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Algebra homework 7
Permutations

Due November 6th, 2019

Please hand in your homework stapled, with your name written on it. All answers have to be justified.
For every $n \geq 1$ we denote by $\mathfrak{S}_n$ the n-th symmetric group.

Exercise 1. Consider the following permutations:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 2 & 1 & 4 & 3 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 3 & 7 & 6 & 1 & 8 & 4 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 6 & 1 & 4 & 5 & 2 \end{pmatrix}.$$

1. Compute $\sigma_1\sigma_3$.

   Solution. Using the fact that for every k, $\sigma_1\sigma_3(k) = \sigma_1(\sigma_3(k))$, you should obtain $\sigma_1\sigma 3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 4 & 5 & 2 & 1 & 6 \end{pmatrix}$

2. Both σ_1 and σ_3 can be seen as elements of $\mathfrak{S}_8$ by putting $\sigma_1(8) = \sigma_3(8) = 8$. Compute the products $\sigma_1\sigma_2$ and $\sigma_2\sigma_3$ in $\mathfrak{S}_8$.

   Solution. You get

   $\sigma_1\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 7 & 3 & 4 & 5 & 8 & 2 \end{pmatrix}$

   and:

   $\sigma_2\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 1 & 2 & 7 & 6 & 5 & 4 \end{pmatrix}$

3. Decompose σ_1, σ_2 and σ_3 into products of disjoint cycles, and then write each of them as a product of transpositions.

   Solution. We have $\sigma_1 = (1, 5)(2, 6, 4)(3, 7) = (1, 5)(2, 6)(6, 4)(3, 7)$.

   $\sigma_2 = (1, 2, 5, 6)(4, 7, 8) = (1, 2)(2, 5)(5, 6)(4, 7)(7, 8)$.

   $\sigma_3 = (1, 7, 2, 3, 6, 5, 4) = (1, 7)(7, 2)(2, 3)(3, 6)(6, 5)(5, 4)$.

Exercise 2. Write the following permutations as products of disjoint cycles:

1. $(2, 3, 7, 5)(2, 6, 1)$

   Solution. $(1, 3, 7, 5, 2, 6)$

2. $(1, 5, 2, 6)(2, 3)(5, 7)(1, 3, 4)$

   Solution. $(1, 6)(2, 3, 4, 5, 7)$

3. $(2, 5, 4)^{122}$

   Solution. A cycle of length 3 is of order 3, so $(2, 5, 4)^{122} = (2, 5, 4)^{120}(2, 5, 4)^2 = (2, 5, 4)^2 = (2, 4, 5)$

4. $(1, 2, 3, 4)^{-1}$

   Solution. $(4, 3, 2, 1)$

Exercise 3. Let $a_1, \dots, a_k$ be distinct elements of $\{1, \dots, n\}$. Compute the inverse in $\mathfrak{S}_n$ of the cycle $(a_1, \dots, a_k)$.

Exercise 4. Compute the sets

$$E = \{\sigma \in \mathfrak{S}_4,\ \sigma(1) = 4\}$$

and

$$F = \{\sigma \in \mathfrak{S}_4,\ \sigma(3) = 3\}.$$

Are they subgroups of $\mathfrak{S}_4$?

Solution. $\mathfrak{S}_4$ is composed of the following permutations (decomposed into products of disjoint cycles):

- The identity:
  id
- 2-cycles:
  (12), (13), (14), (23), (24), (34)
- Products of disjoint 2-cycles:
  (12)(34), (13)(24), (14)(23)
- 3-cycles:
  (123), (124), (132), (134), (142), (143), (234), (243)
- 4-cycles:
  (1234), (1243), (1324), (1342), (1423), (1432)

Therefore:

$$E = \{(14), (14)(23), (142), (143), (1423), (1432)\}$$

$$F = \{\text{id}, (12), (14), (24), (124), (142)\}$$

E can't be a subgroup of $\mathfrak{S}_4$ since it doesn't contain the identity. F is a subgroup of $\mathfrak{S}_4$. Indeed, it's a subset of $\mathfrak{S}_4$ containing the identity. It is closed under composition since if $\sigma(3) = 3$ and $\sigma'(3) = 3$, then $\sigma\sigma'(3) = 3$. It is stable by inversion since if $\sigma(3) = 3$, then we also have that $\sigma^{-1}(3) = 3$.
Remark: In fact, F is isomorphic to $\mathfrak{S}_3$, the isomorphism being given by replacing 4 by 3 in all permutations involving 4: e.g. $(14) \mapsto (13)$, $(24) \mapsto (23)$ etc.

