \times G \to G \times H$, $(h, g) \mapsto (g, h)$.

Exercise 3. Let $\phi : G \to H$ be a group homomorphism.

1. Show that if G is abelian, then $\text{Im}(\phi)$ is also abelian.

   Solution.

   Assume that G is abelian. Let $x, y \in \text{Im}(\phi)$. By definition, there exist $a, b \in G$ such that $x = \phi(a)$, $y = \phi(b)$. Then

   $$xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx,$$

   where we used that fact that ϕ is a homomorphism and commutativity of G. Therefore $\text{Im}(\phi)$ is abelian.

2. Show that if G is cyclic, then $\text{Im}(\phi)$ is also cyclic.

   Solution.

   Let us suppose that G is cyclic, and denote by g a generator of G. We will show that $\text{Im}(\phi)$ is also cyclic, with generator $\phi(g)$. Let $x \in \text{Im}(\phi)$. There exists $a \in G$ such that $x = \phi(a)$. Because G is cyclic with generator g, there exists $m \in \mathbf{Z}$ such that $a = g^m$. Then: $x = \phi(a) = \phi(g^m) = \phi(g)^m$. Therefore $\text{Im}(\phi)$ is cyclic, generated by $\phi(g)$.

Exercise 4. Let T denote the group of invertible upper triangular 2×2 matrices

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad a, b, d \in \mathbf{R}, \ ad \neq 0.$$

1. Show that T is a subgroup of $GL_2(\mathbf{R})$.

   Solution.

   First of all, we check that T is indeed a subset of $GL_2(\mathbf{R})$. For $A \in T$, $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, we have $\det(A) = ad \neq 0$ by assumption. Therefore $A \in GL_2(\mathbf{R})$. So $T \subset GL_2(\mathbf{R})$.

   We now check closure of T. For $a, b, d, e, g, h \in \mathbf{R}$, $ad \neq 0$, $eh \neq 0$, we have:

   $$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} e & g \\ 0 & h \end{pmatrix} = \begin{pmatrix} ae & ag + bh \\ 0 & dh \end{pmatrix} \in T$$

   since $ae \cdot dh = ad \cdot eh \neq 0$. Thus, the product of two elements of $T$is an element of T.

Taking $a, d = 1$ and $b = 0$, we see that $I_2 \in T$.

Finally, T is stable under taking inverses, since for $a, b, d \in \mathbf{R}$, $ad \neq 0$,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & \frac{-db}{a} \\ 0 & d^{-1} \end{pmatrix} \in T$$

2. Let $\phi : T \to \mathbf{R}^\times$ be the map given by sending a matrix A as above to a^2. Show that ϕ is a homomorphism, and give its kernel and image.

   Solution.

   For $a, b, d, e, g, h \in \mathbf{R}$, $ad \neq 0$, $eh \neq 0$, we have:

$$\phi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\phi\begin{pmatrix} e & g \\ 0 & h \end{pmatrix} = a^2e^2 = (ae)^2 = \phi\begin{pmatrix} ae & ag+bh \\ 0 & dh \end{pmatrix} = \phi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} e & g \\ 0 & h \end{pmatrix}\right)$$

   Therefore, ϕ is a homomorphism.

   We have:

$$\mathrm{Ker}(\phi) = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a^2 = 1\right\} = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}; a, b, d \in \mathbf{R}, a \in \{1, -1\}, d \neq 0\right\}$$

   Moreover,

$$\mathrm{Im}(\phi) = \phi\left(\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbf{R}, ad \neq 0\right\}\right) = \left\{a^2 \mid a \neq 0\right\} = \mathbf{R}_{>0}$$

   since every positive real number can be expressed as the square of a non-zero real number.

Exercise 5. Check that the group $(\mathbf{Z}/8\mathbf{Z})^\times$ is of order 4. Is it isomorphic to $\mathbf{Z}/4\mathbf{Z}$? If not, find another group of order 4 it is isomorphic to.

Solution. We have: $(\mathbf{Z}/8\mathbf{Z})^\times = \{1, 3, 5, 7\}$, so it is indeed of order 4. One can check that $3, 5, 7$ are all of order 2, so there is no element of order 4. In particular, this group is not cyclic, and cannot be isomorphic to $\mathbf{Z}/4\mathbf{Z}$. By the lecture notes, we know that up to isomorphism, the only other group of order 4 is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, so we must have $(\mathbf{Z}/8\mathbf{Z})^\times$ isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. Note that if you want to construct an explicit isomorphism, it suffices to send 1 to the identity element $(0, 0)$, and the elements of order 2 in $(\mathbf{Z}/8\mathbf{Z})^\times$ bijectively to the elements of order 2 in $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

Exercise 6. 1. Show that for any $a \in \mathbf{Z}$, the map $\phi : \mathbf{Z} \to \mathbf{Z}$ defined by $\phi(n) = an$ is a group homomorphism. Give its kernel and image.

   Solution.

   For $n, n' \in \mathbf{Z}$, $\phi(n) + \phi(n') = an + an' = a(n + n') = \phi(n + n')$. Therefore, ϕ is a homomorphism. If $a = 0$, then ϕ is the trivial homomorphism, with kernel $\mathbf{Z}$ and image $\{0\}$. If $a \neq 0$, then we have $\phi(n) = 0$ iff $an = 0$ iff $n = 0$, so $\mathrm{Ker}(\phi) = \{0\}$. Moreover, in this case $\mathrm{Im}(\phi) = a\mathbf{Z}$.

2. Conversely, show that a homomorphism $\phi : \mathbf{Z} \to \mathbf{Z}$ is of the form $\phi(n) = an$ for some $a \in \mathbf{Z}$. Thus, the homomorphisms $\mathbf{Z} \to \mathbf{Z}$ are exactly the maps $n \mapsto an$.

   Solution.

If ϕ is such a homomorphism, then for any $n > 0$, we have: $\phi(n) = \phi(\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}) = \phi(1)n = an$ (by induction), with $a = \phi(1) \in \mathbf{Z}$. Then, for every $n < 0$, we have that $m = -n > 0$, and so $\phi(n) = \phi(-m) = -\phi(m) = -am = an$. As a conclusion, for all $n \in \mathbf{Z}$, we have $\phi(n) = an$. Therefore, every homomorphism from $\mathbf{Z}$ to $\mathbf{Z}$ can be written $n \mapsto an$, for a certain $a \in \mathbf{Z}$.

3. An *automorphism* of a group G is an isomorphism from G to itself. Determine all the automorphisms of $\mathbf{Z}$.

   Solution. Let $\phi : \mathbf{Z} \to \mathbf{Z}$ be a automorphism. An automorphism of $\mathbf{Z}$ is in particular a homomorphism from $\mathbf{Z}$ to $\mathbf{Z}$. As a consequence, by the previous question, there exists $a \in \mathbf{Z}$ such that ϕ is given by $n \mapsto an$. Since ϕ is a automorphism, $\mathrm{Im}(\phi) = \mathbf{Z}$. On the other hand, we have seen that $\mathrm{Im}(n \mapsto an) = a\mathbf{Z}$. Therefore $a\mathbf{Z} = \mathbf{Z}$, which implies $a = \pm 1$. Conversely, we can check that $n \mapsto n$ and $n \mapsto -n$ are automorphisms of $\mathbf{Z}$. Therefore, all the automorphisms of $\mathbf{Z}$ are $n \mapsto n$ and $n \mapsto -n$.