Exercise 5. Prove that if σ is a cycle of odd length, then σ^2 is also a cycle. Show that this is not true for cycles of even length by giving a counterexample.

Solution.
Assume that $\sigma = (a_1 a_2 \dots a_{2k+1})$. Then we can check that $\sigma^2 = (a_1 a_3 \dots a_{2k+1} a_2 a_4 \dots a_{2k})$.
A counterexample for a cycle of even length would be (1234), for which $(1234)^2 = (13)(24)$, which is not a cycle. More generally, if $\sigma = (a_1 a_2 \dots a_{2k})$, then $\sigma^2 = (a_1 a_3 \dots a_{2k-1})(a_2 a_4 \dots a_{2k})$.

Exercise 6. Let $\sigma \in \mathfrak{S}_n$.

1. Show that σ can be written as a product of at most $n-1$ transpositions.

   Solution.

   By the theorem seen in lectures, there exist $c_1, \dots, c_k$ disjoint cycles such that $\sigma = c_1 c_2 \dots c_k$.

   Since the cycles are disjoint, the sum of their lengths (n_i for c_i) must be at most n.

   For any cycle of length n_i in this decomposition, we have:

   $$(a_1 a_2 \dots a_{n_i}) = (a_1 a_2)(a_2 a_3) \dots (a_{n_i - 1} a_{n_i}).$$

   This is a decomposition into $n_i - 1$ transpositions.

   At the end of the day, if we perform this on each cycle of the product, σ can be decomposed into a product of $\sum_{i=1}^{k}(n_i - 1) = \sum_{i=1}^{k} n_i - k \leq n - k \leq n - 1$ transpositions.

2. Show that if σ is not a cycle, then σ can be written as a product of at most $n-2$ transpositions.

   Solution.

   If σ is not a cycle then the decomposition of σ into disjoint cycles contains at least two cycles, that is, we have $k \geq 2$ in the previous proof, which shows that σ can be written as a product of at most $n-2$ transpositions.

Exercise 7. 1. Give a list of all possible orders of an element of $\mathfrak{S}_4$.

Solution. An element of $\mathfrak{S}_4$ is either the identity (order 1), or a transposition (order 2), or a product of two transpositions (order 2), or a cycle of length 3 (order 3) or 4 (order 4). Therefore, the possible orders are 1,2,3 or 4.

2. Give a list of all possible orders of an element of $\mathfrak{S}_5$.

   Solution. In the same manner, using decomposition into disjoint cycles, elements of $\mathfrak{S}_5$ are of one of the following forms:

   $$\text{cycle of length 5} \ \ (a, b, c, d, e), \ \text{order 5}$$

   $$\text{cycle of length 4} \ \ (a, b, c, d), \ \text{order 4}$$

   $$\text{cycle of length 3} \ \ (a, b, c), \ \text{order 3}$$

   $$\text{product of a cycle of length 3 and a transposition} \ \ (a, b, c)(d, e), \ \text{order 6}$$

   $$\text{product of two transpositions} \ (a, b)(c, d), \ \text{order 2}$$

   $$\text{transposition} \ (a, b), \ \text{order 2}$$

   $$\text{identity id}, \ \text{order 1}.$$

   Let us check that a product of a cycle of length 3 and of a transposition is indeed of order 6: since disjoint cycles commute, we have

   $[(a, b, c)(d, e)]^2 = (a, b, c)^2(d, e)^2 = (a, c, b)$

   $[(a, b, c)(d, e)]^3 = (a, b, c)^3(d, e)^3 = (d, e)$

   $[(a, b, c)(d, e)]^4 = (a, b, c)^4(d, e)^4 = (a, b, c)$

   $[(a, b, c)(d, e)]^5 = (a, b, c)^5(d, e)^5 = (a, b, c)^2(d, e) = (a, c, b)(d, e)$

   and $[(a, b, c)(d, e)]^6 = (a, b, c)^6(d, e)^6 = \text{id}$